Morita Similar Matrix Rings and their Grothendieck Groups

Syed. Khalid Nauman
Department of Mathematics, Faculty of Science
King Abdulaziz University, P.O.Box 80203
Jeddah 21589, Saudi Arabia

Abstract

In this work sequences of higher order Morita similar matrix rings are constructed from arbitrarily given Morita contexts and their Grothendieck groups are computed.

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Introduction

In the following we have constructed sequences of Morita similar matrix rings by using generalized matrix rings of Morita contexts. The basic technique discussed here is also a generalization of Morita theory of equivalences of module categories.

Historically, for any pair of associative rings A and B with identity, the classical Morita theory completely describes the equivalence of module categories $Mod-A$ and $Mod-B$ via a fixed object M of $B-Mod-A$, termed as a progenerator of $Mod-A$, such that the adjoint functors $Hom_A(M,-)$ and $-\otimes_B M$ determine the equivalence and the inverse equivalence, respectively. This theory is extensively generalized and investigated in several directions. One way is to impose conditions on the fixed object M, thus introducing the terms, like, sigma-quasi projective [10], semi-sigma-quasi projective [22], or a star module [16,7] (= finitely generated self-tilting module [23]). Another way is via representation theory, for instance, generalization of Clifford theory to algebras using twisted modules [5], group-graded rings and modules [8], group algebras over finite groups [2,1]. A more general option is to look at those objects of $Mod-A$ and $Mod-B$ which remain invariant under the composite functors, $Hom_A(M,-)\otimes_B M$ and $Hom_A(M,-\otimes_B M)$, whence the objects are termed as static and adstatic [19,1,21,24,12]. In a different setting, Müller in [18], extended the idea to the quotient categories.

In all above extensions and generalizations the adjoint pair of functors, $(Hom_A(M,-),-\otimes_B M)$ (or at least one of them) play the pivotal rule. The technique introduced here is completely free from the involvement of the adjoint functors and replaced them by the rings of generalized matrices. This may be a sort of migration of the homological treatment of the theory to generalized linear algebra. The idea is very simple and direct and only requires to establish surjections at couple of places.

As an application, after the construction of sequences of these generalized matrix rings, we will compute their Grothendieck groups K_o. In the end, a formula for finding K_o is deduced for a Morita ring in the case when both context maps are monic.

1. Notations and Terminologies

(1.a) Let $T = [A, M, N, B, \langle , \rangle_A, \langle , \rangle_B, I, J]$ be a Morita context (mc) in which, A and B are associative rings with multiplicative identities, M and N are (B, A) and (A, B) bimodules (unital), respectively, $\langle , \rangle_A : N \otimes_B M \to A$ and $\langle , \rangle_B : M \otimes_A N \to B$ are the *mc maps* such that they satisfy the two associative conditions: $m' \langle n, m\rangle_A = \langle m', n\rangle_B m$ and $\langle n, m\rangle_A n' = n \langle m, n'\rangle_B$, and $I = \mathrm{Im} \langle , \rangle_A$ and $J = \mathrm{Im} \langle , \rangle_B$ are the two trace ideals of A and B, respectively. Simultaneously, we denote by

$$R = \begin{bmatrix} A & N \\ M & B \end{bmatrix},$$

the generalized matrix ring of the mc T, in which sum is defined elementwise and the product is defined by the rule:

$$\begin{bmatrix} a & n \\ m & b \end{bmatrix} \begin{bmatrix} a' & n' \\ m' & b' \end{bmatrix} = \begin{bmatrix} aa' + \langle n, m'\rangle_A & an' + nb' \\ ma' + bm' & \langle m, n'\rangle_B + bb' \end{bmatrix}.$$

Note that in above the lower case variables x, x' belong to the upper case variables X.

Let us adopt the following dictionary of abbreviations: T is said to be an injective Morita context (imc) (respt. projective Morita context (pmc)) if the mc maps $\langle , \rangle_A$and $\langle , \rangle_B$ are monic, (respt. epic), and a $semi-imc$ (respt. $semi-pmc$) if one of the mc maps $\langle , \rangle_A$or $\langle , \rangle_B$ is monic (respt. epic). An imc ring is a ring of an imc. Similarly a pmc ring, a $semi-imc$ and a $semi-pmc$ ring are defined.

Note that an mc is an imc iff $N \otimes_B M \cong I$ under $\langle , \rangle_A$ and $M \otimes_A N \cong J$ under $\langle , \rangle_B$ and a pmc iff both Morita maps are epic. In this last case both maps become isomorphisms and $Mod - A \approx Mod - B$ and $A - Mod \approx B - Mod$ and we say that A is Morita similar to B. Note also that pmc's are extensively studied and have abundance of literature, while on the contrary, imc's are not that much attractive, a few applications are outlined in [20,21].

(1.b) Let P be an (A,A)-bimodule and Q an (A,B)-bimodule. Then clearly, the row vector $[P,Q]$ is a left A-module while it may be given a structure of right R-module by using a pair of maps: $f : Q \otimes_B M \to P$ and $g : P \otimes_A N \to Q$ which together with the associativity conditions of mc maps satisfy the commutativity of the diagrams

$$\begin{array}{ccc} Q \otimes_B M \otimes_A N & \xrightarrow{I_Q \otimes \langle , \rangle_B} & Q \otimes_B B \\ f \otimes I_N \downarrow & & \downarrow \cong \\ P \otimes_A N & \xrightarrow[g]{} & Q \end{array}$$

and

$$\begin{array}{ccc} P \otimes_A N \otimes_B M & \xrightarrow{I_P \otimes \langle , \rangle_A} & P \otimes_A A \\ g \otimes I_M \downarrow & & \downarrow \cong \\ Q \otimes_B M & \xrightarrow[f]{} & P \end{array} .$$

The action of R on $[P,Q]$ is given by the rule

$$\begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} a & n \\ m & b \end{bmatrix} = \begin{bmatrix} pa + f(q \otimes m) & g(p \otimes n) + qb \end{bmatrix},$$

where the variables $x \in X$.

Similarly, a column vector $\begin{bmatrix} U \\ V \end{bmatrix}$ in which U is an (A,A)-bimodule and V is a (B,A)-bimodule may be given a structure of an (R,A)-bimodule. Thus, from above, or directly, $\begin{bmatrix} A \\ M \end{bmatrix}$ and $\begin{bmatrix} A & N \end{bmatrix}$ can immediately be given the structures of (R,A) and (A,R) bimodules, respectively. In appearance we will not encounter above situation, but technically, all objects of $Mod - R$ and $R - Mod$ inherit such module structures, where R is the generalized matrix ring of any order of some mc.

2. Construction of a pmc ring from an arbitrary mc.

We do the basic construction iteratively in the following theorem.

Theorem 2.1. For any arbitrary mc there always is

(i) a semi-pmc and a semi-pmc ring,

(ii) a pmc and a pmc ring.

Proof: Let $T = [A, M, N, B, \langle , \rangle_A, \langle , \rangle_B]$ be any mc and $R = \begin{bmatrix} A & N \\ M & B \end{bmatrix}$ its mc ring. We will achieve our goal in the following three iterative steps.

Step-I Consider the set $T' = [R, N', M', A, \langle , \rangle'_R, \langle , \rangle'_A]$, in which we let $M' = \begin{bmatrix} A \\ M \end{bmatrix}$ and $N' = \begin{bmatrix} A & N \end{bmatrix}$. The symbols $\langle , \rangle'_A$ and $\langle , \rangle'_R$ are the maps from $N' \otimes_R M' \to A$ and $M' \otimes_A N' \to R$, respectively, defined by

$$< \begin{bmatrix} a & n \end{bmatrix}, \begin{bmatrix} a' \\ m' \end{bmatrix} >'_A = aa' + < n, m' >_A$$

and

$$< \begin{bmatrix} a \\ m \end{bmatrix}, \begin{bmatrix} a' & n' \end{bmatrix} >'_R = \begin{bmatrix} aa' & an' \\ ma' & \langle m, n' \rangle_B \end{bmatrix}.$$

It is routine work to check the associativity conditions for both maps, hence T' becomes an mc. Moreover, for any $a \in A$,

$$< \begin{bmatrix} a & 0_N \end{bmatrix}, \begin{bmatrix} 1_A \\ 0_M \end{bmatrix} >'_A = a,$$

which shows that $\langle , \rangle'_A$ is epic, hence T' is a semi-pmc.

From T' we arrange the $3 \times 3-$matrix

$$R' = \begin{bmatrix} A & N & A \\ M & B & M \\ A & N & A \end{bmatrix},$$

which is a ring under elementwise addition and multiplication is defined by the rule

$$: R' \times R' \longrightarrow \begin{bmatrix} AA + \langle N, M \rangle_A + AA & AN + NB + AN & AA + \langle N, M \rangle_A + AA \\ MA + BM + MA & \langle M, N \rangle_B + BB + \langle M, N \rangle_B & MA + BM + MA \\ AA + \langle N, M \rangle_A + AA & AN + NB + AN & AA + \langle N, M \rangle_A + AA \end{bmatrix}.$$

In our terminology R' is a semi-pmc ring. Hence (i) holds.

Step-II For (ii), assume temporarily that, in the mc T, the map $\langle , \rangle_B$ is epic, so that every element $b \in B$ is of the form

$$b = \Sigma < m_i, n_i >_B .$$

Then one may express an arbitrary element of R in the form

$$\begin{bmatrix} a & n \\ m & b \end{bmatrix} = <\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a & n \end{bmatrix}>'_R + <\begin{bmatrix} 0 \\ m \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}>'_R$$
$$+\Sigma < \begin{bmatrix} 0 \\ m_i \end{bmatrix}, \begin{bmatrix} 0 & n_i \end{bmatrix}>'_R.$$

This shows that the map $<,>'_R$ is also epic. Using Step-I, we conclude that the context T' is a *pmc*.

Step-III Now again go back to Step-I. One may similarly construct from T' the context

$$T'' = \left[R', M'', N'', R, \langle,\rangle''_{R'}, \langle,\rangle''_R\right],$$

where

$$N'' = \begin{bmatrix} A & N \\ M & B \\ A & N \end{bmatrix}$$

and

$$M'' = \begin{bmatrix} A & N & A \\ M & B & M \end{bmatrix}$$

are (R', R) and (R, R') bimodules, respectively, and the two *mc* maps go smoothly over

$$<,>''_{R'}: N'' \otimes_R M'' \to R' \quad \text{and} \quad <,>''_R: M'' \otimes_{R'} N'' \to R.$$

Again, to verify that $<,>''_R$ and $<,>''_{R'}$ are *mc* maps and that they satisfy the associativity conditions is a routine work. In Inheritance, $<,>''_R$ is epic and according to Step-II, $<,>''_{R'}$ is also epic. Hence we conclude that T'' is a *pmc* and R'' is a *pmc* ring. ¥

Note that R'' can be arranged as a 5×5 matrix

$$R'' = \begin{bmatrix} A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \end{bmatrix}.$$

which is a ring under the similar rules of addition and multiplication as in the case of R or R'.

Following theorem can be compared with the previously known generalizations, for instance, see [7, Theorems 2.2, 3.3].

Theorem 2.2. For any arbitrary pair of rings (A, B) along with an mc $T = [A, M, N, B]$ there exist a pair of rings (A', B') such that and $A \cong A'$ and $B \cong B'$ and another pair of rings (S, R) with the following properties:

(i) $Mod-S$ and $Mod-R$ are full and additive subcategories of $Mod-A'$ and $Mod-B'$, respectively.

(ii) $Mod-S \approx Mod-R$ and $S-Mod \approx R-Mod$.

(iii) There are objects P in $B'-Mod-A'$ and Q in $A'-Mod-B'$ such that $P \in R-Mod-S$ and $Q \in S-Mod-R$ are progenerators in their respective categories.

Proof: Existence: Set $A' = \begin{bmatrix} A & 0_N & 0_A \\ 0_M & 1_B & 0_M \\ 0_A & 0_N & 1_A \end{bmatrix}$ and $B' = \begin{bmatrix} 1_A & 0_N \\ 0_M & B \end{bmatrix}$.
Then $A' \cong A$ and $B' \cong B$.

Also assume that $S = R' = \begin{bmatrix} A & N & A \\ M & B & M \\ A & N & A \end{bmatrix}$ and as before $R = \begin{bmatrix} A & N \\ M & B \end{bmatrix}$.

(i) Clearly, A' is a subring of S and B' is a subring of R. So every $S-$module, including S itself, is an $A'-$module and likewise every $R-$module is a $B'-$module.

(ii) The equivalences hold, since $T'' = [S, M'', N'', R]$ is a pmc as proved in Theorem 2.1.

(iii) Set $Q = N'' = \begin{bmatrix} A & N \\ M & B \\ A & N \end{bmatrix}$ and $P = M'' = \begin{bmatrix} A & N & A \\ M & B & M \end{bmatrix}$. These are the desired progenerators as it is proved above that $N'' \otimes_R M'' \cong R'$ and $M'' \otimes_{R'} N'' \cong R$. ¥

Now we use above theorems to obtain the sequences of semi-pmc and pmc rings.

Corollary 2.3. An arbitrary mc produces a sequence of semi-pmc rings which are generalized matrix rings of orders $n = 3$.

Proof: From $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ we have obtained T' which we rearrange as $T_1' = [A, M', N', R, \langle,\rangle_A', \langle,\rangle_R']$, in which the map $\langle,\rangle_A'$ is epic. Then this semi-*pmc* gives us the $3 \times 3-$matrix ring

$$R_1' = \begin{bmatrix} A & A & N \\ A & A & N \\ M & M & B \end{bmatrix}.$$

By replacing R by R_1' in T_1' we get the context $T_1'' = [A, M_1'', N_1'', R_1', \langle,\rangle_A'', \langle,\rangle_{R_1'}'']$, in which

$$M_1'' = \begin{bmatrix} A \\ A \\ M \end{bmatrix} \quad \text{and} \quad N_1'' = \begin{bmatrix} A & A & M \end{bmatrix}$$

are (R_1', A) and $(A, R_1')-$bimodules, respectively. One can easily deduce that both maps satisfy the associativity properties, and in particular, the map $\langle,\rangle_A''$ is epic. Thus its *mc* ring R_1'' is a semi-*pmc* ring. Consequently, we get the semi-*pmc*

$$T_1^{(n-2)} = [A, M^{(n-2)}, N^{(n-2)}, R_1^{(n-3)}, \langle,\rangle_A^{(n-2)}, \langle,\rangle_{R_1^{(n-3)}}^{(n-2)}]$$

with the epimorphism $\langle,\rangle_A^{(n-2)}$ and hence from this context we get the $n \times n-$ generalized matrix ring

$$R_1^{(n-2)} = \begin{bmatrix} A & A & \cdots & A & N \\ A & A & \cdots & A & N \\ \vdots & \vdots & & \vdots & \vdots \\ A & A & \cdots & A & N \\ M & M & \cdots & M & B \end{bmatrix},$$

which is a semi-*pmc* ring. ¥

Corollary 2.4. In case the *mc* map $\langle,\rangle_B$ in T is epic, then the semi-*pmc* rings $R_1', \cdots, R_1^{(n-2)}$ will become *pmc* rings and hence the rings A and R_1^j are Morita similar, where $j = 1, 2, \cdots, (n-2)$.

Proof: Follows from Step-II of Theorem 2.1 and Corollary 2.3.¥

As a consequence of this corollary we get the following well known result.

Corollary 2.5. Any ring A is Morita similar to its full matrix rings of any order.

Proof: Assume $A = B = M = N$ in the context T, then $R_1^{(n-2)} = M_n(A)$. Hence A and $M_n(A)$ are Morita similar rings.¥

Our next result is related to the famous Fibonacci sequence of natural numbers 1, 1, 2, 3, 5, 8, 13, $\cdots$.

Corollary 2.6. An arbitrary mc produces a sequence of pmc rings. The orders of these generalized matrix rings follow the pattern of the Fibonacci sequence.

Proof: In Step-III of the Theorem 2.1, from any arbitrary mc, $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ we have obtained the pmc $T'' = \left[R', M'', N'', R, \langle,\rangle''_{R'}, \langle,\rangle''_R\right]$ in which the 3×3- and $2\times 2-$matrix rings R' and R, respectively, are Morita similar. Thus the $5 \times 5-$matrix ring R'' is a pmc ring. The next pmc ring in this sequence is composed of R' and R'' will be a square matrix of order *eight*, after that of order *thirteen*, and so and so on. Thus the orders of these matrix rings follow the pattern of the famous Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, $\cdots$. First pmc ring in this sequence is of order 5 and the nth term of this sequence will be the pmc ring $R^{(n-3)}$ of order

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

which is the nth term in the Fibonacci sequence. ¥

Remark 2.7. The higher order generalized matrix rings can conveniently be studied by considering them as $2 \times 2-$ blocked matrix rings, for instance,

$$R_1^{(n-2)} = \begin{bmatrix} A & N_1^{(n-3)} \\ M_1'^{(n-3)} & R_1^{(n-3)} \end{bmatrix}.$$

Module structures over these rings can be studied by using the technique as described in (1.b) above.

3. The Grothendieck Group K_0

Morita context is the main source to compute the Grothendieck groups of various rings of matrices, for example, for any ring R, $K_0(R)$ and $K_0(M_n(R))$ are known to be isomorphic as the rings R and $M_n(R)$ are Morita similar. Bass in [3] described a formula for $K_0(T)$, where T is a Morita context ring in the case in which one of the context map is epic, while recently, in [11] Hao and Shun described a method of finding $K_0(T)$ in the case if one of the context map is monic and T is Noetherian. Using above techniques we will compute Grothendieck groups of higher order generalized matrix rings.

(3.a) The Grothendieck groups of semi-pmc and pmc rings.

In a semi-pmc, one of the mc maps is epic. Assume that $\langle,\rangle_A : N\otimes_B M \rightarrow A$ is epic. In this case A is its own trace ideal. Bass [3] showed that:

Theorem 3.1.(Bass) If $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ is an mc and if $\langle,\rangle_A$ is epic, then

$$K_0(R) \cong K_0(B).$$

Hence if both maps are epic or A and B are Morita similar, then

$$K_0(R) \cong K_0(A) \cong K_0(B).$$

Corollary 3.2. Let $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ be an mc in which $\langle,\rangle_B$ is epic. Then

$$K_0(A) \cong K_0(R_1^j),\ \text{where}\ \ j = 1,\ 2,\ \cdots,\ (n-2)..$$

Proof: In Corollary 2.4, since A and R_1^j, where $j = 1,\ 2,\ \cdots,\ (n-2).$, are Morita similar, their Grothendieck groups are isomorphic. ¥

Corollary 3.3. Let $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ be any mc. Then

$$\begin{aligned}(i)\ K_0(R) &\cong K_0(R_1') \cong K_0(R_1'') \cong \cdots \cong K_0(R_1^{(n-2)}),\\ (ii)\ K_0(R) &\cong K_0(R') \cong K_0(R'') \cong \cdots \cong K_0(R^{(n-2)}).\end{aligned}$$

Proof: (i) In Corollary 2.2 in the context $T_1' = [A, M', N', R, \langle,\rangle_A', \langle,\rangle_R']$, the map $\langle,\rangle_A'$ is epic and R_1' is the mc ring. Hence by the first part of Theorem 3.1, $K_0(R) \cong K_0(R_1')$. The rest follows similarly.

(ii) analogously holds by Corollary 2.5. ¥

(3.b). The Grothendieck group of semi-imc and imc rings.

Hao and Shum in [11] deduced a formula for finding K_0 of a semi-imc ring under finite conditions. An easy extension will lead this formula for finding K_0 of an imc ring. It is proved in [14] that the mc ring R of the context $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B]$ is right Noetherian if and only if each object A, B, M and N are right Noetherian. Let $\langle,\rangle_A : N \otimes_B M \to A$ be an injective map. In this case the trace ideal $I \cong N \otimes_B M$. Thus T is semi-imc if and only if either $I \cong N \otimes_B M$ as $(A, A)-$bimodule or $J \cong M \otimes_B N$ as $(B, B)-$bimodule. T is an imc if and only if both isomorphisms hold.

Theorem 3.4. (Hao & Shum). Let $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B, I, J]$ be a semi-imc with $I \cong N \otimes_B M$. If its mc ring R is right Noetherian and N is projective in $Mod - B$, then

$$K_0(R) \cong K_0(A/I) \oplus K_0(B).$$

From above we deduce a formula for finding K_0 of an imc ring.

Corollary 3.5. Let $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B, I, J]$ be an imc. If its mc ring R is right Noetherian, M is projective in $Mod - A$, and N is projective in $Mod - B$, then

$$K_0(R)^{(2)} \cong K_0(R_0) \oplus K_0(A/I) \oplus K_0(B/J),$$

where R_0 is the ring of the mc $T_0 = [A, 0, 0, B]$.

Proof: The matrix rings $R = \begin{bmatrix} A & N \\ M & B \end{bmatrix}$ and $R^t := \begin{bmatrix} B & M \\ N & A \end{bmatrix}$ are obtained from the same source $T = [A, M, N, B, \langle,\rangle_A, \langle,\rangle_B, I, J]$. Thus clearly we have group isomorphism

$$K_0(R) \cong K_0(R^t).$$

It is obvious that R^t is right Noetherian if and only if R is right Noetherian. Also, T is an *imc* iff both R and R^t are *imc* rings. In either case, $N \otimes_B M \cong I$ and $M \otimes_A N \cong J$. In addition, if M is projective in $Mod - A$ and N is projective in $Mod - B$, then according to Theorem 3.4,

$$K_0(R^t) \cong K_0(A) \oplus K_0(B/J),$$

and

$$K_0(R) \cong K_0(A/I) \oplus K_0(B).$$

Finally, if $R_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, it is proved in [9] that

$$K_0(R_0) \cong K_0(A) \oplus K_0(B).$$

Hence we conclude that:

$$\begin{aligned} K_0(R)^{(2)} &= K_0(R) \oplus K_0(R) \\ &\cong K_0(R) \oplus K_0(R^t) \\ &\cong \{K_0(A/I) \oplus K_0(B)\} \oplus \{K_0(A) \oplus K_0(B/J)\} \\ &\cong K_0(A) \oplus K_0(B) \oplus K_0(A/I) \oplus K_0(B/J) \\ &\cong K_0(R_0) \oplus K_0(A/I) \oplus K_0(B/J). \text{ ¥} \end{aligned}$$

4. Examples.

Examples 4.1 and 4.2 are extensions of Examples 5.1 and 5.3 of [11].

4.1. Any ring R and the non-zero idempotents e and $f = 1 - e$ give the decomposition, $\mathsf{R} = \mathsf{eRe} \oplus e\mathsf{Rf} \oplus \mathsf{fRe} \oplus \mathsf{fRf}$. Then one quickly obtains the *mc* ring $R \cong \begin{bmatrix} \mathsf{eRe} & \mathsf{eRf} \\ \mathsf{fRe} & \mathsf{fRf} \end{bmatrix}$ which gives the semi-*pmc* rings

$$\mathrm{R}' = \begin{bmatrix} \mathsf{eRe} & \mathsf{eRf} & \mathsf{eRe} \\ \mathsf{fRe} & \mathsf{fRf} & \mathsf{fRe} \\ \mathsf{eRe} & \mathsf{eRf} & \mathsf{eRe} \end{bmatrix} \cong \begin{bmatrix} \mathsf{R} & \mathsf{eRe} \oplus \mathsf{fRe} \\ \mathsf{eRe} \oplus \mathsf{eRf} & \mathsf{eRe} \end{bmatrix}$$

with the epimorphism $\langle , \rangle'_{\mathsf{eRe}}$. By Theorem 2.1 and Corollary 3.3, the rings R and R' are Morita similar, and $K_0(R) \cong K_0(R') \cong \cdots$. If the map $\langle , \rangle_{fRf}$ is epic, then $\mathsf{fRe} \otimes_{\mathsf{eRe}} \mathsf{eRf} \cong \mathsf{fRf}$. In this case

$$K_0(\mathsf{eRe}) \cong K_0(R) \cong K_0(R') \cong \cdots .$$

4.2. Let A be a ring and $e \in A$ an idempotent. Then $T = [A, eA, Ae, eAe, <, >_A, <, >_{eAe}]$ is an mc in which the mc map $<, >_{eAe}$: $eA \otimes_A Ae \to eAe$ is always epic and $<, >_A$: $Ae \otimes_A eA \to A$ is epic iff T is a pmc. So assume that $<, >_A$ is not epic and T is a semi-pmc. Then $R = \begin{bmatrix} A & Ae \\ eA & eAe \end{bmatrix}$ is a semi-pmc ring. But then by the Step-II of Theorem 2.1, the ring $R' = \begin{bmatrix} A & Ae & A \\ eA & eAe & eA \\ A & Ae & A \end{bmatrix}$ is a pmc ring and A is Morita similar to R. Hence

$$K_0(A) \cong K_0(R) \cong K_0(R') \cdots .$$

4.3. Let A be a ring, G a finite subgroup of $Aut(A)$, and A^G the fixed subring of A defined by

$$A^G = \{a : a^g = a, \ \forall g \in G\}.$$

Also consider the skew-group ring $A * G$ which consists of all formal sums of the form $\sum_{g \in G} a_g g, \ a_g \in A$, in which, as usual, sum is defined componentwise and product is defined distributively by the rule:

$$ag \cdot bh = ab^{g^{-1}} gh, \ \ \forall a, b \in A, \ g, h \in G.$$

Then with the bimodule structure on A given by A^G and also by $A * G$ one can always construct the mc ring

$$R = \begin{bmatrix} A^G & A \\ A & A * G \end{bmatrix},$$

where the bimodule mc maps are defined by the formulas

$$\langle a, b \rangle_{A*G} = \sum_{g \in G} ab^{g^{-1}} g, \text{ and } \langle a, b \rangle_{A^G} = \sum_{g \in G} (ab)^g, \ \forall a, b \in A.$$

For commutative rings, this context was introduced in [4], while the non-commutative case was studied by Cohen in [6]. We are interested in a different case in which we can obtain the Grothendieck groups directly by using Example 4.2.

Let us assume that the order of G is a unit of A. Then

$$e = |G|^{-1} \sum_{g\in G} g \in A * G$$

is an idempotent and as proved in [17] (see also [13]), that

$$e(A * G)e \cong A^G e \cong A^G,$$

this implies that the map $\langle , \rangle_{A^G}$ is epic, and as in Example 4.2, we can form the semi-*pmc* ring

$$R = \begin{bmatrix} A * G & (A * G)\, e \\ e\,(A * G) & A^G \end{bmatrix}.$$

Hence we conclude that

$$K_0\,(A * G) \cong K_0 \left(\begin{bmatrix} A * G & (A * G)\, e \\ e\,(A * G) & A^G \end{bmatrix} \right) \cong \cdots .$$

If, in addition, the map $\langle , \rangle_{A*G}$ is epic, then

$$K_0\,(A * G) \cong K_0(A^G) \cong \cdots .$$

4.4. The case of crossed product is more simpler. The crossed product of a group G over a ring A, also denoted by $A * G$, is a $G-$graded ring $A * G = \oplus \sum_{g\in G} A_g$ in which $A_g A_{g^{-1}} = A_1$, $\forall g \in G$, and each homogeneous component A_g contains a unit of A. The middle condition implies that $A * G$ is strongly $G-$graded. If any ring R is strongly $G-$graded, then as in [8,15], the graded ring and its 1_G-component which is a subring, are Morita similar. Hence, in this example, $K_0(A * G)$ and K_0 of all higher ordered generalized matrix rings are isomorphic to $K_0(A)$.

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Address: Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia.

Email For Correspondence: sknauman@hotmail.com