A characterization of rational singularities in terms of injectivity of Frobenius maps

Nobuo Hara

American Journal of Mathematics, Volume 120, Number 5, October 1998, pp. 981-996 (Article)

Published by The Johns Hopkins University Press
DOI: 10.1353/ajm.1998.0037

A CHARACTERIZATION OF RATIONAL SINGULARITIES IN TERMS OF INJECTIVITY OF FROBENIUS MAPS

By Nobuo Hara

Abstract. The notions of F-rational and F-regular rings are defined via tight closure, which is a closure operation for ideals in a commutative ring of positive characteristic. The geometric significance of these notions has persisted, and K. E. Smith proved that F-rational rings have rational singularities. We now ask about the converse implication. The answer to this question is yes and no. For a fixed positive characteristic, there is a rational singularity which is not F-rational, so the answer is no. In this paper, however, we aim to show that the answer is yes in the following sense: If a ring of characteristic zero has rational singularity, then its modulo p reduction is F-rational for almost all characteristic p. This result leads us to the correspondence of F-regular rings and log terminal singularities.

1. Introduction. In [HH1], Hochster and Huneke introduced the notion of the tight closure of an ideal in a commutative ring of characteristic $p > 0$. Tight closure enables us to define classes of rings of characteristic p such as F-rational rings [FW] and F-regular rings [HH1], and it turns out that they are closely related with some classes of singularities in characteristic zero defined via resolution of singularity.

By definition, rings in which all "parameter ideals" are tightly closed are said to be F-rational. It was shown by Smith [S] that F-rational rings have rational singularities. More precisely, she proved that F-rational rings are pseudo-rational. Pseudo-rationality is a resolution-free (hence characteristic-free) analogue of the notion of rational singularity [LT], and indeed, Smith's result holds true in arbitrary positive characteristic. But if we consider the converse implication, we soon confront some difficulty arising from pathological phenomena in small positive characteristic. For any fixed characteristic $p > 0$, there are rational singularities which are not F-rational (see [HW]). To avoid such difficulty we will look at "generic behavior" of modulo p reduction from characteristic zero.

Roughly speaking, a ring in characteristic zero is said to have *F-rational type* if its reduction modulo p is F-rational for $p \gg 0$ (see Definition 2.5 for a precise definition). The characteristic zero version of Smith's result says that a ring of F-rational type has at most rational singularity. Unfortunately, except for some special cases (Fedder [F1, 2]), the converse implication remained open, and is considered to be one of the fundamental problems in the tight closure theory.

Manuscript received August 7, 1997.
American Journal of Mathematics 120 (1998), 981–996.

Our main theorem answers this question affirmatively:

THEOREM 1.1. *Let R be a finitely generated algebra over a field k of characteristic zero. If R has at most rational singularity, then R is of F-rational type.*

On the other hand, there is the notion of F-regular rings, which is also defined via tight closure. A ring of characteristic $p > 0$ is said to be F-regular if all ideals are tightly closed in all of its local rings. Works by Watanabe [W3] show the similarity of F-regularity and the class of singularities called log terminal singularity. Among others, Watanabe proved that a ring of characteristic zero has log terminal singularity if it is of F-regular type and **Q**-Gorenstein. Theorem 1.1 also enables us to show the converse of his result (Theorem 5.2).

We shall briefly preview the proof of Theorem 1.1. Our starting point is a ring of characteristic zero, so we can reduce it to characteristic p together with a "good" resolution of singularity. Then, the key point of our proof is the injectivity of Frobenius maps of certain local cohomology groups on the resolution space. To analyze these Frobenius maps we employ logarithmic de Rham complex and the Cartier operator (cf. [C], [Kz]). We see that an obstruction for the maps to be injective lies in certain local cohomology groups. However, a slight generalization of Deligne and Illusie's result on the Akizuki–Kodaira–Nakano vanishing theorem in characteristic p [DI] and the Serre vanishing theorem imply that these local cohomologies vanish if the characteristic p is sufficiently large. Once we have shown that the Frobenius maps are injective, we can prove that the ring is F-rational by an argument which uses test elements as in Fedder and Watanabe's proof for graded rings [FW].

Our exposition is essentially based on the existence of a resolution in characteristic zero [Hi]. It may be interesting if a resolution-free proof of these results is provided.

Acknowledgments. I would like to thank Professor Kei-ichi Watanabe for valuable discussions and comments.

Notation and conventions. Throughout this paper all rings are excellent commutative rings with unity. For a ring R, R^0 will denote the set of elements of R which are not in any minimal prime ideal. We will often work over a field of characteristic $p > 0$. In this case we always use the letter q for a power p^e of p. Also, for an ideal I of R, $I^{[q]}$ will denote the ideal of R generated by the qth powers of elements of I.

A **Q**-divisor on a Noetherian normal scheme X is a linear combination $D = \sum \alpha_i D_i$ of irreducible and reduced closed subschemes $D_i \subset X$ of codimension one, with coefficients $\alpha_i \in \mathbf{Q}$. The integral part of D is defined by $[D] = \sum [\alpha_i] D_i$, where $[\alpha_i]$ is the largest integer which is not greater than α_i. D is said to be **Q**-Cartier (resp. **Q**-ample Cartier) if ND is a Cartier divisor (resp. an ample Cartier divisor) for some positive integer N. We define $\mathcal{O}_X(D) := \mathcal{O}_X([D])$, the sheaf associated with the integral divisor $[D]$.

2. Definitions and preliminaries.

Definition 2.1. [HH1] Let R be a Noetherian ring of characteristic $p > 0$, and $I \subset R$ be an ideal. The *tight closure* I^* of I in R is the ideal defined by $x \in I^*$ if and only if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for $q = p^e \gg 0$. We say that I is *tightly closed* if $I^* = I$.

Definition 2.2. Let R denote a Noetherian ring of characteristic $p > 0$.

(i) [FW] A local ring (R, m) is said to be *F-rational* if some (or, equivalently, every) ideal generated by a system of parameters of R is tightly closed. When R is not local, we say that R is F-rational if every localization is F-rational.

(ii) [HH1] R is said to be *weakly F-regular* if every ideal of R is tightly closed. We say that R is *F-regular* if every localization is weakly F-regular.

Remark 2.2.1. For an R-submodule $N \subseteq M$, we can define the *tight closure* N_M^* of N in M as well [HH1]. The only case we treat here is that (R, m) is a d-dimensional local ring and that $N = (0) \subseteq M = H_m^d(R)$. In this case, an element $\xi \in H_m^d(R)$ lies in $(0)^*$ if and only if there exists $c \in R^0$ such that $c\xi^q = 0$ in $H_m^d(R)$ for $q = p^e \gg 0$, where ξ^q is the image of ξ by the e-times iteration of the induced Frobenius map $F^e\colon H_m^d(R) \to H_m^d(R)$. When (R, m) is Cohen–Macaulay, R is F-rational if and only if $(0)^* = (0)$ in $H_m^d(R)$.

Remark 2.2.2. In characteristic $p > 0$, the following implications are known [HH1]:

$$\text{regular} \Rightarrow F\text{-regular} \Rightarrow F\text{-rational} \Rightarrow \text{Cohen–Macaulay and normal.}$$

Also, a Gorenstein F-rational ring is F-regular.

Definition 2.3. Let R be a Noetherian ring of characteristic $p > 0$ and let $I \subset R$ be an ideal. An element $c \in R^0$ is said to be a *test element* for I if for all $x \in R$, one has

$$x \in I^* \iff cx^q \in I^{[q]} \text{ for all } q = p^e \ (e \geq 0).$$

We say that $c \in R^0$ is a test element if it is a test element for all ideals $I \subset R$.

PROPOSITION 2.4. (Vélez [V]) *Let R be a reduced excellent local ring of characteristic $p > 0$. If $c \in R^0$ is an element such that R_c is F-rational, then some power of c is a test element for all ideals generated by a system of parameters of R.*

The existence of test elements is essential for computing tight closure. The above result implies that if the punctured spectrum of a local ring (R, m) is F-rational, then the "test ideal for parameters" of R is m-primary.

Given a property P defined for rings of characteristic $p > 0$ such as " F-rational" or "F-regular," we will extend the concept to characteristic zero using the technique of reduction modulo p.

Definition 2.5. (cf. [HR]) Let R be a finitely generated algebra over a field k of characteristic zero. We say that R is of *P type* if there exist a finitely generated **Z**-subalgebra A of k and a finitely generated A-algebra R_A satisfying the following conditions:

(i) R_A is flat over A and $R_A \otimes_A k \cong R$.

(ii) $R_\kappa = R_A \otimes_A \kappa(s)$ has property P for every closed point s in a dense open subset of $S = \operatorname{Spec} A$, where $\kappa = \kappa(s)$ denotes the residue field of $s \in S$.

Remark 2.5.1. In condition (ii), as A is finitely generated over **Z**, $\kappa = \kappa(s)$ is a finite field, whence a perfect field of positive characteristic. We sometimes abbreviate the statement in condition (ii) as "the fiber ring R_κ has property P for *general* closed points $s \in S$." However, if R is of P type, we can replace $S = \operatorname{Spec} A$ by a suitable affine open subset so that condition (ii) holds for *every* closed point $s \in S$.

Remark 2.5.2. When P = "F-rational" or "F-regular," condition (ii) does not depend on the choice of A and R_A (see [HH2, 3], [V]).

Now we will recall the following well-known

Definition 2.6. Let Y be a normal variety over a field of characteristic zero. A point $y \in Y$ is said to be a *rational singularity* if for a resolution of singularity $f\colon X \to Y$, one has $(R^i f_* \mathcal{O}_X)_y = 0$ for all $i > 0$. This property does not depend on the choice of a resolution.

Remark 2.6.1. Lipman and Teissier [LT] introduced the notion of pseudo-rational rings as a resolution-free analogue of rational singularity, which we do not define here. We note only that in characteristic zero, Hironaka's resolution theorem [Hi] and the Grauert–Riemenschneider vanishing theorem [GR] guarantee the equivalence of rationality and pseudo-rationality. In particular, rational singularities in characteristic zero are Cohen–Macaulay.

In [S], Smith proved that excellent F-rational rings are pseudo-rational. Our main purpose is to show the converse of the following characteristic zero version.

THEOREM 2.7. (Smith [S]) *Let R be a finitely generated algebra over a field of characteristic zero. If R is of F-rational type, then it has at most rational singularity.*

3. The injectivity of Frobenius and cohomology vanishing. First we shall review some fundamental facts about log de Rham complex and the Cartier operator in characteristic $p > 0$. Concerning these subjects the reader may consult [C] and [Kz] (see also [EV]).

Assumption 3.1. We fix the notaton to be used throughout this section. X will denote a d-dimensional *smooth* variety of finite type over a perfect field k of characteristic $p > 0$, and $E = \sum_{j=1}^{m} E_j$ a reduced simple normal crossing divisor on X, that is, a divisor with smooth irreducible components E_j intersecting transversally.

Let us choose local parameters $t_1, \ldots, t_d$ of X so that E is locally defined by $t_1 \cdots t_s = 0$. Then we can consider the locally free $\mathcal{O}_X$-module $\Omega_X^1(\log E)$ with local basis

$$\frac{\mathrm{d}t_1}{t_1}, \ldots, \frac{\mathrm{d}t_s}{t_s}, \mathrm{d}t_{s+1}, \ldots, \mathrm{d}t_d.$$

We define $\Omega_X^i(\log E) = \bigwedge^i \Omega_X^1(\log E)$ for $i \geq 0$. These sheaves, together with the differential maps d, give rise to a complex $\Omega_X^\bullet(\log E)$ called a log de Rham complex. (In order to define this notion, the base ring k need not be of characteristic $p > 0$, nor even a field.) Note that $\Omega_X^0(\log E) = \mathcal{O}_X$ and that $\Omega_X^d(\log E) = \omega_X \otimes \mathcal{O}_X(E)$, where $\omega_X = \Omega_X^d$ is the dualizing sheaf of X.

3.2. The Cartier operator. ([C], [Kz]) Let $F\colon X \to X$ be the absolute Frobenius morphism of X. F will also denote the associated map $\mathcal{O}_X \to F_*\mathcal{O}_X$, etc. The push-down $F_*\Omega_X^\bullet(\log E)$ of the de Rham complex by $F\colon X \to X$ can be viewed as a complex of $\mathcal{O}_X$-modules via $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$. We denote the ith cohomology sheaf of this complex by $\mathcal{H}^i(F_*\Omega_X^\bullet(\log E))$. Then, there is an isomorphism of $\mathcal{O}_X$-modules

$$C^{-1}\colon\ \Omega_X^i(\log E) \xrightarrow{\sim} \mathcal{H}^i(F_*\Omega_X^\bullet(\log E))$$

for $i = 0, 1, \ldots, d$. In particular, the map $(C^{-1})^{-1}$ for $i = d$ and $E = 0$ induces a map $F_*\omega_X \twoheadrightarrow \mathcal{H}^d(F_*\Omega_X^\bullet) \xrightarrow{\sim} \omega_X$, which is identified with the canonical dual of $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$,

$$F^\vee\colon\ \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X) = \omega_X,$$

via the adjunction formula $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \omega_X) \cong F_*\omega_X$ (cf. [EV, Lemma 9.20]).

Remark 3.2.1. It is usual to use the k-relative Frobenius morphism $F_{\mathrm{rel}}\colon X \to X'$ to define the Cartier operator. In our situation the perfectness of the base field k allows us to use the absolute Frobenius F instead.

LEMMA 3.3. *Let the situation be as in Assumption* 3.1, *and $B = \sum r_j E_j$ be an effective integral divisor supported in E such that $0 \le r_j \le p-1$ for each j. Then we have a naturally induced complex $\Omega_X^\bullet(\log E)(B) = \Omega_X^\bullet(\log E) \otimes \mathcal{O}_X(B)$ of $\mathcal{O}_X^p$-modules, and the inclusion map*

$$\Omega_X^\bullet(\log E) \hookrightarrow \Omega_X^\bullet(\log E)(B)$$

is a quasi-isomorphism.

Proof. For $\imath\colon X - E \hookrightarrow X$, viewing $\Omega_X^i(\log E)(B)$ as a subsheaf of $\imath_*\Omega_{X-E}^i$, we easily see that the differential map in $\Omega_{X-E}^\bullet$ preserves $\Omega_X^\bullet(\log E)(B)$, which is thus a complex.

It is a local question to see whether $\Omega_X^\bullet(\log E) \hookrightarrow \Omega_X^\bullet(\log E)(B)$ is quasi-isomorphic. So let $t_1, \dots, t_d$ be local parameters of X, which form a p-basis of $\mathcal{O}_X$, and let the components $E_1, \dots, E_s$ of E be defined by $t_1, \dots, t_s$, respectively. We consider the complexes

$$\mathcal{K}_j^\bullet := \left[0 \longrightarrow \bigoplus_{i=0}^{p-1} \mathcal{O}_X^p \cdot t_j^i \xrightarrow{\ \mathrm{d}\ } \bigoplus_{i=0}^{p-1} \mathcal{O}_X^p \cdot t_j^i \frac{\mathrm{d}t_j}{t_j^{\varepsilon_j}} \longrightarrow 0 \right],$$

where $\varepsilon_j = 1$ for $1 \le j \le s$ and $\varepsilon_j = 0$ for $s+1 \le j \le d$, and let $\mathcal{L}_j^\bullet = t_j^{-r_j} \cdot \mathcal{K}_j^\bullet$ for $1 \le j \le s$. Then $\Omega_X^\bullet(\log E) = \mathcal{K}_1^\bullet \otimes \cdots \otimes \mathcal{K}_d^\bullet$, $\Omega_X^\bullet(\log E)(B) = \mathcal{L}_1^\bullet \otimes \cdots \otimes \mathcal{L}_s^\bullet \otimes \mathcal{K}_{s+1}^\bullet \otimes \cdots \otimes \mathcal{K}_d^\bullet$ (tensor products are taken over $\mathcal{O}_X^p$), and $\Omega_X^\bullet(\log E) \hookrightarrow \Omega_X^\bullet(\log E)(B)$ is induced by the inclusion maps $\mathcal{K}_j^\bullet \hookrightarrow \mathcal{L}_j^\bullet$, which are easily checked to be quasi-isomorphisms. Hence $\Omega_X^\bullet(\log E) \hookrightarrow \Omega_X^\bullet(\log E)(B)$ is also a quasi-isomorphism by the Künneth formula.

3.4. Key Observation. Let D be a $\mathbf{Q}$-divisor on X such that $\mathrm{Supp}(D-[D]) \subseteq \mathrm{Supp}(E)$. Then $B = -p[-D] + [-pD]$ is an effective divisor supported in E whose coefficient in each component E_j is at most $p-1$. Hence we have a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules $F_*\Omega_X^\bullet(\log E) \hookrightarrow F_*(\Omega_X^\bullet(\log E)(B))$ by Lemma 3.3, and composition with C^{-1} in 3.2 gives an isomorphism

$$\Omega_X^i(\log E) \xrightarrow{\ \sim\ } \mathcal{H}^i(F_*(\Omega_X^\bullet(\log E)(B))).$$

Tensoring with $\mathcal{O}_X(-D) = \mathcal{O}_X([-D])$, we have

$$\Omega_X^i(\log E)(-D) \xrightarrow{\ \sim\ } \mathcal{H}^i(F_*(\Omega_X^\bullet(\log E)(-pD))).$$

Denoting the ith cocycle and the ith coboundary of the complex $F_*(\Omega_X^\bullet(\log E)(-pD))$ by $\mathcal{Z}^i$ and $\mathcal{B}^i$, respectively, we have exact sequences of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{Z}^i \longrightarrow F_*(\Omega_X^i(\log E)(-pD)) \longrightarrow \mathcal{B}^{i+1} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{B}^i \longrightarrow \mathcal{Z}^i \longrightarrow \Omega_X^i(\log E)(-D) \longrightarrow 0$$

for $i = 0, 1, \ldots, d$. Here we note that the upper exact sequence for $i = 0$ is nothing but

$$0 \longrightarrow \mathcal{O}_X(-D) \xrightarrow{F} F_*(\mathcal{O}_X(-pD)) \longrightarrow \mathcal{B}^1 \longrightarrow 0.$$

By considering the local cohomology long exact sequences of these, we have

PROPOSITION 3.5. *Let D be a $\mathbf{Q}$-divisor on X such that* $\mathrm{Supp}(D-[D]) \subseteq \mathrm{Supp}(E)$ *and $Z \subseteq X$ be any closed subset. Then the induced Frobenius map*

$$F\colon H_Z^d(X, \mathcal{O}_X(-D)) \to H_Z^d(X, \mathcal{O}_X(-pD))$$

is injective if the following vanishing of local cohomologies holds:

(a) $H_Z^j(X, \Omega_X^i(\log E)(-D)) = 0$ *for $i + j = d - 1$ and $i > 0$;*

(b) $H_Z^j(X, \Omega_X^i(\log E)(-pD)) = 0$ *for $i + j = d$ and $i > 0$.*

Considering $(p-1)E - B$ instead of B in (3.4), we have $\Omega_X^i(\log E)(-E-[-D]) \xrightarrow{\sim} \mathcal{H}^i(F_*(\Omega_X^\bullet(\log E)(-E-[-pD])))$, from which we obtain the following "dual form."

PROPOSITION 3.6. *Let D be a $\mathbf{Q}$-divisor on X such that* $\mathrm{Supp}(D-[D]) \subseteq \mathrm{Supp}(E)$. *Then the map*

$$F^\vee\colon H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F_*(\mathcal{O}_X(-pD)), \omega_X)) \to H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-D), \omega_X))$$

induced by the canonical dual of the Frobenius is surjective if the following conditions hold:

(a) $H^j(X, \Omega_X^i(\log E)(-E-[-D])) = 0$ *for $i + j = d + 1$ and $j > 1$;*

(b) $H^j(X, \Omega_X^i(\log E)(-E-[-pD])) = 0$ *for $i + j = d$ and $j > 0$.*

Roughly speaking, if D is an ample $\mathbf{Q}$-Cartier divisor, then vanishing (b) holds for "sufficiently large p" by the Serre vanishing theorem. On the other hand, vanishing (a) is a variant of the Akizuki–Kodaira–Nakano vanishing theorem [AN], which is reduced to the following

THEOREM 3.7. (Deligne and Illusie [DI]) *In the situation of Assumption* 3.1, *assume further that $E = \sum_{j=1}^m E_j \subset X$ has a lifting $\tilde{E} = \sum_{j=1}^m \tilde{E}_j \subset \tilde{X}$ to the ring of* *second Witt vectors* $W_2(k)$ *of* k*, i.e., there exist a smooth scheme* $\tilde{X}$ *and a simple normal crossing divisor* $\tilde{E} = \sum_{j=1}^{m} \tilde{E}_j$ *over* $\tilde{S} = \operatorname{Spec} W_2(k)$ *such that* $X = \tilde{X} \times_{\tilde{S}} k$ *and* $E_j = \tilde{E}_j \times_{\tilde{S}} k$*. Then, if* $p > d = \dim X$*, we have an isomorphism*

$$\varphi_{(\tilde{X},\tilde{E})} \colon \bigoplus_{i=0}^{d} \Omega_X^i(\log E)[-i] \xrightarrow{\sim} F_* \Omega_X^\bullet(\log E)$$

in the derived category associated to the category of $\mathcal{O}_X$*-modules.*

Corollary 3.8. *Let* X *be projective over a Noetherian affine scheme and let* D *be an ample* $\mathbf{Q}$*-divisor on* X *such that* $\operatorname{Supp}(D - [D]) \subseteq \operatorname{Supp}(E)$*. Assume that* $E \subset X$ *admits a lifting* $\tilde{E} \subset \tilde{X}$ *to* $W_2(k)$ *as in Theorem* 3.7*. Then, if* $i+j > d = \dim X$ *and if* $p > d$*, we have*

$$H^j(X, \Omega_X^i(\log E)(-E - [-D])) = 0.$$

Proof. Thanks to Theorem 3.7, using Lemma 3.3 as in 3.4, we obtain an isomorphism

$$\bigoplus_{i=0}^{d} \Omega_X^i(\log E)(-E - [-D])[-i] \xrightarrow{\sim} F_*(\Omega_X^\bullet(\log E)(-E - [-pD]))$$

in the derived category of $\mathcal{O}_X$-modules. Taking the hypercohomology we have

$$\bigoplus_{i+j=l} H^j(X, \Omega_X^i(\log E)(-E - [-D])) \cong \mathbf{H}^l(X, \Omega_X^\bullet(\log E)(-E - [-pD]))$$

for each l. To annihilate the right-hand side, in view of the Hodge to de Rham spectral sequence $H^j(X, \Omega_X^i(\log E)(-E-[-pD])) \Longrightarrow \mathbf{H}^l(X, \Omega_X^\bullet(\log E)(-E-[-pD]))$ (see e.g. [EV]), it suffices to show that $H^j(X, \Omega_X^i(\log E)(-E-[-pD])) = 0$ for $i+j = l$. We iterate this procedure replacing D by $pD, \ldots, p^eD$ repeatedly. Consequently, we see that if $H^j(X, \Omega_X^i(\log E)(-E - [-p^eD])) = 0$ for $i+j = l$, then $H^j(X, \Omega_X^i(\log E)(-E - [-D])) = 0$ for $i+j = l$. Hence the conclusion follows from the Serre vanishing theorem applied to $l > d$ and $p^e \gg 0$.

Remark 3.8.1. The above vanishing in characteristic $p > 0$ yields the corresponding vanishing in characteristic zero. The proof uses the standard technique of reduction modulo p and proceeds in the same way as [DI, Corollaire 2.11]. We record in 4.3 the argument thereof for the sake of completeness.

4. Proof of Theorem 1.1. In order to prove the theorem we need to show that certain cohomology vanishing in characteristic zero inherits “uniformly” to characteristic $p > 0$ under the reduction process. For this purpose we will state the following elementary lemma.

LEMMA 4.1. *Let X be a Noetherian separated scheme of finite type over a Noetherian ring A, and let $\mathcal{F}$ be a quasi-coherent sheaf on X, flat over A. Suppose that $H^i(X, \mathcal{F})$ is a flat A-module for each $i > 0$. Then one has an isomorphism*

$$H^i(X, \mathcal{F}) \otimes_A \kappa(s) \cong H^i(X_\kappa, \mathcal{F}_\kappa)$$

for every point $s \in S$ and $i \geq 0$, where $\kappa = \kappa(s)$ is the residue field of $s \in S$, $X_\kappa = X_A \times_A \operatorname{Spec}(\kappa)$, and $\mathcal{F}_\kappa$ is the induced sheaf on X_κ.

Proof. Since $H^i(X_\kappa, \mathcal{F}_\kappa) \cong H^i(X, \mathcal{F} \otimes_A \kappa(s))$ by [Hart, (III, 9.4)], it is sufficient to show that the natural map

$$\varphi^i\colon\ H^i(X, \mathcal{F}) \otimes_A M \longrightarrow H^i(X, \mathcal{F} \otimes_A M)$$

is an isomorphism for any finitely generated A-module M and $i \geq 0$. To see this, it is enough to prove that the functor $T^i = H^i(X, \mathcal{F} \otimes_A -)$ is right exact [Hart, (III, 12.5)]. When $i = \dim X$, T^i is right exact, so φ^i is isomorphic for every M. This allows us to identify T^i with the functor $H^i(X, \mathcal{F}) \otimes_A -$, which is exact since $H^i(X, \mathcal{F})$ is A-flat. Then the (left) exactness of T^i implies the right exactness of T^{i-1}, and descending induction on i completes the proof.

Proof of Theorem 1.1. We may assume without loss of generality that R is an integral domain of dimension $d \geq 2$. We divide the proof into several steps.

4.2. Reduction to characteristic p. First of all, we can blow up $Y = \operatorname{Spec} R$ with respect to some ideal $I \subset R$ to get a "good" resolution of singularity $f\colon\ X \to Y$, that is, a resolution whose exceptional set $E \subset X$ is a simple normal crossing divisor [Hi].

Now, choosing a suitable finitely generated $\mathbf{Z}$-subalgebra A of k, one can construct a finitely generated A-algebra R_A, an ideal $I_A \subset R_A$ and an A-morphism

$$f_A\colon\ X_A = \operatorname{Proj}\left(\bigoplus\nolimits_{n\geq 0} I_A^n\right) \longrightarrow Y_A = \operatorname{Spec} R_A$$

such that by tensoring k over A one gets back R, $I \subset R$ and $f\colon\ X \to Y$. By localizing A at an appropriate element, we may assume that R_A is A-free, X_A is smooth over A, and that the exceptional divisor $E_A \subset X_A$ of f_A is simple normal crossing over A.

Let Γ_A be the effective Cartier divisor on X_A such that $\mathcal{O}_{X_A}(-\Gamma_A) = \mathcal{O}_{X_A}(1)$. We choose a rational number $\varepsilon > 0$ such that the $\mathbf{Q}$-divisor $\varepsilon\Gamma_A$ has no integral part, and set $D_A = -\varepsilon\Gamma_A$. Then D_A is an f_A-ample $\mathbf{Q}$-Cartier divisor supported on E_A with $[-D_A] = 0$.

Given a closed point $s \in S = \operatorname{Spec} A$ with residue field $\kappa = \kappa(s)$, we denote the corresponding fibers over s by $f_\kappa\colon X_\kappa \to Y_\kappa = \operatorname{Spec} R_\kappa$ etc. Obviously, all of the properties mentioned above are preserved in every closed fiber, i.e., X_κ is smooth over κ, E_κ is the simple normal crossing exceptional divisor of f_κ, and $D_\kappa = -\varepsilon\Gamma_\kappa$ is an f_κ-ample **Q**-Cartier divisor supported on E_κ such that $[-D_\kappa] = 0$. Moreover, the fiber ring R_κ over a general closed point $s \in S$ is not only Cohen–Macaulay and normal [HH3, (2.3)], but one also has $H^i(X_\kappa, \mathcal{O}_{X_\kappa}) = 0$ for $i > 0$ (4.1), since this is true for the "generic fiber" R.

4.3. Uniform vanishing of cohomologies. Here we will show that the following vanishing holds for general closed points $s \in S$ with $\operatorname{char}(\kappa(s)) = p$:

(a) $H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - [-p^e D_\kappa])) = 0$ for $i + j > d$ and $e \geq 0$;

(b) $H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - [-p^{e+1} D_\kappa])) = 0$ for $j > 0$ and $e \geq 0$.

To see this, we fix $i \geq 0$ and consider the quasi-coherent sheaf

$$\mathcal{F}_A = \bigoplus_{n \geq 0} \Omega^i_{X_A/A}(\log E_A)(-E_A - [-nD_A])$$

on X_A. For each j, $H^j(X_A, \mathcal{F}_A)$ is a finitely generated module over $R(X_A, D_A) = \bigoplus_{n \geq 0} H^0(X_A, \mathcal{O}_{X_A}(nD_A))$, which is a finitely generated graded A-algebra. So, by generic freeness, further localization of A allows us to assume that $H^j(X_A, \mathcal{F}_A)$ is A-free. Then, its graded piece $H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A - [-nD_A]))$ is a locally free A-module. Hence for every $n \geq 0$ and $s \in S = \operatorname{Spec} A$ with $\kappa = \kappa(s)$, one has

$$\begin{aligned} &H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A - [-nD_A])) \otimes_A \kappa(s) \\ &\quad \cong H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - [-nD_\kappa])) \end{aligned}$$

by Lemma 4.1. In particular, if the right-hand side vanishes for *some* $s \in S$, then it does for *every* $s \in S$. This implies vanishing (a) for every $s \in S$, since there exists a closed point $s \in S$ such that the fibers $E_\kappa \subset X_\kappa$ over s satisfy the lifting property in (3.8) and that $\operatorname{char}(\kappa(s)) > d = \dim X_\kappa$. Indeed, for a closed point t of $\operatorname{Spec}(A \otimes_{\mathbf{Z}} \mathbf{Q})$, the closure T of t in $S = \operatorname{Spec} A$ with reduced scheme structure is generically étale over $\operatorname{Spec} \mathbf{Z}$, so that we can choose a closed point $s \in T$ at which T is étale over $\operatorname{Spec} \mathbf{Z}$ such that $p = \operatorname{char}(\kappa(s)) > d$. Then the maximal ideal $m_{T,s}$ of $\mathcal{O}_{T,s}$ is generated by p, and $E_\kappa \subset X_\kappa$ over $\kappa = \kappa(s)$ admits a lifting to $\mathcal{O}_{T,s}/m^2_{T,s} = W_2(\kappa(s))$ (cf. proof of [DI, Corollaire 2.7]).

On the other hand, by the Serre vanishing, there exists an integer n_0 independent of $s \in S$ such that $H^j(X_\kappa, \Omega^i_{X_\kappa/\kappa}(\log E_\kappa)(-E_\kappa - [-nD_\kappa])) = 0$ for $j > 0$ and $n \geq n_0$. Since the closed points $s \in S$ with $\operatorname{char}(\kappa(s)) \geq n_0$ form a dense open subset of the maximal spectrum of A, we get vanishing (b).

4.4. Notation change. From now on we fix a general closed point $s \in S$ with residue field $\kappa = \kappa(s)$ of characteristic p, and work over κ. Taking this into account, we change the notation so that R, X, D, E and $f\colon X \to Y$ denote the corresponding closed fibers over $s \in S$ (which had been denoted by R_κ, X_κ etc., so far). Then vanishing (a) and (b) in (4.3), together with (3.6) applied to a $\mathbf{Q}$-divisor p^eD, implies that $F^\vee\colon H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F_*(\mathcal{O}_X(-p^{e+1}D)), \omega_X)) \to H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-p^eD), \omega_X))$ is surjective for all $e \geq 0$. Composing these, we see that

$$(F^e)^\vee\colon\ H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X(-qD)), \omega_X)) \to H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X))$$

is surjective for all $q = p^e$.

Now to prove the theorem it is sufficient to show that the new R is F-rational. We will prove this by contradiction, so let $P \in Y = \operatorname{Spec} R$ be a prime ideal such that the localization R_P is not F-rational. We can choose P so that $\operatorname{Spec} R_P \setminus \{P\}$ is F-rational. Then, we further replace R by the local ring R_P and everything by the base change to R_P over R. Note that the surjectivity of the above map $(F^e)^\vee$ implies the surjectivity of the corresponding map after localization.

At last we have come to the following situation.

4.5. The injectivity of Frobenius. We first summarize our new notation. (R, m) is a d-dimensional Cohen–Macaulay normal local ring essentially of finite type over a perfect field κ of positive characteristic p such that $\operatorname{Spec} R \setminus \{m\}$ is F-rational. $f\colon X \to Y = \operatorname{Spec} R$ is a resolution of singularity with simple normal crossing exceptional divisor $E \subset X$ and D is an f-ample $\mathbf{Q}$-Cartier divisor supported on E such that $[-D] = 0$.

Let $Z = f^{-1}(y)$ be the fiber of f over the closed point y of $Y = \operatorname{Spec} R$. Then by duality, the surjectivity of the map $(F^e)^\vee$ in 4.4 implies that the e-times iterated Frobenius map

$$F^e\colon\ H^d_Z(X, \mathcal{O}_X) \to H^d_Z(X, \mathcal{O}_X(-qD))$$

is injective for every $q = p^e$.

Our goal is to show $(0)^* = (0)$ in $H^d_m(R)$ under this setup and the additional condition $H^{d-1}(X, \mathcal{O}_X) = 0$, which comes from the rationality of our singularity (cf. 4.2).

4.6. Conclusion of the proof. Since $R^if_*\mathcal{O}_X = 0$ for $i > 0$, one has

$$H^{d-1}(X - Z, \mathcal{O}_{X-Z}) \cong H^{d-1}(Y - \{y\}, \mathcal{O}_{Y-\{y\}}) \cong H^d_m(R).$$

Now we consider the direct system $\{\cdots \to \mathcal{O}_X(-nD) \to \mathcal{O}_X(-(n+1)D) \to \cdots\}$ indexed by $n \in \mathbf{Z}$. (Note that $-D \geq 0$.) Since E is a Cartier divisor, the inclusion $\imath\colon X - E \hookrightarrow X$ is an affine morphism, and the limit of the direct system is $\imath_*\mathcal{O}_{X-E}$. Therefore, $\varinjlim H^{d-1}(X, \mathcal{O}_X(-nD)) = H^{d-1}(X, \imath_*\mathcal{O}_{X-E}) = H^{d-1}(X - E, \mathcal{O}_{X-E})$. Similarly one has $\varinjlim H^{d-1}(X-Z, \mathcal{O}_X(-nD)|_{X-Z}) = H^{d-1}(X-E, \mathcal{O}_{X-E})$. For $n \geq 0$ let

$$\varphi_n\colon\ H^{d-1}(X, \mathcal{O}_X(-nD)) \to H^{d-1}(X - Z, \mathcal{O}_X(-nD)|_{X-Z})$$

and

$$\psi_n\colon\ H^d_m(R) = H^{d-1}(X - Z, \mathcal{O}_{X-Z}) \to H^{d-1}(X - Z, \mathcal{O}_X(-nD)|_{X-Z})$$

be the natural maps, and define a filtration on $H^d_m(R)$ by

$$\mathrm{Filt}^n(H^d_m(R)) := \psi_n^{-1}(\mathrm{Image}\ \varphi_n).$$

Then $\bigcup_{n\geq 0}\mathrm{Filt}^n(H^d_m(R)) = H^d_m(R)$, since the direct limit map $\varinjlim \varphi_n$ is an isomorphism.

For each $q = p^e$ we consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^{d-1}(X, \mathcal{O}_X) = 0 & \to & H^d_m(R) & \to & H^d_Z(X, \mathcal{O}_X) & \to & 0 \\
 & & \downarrow F^e & & & & \\
\downarrow & & H^d_m(R) & & \downarrow F^e & & \\
 & & \downarrow \psi_q & & & & \\
H^{d-1}(X, \mathcal{O}_X(-qD)) & \stackrel{\varphi_q}{\to} & H^{d-1}(X-Z, \mathcal{O}_X(-qD)) & \to & H^d_Z(X, \mathcal{O}_X(-qD)) & \to & 0,
\end{array}$$

where the Frobenius $F^e\colon\ H^d_Z(X, \mathcal{O}_X) \to H^d_Z(X, \mathcal{O}_X(-qD))$ on the right-hand side is injective (4.5).

Now let $\xi \neq 0 \in H^d_m(R)$. Then $\xi^q := F^e(\xi) \notin \mathrm{Filt}^q(H^d_m(R))$ for all $q = p^e$ from the above diagram. On the other hand, we can choose an integer $N > 0$ such that all nonzero elements of m^N are test elements for parameter ideals (Proposition 2.4). Since $(0 : m^N)$ in $H^d_m(R)$ is a finitely generated R-module, one has $(0 : m^N) \subseteq \mathrm{Filt}^{n_1}(H^d_m(R))$ for some $n_1 \in \mathbf{Z}$. Thus, if we pick a power $q = p^e \geq n_1$, then $\xi^q \notin (0 : m^N)$ in $H^d_m(R)$. Hence there is some test element $c \in m^N$ for parameters such that $c\xi^q \neq 0$. Consequently, $\xi \notin (0)^*$ in $H^d_m(R)$, whence $(0)^* = (0)$ in $H^d_m(R)$. Thus we conclude that R is F-rational, as required. □

Actually, the proof of Theorem 1.1 contains a more general statement, as follows.

THEOREM 4.7. *Let* (R, m) *be a d-dimensional normal local ring which is a "general" modulo p reduction of a (not necessarily rational) singularity in characteristic zero with at most isolated nonrational locus. Let* $f: X \to Y = \operatorname{Spec} R$ *be a resolution of singularity reduced from characteristic zero as in* 4.2–4.5. *Then one has*

$$H^{d-1}(X, \mathcal{O}_X) \cong (0)^* \quad \text{in } H^d_m(R).$$

Proof. Let the notation be as in 4.5 and let $f' = f|_{X-Z}: X - Z \to Y - \{y\}$. From the assumption we have $R^i f'_* \mathcal{O}_{X-Z} = 0$ for $i > 0$, and $Y - \{y\}$ is F-rational by Theorem 1.1. Hence $H^{d-1}(X - Z, \mathcal{O}_{X-Z}) \cong H^{d-1}(Y - \{y\}, \mathcal{O}_{Y-\{y\}}) \cong H^d_m(R)$. Also, the Grauert–Riemenschneider vanishing in characteristic zero [GR] tells us that $H^{d-1}_Z(X, \mathcal{O}_X) = 0$ after passing to general modulo p reduction, so that the natural map $H^{d-1}(X, \mathcal{O}_X) \to H^{d-1}(X - Z, \mathcal{O}_{X-Z})$ is injective. Thus we can view $H^{d-1}(X, \mathcal{O}_X)$ as a submodule of $H^d_m(R)$, and the argument in 4.6 shows $(0)^* \subseteq H^{d-1}(X, \mathcal{O}_X)$ in $H^d_m(R)$.

On the other hand, since $H^{d-1}(X, \mathcal{O}_X)$ has finite length as an R-module, it is killed by some $c \in R^0$. Hence $c\xi^q = 0$ for all $\xi \in H^{d-1}(X, \mathcal{O}_X)$ and $q = p^e$, proving $H^{d-1}(X, \mathcal{O}_X)$
$\subseteq (0)^*$.

Remark 4.7.1. The length of the R-module $H^{d-1}(X, \mathcal{O}_X)$ is an important invariant of the singularity called the "geometric genus" (cf. [Wg]). By the aid of [W2, Corollary 2.6], Theorem 4.7 gives an affirmative answer to the Strong Vanishing Conjecture posed by Huneke and Smith [HS, 3.9].

5. Applications. In this section we shall present two applications of the results obtained so far. First we make use of Theorem 1.1 to establish the correspondence of F-regular rings with so called log terminal singularities. Second we will consider F-rationality (and F-regularity) of some graded rings. It is perhaps unneccessary to say that the graded case is only a special case, but it provides interesting examples.

Definition 5.1. (cf. [KMM]) Let Y be a normal variety over a field of characteristic zero. Y is said to have *log terminal singularity* if the following two conditions hold:

(i) Y is **Q**-Gorenstein, i.e., the canonical divisor K_Y of Y is **Q**-Cartier.

(ii) Let $f: X \to Y$ be a resolution whose exceptional set is a simple normal crossing divisor with irreducible components $E_1, \ldots, E_r$. Condition (i) allows us to write

$$K_X = f^* K_Y + \sum_{i=1}^{r} a_i E_i$$

for some $a_i \in \mathbf{Q}$, where K_X is the canonical divisor of X. Then $a_i > -1$ for each i.

Remark 5.1.1. We have implications similar to Remark 2.2.2:

$$\text{regular} \Rightarrow \text{log terminal} \Rightarrow \text{rational} \Rightarrow \text{Cohen–Macaulay and normal.}$$

Also, a Gorenstein rational singularity is log terminal.

In [W3], Watanabe proved that a ring in characteristic zero has log terminal singularity if it is of F-regular type and $\mathbf{Q}$-Gorenstein (see also [Ha], [Mc]). Conversely we have

THEOREM 5.2. *Let R be a finitely generated algebra over a field of characteristic zero. If R has at most log terminal singularities, then R is of F-regular type.*

Proof. The question is local, so we may replace $\operatorname{Spec} R$ by an affine open subset, and assume that $K_R^{(r)} \cong R$, where r is the order of the canonical class $\operatorname{cl}(K_R)$ of R. Then, by our assumption, the canonical covering $S = \bigoplus_{i=0}^{r-1} K_R^{(-i)}$ of R has Gorenstein rational singularity [Kw, Proposition 1.7], whence has F-rational type by Theorem 1.1. By the standard argument, we can choose a finitely generated $\mathbf{Z}$-algebra A contained in the base field k, and flat, finitely generated A-algebras $R_A \subseteq S_A$ such that one gets back $R \subseteq S$ after tensoring k over A and that R_A is a direct summand of S_A as an R_A-module. Then for a general closed point of $\operatorname{Spec} A$ with residue field κ, S_κ is Gorenstein and F-rational, so, F-regular. This forces R_κ to be F-regular, because a direct summand of an F-regular ring is also F-regular [HH1].

5.3. The graded case. ([FW], [HW]) Through the remainder of this paper we will treat the case that $R = \bigoplus_{n\geq 0} R_n$ is a normal graded ring finitely generated over a perfect field $R_0 = k$ of characteristic $p > 0$ with $d = \dim R \geq 2$. Then R is represented by an ample $\mathbf{Q}$-Cartier divisor D on $X = \operatorname{Proj} R$ as

$$R = R(X, D) := \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n.$$

When $\operatorname{Spec} R$ is F-rational outside the vertex $V(R_+)$, R is F-rational if and only if the following two conditions hold:

(i) R is Cohen–Macaulay and $a(R) < 0$, where $a(R)$ is the a-invariant of R [GW].

(ii) R is F-injective in the sense of [FW], equivalently, the induced Frobenius

$$F\colon\ H^{d-1}(X, \mathcal{O}_X(nD)) \to H^{d-1}(X, \mathcal{O}_X(pnD))$$

is injective for every (negative) integer n.

Condition (i) corresponds to the condition that R has "rational singularity" ([Fl], [W2]), and is easier to check than condition (ii). On the other hand, if X is smooth and $\mathrm{Supp}(D-[D])$ is simple normal crossing, we can use Proposition 3.5 to check if condition (ii) holds (by setting $Z = X$ in Proposition 3.5).

Example 5.4 [HW, Theorem 2.9] When $\dim R(X,D) = 2$, condition (i) holds if and only if $X = \mathbf{P}^1$ and $\deg [nD] > -2$ for every positive integer n. In this case a sufficient condition for (ii) is given by using the "fractional part" D' of D (see [W1]). Namely, condition (ii) holds if p does not divide the denominator of any rational coefficient of D and if $p \deg D > \deg (K_X + D')$. (Note that $\deg K_X = -2$ if $X = \mathbf{P}^1$.) What's more, we can give a necessary and sufficient condition for $R = R(X,D)$ to be F-rational in terms of numerical data involving p and the coefficients of D.

Example 5.5. Let X be a smooth del Pezzo surface (i.e., a smooth projective surface with ample anti-canonical divisor $-K_X$) of characteristic $p > 0$. Then $R = R(X, -K_X)$ has at most isolated Gorenstein rational (whence log terminal) singularity. In this case we can explicitly describe a condition for $R = R(X, -K_X)$ to be F-regular in terms of p and the self intersection number K_X^2. R is F-regular except for the following three cases:

(i) $K_X^2 = 3$ and $p = 2$.

(ii) $K_X^2 = 2$ and $p = 2$ or 3.

(iii) $K_X^2 = 1$ and $p = 2$, 3 or 5.

Moreover, there are both F-regular and non F-regular cases for each of (i), (ii) and (iii). For example, in case (i) R is not F-regular if and only if X is isomorphic to the Fermat cubic surface in $\mathbf{P}^3$.

DEPARTMENT OF MATHEMATICS, WASEDA UNIVERSITY, 3-4-1 OKUBO, SHINJUKU, TOKYO, 169 JAPAN
Electronic mail: NHARA@MN.WASEDA.AC.JP

REFERENCES

[AN] Y. Akizuki and S. Nakano, Note on Kodaira–Spencer's proof of Lefschetz's theorem, *Proc. Japan Acad. Ser. A Math. Sci.* **30** (1954), 266–272.

[C] P. Cartier, Question de rationalité des diviseurs en géometrie algébrique, *Bull. Soc. Math. France* **86** (1958), 177–251.

[DI] P. Deligne and L. Illusie, Relèvements modulo p^2 et décomposition de complexe de de Rham, *Invent. Math.* **89** (1987), 247–270.

[EV] H. Esnault and E. Viehweg, *Lectures on Vanishing Theorems, DMV Sem.*, Band 20, Birkhäuser, Basel, 1992.

[F1] R. Fedder, F-purity and rational singularity in graded complete intersection rings, *Trans. Amer. Math. Soc*. **301** (1987), 47–62.

[F2] ———, A Frobenius characterization of rational singularity in 2-dimensional graded rings, *Trans. Amer. Math. Soc*. **340** (1993), 655–668.

[FW] R. Fedder and K.-i. Watanabe, A characterization of F-regularity in terms of F-purity, *Commutative Algebra, Math. Sci. Res. Inst. Publ*., vol. 15, Springer-Verlag, New York, 1989, pp. 227–245.

[Fl] H. Flenner, Rationale quasi-homogene Singularitäten, *Arch. Math*. **36** (1981), 35–44.

[GW] S. Goto and K.-i. Watanabe, On graded rings I, *J. Math. Soc. Japan* **30** (1978), 179–213.

[GR] H. Grauert and O. Riemenschneider, Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen, *Invent. Math*. **11** (1970), 263–292.

[Ha] N. Hara, Classification of two-dimensional F-regular and F-pure singularities, *Adv. Math*. (to appear).

[HW] N. Hara and K.-i. Watanabe, The injectivity of Frobenius acting on cohomology and local cohomology modules, *Manuscripta Math*. **90** (1996), 301–315.

[Hart] R. Hartshorne, *Algebraic Geometry, Graduate Texts in Math*., vol. 52, Springer-Verlag, New York, 1977.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math*. **79** (1964), 109–326.

[HH1] M. Hochster and C. Huneke, Tight closure, invariant theory and the Briançon-Skoda theorem, *J. Amer. Math. Soc*. **3** (1990), 31–116.

[HH2] ———, *F*-regularity, test elements, and smooth base change, *Trans. Amer. Math. Soc*. **346** (1994), 1–62.

[HH3] ———, Tight closure in equal characteristic zero, preprint.

[HR] M. Hochster and J. L. Roberts, The purity of Frobenius and local cohomology, *Adv. Math*. **21** (1976), 117–172.

[HS] C. Huneke and K. E. Smith, Tight closure and the Kodaira vanishing theorem, *J. Reine Angew. Math*. **484** (1997), 127–152.

[Kz] N. Katz, Nilpotent connections and monodromy theorem, *Inst. Hautes Études Sci. Publ. Math*. **39** (1970), 175–232.

[Kw] Y. Kawamata, The cone of curves of algebraic varieties, *Ann. of Math*. **119** (1984), 603–633.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problem, *Adv. Stud. Pure Math*., vol. 10, Academic Press, Boston, 1987, pp. 283–360.

[LT] J. Lipman and B. Teissier, Pseudo-rational local rings and a theorem of Briançon-Skoda on the integral closure of ideals, *Michigan Math. J*. **28** (1981), 97–116.

[Mc] B. MacCrimmon, Weak *F*-regularity is strong *F*-regularity for rings with isolated non-**Q**-Gorenstein points, preprint.

[S] K. E. Smith, F-rational rings have rational singularities, *Amer. J. Math*. **119** (1997), 159–180.

[V] J. D. Vélez, Openness of the F-rational locus and smooth base change, *J. Algebra* **172** (1995), 425–453.

[Wg] P. Wagreich, Elliptic singularities of surfaces, *Amer. J. Math*. **92** (1970), 419–454.

[W1] K.-i. Watanabe, Some remarks concerning Demasure's construction of normal graded rings, *Nagoya Math. J*. **83** (1981), 203–211.

[W2] ———, Rational singularities with k^*-action, *Commutative Algebra, Proc. Trento Conf., Lecture Notes in Pure and Appl. Math*., vol. 84, Marcel Dekker, 1983, pp. 331–359.

[W3] ———, F-regular and F-pure rings vs. log-terminal and log-canonical singularities, preprint.