# Repair-Limit Risk-Free Warranty Policies With Imperfect Repair

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July 25, 2004

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**Abstract**

Current competitive market environment requires manufacturers to continuously provide better service and support. As a result, warranty considerations emerge as a significant instrument for increasing product marketability. In this paper, we propose a new warranty policy, the repair-limit risk-free warranty with a threshold point on the number of repairs, where replacement is deemed to be more cost effective thereafter. Consumers are better off than with a traditional free repair policy since they could be compensated with a new product in case of premature failures. As for the manufacturers, it not only offers extra marketing incentives, but also reduces the possibility of high cost lawsuits due to the products with 'proven' bad quality. Some useful results of the warranty cost of imperfectly repaired products are derived through a censored quasi-renewal process.

*keywords:* Warranty cost; Repair limit; Imperfect repair; Quasi-renewal processes; Renewal processes.

**Acronyms**

| | |
|---|---|
| cdf | cumulative distribution function |
| i.i.d. | identically and independently distributed |
| pdf | probability density function |
| pmf | probability mass function |
| FRW | free repair warranty |
| PRW | pro-rata warranty |
| r.v. | random variables |

**Notation**

| | |
|---|---|
| $w$ | length of a warranty period ($w > 0$) |
| $m$ | threshold point of the number of repairs under warranty |
| $\alpha$ | parameter for a (censored) quasi-renewal process |
| $N_a(w), N_b(w)$ | number of free repairs and replacements within $w$ respectively |
| $c_a, c_b$ | unit repair and replacement cost respectively, both constant |
| $T_p, t_p$ | pivot points. Capital letter indicates an r.v. |
| $X_i$ | inter-occurrence times of a (censored) quasi-renewal process |
| $S_i$ | occurrence times of a (censored) quasi-renewal process |
| $F, f$ | cdf and pdf of the first failure time of a new product |
| $F_i, f_i$ | cdf and pdf of the $i^{th}$ failure time of a (censored) quasi-renewal process |
| $G^{(n)}$ | cdf of the $n^{th}$ occurrence time of a (censored) quasi-renewal process |
| $C(w)$ | warranty cost per product sold |
| $M_q(\cdot), M_{q,2}(\cdot)$ | first and second moments of a (censored) quasi-renewal process |
| $M_d(\cdot), M_{d,2}(\cdot)$ | first and second moments of a delayed renewal process |

# 1 Introduction

Numerous warranties have been studied extensively in warranty literature. In general, one can divide them into three categories: free repair/replacement warranty

(FRW), pro-rata warranty (PRW) and combination warranty that contains both the features of a FRW and a PRW. For a complete categorization of warranties and the cost analysis of these policies, we refer to [2].

In this paper we propose and study a repair-limit risk-free warranty of a fixed period $w$. Different from the traditional FRW policies, this policy has a threshold point $m$ on the number of repairs. In case there are more than $m$ system failures within $w$, the failed product will be replaced instead of being repaired again. Such a policy is desirable for both manufacturers and consumers. Consumers should prefer such a policy to a simple free repair policy because there are chances that they could own another new product for free. From manufacturers' point of view, first of all, such a policy offers extra marketing incentives. Secondly, if a single product has failed $m$ times within a period of $w$, this might have provided sufficient information that the particular product is indeed of low quality. So it could be cost effective for the manufacturer to simply provide replacements without wasting more time on repairs. In addition, such extra compensation for those 'unlucky' consumers may effectively reduce the chance of high-cost lawsuits due to the products with 'proven' bad quality.

In the literature of maintenance, many researchers have studied various repair limit problems that can be categorized into three groups: repair-number limit problems, repair-time limit problems and repair-cost limit problems. Park [14] determined the optimal number of repairs before replacement based on the assumptions of minimal repair and Weibull failure time distribution. Nguyen and Murthy [13] and Kaio and Osaki [6] discussed several repair-time limit problems. They established the threshold time point after which the failed products should no longer be repaired. When repair cost per failure is considered random, it is possible to set up a maintenance policy based on the estimated repair cost to determine whether to repair or replace a failed product. This problem was investigated by Dohi [4], Koshimae, et al. [8] and others. Almost all researchers dealing with repair-limit problems assume infinite horizon and perform the analysis based on asymptotic cost measures such as the long-run average cost. In this study, we consider finite horizon for the proposed repair-number limit warranty policy. Exact expressions of the warranty cost moments are derived based on a censored quasi-renewal process and other probability methodologies.

The key analytical tool used in this paper is the censored quasi-renewal processes. Wang and Pham [23] studied the quasi-renewal processes and applied to imperfect maintenance problems. They also successfully applied the methodology to the modeling of software reliability growth and testing costs [16]. Censored quasi-renewal processes is an extension of regular quasi-renewal processes. The concept and some important properties will be discussed in section 2.

Another important issue in warranty cost analysis is the variability of warranty cost. It is often not sufficient for warranty managers to simply obtain an estimate of the expected warranty cost. Additional information about the variability of warranty cost is essential to evaluate the risks involved in warranty programs. Discussions on warranty cost variation can be seen in [1, 5, 11, 12, 15].

The rest of this paper is organized as follows: Section 2 introduces the censored quasi-renewal processes and presents some important properties. Section 3 provides the cost analysis of the repair-limit risk-free warranty policy. Several special cases are discussed in section 4. Section 5 presents the sensitivity analysis for various policy parameters. We conclude this paper in section 6.

**Assumptions**

1. All warranty service is instant.
2. Repairs are imperfect and the repair process can be modelled by a quasi-renewal process.
3. All warranty claims are executed and all claims are valid.
4. Both repair cost $c_a$ and replacement cost $c_b$ are constant and $c_a < c_b$ to avoid

triviality.

# 2 Censored Quasi-Renewal Processes

In this section, we first introduce the concept of quasi-renewal processes following Pham and Wang [16], then discuss the censored and truncated quasi-renewal processes and several important properties.

There are extensive discussion on truncation and censoring in survival analysis literature, for example, [7]. However, to our knowledge, no explicit study on censored distributions (processes) have been done in warranty literature. Censored quasi-renewal processes arise naturally in the study of warranty policies involving imperfect repair. There are also potential applications in reliability and maintenance modelling.

## 2.1 Quasi-renewal Processes

Let $\{N(t), t > 0\}$ be a counting process and $X_n$ be the inter-occurrence time between the $(n-1)^{th}$ and $n^{th}$ events of the process. We say $\{N(t), t > 0\}$ is a quasi-renewal process associated with the distribution $F$ and the parameter $\alpha, \alpha > 0$, a constant, if $X_n = \alpha^{n-1} Z_n,\ n = 1, 2, \cdots$, where $Z_n$s are i.i.d. and $Z_n \sim F$.

Denote $f_i$ and $F_i$ as the pdf and cdf of $X_i$. It is easy to see that for the quasi-renewal process,

$$f_i(x) = \alpha^{1-i} f(\alpha^{1-i} x)$$

$$F_i(x) = F(\alpha^{1-i} x).$$

The pmf of $N(t)$ can be easily derived through the relationship that $N(t) \geq n \Longleftrightarrow S_n \leq t$, where $S_n$ is the occurrence time of the $n^{th}$ event in the process.

$$\mathrm{P}[N(t) = n] = G^{(n)}(t) - G^{(n+1)}(t), \quad n = 0, 1, 2, \cdots$$

where $G^{(n)}(t)$ is the convolution of the inter-occurrence times $X_1, X_2, \cdots, X_n$, and $G^{(0)}(t) = 1$ .

The expected value of $N(t)$, or the renewal function, $M_q(t)$, for the quasi-renewal process is given by :

$$M_q(t) = \sum_{n=1}^{\infty} G^{(n)}(t).$$

Denote $M_{q,2}(t)$ as the second non-centered moment of $N(t)$. It is well known (i.e. see [22]) that

$$M_{q,2}(t) = \sum_{n=1}^{\infty} (2n-1) G^{(n)}(t).$$

## 2.2 Truncated Quasi-Renewal Processes

We now consider an extension of quasi-renewal processes - the truncated quasi-renewal processes by omitting from the range of possible values some of non-negative integers. Depending on the values omitted from the underlying distribution of a quasi-renewal process, there are three types of truncation: truncation above, truncation below and double truncation.

In particular, a quasi-renewal process truncated above $m$ means that for a given $t$, $N(t)$ can only take values of $0, 1, \cdots, m$. For such $N(t)$, let $P_i(t) \equiv \mathrm{P}[N(t) = i]$, then

$$P_i(t) = \frac{G^{(i)}(t) - G^{(i+1)}(t)}{1 - G^{(m+1)}(t)}, \qquad i = 0, 1, \cdots, m \tag{1}$$

The scaling of the pmf by $1 - G^{(m+1)}(t)$ is to ensure the distribution integrates to unity.

As a result, the first and second moments of $N(t)$ are given by:

$$\begin{aligned} \mathbf{E}[N(t)] &= \sum_{i=0}^{m} i(G^{(i)}(t) - G^{(i+1)}(t))/(1 - G^{(m+1)}(t)) \\ &= \frac{1}{1 - G^{(m+1)}(t)} [\sum_{i=1}^{m} i G^{(i)}(t) - \sum_{j=2}^{m+1} (j-1) G^{(j)}(t)] \qquad (\text{let } j = i+1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-G^{(m+1)}(t)}[\sum_{i=1}^{m} iG^{(i)}(t) - \sum_{j=2}^{m+1} jG^{(j)}(t) + \sum_{j=2}^{m+1} G^{(j)}(t)] \\
&= \frac{\sum_{i=1}^{m+1} G^{(i)}(t) - (m+1)G^{(m+1)}(t)}{1-G^{(m+1)}(t)} \\
&= \frac{\sum_{i=1}^{m} G^{(i)}(t) - mG^{(m+1)}(t)}{1-G^{(m+1)}(t)} \qquad (2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}[N^2(t)] &= \frac{1}{1-G^{(m+1)}(t)} \sum_{i=0}^{m} i^2(G^{(i)}(t) - G^{(i+1)}(t)) \\
&= \frac{1}{1-G^{(m+1)}(t)}[\sum_{i=1}^{m} i^2 G^{(i)}(t) - \sum_{j=2}^{m+1}(j-1)^2 G^{(j)}(t)] \qquad (\text{let } j = i+1) \\
&= \frac{\sum_{i=1}^{m} i^2 G^{(i)}(t) - (\sum_{j=2}^{m+1} j^2 G^{(j)}(t) - \sum_{j=2}^{m+1} 2jG^{(j)}(t) + \sum_{j=2}^{m+1} G^{(j)}(t))}{1-G^{(m+1)}(t)} \\
&= \frac{\sum_{i=1}^{m+1}(2i-1)G^{(i)}(t) - (m+1)^2 G^{(m+1)}}{1-G^{(m+1)}(t)} \\
&= \frac{\sum_{i=1}^{m}(2i-1)G^{(i)}(t) - m^2 G^{(m+1)}(t)}{1-G^{(m+1)}(t)}. \qquad (3)
\end{aligned}
$$

## 2.3 Censored Quasi-Renewal Processes

Censoring refers to the fact that any observations above (below) a certain value are reported or transformed into a single value. For example, the observation rule of $y$, given an underlying $y^*$, is as follows:

$$
y = \begin{cases} a, & \text{if } y^* > a \\ y^*, & \text{if } y^* \le a \end{cases}
$$

Similar to the case of truncation, there are three types of censoring: censoring above, censoring below and double censoring. Here we focus on the discussion of the case of censoring above because it is directly related to the proposed warranty policy.

Different from the truncated quasi-renewal processes, in a censored quasi-renewal process, the relative magnitude of $P_i(t)$ changes. In particular, let $\{N(t), t > 0\}$ be a quasi-renewal process censored above $m$, and again denote $P_i(t) \equiv \mathrm{P}[N(t) = i]$, then

$$
\begin{aligned}
P_i(t) &= G^{(i)}(t) - G^{(i+1)}(t), \qquad \text{for } i = 0, 1, \cdots, m-1 \\
P_m(t) &= 1 - \sum_{j=0}^{m-1} P_j(t) = G^{(m)}(t).
\end{aligned} \tag{4}
$$

Consequently, the first and second moments of $N(t)$ are:

$$
\begin{aligned}
\mathbf{E}[N(t)] &= \sum_{i=0}^{m-1} i(G^{(i)}(t) - G^{(i+1)}(t)) + mG^{(m)}(t) \\
&= \sum_{i=1}^{m-1} iG^{(i)}(t) - \sum_{j=2}^{m} (j-1)G^{(j)}(t) + mG^{(m)}(t) \qquad (\text{let } j = i+1) \\
&= \sum_{i=1}^{m} G^{(i)}(t)
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
\mathbf{E}[N^2(t)] &= \sum_{i=0}^{m-1} i^2(G^{(i)}(t) - G^{(i+1)}(t)) + m^2G^{(m)}(t) \\
&= \sum_{i=1}^{m-1} i^2G^{(i)}(t) - \sum_{j=2}^{m} (j-1)^2G^{(j)}(t) + m^2G^{(m)}(t) \qquad (\text{let } j = i+1) \\
&= \sum_{i=1}^{m-1} i^2G^{(i)}(t) - (\sum_{j=2}^{m} j^2G^{(j)}(t) - \sum_{j=2}^{m} 2jG^{(j)}(t) \\
&\quad + \sum_{j=2}^{m} G^{(j)}(t)) + m^2G^{(m)}(t) \\
&= \sum_{i=1}^{m} (2i-1)G^{(i)}(t).
\end{aligned} \tag{6}
$$

It should be noted that for the purpose of analyzing the repair-limit risk free warranty, one should apply the censored quasi-renewal processes because the policy in essence specifies a censoring rule on the number of repairs, and the threshold point $m$ has no impact on the probabilities of the number of repairs except when it is equal to $m$.

# 3 Analysis of Repair-Limit Risk-Free Warranties

In this section we study the warranty cost of the repair-limit risk-free policy with parameters $w$ and $m$. We assume that repairs are imperfect such that after each repair, the system is between the states of new and old. In particular, the imperfect repair can be modelled by a quasi-renewal process associated with $F$ and $\alpha$ ($0 < \alpha < 1$).

According to the definition of the policy, any warranted products will be repaired no more than $m$ times. Consequently, the failure process before the first replacement is actually a censored quasi-renewal process (censored above $m$). As a reminder, by the definition of a quasi-renewal processes, the inter-failure times in the process are independent and follow the distributions $F(\alpha^{1-n}x)$.

Let $N_a(w)$ and $N_b(w)$ be the number of repairs and the number of replacement under the warranty respectively. Since $w$ is predetermined, we will suppress it later on for simplicity. Denote $c_a$ as the repair cost per failure and $c_b$ the replacement cost per unit, then for the warranty cost per product sold $C$, we have

$$\mathrm{E}[C] = c_a \mathrm{E}[N_a] + c_b \mathrm{E}[N_b] \tag{7}$$

and

$$\mathrm{V}[C] = c_a^2 \mathrm{V}[N_a] + c_b^2 \mathrm{V}[N_b] + 2c_a c_b \mathrm{cov}(N_a, N_b). \tag{8}$$

It is worth noting that $N_a$ and $N_b$ are correlated and $\mathrm{cov}(N_a, N_b) \neq 0$. The relationship between them can be summarized as follows:

- $N_a < m$ implies $N_b = 0$
- $N_b > 0$ implies $N_a = m$
- $N_b = 0$ implies $N_a \leq m$
- $N_a = m$ implies $N_b \geq 0$

**Moments of $N_a$**

Since under the warranty replacement instead of repair will be performed if there are more than $m$ failures within $w$, it is obvious that $N_a$ is a realization of a quasi-renewal process censored above $m$. Based on the results in section 2.3, we have

$$\mathbf{E}[N_a] = \sum_{i=1}^{m} G^{(i)}(w) \tag{9}$$

$$\mathbf{E}[N_a^2] = \sum_{i=1}^{m}(2i-1)G^{(i)}(w). \tag{10}$$

**The Pivot Point $S_m$**

A pivot point is the time epoch that indicates the change of the type of warranty service. For the repair-limit risk-free policy, clearly $S_m$ is the pivot point because any failed products will be replaced instead of being repaired again afterward. Let $H(t_p)$ be the cdf of $S_m$, then

$$\begin{aligned} H(t_p) &\equiv \mathrm{P}[S_m \le t_p] \\ &= G^{(m)}(t_p), \quad t_p \ge 0. \end{aligned} \tag{11}$$

**Moments of $N_b$**

$N_b \ge 0$ if $N_a = m$, the latter holds if and only if $S_m \le w$. Suppose $S_m = t_p, 0 \le t_p \le w$, then starting from $t_p$, the system failure process becomes a delayed renewal process with the first failure time having the distribution $F_{m+1}$ ($F_{m+1}(x) = F(\alpha^{-m}x)$) , and all the following failure times are i.i.d. with distribution $F$.

Conditioning on $S_m = t_p, t_p \le w$, from the theory of renewal processes,

$$\mathrm{E}[N_b|S_m = t_p, t_p \le w] = M_d(w - t_p),$$

where $M_d(\cdot)$, the renewal function for the delayed renewal process, is given by

$$M_d(t) = \sum_{i=0}^{\infty} F_{m+1} * F^{(i)}(t).$$

Here $F^{(i)}(\cdot)$ is the $i$-fold convolution of $F(\cdot)$ itself and $F^0(\cdot) = 1$ and

$$F_{m+1} * F^{(i)}(t) = \int_0^t F_{m+1}(t-x)\mathrm{d}F^{(i)}(x), \quad i \geq 0$$

is the convolution of $F_{m+1}$ and $F^{(i)}$.

After un-conditioning on $S_m$,

$$\mathrm{E}[N_b] = \int_0^w M_d(w - t_p)\mathrm{d}H(t_p). \tag{12}$$

Similar techniques can be used to obtain $\mathrm{E}[N_b^2]$. Let $M_{d,2}(w-t_p) \equiv \mathrm{E}[N_b^2|S_m = t_p, t_p \leq w]$, then

$$\mathrm{E}[N_b^2] = \int_0^w M_{d,2}(w - t_p)\mathrm{d}H(t_p), \tag{13}$$

where $M_{d,2}(\cdot)$ can obtained by

$$M_{d,2}(t) = \sum_{i=0}^{\infty} (2i+1) F_{m+1} * F^{(i)}(t).$$

For more discussion on delayed renewal processes, we refer to [18].

**Covariance of** $(N_a, N_b)$

Next we determine $\mathrm{cov}(N_a, N_b)$. Since $\mathrm{cov}(N_a, N_b) = \mathrm{E}[N_a N_b] - \mathrm{E}[N_a]\mathrm{E}[N_b]$, it is sufficient to know $\mathrm{E}[N_a N_b]$.

Since

$$\begin{aligned}
\mathrm{E}[N_a, N_b] &= \sum_{n_b=0}^{\infty} \sum_{n_a=0}^{m} n_a n_b \mathrm{P}[N_a = n_a, N_b = n_b] \\
&= m \sum_{n_b=0}^{\infty} n_b \mathrm{P}[m, n_b] \\
&= m \sum_{n_b=1}^{\infty} n_b \int_0^w \mathrm{P}[n_b|S_m = t_p]\mathrm{d}H(t_p)
\end{aligned}$$

and for $t_p \leq w$,

$$\mathrm{P}[N_b = 0 | S_m = t_p] = 1 - F_{m+1}(w - t_p)$$

$$\mathrm{P}[N_b = n_b | S_m = t_p] = F_{m+1} * G^{(n_b - 1)}(w - t_p) - F_{m+1} * G^{(n_b)}(w - t_p), \quad n_b \geq 1,$$

we obtain

$$\mathrm{E}[N_a, N_b] = m \sum_{n_b=1}^{\infty} n_b \int_0^w [F_{m+1} * G^{(n_b-1)}(w - t_p) - F_{m+1} * G^{(n_b)}(w - t_p)] \mathrm{d}H(t_p). \quad (14)$$

Consequently,

$$\begin{aligned} \mathrm{cov}[N_a, N_b] \quad = \quad & m \sum_{n_b=1}^{\infty} n_b \int_0^w [F_{m+1} * G^{(n_b-1)}(w - t_p) - F_{m+1} * G^{(n_b)}(w - t_p)] \\ & - \int_0^w M_d(w - t_p) \mathrm{d}H(t_p) \sum_{i=1}^{m} G^{(i)}(w). \end{aligned} \quad (15)$$

**First and Second Warranty Cost Moments**

Substituting equations (9) and (12) into (7), we obtain the expected warranty cost per unit sold:

$$\mathrm{E}[C] = c_a \sum_{i=1}^{m} G^{(i)}(w) + c_b \int_0^w M_d(w - t_p) \mathrm{d}H(t_p) \quad (16)$$

The variance of the warranty cost per unit sold can be obtained through equations (10), (13) and (15):

$$\begin{aligned} \mathrm{V}[C] \quad = \quad & c_a^2 \{ \sum_{i=1}^{m} (2i - 1) G^{(i)}(w) - [\sum_{i=1}^{m} G^{(i)}(w)]^2 \} \\ & + c_b^2 \{ \int_0^w M_{d,2}(w - t_p) \mathrm{d}H(t_p) - [\int_0^w M_d(w - t_p) \mathrm{d}H(t_p)]^2 \} \\ & + 2 c_a c_b \{ m \sum_{n_b=1}^{\infty} n_b \int_0^w [F_{m+1} * G^{(n_b-1)}(w - t_p) - F_{m+1} * G^{(n_b)}(w - t_p)] \\ & - \int_0^w M_d(w - t_p) \mathrm{d}H(t_p) \sum_{i=1}^{m} G^{(i)}(w) \} \end{aligned} \quad (17)$$

# 4 Special Cases

**Case I**:

Suppose for finite $w$, $m = 0$. In this case, no repair is allowed so all failed products within $w$ will always be replaced. This implies that $F_i \sim F, \forall i, i \geq 1$. So the warranty policy degenerates to the regular free replacement policy. As a result, equation (16) becomes

$$\mathrm{E}[C] = c_b M(w) \tag{18}$$

And equation (17) changes to

$$\mathrm{V}[C] \quad = \quad c_b^2\{M_2(w) - (M(w))^2\} \tag{19}$$

where $M(t)$ and $M_2(w)$ are the first and the second moments of the number of renewals in a renewal process associated with $F$.

These are the well-known results for the FRW policy [2].

**Case II**:

Consider $m = \infty$ and $w$ is finite. Thus no change of warranty service will ever happen and all failed products within $w$ will be repaired. Consequently, we have:

$$\mathrm{E}[C] = c_a \sum_{i=1}^{\infty} G^{(i)}(w) \tag{20}$$

and

$$\mathrm{V}[C] \quad = \quad c_a^2\{\sum_{i=1}^{\infty}(2i-1)G^{(i)}(w) - [\sum_{i=1}^{\infty} G^{(i)}(w)]^2\} \tag{21}$$

These results agree with the study in [23].

**Case III**:

Suppose for finite positive integer valued $m$, $w$ is large such that it can be treated as infinity. In this case, $\mathrm{E}[C] \rightarrow \infty$ since it is strictly increasing in $w$. One may be interested in determining the warranty cost per unit time (long-run average cost), another cost measure that is commonly used in the warranty and maintenance literature. It is not difficult to see that the long-run average cost $\mathrm{E}[C']$ is given by

$$\mathrm{E}[C'] = \frac{c_b}{\int_{-\infty}^{\infty} x f(x) \mathrm{d}x} \tag{22}$$

It is worth noting that this measure is only a crude approximation for the true warranty cost per unit time, and the accuracy heavily depends on the magnitude of $w$ compared to the product life times.

**Case IV**:

Assume $F$ has a normal distribution with mean $\mu$ and variance $\sigma^2$ for finite positive integer-valued $m$ and finite $w$. That is, $F \sim N(\mu, \sigma^2)$. As a result, the inter-occurrence failure times under the imperfect repairs are independent and also follow normal distribution. In particular, it is easy to see that $F_i \sim N(\alpha^{i-1}\mu, \alpha^{2(i-1)}\sigma^2)$. Thus $G^{(i)} \sim N(\frac{1-\alpha^i}{1-\alpha}\mu, \frac{1-\alpha^{2i}}{1-\alpha^2}\sigma^2)$. The pivot point distribution $H(t_p)$ is given by

$$H(t_p) = \Psi((t_p - \frac{1-\alpha^m}{1-\alpha}\mu)/\sqrt{(1-\alpha^{2m})/(1-\alpha^2)\sigma^2}\ ) \tag{23}$$

where $\Psi(\cdot)$ is the cdf of the standard normal distribution.

To compute $M_d(w - t_p)$, we need $F_{m+1} * F^{(i)}$, which obeys the distribution $N((\alpha^m + i)\mu, (\alpha^{2m} + i)\sigma^2)$. So

$$M_d(w - t_p) = \sum_{i=0}^{\infty} \Psi(\frac{w - t_p - (\alpha^m + i)\mu}{\sigma\sqrt{\alpha^{2m} + i}}) \tag{24}$$

Similarly,

$$M_{d,2}(w - t_p) = \sum_{i=0}^{\infty} (2i+1)\Psi(\frac{w - t_p - (\alpha^m + i)\mu}{\sigma\sqrt{\alpha^{2m} + i}}) \tag{25}$$

It is also necessary to obtain $F_{m+1} * G^{n_b - 1}$ and $F_{m+1} * G^{n_b}$. Clearly that they again follow normal distribution with parameters $((\alpha^m + \frac{1-\alpha^{n_b-1}}{1-\alpha})\mu, (\alpha^{2m} + \frac{1-\alpha^{2(n_b-1)}}{1-\alpha^2})\sigma^2)$ and $((\alpha^m + \frac{1-\alpha^{n_b}}{1-\alpha})\mu, (\alpha^{2m} + \frac{1-\alpha^{2n_b}}{1-\alpha^2})\sigma^2)$ respectively.

To obtain the expected warranty cost, combining the previous results together, equation (16) is simplified to

$$
\begin{align*}
\mathrm{E}[C] &= c_a \sum_{i=1}^{m} \Psi(\frac{w - \frac{1-\alpha^i}{1-\alpha}\mu}{\sigma\sqrt{\frac{1-\alpha^{2i}}{1-\alpha^2}}}) \\
&+ c_b \int_0^w \sum_{i=0}^{\infty} \Psi(\frac{w - t_p - (\alpha^m + i)\mu}{\sigma\sqrt{\alpha^{2m}+i}}) \mathrm{d}\Psi(\frac{t_p - \frac{1-\alpha^m}{1-\alpha}\mu}{\sigma\sqrt{(1-\alpha^{2m})/(1-\alpha^2)}}) \tag{26}
\end{align*}
$$

The variance can be simplified in a similar way:

$$
\begin{align*}
\mathrm{V}[C] &= c_a^2\{\sum_{i=1}^{m}(2i-1)\Psi(\frac{w - \frac{1-\alpha^i}{1-\alpha}\mu}{\sigma\sqrt{\frac{1-\alpha^{2i}}{1-\alpha^2}}}) - [\sum_{i=1}^{m}\Psi(\frac{w - \frac{1-\alpha^i}{1-\alpha}\mu}{\sigma\sqrt{\frac{1-\alpha^{2i}}{1-\alpha^2}}})]^2\} \\
&+ c_b^2\{\int_0^w \sum_{i=0}^{\infty}(2i+1)\Psi(\frac{w - t_p - (\alpha^m+i)\mu}{\sigma\sqrt{(\alpha^{2m}+i)}})\mathrm{d}\Psi(\frac{t_p - \frac{1-\alpha^m}{1-\alpha}\mu}{\sigma\sqrt{(1-\alpha^{2m})/(1-\alpha^2)}}) \\
&- [\int_0^w \sum_{i=0}^{\infty}\Psi(\frac{w - t_p - (\alpha^m+i)\mu}{\sigma\sqrt{\alpha^{2m}+i}})\mathrm{d}\Psi(\frac{t_p - \frac{1-\alpha^m}{1-\alpha}\mu}{\sigma\sqrt{(1-\alpha^{2m})/(1-\alpha^2)}}) \\
&+ 2c_ac_b\{m\sum_{n_b=1}^{\infty} n_b \int_0^w [\Psi(\frac{w - t_p - (\alpha^m + \frac{1-\alpha^{n_b-1}}{1-\alpha})\mu}{\sigma\sqrt{\alpha^{2m} + \frac{1-\alpha^{2(n_b-1)}}{1-\alpha^2}}}) \\
&- \Psi(\frac{w - t_p - (\alpha^m + \frac{1-\alpha^{n_b}}{1-\alpha})\mu}{\sigma\sqrt{\alpha^{2m} + \frac{1-\alpha^{2n_b}}{1-\alpha^2}}})]\mathrm{d}\Psi(\frac{t_p - \frac{1-\alpha^m}{1-\alpha}\mu}{\sigma\sqrt{(1-\alpha^{2m})/(1-\alpha^2)}}) \\
&- \int_0^w \sum_{i=0}^{\infty}\Psi(\frac{w - t_p - (\alpha^m+i)\mu}{\sigma\sqrt{\alpha^{2m}+i}})\mathrm{d}\Psi(\frac{t_p - \frac{1-\alpha^m}{1-\alpha}\mu}{\sigma\sqrt{(1-\alpha^{2m})/(1-\alpha^2)}}) \\
&* \sum_{i=1}^{m}\Psi(\frac{w - \frac{1-\alpha^i}{1-\alpha}\mu}{\sigma\sqrt{\frac{1-\alpha^{2i}}{1-\alpha}}})\} \tag{27}
\end{align*}
$$

# 5 Sensitivity Analysis

For illustration purpose, let us consider a simple numerical example. Suppose

$$F \sim N(4,1), m = 1, w = 2.5 \text{ year}, c_a = \$100, c_b = \$5,000, \text{ and } \alpha = 0.70.$$

Using equations (26) and (27), we obtain that $\mathrm{E}[C] = \$6.69$, which accounts for 3.35% of the unit production (replacement) cost. By looking into the components of $\mathrm{E}[C]$, we find that the repair cost is the dominant source of the warranty cost as

Table 1: E[$C$] and V($C$) for $\alpha = 0.9$ and $c_a = 100$, $c_b = 200$

| w | E[C] | | | V(C) | | |
|---|---|---|---|---|---|---|
| | m=1 | m=2 | m=3 | m=1 | m=2 | m=3 |
| 1.0 | 0.1350 | 0.1350 | 0.1350 | 13.4993 | 13.4947 | 13.4947 |
| 1.5 | 0.6214 | 0.6213 | 0.6213 | 61.8705 | 61.7975 | 61.7975 |
| 2.0 | 2.2776 | 2.2766 | 2.2766 | 223.3563 | 222.7911 | 222.7912 |
| 2.5 | 6.6944 | 6.6882 | 6.6882 | 628.7408 | 625.5939 | 625.5947 |
| 3.0 | 15.9258 | 15.8970 | 15.8970 | 1357.0441 | 1343.2721 | 1343.2765 |

it contributes 99.80% to the total cost. This is what one should expect since the probability of more than one failures within $w$ is truly small (0.0075%), indicating that most time no replacement will ever happen within a warranty period.

The standard deviation of the warranty cost is 25.07, indicating a moderate risk contained in this warranty policy. When decomposing V($C$), not surprisingly we find that the dominant source of the variation in the warranty cost per unit sold is from the repair cost. The contributions from repair, replacement and the interaction between them are $99.16\%, 0.44\%$ and $0.41\%$ respectively.

It is of interest to know how the warranty cost measures E[$C$] and V($C$) change with regard to parameters $m, w, \alpha, c_a$ and $c_b$. We first vary $m$ in $\{1, 2, 3\}$ and $w$ in $\{1.0, 1.5, 2.0, 2.5, 3.0\}$ while keeping other parameters unchanged. The corresponding reliability of a new warranted product evaluated at $w$ are within the range of 84.13% to 99.87%.

From table 1, it is clear that both E[$C$] and V($C$) are monotonically increasing in $w$. This is expected since the chance of failures increases as the warranty period becomes longer. As to the impact from $m$ on the expected warranty cost, it seems that such impact is very small for all values of $w$ investigated. Therefore, manufacturers could simply choose $m = 1$ as long as the warranted products are reasonably

Table 2: Moments of Warranty Cost for Various $\alpha$

| w | E[C] | | | V(C) | | |
|---|---|---|---|---|---|---|
| | $\alpha$=0.9 | $\alpha$=0.7 | $\alpha$=0.5 | $\alpha$=0.9 | $\alpha$=0.7 | $\alpha$=0.5 |
| 1.0 | 0.1350 | 0.1351 | 0.1354 | 13.4947 | 13.5110 | 13.6031 |
| 1.5 | 0.6213 | 0.6217 | 0.6240 | 61.7975 | 61.9226 | 62.6523 |
| 2.0 | 2.2766 | 2.2792 | 2.2936 | 222.7911 | 223.5735 | 228.2000 |
| 2.5 | 6.6882 | 6.7023 | 6.7765 | 625.5939 | 629.5976 | 653.4632 |
| 3.0 | 15.8970 | 15.9593 | 16.2781 | 1343.2721 | 1359.8630 | 1459.7480 |

reliable.

To investigate how the effort of repair affects the warranty cost, we consider three different levels of $\alpha$ (higher $\alpha$ indicates better repair). All other parameters are kept the same and $m$ is fixed at 2. As expected, as $\alpha$ increases, the moments of $C$ decreases (see table 2). This implies that one way to reduce the warranty cost and mitigate the warranty cost risk is by improving the repair quality.

Replacement cost to repair cost ratio is another important factor in determining the warranty cost. In table 3, we report the results for various cost ratios for $w = 3.0$, $\alpha = 0.5$ and $m = 1$. It is clear that the cost ratio is positively related to both E($C$) and V($C$), implying that as replacement becomes more expensive compared to repair, the repair-limit warranty policy will be more costly with higher warranty cost risk.

Table 3: Moments of Warranty Cost for Various Cost Parameters

| $(c_a, c_b)$ | (100,200) | (100,500) | (100,1000) | (100,2000) |
|---|---|---|---|---|
| E[C] | 16.5884 | 17.6726 | 19.4797 | 23.0939 |
| V(C) | 1600.5120 | 2539.5770 | 5546.1550 | 16964.8300 |

# 6 Conclusion Remarks

In this paper, we studied a repair-limit risk-free warranty policy and provided the first and second moments of the warranty cost per unit sold through censored quasi-renewal processes. Warranty designers such as manufacturers are constantly in search of novel ideas to promote their products due to the more than ever fierce competition. Based on our research, the proposed repair-limit risk-free warranty may be a good candidate for marketing purpose since it provides extra compensation to consumers suffering from low quality products with a relatively low cost.

In the numerical example, we assume the failure time of a new product follows a normal distribution to simplify the computational work. One may improve the study by considering non-negative failure time distributions such as Weibull distribution, Gamma distribution, or truncated normal distribution based on Tobit models [20].

**Acknowledgements**

We would like to thank the referees and the editor for their helpful suggestions and constructive comments on the previous version of this paper.

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