Achievable Degrees of Freedom of MIMO Multiway Relaying with Pairwise Data Exchange

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***Abstract*—We study achievable degrees of freedom (DoF) of a multi-input multi-output (MIMO) multiway relay channel (mRC) where K users, each equipped with M antennas, exchange messages in a pairwise manner with the help of a single N-antenna relay node. A novel and systematic way of designing beamforming vectors and matrices at the user and at the relay is proposed to realize signal alignment and to implement physical-layer network coding (PNC). It is shown that, for the considered mRC with $K = 3$ users, the proposed beamforming design achieves the DoF capacity for any (M, N) setups. For the scenarios with $K > 3$, we show that the proposed scheme can be improved by disabling a portion of relay antennas so as to align signals more efficiently. Our analysis reveals that the obtained achievable DoF is always piecewise linear, and is bounded either by the number of user antennas M or by the number of relay antennas N. Asymptotic DoF as $K \to \infty$ is also derived based on the proposed signal alignment scheme.**

I. Introduction

Physical-layer network coding (PNC) for two-way relaying has been proven a promising technique to enhance the spectral efficiency of relay-assisted bidirectional communications. A natural generalization of the two-way relay channel is the multiway relay channel (mRC), in which multiple users exchange messages with the help of a common relay. Some special mRCs, such as multipair PNC-based multiuser transmission [1] and cellular PNC-based multiuser transmission [2], have been studied in the literature.

The capacity characterization has been studied for mRCs in [3], [4]. These initial results on mRCs are limited to the single-antenna setup. The multi-input multi-output (MIMO) technique has been introduced into mRCs to allow spatial multiplexing [5]–[11]. In this regard, the authors in [6]–[11] studied the MIMO mRC from the perspective of degrees of freedom (DoF). Particularly, the authors in [6] investigated the DoF capacity of the MIMO Y channel (a special case of the MIMO mRC with three users) and showed that the DoF capacity can be achieved when $\frac{M}{N} \geq \frac{2}{3}$, where M denotes the number of antennas at each user and N denotes the number of antennas at the relay. Later, the work in [7] extended the channel model in [6] to the case of K users and studied the achievability conditions of the DoF capacity. The authors in [8], [9] generalized the results in [6] by considering a three-user asymmetric MIMO Y channel and a four-user symmetric MIMO Y channels, respectively, and proved that the DoF capacities can be achieved. Further, the authors in [10], [11] studied more general scenarios in which the users in the network are grouped into clusters, and each user in a cluster exchanges information only with the other users in the same cluster. In particular, the authors in [10] derived sufficient conditions on antenna configuration to achieve the DoF capacity of a clustered mRC with pairwise data exchange model, in which each user in a cluster sends a different message to each of the other users in the same cluster. Note that the data exchange models considered in [6]–[9] can be regarded as the one-cluster case of the model studied in [10]. Moreover, the author in [11] derived an achievable DoF for a clustered MIMO mRC with full data exchange, i.e., each user in a cluster delivers a common message to all the other users in the same cluster.

In this work, we study the MIMO mRC with pairwise data exchange, i.e., each user delivers a private message to each of the other users, and derive an achievable DoF for an arbitrary configuration of the antenna numbers (M, N) and the user number K. In general, an achievable DoF of a network can be seen as the number of independent data streams that can be supported by the network. In the considered MIMO mRC, as multiple users are simultaneously served by a common relay, the number of relay antennas is usually the bottleneck to achieve a higher DoF. Therefore, the challenge is how to align the user and relay signals to efficiently utilize the relay's signal space. To achieve this goal, we propose a novel and systematic way of beamforming design at the users and at the relay to align signals, and then to efficiently implement PNC. We show that, when the user number is $K = 3$, the proposed signal alignment scheme achieves the DoF capacity of the considered MIMO mRC for any (M, N) setup. The obtained result coincides with the DoF result in [8] and improves the existing DoF capacity result in [6]. For the case of $K > 3$, we show that the DoF capacity can be obtained when $\frac{M}{N} \in (0, \frac{K-1}{K(K-2)}]$ and $\frac{M}{N} \in [\frac{1}{K(K-1)} + \frac{1}{2}, \infty)$, which provides a broader DoF capacity range than the existing result in [10]. For $\frac{M}{N} \in (\frac{K-1}{K(K-2)}, \frac{1}{K(K-1)} + \frac{1}{2})$, we derive an achievable DoF of the MIMO mRC for an arbitrary setup of (M, N, K). Our analysis reveals that the obtained achievable DoF is always piecewise linear, and is bounded either by the number of user antennas M or by the number of relay antennas N. An asymptotic achievable DoF as $K \to \infty$ is also derived based on the proposed signal alignment scheme.

Notation: $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and the Hermitian transpose, respectively; $\text{Tr}(\cdot)$ and $(\cdot)^{-1}$ stand for the

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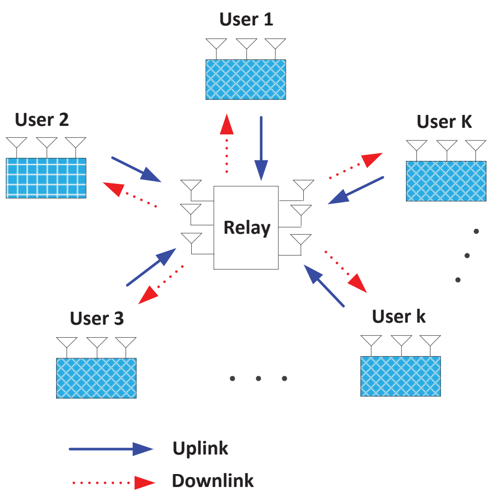

Fig. 1. An illustration of the MIMO mRC with K users operating in pairwise exchange.

trace and inverse, respectively. For any matrix $\mathbf{A}$, $\text{span}(\mathbf{A})$ and $\text{null}(\mathbf{A})$ denote the column space and the null space of $\mathbf{A}$, respectively. $\mathbb{C}^{n\times m}$ denotes the $n\times m$ dimensional complex space.

II. System Model

A. Channel Model

Consider a discrete memoryless symmetric MIMO mRC, where K users, each equipped with M antennas, exchange messages in a pairwise manner with the help of a common N-antenna relay node, as illustrated in Fig. 1. Specifically, each user k delivers a private message to user k' for $\forall k' \neq k$. Full-duplex communication is assumed, i.e., all the nodes transmit and receive signal simultaneously.[1] Moreover, we assume that no direct link exists between the users due to deep channel degradation. Denote by $\mathbf{H}_k \in \mathbb{C}^{N\times M}$ and $\mathbf{G}_k \in \mathbb{C}^{M\times N}$ the channel matrix from user k to the relay and the channel matrix from the relay to user k, respectively. We assume that $\mathbf{H}_k$ and $\mathbf{G}_k$, $k \in \mathcal{I}_K \triangleq \{1,2,\cdots,K\}$, are drawn from a continuous distribution and are perfectly known at all nodes.

Each round of information exchange is implemented in two phases, namely, an uplink phase and a downlink phase. The received signal at the relay node and at user k can be expressed respectively as

$$\mathbf{Y}_R = \sum_{k=1}^{K} \mathbf{H}_k \mathbf{X}_k + \mathbf{Z}_R \tag{1a}$$

$$\mathbf{Y}_k = \mathbf{G}_k \mathbf{X}_R + \mathbf{Z}_k, \quad k \in \mathcal{I}_K, \tag{1b}$$

where $\mathbf{X}_k \in \mathbb{C}^{M\times T}$ is the transmit signal from user k in a transmission frame of T channel uses; $\mathbf{Z}_R \in \mathbb{C}^{N\times T}$ is the additive white Gaussian noise (AWGN) matrix at the relay with each element following the distribution of $\mathcal{CN}(0,1)$; the power constraint at each user k is $\frac{1}{T}\text{Tr}(\mathbf{X}_k\mathbf{X}_k^H) \leq P$, where P is the maximum transmission power; $\mathbf{X}_R \in \mathbb{C}^{N\times T}$ is the transmit signal at the relay node; $\mathbf{Z}_k$ is the AWGN noise matrix at user k with the elements independently drawn from $\mathcal{CN}(0,1)$; the transmitted signal $\mathbf{X}_R$ satisfies the power constraint of $\frac{1}{T}\text{Tr}(\mathbf{X}_R\mathbf{X}_R^H) \leq P_R$, where P_R is the maximum transmission power at the relay. Note that for simplicity, we assume that $P_R = P$, which does not compromise the generality of the obtained DoF results in this paper.

We note that, when $K = 2$ and $K = 3$, the considered MIMO mRC reduces to the MIMO two-way relay channel (TWRC) and the MIMO Y channel, respectively. As the DoF capacity of the MIMO TWRC is well understood, we henceforth focus on the case of $K \geq 3$.

B. Degrees of Freedom

Let $W^{(k,k')}$ denote the private message sent from user k to user k' for $\forall k' \neq k$, and $R^{(k,k')}$ be the information rate carried in $W^{(k,k')}$. Denote $\hat{W}^{(k,k')}$ the estimate of $W^{(k,k')}$ at user k'. We say that user k achieves a sum rate of $R_k(P) = \sum_{k=1,k\neq k'}^{K} R^{(k,k')}$, if $\Pr\{\hat{W}^{(k,k')} \neq W^{(k,k')}\}$ tends to zero as $T \to \infty$. The total achievable DoF of the considered MIMO mRC is defined as

$$d_{\text{sum}} \triangleq \lim_{P\to\infty} \frac{\sum_{k=1}^{K} R_k(P)}{\log P}.$$

Also, we define the corresponding achievable DoF per user and achievable DoF per relay dimension respectively as

$$d_{\text{user}} \triangleq \frac{1}{K} d_{\text{sum}} \text{ and } d_{\text{relay}} \triangleq \frac{1}{N} d_{\text{sum}}. \tag{2}$$

In the above, P is the signal-to-noise ratio of the mRC due to the unit noise variance, and hence the above definition of DoF is consistent with those in the literature.

C. A DoF Outer Bound

An outer bound on the sum DoF of the MIMO mRC is given in [10] as

$$d_{\text{sum}} \leq \min(KM, 2N), \tag{3a}$$

or equivalently

$$d_{\text{user}} \leq \min\left(M, \frac{2N}{K}\right). \tag{3b}$$

The above outer bound can be intuitively explained as follows. On one hand, the total number of independent spatial data streams supported by the MIMO mRC cannot exceed $2N$, as the relay's signal space has N dimensions and thus the relay can only decode and forward N network-coded messages. On the other hand, the number of independent spatial data streams transmitted or received by each user cannot exceed M, as each user only has M antennas. The outer bound in (3) will be used as a benchmark in the following system design.

III. MIMO mRC with $K = 3$

The main contribution of this paper is a novel signal alignment technique to efficiently utilize the relay's signal space. This technique generally applies to the considered MIMO mRC with an arbitrary number of users. In this section, we focus on the case of three users, i.e., $K = 3$.

[1] All the DoF results obtained in this paper directly hold for the half-duplex case by multiplying a factor of $\frac{1}{2}$.

A. Preliminaries

We first give two important definitions (for a general K) used in this work. We call a bunch of $K(K-1)$ spatial streams as a *unit* if these streams are from K users, and every two users exchange one pair of such streams. Each spatial stream corresponds to a directed vector in the relay's signal space. The spatial data streams in a unit are aligned to form a certain spatial structure, called a *pattern*. Note that multiple units can be constructed following a single pattern. The efficiency of a pattern is measured by d_{relay} in (2), which can be explained as the ratio of the number of spatial streams involved in a pattern to the dimension of the relay's subspace spanned by spatial streams contained in this pattern. As one unit may not occupy the whole relay's signal space, to achieve a higher DoF, we need to construct multiple units with the most efficient pattern so that the overall relay's signal space is occupied. Then, an achievable DoF can be obtained by counting the number of units constructed for a given antenna setup of (M, N).

We now give some intuitions on the proposed signal alignment technique by considering *one unit*. In addition to achieving a higher d_{relay}, another key criterion in signal alignment design is to ensure that the signal is decodable at the user end. Let $\mathbf{s}^{(k,k')}$ be the spatial data stream from user k to user k', $\mathbf{u}^{(k,k')}$ be the beamformer at user k corresponding to the spatial data stream $\mathbf{s}^{(k,k')}$, and $\mathbf{v}^{(k,k')}$ be the receiving vector at user k corresponding to the spatial data stream $\mathbf{s}^{(k',k)}$. Denote by $\mathbf{h}^{(k,k')} = \mathbf{H}_k\mathbf{u}^{(k,k')}$ and $\mathbf{g}^{(k,k')} = \mathbf{G}_k^T\mathbf{v}^{(k,k')}$ the equivalent channels in the uplink and downlink, respectively. As one unit is considered, the system model in (1) reduces to

$$\mathbf{Y}_R = \sum_{k=1}^{K}\sum_{k'=1,k'\neq k}^{K}\mathbf{h}^{(k,k')}\mathbf{s}^{(k,k')T} + \mathbf{Z}_R \quad (4a)$$

$$\mathbf{y}_k^{(k,k')T} = \mathbf{g}^{(k,k')T}\mathbf{X}_R + \mathbf{z}_k^{(k,k')T}, \quad k \in \mathcal{I}_K. \quad (4b)$$

The principle of PNC is applied in relay decoding. Specifically, for each user pair (k, k'), the relay decodes a linear mixture of $\mathbf{s}^{(k,k')}$ and $\mathbf{s}^{(k',k)}$ as follows. Denote by $\mathbf{H}^{(kk')} \in \mathbb{C}^{N\times 4}$ a matrix formed by all the channel vectors except $\mathbf{h}^{(k,k')}$ and $\mathbf{h}^{(k',k)}$. Then, define the projection matrix of pair (k, k') as $\mathbf{P}^{(kk')} = \mathbf{I}_N - \mathbf{H}^{(kk')}(\mathbf{H}^{(kk')H}\mathbf{H}^{(kk')})^{-1}\mathbf{H}^{(kk')H} \in \mathbb{C}^{N\times N}$. For each user pair (k, k'), the relay projects the received signal vector $\mathbf{Y}_R$ onto the nullspace of $\mathbf{H}^{(kk')}$, yielding

$$\begin{aligned}\mathbf{P}^{(kk')}\mathbf{Y}_R &= \mathbf{P}^{(kk')}\Big(\sum_{n=1}^{K}\sum_{n'=1,n'\neq n}^{K}\mathbf{h}^{(n,n')}\mathbf{s}^{(n,n')T} + \mathbf{Z}_R\Big)\\ &= \mathbf{P}^{(kk')}\big(\mathbf{h}^{(k,k')}\mathbf{s}^{(k,k')T} + \mathbf{h}^{(k',k)}\mathbf{s}^{(k',k)T} + \mathbf{Z}_R\big). \quad (5)\end{aligned}$$

We now move to the relay-to-user phase modeled in (1b). Similar to $\mathbf{H}^{(kk')}$, we denote $\mathbf{G}^{(kk')} \in \mathbb{C}^{N\times 4}$ as a matrix formed by all the channel vectors except for $\mathbf{g}^{(k,k')}$ and $\mathbf{g}^{(k',k)}$. The projection matrix of pair (k, k') in the downlink is then defined as $\mathbf{W}^{(kk')} = \mathbf{I}_N - \mathbf{G}^{(kk')}(\mathbf{G}^{(kk')H}\mathbf{G}^{(kk')})^{-1}\mathbf{G}^{(kk')H} \in \mathbb{C}^{N\times N}$. The relay sends out $\mathbf{F}\mathbf{Y}_R$ with $\mathbf{F}$ given by

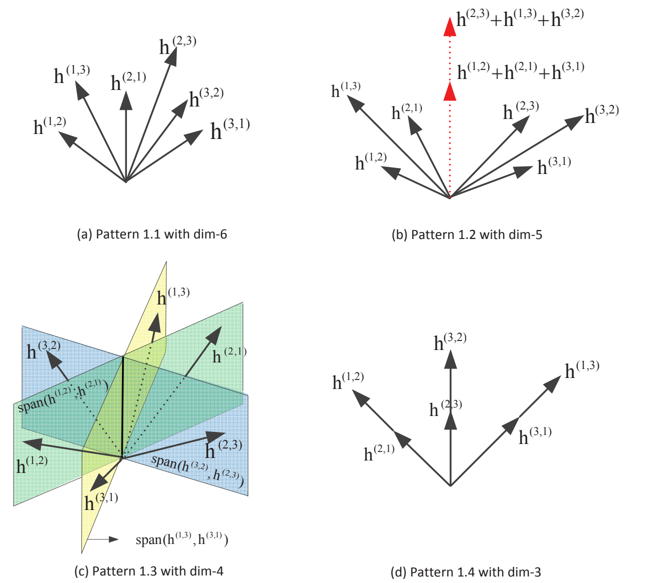

Fig. 2. A geometric illustration of Patterns 1 to 4.

$$\mathbf{F} = \alpha\sum_{k=1}^{K}\sum_{k'=k+1}^{K}\mathbf{W}^{(kk')}\mathbf{P}^{(kk')}, \quad (6)$$

where α is a scaling factor used to satisfy the power constraint at the relay node. The received signal at user k is

$$\begin{aligned}\mathbf{y}_k^{(k,k')T} &= \mathbf{g}^{(k,k')T}\sum_{n=1}^{K}\sum_{n'=n+1}^{K}\mathbf{W}^{(nn')}\mathbf{P}^{(nn')}\mathbf{Y}_R + \mathbf{z}_k^{(k,k')T}\\ &= \mathbf{g}^{(k,k')T}\mathbf{W}^{(kk')}\mathbf{P}^{(kk')}\Big(\mathbf{h}^{(k,k')}\mathbf{s}^{(k,k')T}\\ &\quad + \mathbf{h}^{(k',k)}\mathbf{s}^{(k',k)T} + \mathbf{Z}_R\Big) + \mathbf{z}_k^{(k,k')T}. \quad (7)\end{aligned}$$

We note that $\mathbf{g}^{(k,k')}$, $\mathbf{W}^{(kk')}$, $\mathbf{P}^{(kk')}$, and $\mathbf{h}^{(k',k)}$ are independent of each other. Therefore, the equivalent user-to-user channel ${\mathbf{g}^{(k,k')}}^T\mathbf{W}^{(kk')}\mathbf{P}^{(kk')}\mathbf{h}^{(k',k)}$ is non-zero with probability one, provided that $\mathbf{W}^{(kk')}$ and $\mathbf{P}^{(kk')}$ are of at least rank one. Then, each user k receives one linear combination of the signals in pair (k, k'). By subtracting the self-interference, each user can decode the desired signals from the other two users, which achieves a per-user DoF $d_{\text{user}} = 2$ for each user, or equivalently, a total DoF of $d_{\text{sum}} = 6$ can be achieved.

From (7), we see that the symmetry exists between the signal alignment design in the uplink and the signal alignment design in the downlink. Given the design of the beamformer $\mathbf{u}^{(k,k')}$ and the projection matrix $\mathbf{P}^{(kk')}$, the receive vectors $\mathbf{v}^{(k,k')}$ and the projection matrix $\mathbf{W}^{(kk')}$ in the downlink can be designed similarly, since ${\mathbf{g}^{(k,k')}}^T\mathbf{W}^{(kk')}$ can be simply regarded as the transpose of $\mathbf{P}^{(kk')}\mathbf{h}^{(k',k)}$. Therefore, it suffices to focus on design of the uplink in what follows.

We now describe four patterns involved (with different d_{relay}) in achieving the DoF capacity of the MIMO mRC with $K = 3$. Denote $\mathcal{U} \triangleq \{\mathbf{h}^{(k,k')}|k \in \mathcal{I}_K, k' \in \mathcal{I}_K, k \neq k'\}$. Let

TABLE I
PATTERNS FOR MIMO MRC WITH $K = 3$

Pattern	Dimension	d_{sum}	d_{relay}	Requirement
1	6	6	1	$\frac{M}{N} > 0$
2	5	6	$\frac{6}{5}$	$\frac{M}{N} > \frac{1}{3}$
3	4	6	$\frac{3}{2}$	$\frac{M}{N} > \frac{1}{3}$
4	3	6	2	$\frac{M}{N} > \frac{1}{2}$

$\mathcal{U}\backslash\{\mathbf{h}^{(k,k')}, \mathbf{h}^{(k',k)}\}$ be the vector set obtained by excluding $\mathbf{h}^{(k,k')}$ and $\mathbf{h}^{(k',k)}$ from $\mathcal{U}$.

1) **Pattern 1:** $\mathcal{U}$ spans a subspace with dimension 6 (dim-6).
2) **Pattern 2:** $\mathcal{U}$ spans a subspace with dim-5; for any pair (k, k'), $\mathcal{U}\backslash\{\mathbf{h}^{(k,k')}, \mathbf{h}^{(k',k)}\}$ spans a subspace of dim-4.
3) **Pattern 3:** $\mathcal{U}$ spans a subspace with dim-4; the intersection of $\text{span}(\mathbf{h}^{(1,2)}, \mathbf{h}^{(2,1)})$, $\text{span}(\mathbf{h}^{(2,3)}, \mathbf{h}^{(3,2)})$, and $\text{span}(\mathbf{h}^{(1,3)}, \mathbf{h}^{(3,1)})$ is of dim-1, and $\mathcal{U}$ spans a subspace of dim-4.
4) **Pattern 4:** $\mathcal{U}$ spans a subspace with dim-3; for any pair (k, k'), $\mathbf{h}^{(k,k')}$ and $\mathbf{h}^{(k',k)}$ span a subspace with dim-1.

The above four patterns are geometrically illustrated in Fig. 2. It can be verified that the projection matrices $\mathbf{P}^{(kk')}$, $\forall k, k', k' \neq k$, for Patterns $1-4$ are of at least rank one for sure. For example, $\mathbf{P}^{(kk')}$ for Pattern 1 is of rank two. Hence the proposed signaling scheme achieves a total DoF of 6. However, different patterns occupy different dimensions over the relay's signal space, which yields a different d_{relay} as shown in Table I. In general, a pattern with a greater d_{relay} is more efficient in utilizing the relay's signal space, and hence is more desirable in signaling design. In Table I, the requirements on $\frac{M}{N}$ to realize different patterns are also given in the last column, and the corresponding requirements will be discussed in details in Subsection III-C. As mentioned earlier, we will mainly focus on the uplink in what follows due to the symmetry between designs in downlink and uplink.

B. Main Result

We now consider the general case that multiple units are transmitted over the considered MIMO mRC. Our goal is to construct units with the most efficient patterns to occupy the relay's signal space.

THEOREM 1. *For the considered MIMO mRC with $K = 3$, the per-user DoF capacity is given by*

$$d_{\text{user}} = \begin{cases} M, & \frac{M}{N} < \frac{2}{3} \\ \frac{2N}{3}, & \frac{M}{N} \geq \frac{2}{3}. \end{cases} \tag{8}$$

The per-user DoF capacity with respect to $\frac{M}{N}$ is shown in Fig. 3. We see that for $\frac{M}{N} < \frac{2}{3}$, the per-user DoF $d_{\text{user}} = M$, which means that the DoF is bounded by the number of antennas at the user ends. On the other hand, when $\frac{M}{N} \geq \frac{2}{3}$, the DoF is bounded by the number of relay antennas.

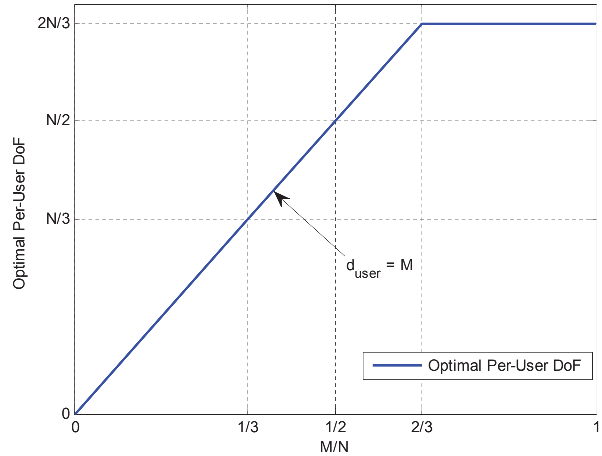

Fig. 3. The optimal DoF with respect to the antenna ratio $\frac{M}{N}$ for MIMO mRC with $K = 3$.

C. Proof of Theorem 1

We first note that d_{user} in (8) coincides with the DoF outer bound in (3) with $K = 3$. Therefore, to prove Theorem 1, it is sufficient to show the achievability of (8). We start with a brief description of the overall transceiver design. We need to jointly design the transmit beamformers $\mathbf{u}_l^{(k,k')}$, the receive vectors $\mathbf{v}_l^{(k,k')}$ and the relay projection matrices $\mathbf{W}_l^{(kk')}$ and $\mathbf{P}_l^{(kk')}$ (here l denotes the unit index) to efficiently utilize the relay's signal space so that the DoF given in (8) can be achieved. Different from Subsection III-A, the relay projection matrices $\mathbf{W}_l^{(kk')}$ and $\mathbf{P}_l^{(kk')}$ should null the interference from not only the other pairs but also the other units. Hence the relay's beamforming matrix $\mathbf{F}$ given in (6) is expressed by

$$\mathbf{F} = \alpha \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{k'=k+1}^{K} \mathbf{W}_l^{(kk')} \mathbf{P}_l^{(kk')},$$

where L denotes the number of unit. Based on that, in each unit, each user can achieve a DoF of two, provided that the projection matrices $\mathbf{W}_l^{(kk')}$ and $\mathbf{P}_l^{(kk')}$ are at least of rank one.

1) Case of $\frac{M}{N} \leq \frac{1}{3}$: In this case, the relay's antenna number N is no less than the number of total antennas of all the users, i.e., $\text{span}(\{\mathbf{H}_k|\forall k\})$ is of dim-$3M$ with probability one. This implies that the relay has enough dimensions to support full multiplexing at the user end, i.e., each user transmits M spatial data streams. M units with Pattern 1 can be constructed. Clearly, the projection matrix $\mathbf{P}_l^{(kk')}$ is of at least rank one. Thus, each unit achieves a DoF of 6. Considering all the M units, we obtain that the achievable per-user DoF is M.

2) Case of $\frac{1}{3} < \frac{M}{N} \leq \frac{1}{2}$: From Table I, this case corresponds to Patterns 2 and 3. We first consider Pattern 2. Since $N \leq 3M$, we can design the transmit beamforming vectors satisfying the condition of

$$\mathbf{H}_1(\mathbf{u}^{(1,2)} + \mathbf{u}^{(1,3)}) + \mathbf{H}_2(\mathbf{u}^{(2,1)} + \mathbf{u}^{(2,3)}) + \mathbf{H}_3(\mathbf{u}^{(3,1)} + \mathbf{u}^{(3,2)}) = \mathbf{0}, \tag{9}$$

which implies that $\mathcal{U}$ occupies a subspace of dim-5 and $\mathcal{U}\backslash\{\mathbf{h}^{(k,k')}, \mathbf{h}^{(k',k)}\}$ occupies a subspace of dim-4. By that, we can always extract the data steams of each pair by nulling the interference from other pairs. In this way, we can construct $3M-N$ units with Pattern 2, which totally occupy $5(3M-N)$ dimensions in the relay's signal space. The remaining $N-5(3M-N)$ dimensions relay's signal space are used to construct $\frac{N-5(3M-N)}{6}$ units with Pattern 1. The achievable DoF per user is given by $d_{\text{user}} = 2(3M-N) + \frac{2(N-5(3M-N))}{6} = M$.

The above DoF per user reaches the maximum when the overall relay's signal space is wholly occupied by units of Pattern 2. This corresponds to a maximum DoF per user of $\frac{N}{5} \times 2 = \frac{2N}{5}$. The achievable per-user DoF is thus obtained as $\min(M, \frac{2N}{5})$.

Now we consider the construction of Pattern 3. Denote the intersection of $\text{span}(\mathbf{H}_1, \mathbf{H}_2)$ and $\text{span}(\mathbf{H}_3)$ by $\mathcal{S}^{(3)}$. The dimension of $\mathcal{S}^{(3)}$ is $3M - N > 0$. We can design $\mathbf{u}_3$ and $\mathbf{u}^{(3,1)}$ such that $\mathbf{H}_3\mathbf{u}_3$ and $\mathbf{H}_3\mathbf{u}^{(3,1)}$ are two independent vectors in $\mathcal{S}^{(3)}$. By definition, $\mathbf{H}_3\mathbf{u}_3$ and $\mathbf{H}_3\mathbf{u}^{(3,1)}$ belong to $\text{span}(\mathbf{H}_1, \mathbf{H}_2)$. Thus, there exist $\{\mathbf{u}^{(1,3)}, \mathbf{u}^{(2,3)}\}$ and $\{\mathbf{u}_1, \mathbf{u}^{(2,1)}\}$ satisfying

$$\mathbf{H}_1\mathbf{u}^{(1,3)} + \mathbf{H}_2\mathbf{u}^{(2,3)} + \mathbf{H}_3\mathbf{u}_3 = \mathbf{0} \quad (10a)$$
$$\mathbf{H}_1\mathbf{u}_1 + \mathbf{H}_2\mathbf{u}^{(2,1)} + \mathbf{H}_3\mathbf{u}^{(3,1)} = \mathbf{0}. \quad (10b)$$

Let $\mathbf{u}^{(3,2)} = \mathbf{u}_3 - \mathbf{u}^{(3,1)}$ and $\mathbf{u}^{(1,2)} = \mathbf{u}_1 - \mathbf{u}^{(1,3)}$. Together with (10), we obtain

$$\mathbf{H}_1\mathbf{u}^{(1,3)} + \mathbf{H}_2\mathbf{u}^{(2,3)} + \mathbf{H}_3(\mathbf{u}^{(3,2)} + \mathbf{u}^{(3,1)}) = \mathbf{0} \quad (11a)$$
$$\mathbf{H}_1(\mathbf{u}^{(1,2)} + \mathbf{u}^{(1,3)}) + \mathbf{H}_2\mathbf{u}^{(2,1)} + \mathbf{H}_3\mathbf{u}^{(3,1)} = \mathbf{0}. \quad (11b)$$

Subtracting (11a) by (11b), we obtain

$$\mathbf{H}_1\mathbf{u}^{(1,2)} + \mathbf{H}_2(\mathbf{u}^{(2,1)} - \mathbf{u}^{(2,3)}) - \mathbf{H}_3\mathbf{u}^{(3,2)} = \mathbf{0}. \quad (11c)$$

From (11), $\{\mathbf{H}_k\mathbf{u}^{(k,k')} | \forall k, k' \neq k\}$ span a subspace of dim-4 following Pattern 3. Further, from (11a), two signal pairs $(1,3)$ and $(2,3)$ span a subspace with dim-3. Thus, the null space of pairs $(1,3)$ and $(2,3)$ is of dim-1. Similarly, from (11b) and (11c), the null space of any two of the three pairs is of dim-1. Therefore, one linear combination for each signal pair can be decoded at the relay. In this way, we can construct $\frac{3M-N}{2}$ units with Pattern 3,[2] occupying a signal space of dim-$2(3M-N)$. The remaining $N-2(3M-N)$ dimensions are used to construct $\frac{N-2(3M-N)}{6}$ units following Pattern 1. Thus, the achievable DoF per user is given by $d_{\text{user}} = 3M - N + \frac{2(N-2(3M-N))}{6} = M$.

If the overall relay's signal space is occupied by Pattern 3, the maximum per-user DoF is $\frac{N}{4} \times 2 = \frac{N}{2}$. Therefore, the achievable per-user DoF is given by $\min(M, \frac{N}{2}) = M$. It is worth noting that, compare to Pattern 2, Pattern 3 achieves a higher per-user DoF.

[2]Here we assume that $\frac{3M-N}{2}$ is an integer. Otherwise, we use symbol extension to ensure that the dimension of the intersection concerned is dividable by two.

3) Case of $\frac{M}{N} > \frac{1}{2}$: In this case, $\frac{M}{N}$ is large enough to construct units with Pattern 4. The intersection of $\text{span}(\mathbf{H}_k)$ and $\text{span}(\mathbf{H}_{k'})$ is of $2M - N > 0$. The signal alignment can be realized between the signals within each pair. Let $\mathbf{h}^{(k,k')}$ be the intersection of $\text{span}(\mathbf{H}_k)$ and $\text{span}(\mathbf{H}_{k'})$. There exists $\{\mathbf{u}^{(k,k')}, \mathbf{u}^{(k',k)}\}$ satisfying

$$\mathbf{H}_k\mathbf{u}^{(k,k')} = \mathbf{H}_{k'}\mathbf{u}^{(k',k)} = \mathbf{h}^{(k,k')}, \ \forall k, k \neq k',$$

which further implies that the signals of each pair occupy one dimension. In this way, we can construct $2M - N$ units with Pattern 4, in total spanning a subspace of dim-$3(2M - N)$. Again, the remaining $N - 3(2M - N)$ dimensions of relay's signal space can be used for constructing $\frac{N-3(2M-N)}{4}$ units with Pattern 3. Thus an achievable per-user DoF is given by $d_{\text{user}} = 2(2M - N) + \frac{2(N-3(2M-N))}{4} = M$.

When the overall relay's signal space is occupied by the units with Pattern 4, we have the maximum per-user DoF as $\frac{N}{3} \times 2 = \frac{2N}{3}$. Therefore, the maximum per-user DoF is given by $\min(M, \frac{2N}{3})$, which is equivalent to (8). This completes the proof of Theorem 1.

IV. MIMO MRC WITH $K > 3$

We now generalize Theorem 1 to the case of an arbitrary number of users. Due to space limitation, we omit proofs here.

THEOREM 2. *For the considered MIMO mRC with K users, the per-user DoF capacity of $d_{\text{user}} = M$ is achieved when $\frac{M}{N} \in (0, \frac{K-1}{K(K-2)}]$ and the per-user DoF capacity of $d_{\text{user}} = \frac{2N}{K}$ is achieved when $\frac{M}{N} \in [\frac{1}{K(K-1)} + \frac{1}{2}, \infty)$. For $\frac{M}{N} \in (\frac{K-1}{K(K-2)}, \frac{1}{K(K-1)} + \frac{1}{2})$, an achievable per-user DoF is given by*

$$d_{\text{user}} = \begin{cases} \dfrac{N\alpha_{t+1}}{\beta_{t+1}} & \dfrac{M}{N} \in \left(\dfrac{t}{(t+1)\beta_{t+1}} + \dfrac{1}{t+1}, \gamma_t\right] \\ \dfrac{Mt\alpha_t}{t-1+\beta_t}, & \dfrac{M}{N} \in \left(\gamma_t, \dfrac{t-1}{t\beta_t} + \dfrac{1}{t}\right] \end{cases}$$

where $\alpha_t = \binom{K-1}{t-1}(t-1)$, $\beta_t = \binom{K}{t}(t-1)^2$, $\gamma_t = \frac{\alpha_{t+1}(t-1+\beta_t)}{t\alpha_t\beta_{t+1}}$, and $t \in [2, K-2]$ is an integer.

COROLLARY 1. *For the considered MIMO mRC with $K = 4$, the per-user DoF capacity of $d_{\text{user}} = M$ is achieved when $0 < \frac{M}{N} \leq \frac{3}{8}$, and the per-user DoF capacity of $d_{\text{user}} = \frac{N}{2}$ is achieved when $\frac{M}{N} \geq \frac{7}{12}$. For $\frac{M}{N} \in (\frac{3}{8}, \frac{7}{12})$, an achievable per-user DoF is given by*

$$d_{\text{user}} = \begin{cases} \dfrac{3N}{8} & \dfrac{3}{8} < \dfrac{M}{N} \leq \dfrac{7}{16} \\ \dfrac{6M}{7}, & \dfrac{7}{16} < \dfrac{M}{N} < \dfrac{7}{12}. \end{cases}$$

Corollary 1 is obtained from Theorem 2 by letting $K = 4$, with the DoF curve illustrated in Fig. 4. It is interesting to see that the obtained DoF curve is piecewise linear, depending on the number of user antennas M and the number of relay antennas N alternately. The piecewise linearity implies antenna redundancy. Specifically, for $\frac{M}{N} \in (0, \frac{3}{8}] \cup (\frac{7}{16}, \frac{7}{12})$, the DoF is bounded by the number of relay antennas, implying

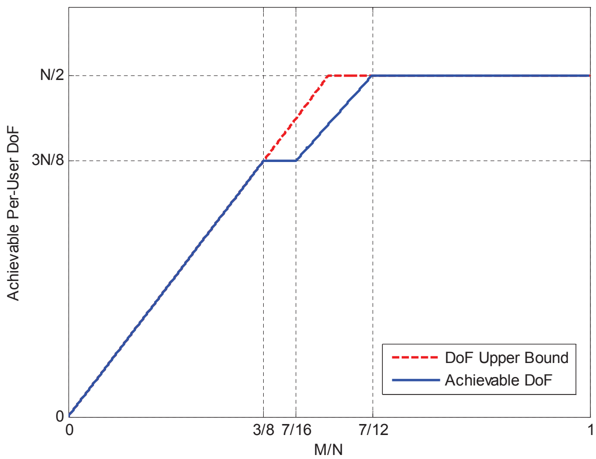

Fig. 4. An achievable per-user DoF for MIMO mRC with $K = 4$ at different (M, N) configurations.

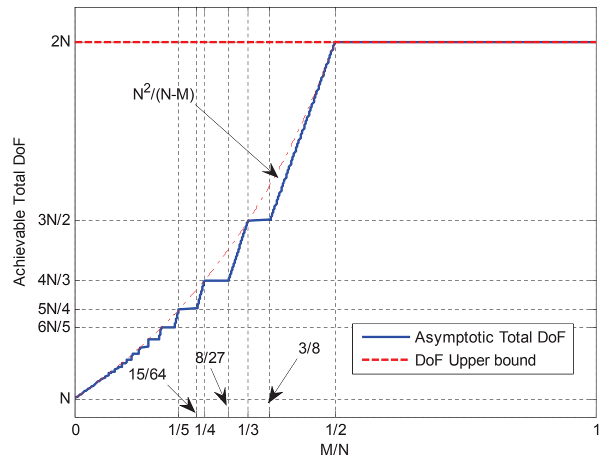

Fig. 5. An achievable DoF for the MIMO mRC with an infinite number of users at different (M, N) configurations.

antenna redundancy at the user side; for $\frac{M}{N} \in (\frac{3}{8}, \frac{7}{16}] \cup (\frac{7}{12}, 1]$, the DoF is bounded by the number of user antennas, which implies that the antenna redundancy occurs at the relay. It is worth noting that recently a parallel work [9] established the DoF for the four-user MIMO Y channel. The result in [9] slightly outperforms the achievable DoF in Corollary 1 in the interval of $\frac{M}{N} \in \left(\frac{3}{8}, \frac{1}{2}\right)$. As the signal alignment technique in [9] is only applicable for the case of $K = 4$, the DoF capacity of the MIMO mRC with an arbitrary K remains an open problem.

We now study the asymptotic DoF behavior as $K \to \infty$. Note that when $K \to \infty$, we have $d_{\text{user}} \to 0$. Thus, we only consider d_{sum} here.

COROLLARY 2. *For the considered MIMO mRC with $K \to \infty$, the total DoF of $d_{\text{sum}} = 2N$ is achieved when $\frac{M}{N} > \frac{1}{2}$ and $d_{\text{sum}} = N$ is achieved as $\frac{M}{N} \to 0$. For $\frac{M}{N} \in (0, \frac{1}{2}]$, the achievable total DoF is piecewise linear with respect to either M or N. Specifically, for $t = 2, 3, \cdots, \infty$, we have $d_{\text{sum}} = \frac{(t+1)N}{t}$ for $\frac{M}{N} \in \left(\frac{1}{t+1}, \frac{(t+1)(t-1)}{t^3}\right]$, and $d_{\text{sum}} = \frac{Mt^2}{t-1}$ for $\frac{M}{N} \in \left(\frac{(t+1)(t-1)}{t^3}, \frac{1}{t}\right]$.*

The DoF curve for Corollary 2 is illustrated in Fig. 5. We see that the achievable DoF for $\frac{M}{N} \in (0, \frac{1}{2}]$ is enveloped by the curve of $y = \frac{N^2}{N+M}$. Further, we can partition the range of $\frac{M}{N} \in (0, \frac{3}{8}]$ into an infinite number of intervals, namely, $\frac{M}{N} \in \left(\frac{(t+2)t}{(t+1)^3}, \frac{(t+1)(t-1)}{t^3}\right]$ for $t = 2, 3, \cdots, \infty$; implying that a new pattern arises for efficient signal alignment. Similarly, the achievable DoFs in the ranges of $\frac{M}{N} \in \left(\frac{(t+2)t}{(t+1)^3}, \frac{1}{t+1}\right]$ and $\frac{M}{N} \in \left(\frac{1}{t+1}, \frac{(t+1)(t-1)}{t^3}\right]$ indicate antenna redundancy at the user side and at the relay side, respectively.

V. CONCLUSION

In this paper, we studied the achievable DoF of the MIMO mRC with arbitrary number of antennas and number of users. A novel and systematic beamforming design approach was proposed to align signals, so as to facilitate the implementation of PNC. It was shown that the proposed signal alignment scheme achieves the DoF capacity of the MIMO mRC with $K = 3$. The asymptotic achievable DoF was also analyzed when the number of users tends to infinity.

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