

A new discrete-time robust stability condition [☆]

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Received 6 February 1999; received in revised form 28 March 1999

Abstract

A new robust stability condition for uncertain discrete-time systems with convex polytopic uncertainty is given. It enables to check stability using parameter-dependent Lyapunov functions which are derived from LMI conditions. It is shown that this new condition provides better results than the classical quadratic stability. Besides the use of a parameter-dependent Lyapunov function, this condition exhibits a kind of decoupling between the Lyapunov and the system matrices which may be explored for control synthesis purposes. A numerical example illustrates the results. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Robust stability; Parameter-dependent Lyapunov functions; Linear matrix inequalities

1. Introduction

The domain of robust analysis and robust control synthesis for uncertain linear systems has been thoroughly investigated in the last 15 years. Important results are scattered in the vast literature related to this area which addresses interesting practical topics for control synthesis. These statements are supported by a number of recent books as [4,2,11] and their references, to mention a few.

This paper focuses on the robust analysis problem in an attempt to answer the following question: “Given a compact matrix set, decide whether each element of this set is asymptotically stable in the discrete-time

setting, i.e. each matrix element has its eigenvalues with magnitude less than one, or not”. In fact, in most cases, robust analysis is performed for convex, generally bounded, uncertainty domains.

Among different robust stability tests, we can first quote frequency domain oriented ones, such as μ analysis [5]. Being fairly general, these approaches address real parametric as well as dynamic complex uncertainties. However, from a numerical point of view, they are quite involved and generally make intensive use of scaling factors or multipliers to get precise results. It is also worth to mention some algebraic approaches which followed the seminal work of Kharitonov. Applicable only to the real parametric uncertainty case, they suffer from being difficult to apply to matrix polytopic analysis and being almost of no use for synthesis problems [1]. Finally, the last approach to be mentioned is the Lyapunov one, the class of methods to which this paper belongs. The basic approach is termed “quadratic stability”. In its simplest version, its main drawback comes from the fact that it is based on the use of a *single* Lyapunov

[☆] This work has been supported in part by grants from “Fundação de Amparo à Pesquisa do Estado de São Paulo — FAPESP” and “Conselho Nacional de Desenvolvimento Científico e Tecnológico — CNPq” — Brazil.

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quadratic function (i.e. it uses a single Lyapunov matrix) for testing stability over the whole uncertainty domain [3,8]. There have been also a few attempts to imply parameter-dependent Lyapunov matrices in a will to reduce the conservativeness of the quadratic framework [6,7], most of them in the continuous-time domain. In the discrete-time domain we can mention some works on Lur'e like systems using the passivity and real-positiveness approaches [9,10].

The scope of this paper is robust stability of uncertain discrete-time systems with polytopic convex uncertainty domain. It is shown how to expand the discrete Lyapunov condition for stability analysis by introducing a new matrix variable. The new extended matrix inequality is linear with respect to the variable, in fact a linear matrix inequality (LMI), and does not involve any product of the Lyapunov matrix and the system dynamic matrix. This enables us to derive a sufficient condition for robust stability which encompasses the basic quadratic results and provides a new way for practical determination of parameter-dependent Lyapunov functions by solving LMI problems. We claim that due to the above *decoupling* between the Lyapunov matrix and the system dynamic matrix this condition may be of use in the solution of many difficult control synthesis problems. This fact is illustrated by the simple state feedback robust stabilizability control problem. Of course, such a discrete-time condition is of some use for continuous-time systems when considering the pole location in a disk. Considering a disk tangent to the imaginary axis at the origin with a large radius would, certainly, result in a quite accurate robust stability condition in the continuous time domain.

The paper is organized as follows. In Section 2 the robust analysis problem is presented along with well-known results on quadratic stability. Section 3 presents the main results, namely a new robust stability condition and a new state feedback stabilizability LMI problem based on this condition. Numerical experiments are provided in Section 4 with a comparison with existing results. Finally, a conclusion and some perspectives are given in the end.

2. The robust analysis problem

Let the linear discrete-time uncertain system be

$$x_{k+1} = A(\alpha)x_k, \quad (1)$$

where the dynamic matrix $A(\alpha)$ belongs to a convex polytopic set defined as

$$\mathcal{A} := \left\{ A(\alpha): A(\alpha) = \sum_{i=1}^N \alpha_i A_i, \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0 \right\}. \quad (2)$$

We begin our discussion by defining robust stability with respect to system (1) and the structured uncertainty model (2).

Definition 1. System (1) is robustly stable in the uncertainty domain (2) if all eigenvalues of matrix $A(\alpha)$ have magnitude less than one for all values of α such that $A(\alpha) \in \mathcal{A}$.

Using Lyapunov stability it is possible to convert this definition into the equivalent condition given in the following lemma.

Lemma 1. System (1) is robustly stable in the uncertainty domain (2) if, and only if, there exists a matrix $P(\alpha) = P(\alpha)' > 0$ such that

$$A(\alpha)'P(\alpha)A(\alpha) - P(\alpha) < 0 \quad (3)$$

for all α such that $A(\alpha) \in \mathcal{A}$.

To the authors knowledge, there is no general and systematic way to formally determine $P(\alpha)$ as a function of the uncertain parameter α . Such a matrix $P(\cdot)$ is called a parameter-dependent Lyapunov matrix. Based on an equivalent structured uncertainty description, the search for $P(\alpha)$ may be performed with respect to a first-order series expansion in terms of the uncertain parameter, which enables to establish sufficient conditions for robust stability in the continuous-time domain [6,7]. Another effective way of addressing such problem is to look for a single Lyapunov matrix $P(\alpha) = P$ which solves inequality (3). Unfortunately, this approach generally provides quite conservative results. However, it constitutes one of the first results in the quadratic approach: the stability assessment over compact set (2) may be determined by testing the discrete, enumerable and bounded set of the vertices of the polyhedron (2). Then a single matrix P which satisfies the condition given in Lemma 1 may be found by the use of now standard efficient LMI tools. The test for this kind of stability also known as a quadratic stability test is summarized in the following lemma.

Lemma 2. *Uncertain system (1) is robustly stable in uncertainty domain (2) if*

$$A_i'PA_i - P < 0 \tag{4}$$

for all $i = 1, \dots, N$.

3. The main result

We begin this section by stating the following equivalence.

Theorem 1. *The following conditions are equivalent:*

(i) *There exists a symmetric matrix $P > 0$ such that*

$$A'PA - P < 0. \tag{5}$$

(ii) *There exist a symmetric matrix P and a matrix G such that*

$$\begin{bmatrix} P & A'G' \\ GA & G + G' - P \end{bmatrix} > 0. \tag{6}$$

Proof. We prove this theorem by noticing that if we apply Schur complement with respect to the block (2,2) in (6) we recover (5) by choosing $G = G' = P > 0$, hence (i) implies (ii). On the other hand, from the first block of (6) we have $P > 0$. Then multiplying (6) by $T := [I \quad -A']$ on the left and by T' on the right we get (5) which establishes that (ii) implies (i) and concludes this proof. \square

Condition (ii) appears as a direct expansion of condition (i) via its ‘‘Schur complement’’ formulation where, with the introduction of a new additional matrix G , we obtain a linear matrix inequality in which the Lyapunov matrix P is not involved in any product with the dynamic matrix A . This feature enables one to write a new robust stability condition which, although sufficient, is assumed not too conservative due to the presence of the extra degree of freedom provided by the introduction of matrix G (see the numerical example). Note that this extra matrix is not even constrained to be symmetric. The condition is given in the following theorem.

Theorem 2. *Uncertain system (1) is robustly stable in uncertainty domain (2) if there exist symmetric matrices P_i and a matrix G such that*

$$\begin{bmatrix} P_i & A_i'G' \\ GA_i & G + G' - P_i \end{bmatrix} > 0 \tag{7}$$

for all $i = 1, \dots, N$.

By following the same lines as in the proof of Theorem 1 it is possible to show that if (7) holds then there exists a parameter-dependent Lyapunov matrix

$$P(\alpha) = \sum_{i=1}^N \alpha_i P_i \tag{8}$$

which is positive definite for all values of α such that $A(\alpha) \in \mathcal{A}$. Since inequality (7) is linear on P_i and A_i , robust stability may be once again established by LMI tests over the discrete, enumerable and bounded set of the polytope vertices which define the uncertainty domain (2). Hence, the determination of feasible matrices P_i , $i = 1, \dots, N$ and G may be also easily performed using standard LMI solvers. It is clear that this result encompasses the well-known quadratic stability test since if (7) holds for a single matrix $P_i = P$ then necessity with respect to Lemma 2 can be proved by imposing $G = G' = P$ as in Theorem 1.

Perhaps the most interesting aspect of the given stability condition is that it can be easily extended to cope with the stabilizability problem in the following way. Let us consider the linear discrete-time system

$$x_{k+1} = A(\alpha)x_k + B(\beta)u_k, \tag{9}$$

where the dynamic matrix $A(\alpha)$ belongs to \mathcal{A} as defined in (2) and $B(\beta)$ is in the convex polytope defined by

$$\mathcal{B} := \left\{ B(\beta) : B(\beta) = \sum_{i=1}^M \beta_i B_i, \sum_{i=1}^M \beta_i = 1, \beta_i \geq 0 \right\}. \tag{10}$$

We look for a single state feedback gain K such that $A(\alpha) + B(\beta)K$ is asymptotically stable for every $A(\alpha) \in \mathcal{A}$ and $B(\beta) \in \mathcal{B}$. A new sufficient condition is stated in the following theorem.

Theorem 3. *Uncertain system (9) is robustly stable in uncertainty domains (2) and (10) if there exist symmetric matrices P_{ij} and a matrix G such that*

$$\begin{bmatrix} P_{ij} & A_iG + B_jL \\ G'A_i' + L'B_j' & G + G' - P_{ij} \end{bmatrix} > 0 \tag{11}$$

for all $i = 1, \dots, N$, $j = 1, \dots, M$. If (11) is feasible then a robust state feedback control is given by

$$K = LG^{-1}. \tag{12}$$

Proof. This theorem is proved by using a *transposed* version of Theorem 2, i.e., by transposing A_i and G , along with the change of variable $L = KG$. Notice that

the regularity of G is implied by the diagonal blocks of (11) since $G' + G > P_{ij} > 0$. \square

Hence it is possible to deal with the state feedback robust stabilizability problem by finding a feasible solution to a finite set of linear matrices inequalities. It is interesting to notice that, in contrast with the quadratic stability synthesis, the determination of the control (12) does not directly depend on the Lyapunov matrices P_{ij} which may be used to build a parameter-dependent Lyapunov matrix

$$P(\alpha, \beta) = \sum_{i=1}^N \alpha_i \left(\sum_{j=1}^M \beta_j P_{ij} \right). \tag{13}$$

We conjecture that such a property can be explored in several ways in control design. Notice that the extra degree of freedom introduced represented by matrix G , which is not even constrained to be symmetric, is fully incorporated in the control gain variable transformation. The application of this kind of conditions to other difficult control synthesis problems is currently under investigation.

4. Numerical example

In order to illustrate the results of this paper we consider the example given in [10]. The problem to be solved is to find the largest value of the scalar γ such that matrix

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} [0.8 \quad -0.5 \quad 0 \quad 1],$$

is robustly stable for all $|\alpha| < \gamma$. The stability bounds given by several known conditions are in Table 1. In this table the exact maximum value was found iteratively by root-locus. Notice that in this case we were able to get to the maximum bound even constraining G to be symmetric. Also notice that with the results in [10] we were also able to find the maximum value of γ . This method however, which tests stability with the help of two free parameters, is not amenable to synthesis.

Table 1
Stability bounds

Method	γ	
	Analysis	Synthesis
Exact maximum value	0.4619	—
Theorem 2	0.4619	0.8892
Theorem 2 with $G = G'$	0.4619	0.8878
Theorem 3 in [10]	0.4619	—
Quadratic Stability	0.4279	0.5282
RH_∞	0.2956	—

In the third column of this same table we give stability bounds achieved under state feedback for a system in the form (9) with input matrix

$$B(\beta) = \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (1 - \beta) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad 0 \leq \beta \leq 1.$$

In this case, only quadratic stability and Theorem 3 enable us to synthesize a robust state feedback gain by solving LMI problems. Also notice that in the synthesis problem, even with the additional symmetric constraint over G it was possible to find a stability bound which is very close to the one achieved with G nonsymmetric and which is much better than the closed-loop stability bound given by the standard quadratic approach.

5. Conclusion

A new robust sufficient stability condition for uncertain discrete-time systems has been given. Stated as a set of linear matrix inequalities, this condition enables the determination of parameter-dependent Lyapunov matrices and encompasses quadratic stability as a particular case.

We conjecture that the proposed approach will provide solution to several control design problems which have not been given a definitive answer. This claim is supported by the fact that using the given controller parametrization the control gain does not directly depend on the Lyapunov matrix. This point will be the subject of future research.

References

[1] B.R. Barmish, A generalization of kharitonov four-polynomial concept for robust stability problems with linearly dependent

- coefficient perturbations, *J. Optim. Theory Appl.* 46 (1985) 399–408.
- [2] S.P. Battacharyya, H. Chapellat, L.H. Keel, *Robust Control: the Parametric Approach*, Prentice-Hall, Upper Saddle River, NJ, 1997.
- [3] J. Bernussou, J.C. Geromel, P.L.D. Peres, A linear programming oriented procedure for quadratic stabilization of uncertain systems, *Systems Control Lett.* 13 (1) (1989) 65–72.
- [4] P. Colaneri, J.C. Geromel, A. Locatelli, *Control Theory and Design: An RH_2 – RH_∞ Viewpoint*, Academic Press, New York, 1997.
- [5] M.K.H. Fans, A.L. Tits, J.C. Doyle, Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics, *IEEE Trans. Automat. Control* 36 (1) (1991) 25–38.
- [6] E. Feron, P. Apkarian, P. Gahinet, Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions, *IEEE Trans. Automat. Control* 41 (7) (1996) 1041–1046.
- [7] P. Gahinet, P. Apkarian, M. Chilaly, Affine parameter-dependent Lyapunov functions and real parametric uncertainty, *IEEE Trans. Automat. Control* 41 (3) (1996) 436–442.
- [8] F. Garofalo, G. Celentano, L. Glielmo, Stability robustness of interval matrices via Lyapunov quadratic forms, *IEEE Trans. Automat. Control* 38 (2) (1993) 281–284.
- [9] W.M. Haddad, V. Kapila, Robust stabilization for discrete-time systems with slowly time-varying uncertainty, *J. Franklin Inst. — Eng. Appl. Math.* 333 B (1) (1996) 71–84.
- [10] M.C. de Oliveira, J.C. Geromel, L. Hsu, LMI characterization of structural and robust stability: the discrete-time case, *Linear Alg. Appl.*, to appear.
- [11] K. Zhou, J.C. Doyle, K. Glover, *Robust and Optimal Control*, Prentice-Hall, Upper Saddle River, NJ, 1996.