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Clean Property in Subring Retracts

Chahrazade Bakkari

Department of Mathematics, Faculty of Science and Technology of Fez
Box 2202, University S. M. Ben Abdellah Fez, Morocco
cbakkari@hotmail.com

Abstract

In this work, we investigate the transfer of clean property between a commutative ring and its subring retract. Also, we study the transfer of $\hbar$-rings property in trivial ring extensions. The article includes a brief discussion of the scope and precision of our results.

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1 Introduction

Throughout this paper all rings are assumed to be commutative with identity element. The ring A is called clean if every element is the sum of an idempotent and a unit. Some examples of clean rings include all Von Neumann regular rings and all local rings. A basic property of clean rings is that any homomorphic image of a clean ring is again clean. This leads to our definition of a neat ring. We say a ring A is a neat ring if every nontrivial homomorphic image is clean. For instance, any clean ring is neat and the converse is false (for example, the ring of integers is a neat ring which is not clean). See for instance [6, 7].

For two rings $A \subseteq B$, we say that A is a module retract (or a subring retract) of B if there exists an A-module homomorphism $\phi : B \longrightarrow A$ such that $\phi|_A = id|_A$; ϕ is called a module retraction map. If such a map ϕ exists, B contains A as an A-module direct summand. In this work, we set $V = Ker(\phi)$. See for instance [4].

A special application of subring retract is the notion of trivial ring extension. Let A be a ring, E an A-module and $R = A \propto E$, the set of pairs (a, e) with $a \in A$ and $e \in E$, under coordinatewise addition and under an adjusted multiplication defined by $(a, e)(a', e') = (aa', ae' + a'e)$, for all $a, a' \in A, e, e' \in E$. Then R is called the trivial ring extension of A by E. It is clear that A is a module retract of R, where the module retraction map ϕ is defined by $\phi(x, e) = x$ and so $V^2 = 0$. See for instance [2, 3, 4].

In this work, we study the transfer of clean property to subring retract. Also, we study the transfer of $\hbar$-ring property to trivial ring extensions. The article includes a brief discussion of the scope and precision of our results.

2 Main Results

This is the first main results of the paper.

Theorem 2.1 *Let R be a ring and A be a subring retract of R such that $V^2 \subseteq V$. Then:*
1) *If R is a clean ring then so is A.*
2) *Assume that $(a+v)$ is invertible in R if and only if a is invertible in A for each $a \in A$ and $v \in V$. Then, R is a clean ring if and only if so is A.*

Proof. 1) Let $x \in A$. Then $x \in R$ and so $x = a + e$, where $a \in R$ is invertible in R and $e \in R$ is idempotent in R (since R is clean). But $a = a_A + a_V$ and $e = e_A + e_V$ for some $a_A, e_A \in A$ and $a_V, e_V \in V$ since $R = A \oplus V$. Then $x = (a_A + e_A) + (a_V + e_V)$ and so $x = a_A + e_A$ and $a_V + e_V = 0$. It remains to show that a_A is invertible in A and e_A is idempotent.
Since $a (= a_A + a_V)$ is invertible in R, then there exists $(b_A + b_V) \in R$ (where $b_A \in A$ and $b_V \in V$) such that $1 = (a_A + a_V)(b_A + b_V) = (a_A b_A) + [a_A b_V + a_V b_A + a_V b_V]$. Therefore, $a_A b_A = 1$ since $a_A b_A \in A$ and $a_A b_V + a_V b_A + a_V b_V \in V$. Hence, a_A is invertible in A.
Now, we show that e_A is idempotent in A. Since $e (= e_A + e_V)$ is idempotent in R, then $e_A + e_V = (e_A + e_V)(e_A + e_V) = (e_A^2) + [e_A e_V + e_V e_A + e_V^2]$. Therefore, $e_A^2 = e_A$ since $e_A^2 \in A$ and $e_A e_V + e_V e_A + e_V^2 \in V$. Hence, e_A is idempotent in A.
2) If R is clean, then so is A by 1). Conversely, assume that A is clean and $(a + v)$ is invertible in R if and only if a is invertible in A for each $a \in A$ and $v \in V$. Our aim is to show that R is clean.
Let $x = a + v$ be an element of R, where $a \in A$ and $v \in V$. Then, we may write $a = a_{inv} + a_{id}$ (where a_{inv} is an invertible element of A and a_{id} is an idempotent element of A) since A is clean. Therefore, $x = a + v = (a_{inv} + v) + a_{id}$, where $a_{inv} + v$ is an invertible element of R by hypothesis and a_{id} is an idempotent

element of R; this means that R is clean and this completes the proof of Theorem 2.1.

Corollary 2.2 *Let A be a ring, E be an A-module and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is clean if and only if so is A.*

Proof. Clear by Theorem 2.1 and since $(a, e) \in R$ is invertible if and only if a is invertible in A (by [3, Theorem 25.1].

The second application is devoted to the amalgamated duplication of a ring A along an ideal I , and denoted by $A \bowtie I$. When $I^2 = 0$, $A \bowtie I = A \propto I$. More precisely, the amalgamated duplication of A along an ideal I is a ring that is defined as the following subring of $A \times A$: $A \bowtie I = \{(a, a+i)/a \in A, i \in I\}$. See for instance [5].

Corollary 2.3 *Let A be a ring, I be an ideal of A and let $R := A \bowtie E$. Then:*
1) *If R is clean, then so is A.*
2) *Assume that (A, I) is a local ring, where I is its maximal ideal. Then R is clean if and only if so is A.*

Proof. By Theorem 2.1, it remains to show that if (A, I) is a local clean ring, where I is its maximal ideal, then R is clean.
Assume that (A, I) is a local clean ring, where I is its maximal ideal, and let $(a, a+i) \in R$, where $a \in A$ and $i \in I$. But $a = a_{inv} + a_{id}$, where a_{inv} is an invertible element of A and a_{id} is an idempotent element of A since A is clean. Therefore, $(a, a+i) = (a_{inv}, a_{inv} + i) + (a_{id}, a_{id})$ and it is clear that (a_{id}, a_{id}) is an idempotent element of R. We claim that $(a_{inv}, a_{inv}+i)$ is invertible element of R.
Indeed, $a_{inv} + i \notin I$ since $a_{inv} \notin I$ (since a_{inv} is invertible), $i \in I$ and (A, I) is a local ring. Hence, $a_{inv}+i$ is invertible in A and so $(a_{inv}, a_{inv}+i)$ is invertible in R as desired.

Now, we construct an example showing that, even if $V^2 \subseteq V$, the condition imposed in Theorem 2.1(2) cannot be removed.

Example 2.4 *Let K be a field. The ring $R := K[X](= K + XK[X])$ is not clean (since it is a non local domain) even if the field K is clean and $(XK[X])^2 = X^2K[X] \subseteq XK[X]$.*

Now, we construct a class of rings such that the neat and clean properties coincident.

Proposition 2.5 *Let A be a ring, E be an A-module and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is clean if and only if it is neat.*

Proof. If R is clean, then R is neat in general. Conversely, assume that $R := A \propto E$ is neat. Then $A(= R/(0 \propto E))$ is clean as nontrivial homomorphic image. Hence R is clean by Corollary 2.2, as desired.

Our second main result is the transfer of $\hbar$-rings in trivial ring extensions. Recall that a ring A is a $\hbar$-ring if every pure ideal is generated by idempotents (Recall that the ideal I is said to be pure if for each $a \in I$ there is an element $b \in I$ such that $ab = a$). For instance, any clean ring is an $\hbar$-ring by [6, Theorem 1.7].

Now, we study the transfer of $\hbar$-property in particular trivial extensions.

Theorem 2.6 *Let A be a ring which does not contain any proper pure ideal (in particular, if A is a domain), E be an A-module and let $R := A \propto E$ be the trivial ring extension of A by E. Then R does not contain any proper pure ideal. In particular, R is a $\hbar$-ring.*

Proof. We claim that R does not contained any proper pure ideal. Deny. Let J be a proper pure ideal of R and set $I = \{a \in A/(a, e) \in J$ for some $e \in E\}$. Two cases are then possibles:
Case 1. $I \neq 0$. We claim that I is a pure ideal of A. Indeed, let I_1 be an ideal of A and set $J_1 = I_1 \propto E$ which is an ideal of R. But $J_1 \cap J = J_1 J$ by [2, Theorem 1.2.15] since J is a pure ideal of R. Hence, $I_1 \cap I = I_1 I$ and so I is a pure ideal of A, a desired contradiction.
Case 2. $I = 0$. In this case, $J = 0 \propto E^{'}$, where $E^{'}$ is an A- submodule of E. Hence, $0 = J^2 = J \cap J = J$ by [2, Theorem 1.2.15] since $J \neq 0$.
Therefore, there is no proper pure ideal of R and so R is a $\hbar$-ring.

Now, we give a class of $\hbar$-rings which are neither clean rings nor neat rings.

Example 2.7 *Let A be a non local domain, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then:*
1) *R is an $\hbar$-ring by Theorem 2.6.*

2) *R is not clean since A is not clean by Corollary 2.1 (since a domain is clean if and only if it is local by [6, Example 1.1]).*
3) *R is not neat by Proposition 2.5 since it is not clean.*

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