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PRIME BASES OF WEAKLY PRIME SUBMODULES AND THE WEAK RADICAL OF SUBMODULES

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Abstract. We will introduce and study the notion of prime bases for weakly prime submodules and utilize them to derive some formulas on the weak radical of submodules of a module. In particular, we will show that every one dimensional integral domain weakly satisfies the radical formula and state some necessary conditions on local integral domains which are semi-compatible or satisfy the radical formula and also on Noetherian rings which weakly satisfy the radical formula.

1. Introduction

In this paper all rings are commutative and with identity, all modules are unitary, R denotes a ring and M denotes an R-module. Also by $\mathbb{N}$ we mean the set of positive integers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. We indicate the relation of containment and strict containment by $\subseteq$ and $\subset$, respectively. Furthermore $N \leq M$ ($N < M$) means that N is a submodule (proper submodule) of M.

Prime ideals of rings play an important role in commutative ring theory, hence many have tried to generalize this concept to modules. A proper submodule P of M is called *prime*, when from $rm \in P$ for some $r \in R$ and $m \in M$, we can conclude either $m \in P$ or $rM \subseteq P$ (see for example [2, 9, 10, 11]). Let $(P : M)$ be the set of all $r \in R$ such that $rM \subseteq P$. If P is a prime submodule, then $\mathfrak{P} = (P : M)$ is a prime ideal of R and we say that P is $\mathfrak{P}$-prime.

Another generalization of prime ideals was proposed in [6]. There a proper submodule W of M is said to be *weakly prime*, if from $rsm \in W$ for $r, s \in R$ and $m \in M$, we can conclude either $rm \in W$ or $sm \in W$. One can easily see that it is equivalent to asserting that $(W : m)$ is a prime ideal for every $m \in M \setminus W$.

If W is weakly prime, then we consider $\mathcal{C}(W)$ (or just $\mathcal{C}$) to be

$$\mathcal{C}(W) = \{(W : m) \mid m \in M \setminus W\},$$

and we say that W is $\mathcal{C}$-weakly prime.

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If W is $\mathcal{C}$-weakly prime, then by [1, Lemma 2.1], $\mathcal{C}$ is a chain of prime ideals with respect to inclusion, hence $\bigcap \mathcal{C} = (W : M)$ and $\bigcup \mathcal{C} = \mathrm{Z}\left(\frac{M}{W}\right)$ are prime ideals, where $\mathrm{Z}(A) = \{r \in R \mid \exists\ 0 \neq x \in A,\ rx = 0\}$ denotes the set of zero divisors of an R-module A. Also obviously W is $\mathfrak{P}$-prime if and only if $\mathcal{C} = \{\mathfrak{P}\}$.

Recall that for an ideal $\mathfrak{I}$ of R, the intersection of all prime ideals of R containing $\mathfrak{I}$ is called the radical of $\mathfrak{I}$ and is denoted by $\sqrt{\mathfrak{I}}$. Similarly if N is a submodule of M, the intersection of prime (weakly prime) submodules of M containing N is called the *radical* (*weak radical*) *of* N and we denote it by $\mathrm{rad}_M(N)$ $[\mathrm{wrad}_M(N)]$ (or $\mathrm{rad}(N)$ $(\mathrm{wrad}(N))$ if there is no subtlety). If M has no prime (weakly prime) submodule containing N, then we say $\mathrm{rad}_M(N) = M$ $(\mathrm{wrad}_M(N) = M)$.

A well-known and very useful theorem in commutative ring theory is that $\sqrt{\mathfrak{I}} = \left\{r \in R \mid r^k \in \mathfrak{I} \text{ for some k} \in \mathbb{N}\right\}$. To find a similar characterization for the radical of a submodule, the notion of envelope of a submodule was introduced in [9]. The *envelope* of a submodule N of M, $\mathrm{E}_M(N)$ (or $\mathrm{E}(N)$ if no subtlety), is the set of all $x \in M$ for which, there exist $r \in R$, $m \in M$ and $k \in \mathbb{N}$ such that $x = rm$ and $r^k m \in N$. The envelope of a submodule is not necessarily itself a submodule, so we usually use the submodule it generates, denoted by $R\mathrm{E}(N)$.

One can easily verify that for every submodule N of M, we have $N \subseteq R\mathrm{E}(N) \subseteq \mathrm{wrad}(N) \subseteq \mathrm{rad}(N)$. Now if $\mathrm{rad}(N) = R\mathrm{E}(N)$ $(\mathrm{wrad}(N) = R\mathrm{E}(N))$, it is said that N *satisfies* (*weakly satisfies*) *the radical formula* (*s.t.r.f.* (*weakly s.t.r.f.*)) in M. A module M (weakly) s.t.r.f., when every submodule of M (weakly) s.t.r.f. in M. Also we say that R (weakly) s.t.r.f., if every R-module (weakly) s.t.r.f. If for every submodule N of M, we have $\mathrm{wrad}(N) = \mathrm{rad}(N)$, it is said that M is *semi-compatible*. Also if every R-module is semi-compatible, we say the ring R is *semi-compatible*. Clearly a ring s.t.r.f., if and only if it is semi-compatible and weakly s.t.r.f.

Many have studied when a ring or a module s.t.r.f. (see [2, 7, 9, 10, 11]). For example in [7], Noetherian rings which s.t.r.f. are characterized and in [10] it is proved that every finite dimensional *arithmetic ring* (that is, a ring in which for every three ideals I, J and K, we have $I + (J \cap K) = (I + J) \cap (I + K)$) s.t.r.f.

In [4] a simpler form of rings and modules which s.t.r.f. are introduced and studied.

For some references on semi-compatible rings and modules, one may study [1, 3, 5].

In Section 2, we will introduce and study the concept of prime bases and the standard prime basis of a submodule. Particularly we will prove that a submodule is weakly prime if and only if it has a prime basis. Also we will study when the standard prime basis of a weakly prime submodule is finite.

In Section 3, we will utilize the concept of prime bases to investigate rings which weakly s.t.r.f. or are semi-compatible. In particular, we will show that every one dimensional integral domain weakly s.t.r.f. and show that a Noetherian domain weakly s.t.r.f. if and only if it has Krull dimension one. Furthermore we will state conditions on local domains which s.t.r.f. or are semi-compatible and conditions on Noetherian rings which weakly s.t.r.f.

2. Prime bases of weakly prime submodules

Let W be a weakly prime submodule of an R-module M and S a multiplicatively closed subset (MCS) of R. It is easy to see that if $W_S \neq M_S$, then W_S is a weakly prime submodule of M_S and conversely if W is a weakly prime submodule of M_S, then W^c (by which, in this paper we denote $W \cap M = \{x \mid \frac{x}{1} \in W\}$) is a weakly prime submodule of M (cf. [1, Section 2]). But unlike prime submodules, it is quite possible that $W \neq {W_S}^c$, for a weakly prime submodule W of M with $W_S \neq M_S$. For example if $\mathfrak{P} \subset \mathfrak{Q}$ are two distinct prime ideals of R, then for the R-module $M = R \oplus R$, we have ${(\mathfrak{P} \oplus \mathfrak{Q})_{\mathfrak{P}}}^c = \mathfrak{P} \oplus R$, although $\mathfrak{P} \oplus \mathfrak{Q}$ is a weakly prime submodule of M. Two sorts of submodules of the form ${W_S}^c$ prove to be useful in the study of weakly prime submodules, which are introduced in the following.

Notation 1. Let W be a weakly prime submodule of M, and suppose $m \in M \setminus W$. We consider $\mathcal{A}(W)$, W_m and $\overline{W_m}$ as follows:

$$\mathcal{A}(W) = \{{W_S}^c \mid {W_S}^c \neq M,\ S \text{ is an MCS of } R\},$$
$$W_m = \{x \in M \mid (W : m) \subset (W : x)\},$$
$$\overline{W_m} = \{x \in M \mid (W : m) \subseteq (W : x)\}.$$

We denote $\mathcal{A}(W)$, just by $\mathcal{A}$ if there is no ambiguity.

It is clear that W_m's and $\overline{W_m}$'s are submodules of M.

Suppose W is weakly prime and $m \in M \setminus W$. Then $\mathfrak{P} = (W : m)$ is a prime ideal and one can easily see that $W_m = {W_{\mathfrak{P}}}^c$, so $W_m \in \mathcal{A}$. Also recall that $\mathcal{C}(W) = \{(W : x) \mid x \in M \setminus W\}$ is a chain of prime ideals, hence the set $S(m) = R \setminus \bigcup\{(W : x) \mid (W : x) \subset (W : m)\}$ is a multiplicatively closed subset of R, and it is easy to observe that $\overline{W_m} = {W_{S(m)}}^c$. Therefore $\overline{W_m} \in \mathcal{A}$. Furthermore obviously $m \in \overline{W_m} \setminus W_m$ and $W \subseteq W_m \subset \overline{W_m}$.

Note that we take the intersection of an empty family of submodules of M to be M.

Lemma 2.1 ([8, Proposition 2.5]). *For every submodule N of M and each prime ideal $\mathfrak{P}$ of R, the intersection of all $\mathfrak{P}$-prime submodules of M containing N is ${(N + \mathfrak{P}M)_{\mathfrak{P}}}^c$ and it is itself a $\mathfrak{P}$-prime submodule, if it is a proper submodule.*

Next proposition states some preliminary properties of W_m and $\overline{W_m}$.

Proposition 2.2. *Suppose that W is a weakly prime submodule of M, S is an MCS of R, $\mathfrak{P} = (W : M)$ and $\mathfrak{Q} = Z\left(\frac{M}{W}\right)$.*

(i) $W_S{}^c = \bigcap\{W_m \mid m \in M \setminus W, S \cap (W : m) = \emptyset\} = \bigcup\{\overline{W_m} \mid m \in M \setminus W,\ S \cap (W : m) \neq \emptyset\}$

(ii) *W is contained in a $\mathfrak{P}$-prime submodule, if and only if $\overline{W_x} = M$ for some $x \in M$, if and only if $\mathfrak{P} = (W : x)$ for some $x \in M$, if and only if $\bigcup_{m \in M\setminus W} W_m \neq M$.*

(iii) *$\bigcap_{m\in M\setminus W} \overline{W_m} \neq W$ if and only if $W = W_x$ for some $x \in M$, if and only if $\mathfrak{Q} = (W : x)$ for some $x \in M$.*

(iv) *$\bigcup_{m\in M\setminus W} \overline{W_m} = M$ and*

$$\bigcup_{m\in M\setminus W} W_m = W_{\mathfrak{P}}{}^c = \bigcap\{P \mid W \subseteq P \text{ is a } \mathfrak{P}\text{-prime submodule of } M\}.$$

(v) *$\bigcap_{m\in M\setminus W} W_m = W$ and $\bigcap_{m\in M\setminus W} \overline{W_m} = (W_T)^c$, where $T = R \setminus \bigcup\{(W : m) \mid m \in M \setminus W, (W : m) \subset \mathfrak{Q}\}$.*

Proof. **(i)** Assume $x \in W_S{}^c \setminus W$. Then $(W : x) \cap S \neq \emptyset$ and since $x \in \overline{W_x}$, we have $W_S{}^c \subseteq \bigcup\{\overline{W_m} \mid m \in M \setminus W,\ S \cap (W : m) \neq \emptyset\} \cup W$. Now let $m, m' \in M \setminus W$ be such that $(W : m) \cap S = \emptyset$ and $(W : m') \cap S \neq \emptyset$, say $s \in (W : m') \cap S$. Then $(W : m) \subset (W : m')$, because $\{(W : y) \mid y \in M \setminus W\}$ is a chain and $s \in (W : m') \setminus (W : m)$. Therefore $\overline{W_{m'}} \subseteq W_m$ and thus $\bigcup\{\overline{W_m} \mid S \cap (W : m) \neq \emptyset\} \cup W \subseteq \bigcap\{W_m \mid S \cap (W : m) = \emptyset\}$. Now suppose $x \in M \setminus W_S{}^c$. Then $(W : x) \cap S = \emptyset$ and $x \notin W_x$. Hence $\bigcap\{W_m \mid S \cap (W : m) = \emptyset\} \subseteq W_S{}^c$ and the result follows.

(ii) One can easily show that for every $x \in M$, $\overline{W_x} = M$ if and only if $(W : x) = \mathfrak{P}$. If $(W : x) = \mathfrak{P}$, then $x \notin \bigcup_{m\in M\setminus W} W_m = W_x = W_{\mathfrak{P}}{}^c \neq M$ and W is contained in a $\mathfrak{P}$-prime submodule of M by **(2.1)**. If P is $\mathfrak{P}$-prime submodule containing W and $x \in W_m$, then since $\mathfrak{P} \subseteq (W : m) \subset (W : x)$, there is an $r \in (W : x) \setminus \mathfrak{P}$. So $rx \in W \subseteq P$ and hence $x \in P$. Therefore each W_m (and consequently their union) falls in $P \neq M$. Finally assume that $(W : x) \neq \mathfrak{P}$ for all $x \in M$. Then for all $x \in M$ there is an $m \in M$ such that $(W : m) \subset (W : x)$ (else $\mathfrak{P} = \bigcap_{m\in M}(W : m) = (W : x)$, a contradiction). Whence $\bigcup_{m\in M\setminus W} W_m = M$.

(iii) Similar to **(ii)**. Using **(i)**–**(iii)** and their proofs one can readily prove **(iv)** and **(v)**. □

Corollary 2.3. *For every weakly prime submodule W of M, $\mathcal{A}(W)$ is totally ordered with respect to inclusion.*

Proof. Clearly the inclusion relation induces a total order on $\mathcal{A}' = \{W_m \mid m \in M \setminus W\} \cup \{M\}$ and every element of $\mathcal{A}$ is an intersection of elements of $\mathcal{A}'$ by **(2.2)(i)**, so $\mathcal{A}$ is totally ordered, too. □

Let $(A, <)$ be a partially ordered set. If $a \in A$ and $a^+ = \min\{x \in A \mid a < x\}$ exists, we say that a has a next element and call a^+ the next element of a. Similarly a^- denotes the previous element of a, that is, $\max\{x \in A \mid x < a\}$, if it exists.

Corollary 2.4. *Let W be a weakly prime submodule of M. A submodule $A \in \mathcal{A}(W)$ has a next (previous) element (with respect to inclusion), if and only if $A = W_m$ ($A = \overline{W_m}$) for some $m \in M \setminus W$, and in this case $A^+ = \overline{W_m}$ ($A^- = W_m$).*

Proof. We prove the claim about the next element, the proof for the previous element follows from the fact that in the totally ordered set $\mathcal{A}$, if A has a previous element, then A is the next element of its previous element.

Let $A \neq W_m$ for any $m \in M \setminus W$ and assume that A^+ exists. Then according to **(2.2)(i)**, $A = \bigcap \{W_m \mid A \subseteq W_m,\ m \in M \setminus W\}$. But each such W_m in the intersection (and hence A) contains A^+ by definition of A^+. So $A = A^+$ which is against the definition of A^+.

Now suppose that $A = W_m$ for an $m \in M \setminus W$. Since $A \subset \overline{W_m}$, it suffices to show that if $B \in \mathcal{A}$ and $A \subset B$, then $\overline{W_m} \subseteq B$. Suppose not, then by the previous corollary we must have $B \subset \overline{W_m}$. But by **(2.2)(i)**, B is a union of $\overline{W_x}$'s. So there exist $x \in M \setminus W$ such that $W_m \subset \overline{W_x} \subseteq B \subset \overline{W_m}$. Therefore $x \in \overline{W_m} \setminus W_m$, thus $(W : x) = (W : m)$, a contradiction to $\overline{W_m} \neq \overline{W_x}$. $\square$

Next we state a theorem which investigates the existence and the uniqueness of what we will call *the prime basis of a weakly prime submodule* (see Definition 2).

Theorem 2.5. *Let W be a proper submodule of M. The following are equivalent.*

- **(i)** *W is weakly prime.*
- **(ii)** *There is a family $\{(N_\alpha, M_\alpha)\}_{\alpha \in A}$ of pairs of submodules of M containing W such that:*
  - (a) *N_α is a prime submodule of M_α for all $\alpha \in A$,*
  - (b) *$(N_\alpha : M_\alpha) = (W : M_\alpha)$ for all $\alpha \in A$,*
  - (c) *$(M \setminus W) \subseteq \bigcup_{\alpha \in A} (M_\alpha \setminus N_\alpha)$.*
- **(iii)** *There is a family $\{(N_\alpha, M_\alpha)\}_{\alpha \in A}$ of pairs of submodules of M containing W satisfying* **(a)**, **(b)**, **(c)** *above such that it also has the following property:*
  - (d) *$(N_\alpha : M_\alpha) \neq (N_\beta : M_\beta)$ for all $\alpha \neq \beta \in A$.*

Also if W is weakly prime, then there exists exactly one family $\{(N_\alpha, M_\alpha)\}_{\alpha \in A}$ of pairs of submodules of M containing W, which satisfies all four of the above conditions.

Proof. **(i) $\Rightarrow$ (iii)**: We show that the family $\{(W_x, \overline{W_x}) \mid x \in M \setminus W\}$ satisfies all of the above four conditions, where the index set of this family is $A = \{[x] \mid x \in M \setminus W\}$, where $[x]$ denotes the equivalency class of x under the relation $x \equiv y \iff (W : x) = (W : y)$.

Suppose $x, y \in M \setminus W$. If $m \in \overline{W_x} \setminus W_x$ and $rm \in W_x$ for some $r \in R$, then $(W : x) \subset (W : rm)$. Let $s \in (W : rm) \setminus (W : x)$, so $rs \in (W : m) = (W : x)$, because $m \in \overline{W_x} \setminus W_x$. But $(W : x)$ is a prime ideal and $s \notin (W : x)$, thus

$r \in (W : x) = (W : \overline{W_x}) = (W_x : \overline{W_x})$. Consequently W_x is a $(W : \overline{W_x})$-prime submodule of $\overline{W_x}$ and **(a)** and **(b)** holds. Clearly $x \in \overline{W_x} \setminus W_x$ and $(W_x : \overline{W_x}) = (W_y : \overline{W_y})$ if and only if $(W : x) = (W : y)$ if and only if $[x] = [y]$, whence **(c)** and **(d)** follow.

(iii) $\Rightarrow$ **(ii)**: Trivial.

(ii) $\Rightarrow$ **(i)**: We must show that $(W : x)$ is a prime ideal, for all $x \in M \setminus W$. Let $\alpha \in A$ be such that $x \in M_\alpha \setminus N_\alpha$. Now

$$(*) \qquad (W : M_\alpha) \subseteq (W : x) \subseteq (N_\alpha : x) = (N_\alpha : M_\alpha) = (W : M_\alpha),$$

where the equalities hold by **(a)** and **(b)**. Hence $(W : x) = (N_\alpha : M_\alpha)$, and it is a prime ideal of R, because N_α is a prime submodule of M_α. Consequently W is a weakly prime submodule of M.

To prove the final assertion suppose that W is weakly prime and the family $\mathcal{B} = \{(N_\alpha, M_\alpha)\}$ satisfies all of the four conditions. We will show that $\mathcal{B} = \{(W_x, \overline{W_x})\}_{x \in M \setminus W}$. Set $\mathfrak{P}_\alpha = (N_\alpha : M_\alpha)$.

We will show that for each $x \in M$,

$$(**) \qquad x \in M_\alpha \setminus N_\alpha \Longleftrightarrow (W : x) = (N_\alpha : M_\alpha).$$

Note that $W \subseteq N_\alpha$, so if $x \in M_\alpha \setminus N_\alpha$, then $x \in M \setminus W$, and the $(\Longrightarrow)$ implication is given by $(*)$. For the converse implication, suppose $(W : x) = (N_\alpha : M_\alpha)$. Since $(N_\alpha : M_\alpha)$ is a prime ideal, $x \in M \setminus W$. Then from (c), for some $\beta \in A$, we have $x \in M_\beta \setminus N_\beta$. Hence by $(*)$, $(N_\alpha : M_\alpha) = (W : x) = (N_\beta : M_\beta)$, which implies that $\beta = \alpha$, and so $x \in M_\alpha \setminus N_\alpha$.

Now by $(**)$, $M_\alpha = N_\alpha \cup \{x \in M \mid (W : x) = \mathfrak{P}_\alpha\}$, and so $M_\alpha = N_\alpha \cup \langle\{x \in M \mid (W : x) = \mathfrak{P}_\alpha\}\rangle$. Thus either $M_\alpha = N_\alpha$ which is impossible, or $M_\alpha = \langle\{x \in M \mid (W : x) = \mathfrak{P}_\alpha\}\rangle$.

Fix an $m \in M_\alpha \setminus \mathrm{N}_\alpha$. Then $m \in \overline{W_m} \setminus W_m$ and $(W_m : \overline{W_m}) = (W : m) = \mathfrak{P}_\alpha$. Since $\{(W_x, \overline{W_x})\}_{x \in M \setminus W}$ satisfies the four conditions of the statement, as a special case of the above argument, we get $\overline{W_m} = \langle\{x \in M \mid (W : x) = \mathfrak{P}_\alpha\}\rangle$, whence $M_\alpha = \overline{W_m}$. Also

$$N_\alpha = M_\alpha \setminus \{x \in M \mid (W : x) = \mathfrak{P}_\alpha\} = \overline{W_m} \setminus \{x \in M \mid (W : x) = (W : m)\} = W_m.$$

$\square$

Definition 2. Let W be a submodule of M. Then we call a nonempty family $\mathcal{B} = \{(N_\alpha, M_\alpha)\}_{\alpha \in A}$ of pairs of submodules of M containing W which satisfies **(a)**, **(b)** and **(c)** of **(2.5)**, a prime basis for W (in M). If moreover $\mathcal{B}$ satisfies **(d)**, we say it is the standard prime basis of W (in M).

In the proof of **(2.5)**, we mentioned that W_m is a prime submodule of $\overline{W_m}$. This raises the question answered below, that if $W_1, W_2 \in \mathcal{A}$, when W_1 is prime (or weakly prime) considered as a submodule of W_2.

Remark 2.6. **(i)** According to Theorem **(2.5)**, a submodule W of M is weakly prime if and only if it has a prime basis. Also its proof shows that when W is $\mathcal{C}$-weakly prime, $\mathcal{B} = \{(W_x, \overline{W_x}) \mid x \in M \setminus W)\}$ is the standard prime basis for W in M. Now the map $(W_x : \overline{W_x}) \mapsto \mathfrak{P}_x = (W_x : \overline{W_x}) = (W : x)$ is a one to one correspondence between $\mathcal{B}$ and $\mathcal{C}$. Therefore W is prime if and only if $|\mathcal{B}| = |\mathcal{C}| = 1$.

(ii) Let W be a weakly prime submodule of M and $\mathcal{B}$ be the standard prime basis of W in M. Suppose that $W_1, W_2 \in \mathcal{A}(W)$. Then using **(i)**, **(2.3)** and **(2.4)**, it is easy to see that $\mathcal{B}' = \{(N, N') \in \mathcal{B} \mid W_1 \subseteq N,\ N' \subseteq W_2\}$ is the standard prime basis for W_1 in W_2. Therefore W_1 is prime in W_2, if and only if $|\mathcal{B}'| = 1$, if and only if $\mathcal{B}' = \{(W_1, W_2)\}$, if and only if $(W_1, W_2) \in \mathcal{B}$, if and only if $W_1 = W_m$ and $W_2 = \overline{W_m}$ for some $m \in M \setminus W$.

By **(2.5)**, the standard prime basis of a weakly prime submodule is unique. But a weakly prime submodule may have distinct prime bases, as the following example shows.

Example 2.7. Let $M = R \oplus R$ and $W = \mathfrak{P} \oplus R$, where $\mathfrak{P}$ is a non-maximal prime ideal of R contained in the maximal ideal $\mathfrak{M}$. Then both $\mathcal{B} = \{(W, M)\}$ and $\mathcal{B}' = \mathcal{B} \cup \{(W, \mathfrak{M} \oplus R)\}$ are prime bases for W in M.

It is not difficult to prove the next result without any use of **(2.5)**, but we prove it using **(2.5)**, to show how prime bases can be handled.

Recall that in the following result, $\mathrm{wrad}_{(\mathfrak{P}M)_{\mathfrak{P}}{}^c}(\mathfrak{P}M)$ is the intersection of all weakly prime submodule of the module $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ containing the submodule $\mathfrak{P}M$.

If we set $\mathrm{T}(M) = \{m \in M \mid \exists 0 \neq r \in R,\ rm = 0\}$, then by applying the following result in the case that R is an integral domain, we get $\mathrm{wrad}_M(0) = \mathrm{wrad}_{(0_0)^c}(0) = \mathrm{wrad}_{\mathrm{T}(M)}(0)$, which is [5, Corollary 1.4].

Corollary 2.8. *For every R-module M, we have*

$$wrad_M(0) = \bigcap \Big\{ wrad_{(\mathfrak{P}M)_{\mathfrak{P}}{}^c}(\mathfrak{P}M) \mid \mathfrak{P} \text{ is a minimal prime ideal of } R \Big\}.$$

Proof. To show $\mathrm{wrad}_M(0) \subseteq \bigcap \Big\{ \mathrm{wrad}_{(\mathfrak{P}M)_{\mathfrak{P}}{}^c}(\mathfrak{P}M) \mid \mathfrak{P}$ is a minimal prime ideal of $R\}$, we prove that if for some prime ideal $\mathfrak{P}$, we have $\mathfrak{P}M \subseteq W$ and W is weakly prime in $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$, then it is weakly prime in M. If $(\mathfrak{P}M)_{\mathfrak{P}}{}^c = M$, then the claim is clear, so suppose $(\mathfrak{P}M)_{\mathfrak{P}}{}^c \neq M$. Then by **(2.1)**, $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ is a $\mathfrak{P}$-prime submodule of M. Let $\mathcal{B}$ be a prime basis for W in $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ and set $\mathcal{B}' = \mathcal{B} \cup \big\{ \big((\mathfrak{P}M)_{\mathfrak{P}}{}^c, M \big) \big\}$. It is a routine task to check that $\mathcal{B}'$ is a prime basis for W in M, which concludes the claim.

Now to prove the converse inclusion, assume that W is a weakly prime submodule of M. Let $\mathfrak{P}$ be a minimal prime ideal contained in $(W : M)$. It suffices to show that either W contains $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ or $W' = W \cap (\mathfrak{P}M)_{\mathfrak{P}}{}^c$ is a weakly prime submodule of $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ (since in both cases $\mathrm{wrad}_{(\mathfrak{P}M)_{\mathfrak{P}}{}^c}(\mathfrak{P}M) \subseteq W$). Suppose $W' \neq (\mathfrak{P}M)_{\mathfrak{P}}{}^c$. Let $\mathcal{B}$ be a prime basis for W in M and set $\mathcal{B}' = \{(A, A') \mid A = B \cap (\mathfrak{P}M)_{\mathfrak{P}}{}^c,\ A' = B' \cap (\mathfrak{P}M)_{\mathfrak{P}}{}^c$ for some $(B, B') \in \mathcal{B}$

and $A \neq A'\}$. Again it is easy to see that **(a)**, **(b)** and **(c)** of **(2.5)**, hold for $\mathcal{B}'$ (with M replaced with $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ and W with W'). Consequently W' is a weakly prime submodule of $(\mathfrak{P}M)_{\mathfrak{P}}{}^c$ and the result is established. □

Next we investigate when weakly prime submodules have finite standard prime bases.

Proposition 2.9. *Let W be a weakly prime submodule of M, $\mathcal{A} = \mathcal{A}(W)$ and $\mathcal{B}$ be the standard prime basis of W. Then $|\mathcal{A}| < \infty$, if and only if $|\mathcal{B}| < \infty$, if and only if every element of $\mathcal{A}$ except possibly M, has a next element and each element of $\mathcal{A}$ except possibly W, has a previous element.*

Proof. Using **(2.2)(i)** and **(2.6)**, one can easily see that $|\mathcal{A}| < \infty$, if and only if $|\mathcal{B}| < \infty$. Also it is clear that if $|\mathcal{A}| < \infty$, then every element except the first one (which is W) has a previous element and every element except the last one (which is M) has a next element.

Conversely, assume that every element of $\mathcal{A}$ except possibly M, has a next element and each element of $\mathcal{A}$ except possibly W, has a previous element but $|\mathcal{A}| = \infty$. Let $A_0 \in \mathcal{A}$. Then either there exist infinite elements of $\mathcal{A}$ containing A_0 or infinite elements of $\mathcal{A}$ are contained in A_0. Assume that the former holds.

Set $A_i = A_{i-1}^+$ for each $i \in \mathbb{N}$. Note that by **(2.4)**, for each $i \in \mathbb{N}$ there is an $m_i \in M \setminus W$ such that $A_i = W_{m_i}$ and $A_{i+1} = \overline{W_{m_i}}$. Let $\mathfrak{P}_i = (W : m_i)$ and set $\mathfrak{P} = \bigcap_{i \in \mathbb{N}} \mathfrak{P}_i$. Now by **(2.2)(i)**, $W_{\mathfrak{P}}{}^c = \bigcup\{\overline{W_m} \mid m \in M \setminus W,\ \overline{W_m} \subseteq W_{\mathfrak{P}}{}^c\}$. But if $\overline{W_m} \subseteq W_{\mathfrak{P}}{}^c$, then $(W : m) \not\subseteq \mathfrak{P}$. Thus for some $i \in \mathbb{N}$, we must have $(W : m) \not\subseteq \mathfrak{P}_i$, therefore $(W : m_i) \subset (W : m)$. So $m \in A_i$. Consequently $W_{\mathfrak{P}}{}^c = \bigcup_{i \in \mathbb{N}} A_i$.

But by our assumption, $W_{\mathfrak{P}}{}^c$ has a previous element in $\mathcal{A}$, say B. If each $A_i \neq W_{\mathfrak{P}}{}^c$, then by **(2.3)**, each $A_i \subseteq B$ and hence $W_{\mathfrak{P}}{}^c = \bigcup A_i \subseteq B$, which is impossible. So for some $k \in \mathbb{N}$, we have $A_k = A_{k+1} = \cdots = W_{\mathfrak{P}}{}^c$, which is contrary to the definition of the next element. By a similar argument, we get a contradiction in the case that infinite elements of $\mathcal{A}$ are contained in A_0. From this contradiction, we deduce that $|\mathcal{A}| < \infty$. □

If W is a weakly prime submodule with the finite standard basis $\{(N_i, M_i) \mid 1 \leq i \leq n\}$, then by the previous result and **(2.2)**, one of the M_i's, say M_1, must be M. Now $N_1 = W$ or it is of the form $\overline{W_m}$ for some $m \in M \setminus W$ and hence it ought to show up in the standard basis as one of the M_i's, say M_2. Continuing this way, we can assume that $M_{i+1} = N_i$ for each $1 \leq i < n$. Finally we have $N_n = W$ by **(2.2)**. This leads us to a chain

$$(*) \qquad W = N_n < \cdots < N_1 < N_0 = M$$

of submodules of M with the property that for all $1 \leq i \leq j \leq n$, N_i is a prime submodule of N_{i-1}, $\mathfrak{P}_i = (N_i : N_{i-1}) = (W : N_{i-1})$ and $\mathfrak{P}_i \subset \mathfrak{P}_j$. Conversely, if we have chain of submodules of M, such as $(*)$, satisfying these conditions, then clearly $\{(N_i, N_{i-1}) \mid 1 \leq i \leq n\}$ is the standard basis for W

in M. Consequently, in the rest of this paper, if W has a finite standard basis, we represent this basis by $(*)$ and say that this chain *represents* the standard prime basis of W.

Note that by **(2.6)(i)** for every weakly prime submodule of M, such as W, the standard basis of W has the same cardinality as $\mathcal{C}(W)$. Therefore if R has no infinite chain of prime ideals (or equivalently has both ACC and DCC on prime ideals), then the standard basis of every weakly prime submodule of M is finite. In particular, this happens when R has finite Krull dimension. According to the following result, another condition under which every weakly prime submodule of M has a finite standard prime basis, is M being Laskerian.

Here if $N = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition for N, where Q_i is a $\mathfrak{P}_i$-primary submodule of M, then by the primary component of N corresponding to an isolated subset $\mathcal{P}$ of $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_n\}$, we mean $\bigcap\{Q_i \mid 1 \leq i \leq n,\ \mathfrak{P}_i \in \mathcal{P}\}$. Note that by the second uniqueness theorem, this component is independent of the specific primary decomposition.

Theorem 2.10. *Let W be a weakly prime submodule of M and assume that $W = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition for W, where Q_i is a $\mathfrak{P}_i$-primary submodule of M. Then, after a possible reordering of Q_i's, we have $\mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \cdots \subset \mathfrak{P}_n$ and the standard prime basis for W in M is represented by $W = W_n < W_{n-1} < \cdots < W_1 < M$, where W_i is the primary component of W corresponding to the isolated set $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_i\}$ of prime ideals belonging to W.*

Proof. We prove the statement by induction on n. If $n = 1$ then W is both weakly prime and primary and hence it is a prime submodule and the claim turns trivial. Suppose $n > 1$ and let $\mathfrak{Q}_i = (Q_i : M)$. Since $(W : M) = \bigcap_{i=1}^{n} \mathfrak{Q}_i$ is a prime ideal, $\mathfrak{Q}_i = (W : M)$, for some $1 \leq i \leq n$, say $i = 1$. Thus $\mathfrak{Q}_1 = \mathfrak{P}_1$ is a prime ideal and therefore Q_1 is a prime submodule of M and $(W : M) = (Q_1 : M)$.

Now consider W as a submodule of Q_1. It is easy to see that $Q'_i = Q_i \cap Q_1$ is a $\mathfrak{P}_i$-primary submodule of Q_1 and $W = \bigcap_{i=2}^{n} Q'_i$ is a minimal primary decomposition for W in Q_1. So by the induction hypothesis after a reordering of Q_i's for $2 \leq i \leq n$, we can assume that $\mathfrak{P}_2 \subset \cdots \subset \mathfrak{P}_n$ and $W = W_n < W_{n-1} < \cdots < W_2 < Q_1$ represents the standard basis of W in Q_1, where $W_j = \bigcap_{i=2}^{j} Q'_i = \bigcap_{i=1}^{j} Q_i$.

But $\mathfrak{P}_1 = (W : M) \subset \mathfrak{P}_i$ for each $1 < i$. Therefore if we set $W_1 = Q_1$, then for each $1 \leq i \leq n$, W_i is the component of W corresponding to the isolated set $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_i\}$ and $W = W_n < W_{n-1} < \cdots < W_1 < M$, represents the standard prime basis of W in M, as required. □

Now we give an example of a weakly prime submodule with an infinite standard prime basis. For this we need a lemma.

Lemma 2.11. *Suppose that M_i is an R-module and W_i is a $\mathcal{C}_i$-weakly prime submodule of M_i, for each $i \in I$. If the inclusion relation induces a total* *order on* $\mathcal{C} = \bigcup_{i\in I} \mathcal{C}_i$*, then* $W = \bigoplus_{i\in I} W_i$ $(W = \prod_{i\in I} W_i)$ *is a weakly prime submodule of* $M = \bigoplus_{i\in I} M_i$ $(M = \prod_{i\in I} M_i)$.

Proof. Let $m = (m_i)_{i\in I}$ be in M, then $(W : m) = \bigcap_{i\in I}(W_i : m_i)$. But $(W_i : m_i)$'s form a chain of prime ideals and hence their intersection is either R or a prime ideal. □

Example 2.12. Let R be a ring containing an infinite chain of prime ideals $\mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \cdots$ and consider $W = \bigoplus_{i\in\mathbb{N}} \mathfrak{P}_i$ as a submodule of $M = \bigoplus_{i\in\mathbb{N}} R$. Particularly let K be a field and $R = K[x_1, x_2, \ldots]$ and set $\mathfrak{P}_i = \langle x_1, x_2, \ldots, x_i\rangle$, for each $i \in \mathbb{N}$.

Then by the previous lemma, W is a weakly prime submodule of M. Also $\mathfrak{M} = \mathrm{Z}\left(\frac{M}{W}\right) = \bigcup_{i\in\mathbb{N}} \mathfrak{P}_i$, and it is easy to see that for every $m \in M \setminus W$, there exists a $\mathfrak{P}_i$ with $(W : m) = \mathfrak{P}_i$. Hence $\mathfrak{M} \neq (W : m)$ for every $m \in M$. Therefore by **(2.2)(iii)** and **(2.4)**, $W = W_{\mathfrak{M}}{}^c$ does not have a next element, hence according to **(2.9)**, the standard basis of W is infinite.

Indeed if we put $N_0 = M$ and for each $n \in \mathbb{N}$, consider $N_n = (\oplus_{k=1}^{n}\mathfrak{P}_k) \oplus (\oplus_{k>n} R)$, then

$$\cdots \subset N_2 \subset N_1 \subset N_0 = M$$

is a standard prime basis of W.

3. The weak radical of submodules

The following theorem helps us to characterize the weak radical of a submodule as we will see in **(3.3)**.

Notation 3. Let $\mathcal{C} = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\}$, where $\mathfrak{P}_1 \subset \mathfrak{P}_2 \cdots \subset \mathfrak{P}_n$ are prime ideals of R, and let N be a submodule of M. Set $M_0 = M$, $N_0 = N$ and inductively let $N_i = N_{i-1} + \mathfrak{P}_i M_{i-1}$ and $M_i = (N_i)_{\mathfrak{P}_i}{}^c$ for each $1 \leq i \leq n$. Then we denote M_n by $\mathcal{C} - w_M(N)$.

Theorem 3.1. *Let* $\mathcal{C} = \{\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n\}$*, where* $\mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \cdots \subset \mathfrak{P}_n$ *are prime ideals of* R*, and suppose* N *is a submodule of* M*. Then the intersection of all* $\mathcal{C}'$*-weakly prime submodules of* M *containing* N *with* $\mathcal{C}' \subseteq \mathcal{C}$ *is* $\mathcal{C} - W_M(N)$ *(if there is no such a submodule, then* $\mathcal{C} - w_M(N) = M$*).*

Proof. Suppose that N_i's and M_i's are as in Notation **(3)**. First we will show that $M_n = M$ or M_n is a $\mathcal{C}'$-weakly prime submodule of M with $\mathcal{C}' \subseteq \mathcal{C}$. Suppose that $1 \leq i \leq n$. Note that if $x \in M_i$, then for some $s \in R \setminus \mathfrak{P}_i$, $sx \in N_i = N_{i-1} + \mathfrak{P}_i M_{i-1}$. Since $M_{i-1} = (N_{i-1})_{\mathfrak{P}_{i-1}}{}^c$ there is an $s' \in R \setminus \mathfrak{P}_{i-1}$ such that $s'sx \in N_{i-1}$. But $\mathfrak{P}_{i-1} \subseteq \mathfrak{P}_i$ and so $ss' \in R \setminus \mathfrak{P}_{i-1}$. Therefore $x \in M_{i-1}$ and hence $M_i \subseteq M_{i-1}$.

Assume that $M_i \neq M_{i-1}$. Then $(M_i)_{\mathfrak{P}_i} \neq (M_{i-1})_{\mathfrak{P}_i}$, for M_i is a contracted submodule. Also because $(M_i)_{\mathfrak{P}_i} = (N_i)_{\mathfrak{P}_i} = (N_{i-1})_{\mathfrak{P}_i} + (\mathfrak{P}_i)_{\mathfrak{P}_i}(M_{i-1})_{\mathfrak{P}_i}$, we must have $\left((M_i)_{\mathfrak{P}_i} : (M_{i-1})_{\mathfrak{P}_i}\right) = (\mathfrak{P}_i)_{\mathfrak{P}_i}$. But since $(\mathfrak{P}_i)_{\mathfrak{P}_i}$ is the maximal ideal of $R_{\mathfrak{P}_i}$, $(M_i)_{\mathfrak{P}_i}$ is a $(\mathfrak{P}_i)_{\mathfrak{P}_i}$-prime submodule of $(M_{i-1})_{\mathfrak{P}_i}$ and thus M_i is a $\mathfrak{P}_i$-prime submodule of M_{i-1}. Moreover $(M_i : M_{i-1}) = \mathfrak{P}_i \subseteq (N_i : M_{i-1}) \subseteq$

$(N_n : M_{i-1}) \subseteq (M_n : M_{i-1}) \subseteq (M_i : M_{i-1})$, that is, $(M_i : M_{i-1}) = (M_n : M_{i-1})$. Therefore if $M_n \neq M$, then the chain $M_n \leq M_{n-1} \leq \cdots \leq M_0 = M$ (deleting the possible repeated terms) represents the standard basis for M_n in M. Consequently M_n is weakly prime and from the above argument it is clear that $\mathcal{C}(M_n) \subseteq \mathcal{C}$.

If $M_n = M$, then $M_i = M$ and $N_i = N + \mathfrak{P}_i M$, for each $1 \leq i \leq n$. Thus $(N + \mathfrak{P}_i M)_{\mathfrak{P}_i}{}^c = M$ for all $1 \leq i \leq n$, whence it follows from **(2.1)** that M has no $\mathfrak{P}_i$-prime submodule containing N, for each $1 \leq i \leq n$. Therefore if $\mathcal{C}' \subseteq \mathcal{C}$, there is no $\mathcal{C}'$-weakly prime submodule of M which contains N, else if $W_k < \cdots < W_1 < W_0 = M$ represents the standard basis of such a submodule, then W_1 is a $\mathfrak{P}_i$-prime submodule of M containing N, for some $\mathfrak{P}_i \in \mathcal{C}'$. Thus in this case the claim comes true.

Now suppose that $M_n \neq M$. We will prove by induction on n that every $\mathcal{C}'$-weakly prime submodule of M which contains N, contains M_n too, assumed $\mathcal{C}' \subseteq \mathcal{C}$. The base case for $n = 1$ follows **(2.1)**. Suppose that $n > 1$ and $W_k < \cdots < W_1 < M$ represents the standard basis for a $\mathcal{C}'$-weakly prime submodule $W = W_k$ of M, where $\mathcal{C}' \subseteq \mathcal{C}$. If $M_1 \subseteq W$, then the claim is clearly true, so assume that $W' = W \cap M_1 \neq M_1$.

It is clear that W' is a weakly prime submodule of M_1 and $\mathcal{C}(W') \subseteq \mathcal{C}' \subseteq \mathcal{C}$. We will show that $\mathfrak{P}_1 \notin \mathcal{C}(W')$. To this end, it suffices to show that $(W' : M_1) \nsubseteq \mathfrak{P}_1$, because $(W' : M_1)$ is contained in every element of $\mathcal{C}(W')$ (note that W' here is considered as a submodule of M_1). If $\mathfrak{P}_1 \notin \mathcal{C}'$, then clearly $\mathfrak{P}_1 \notin \mathcal{C}(W')$, so we suppose that $\mathfrak{P}_1 \in \mathcal{C}'$. In this case W_1 is a $\mathfrak{P}_1$-prime submodule of M and by **(2.1)**, $M_1 \subseteq W_1$. Furthermore $(W_1 : M)$ is strictly contained in $(W : W_1)$ by definition of standard prime basis. Therefore $\mathfrak{P}_1 = (W : M) \subset (W : W_1) \subseteq (W : M_1) = (W' : M_1)$, which completes the proof of $\mathfrak{P}_1 \notin \mathcal{C}(W')$.

Let $\mathcal{C}_2 = \{\mathfrak{P}_2, \mathfrak{P}_3, \ldots, \mathfrak{P}_n\}$. From the construction of $\mathcal{C} - w_M(N)$, it is clear that $\mathcal{C} - w_M(N) = \mathcal{C}_2 - w_{M_1}(N + \mathfrak{P}_1 M)$. Now since $(W : M) \subseteq \mathfrak{P}_1$, we have $\mathfrak{P}_1 M \subseteq W$. Also clearly $\mathfrak{P}_1 M \subseteq M_1$, thus $N + \mathfrak{P}_1 M \subseteq W'$. Hence applying the induction hypothesis we get that $\mathcal{C}_2 - w_{M_1}(N + \mathfrak{P}_1 M) \subseteq W' \subseteq W$ and the result follows. $\square$

The following lemma, which is of much use in the rest of this paper, states some preliminary properties of envelopes and radicals of submodules. Here by $\frac{L}{K}$, where K is a submodule of M and L is a subset of M containing K, we mean $\{l + K \mid l \in L\}$.

Lemma 3.2. *Let M be an R-module, N and K submodules of M with $K \subseteq N$. Also suppose that S is a multiplicatively closed subset of R and L is a subset of M containing K.*

(i) $E_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{E_M(N)}{K}$; $rad_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{rad_M(N)}{K}$; $wrad_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{wrad_M(N)}{K}$.

(ii) $\frac{N}{K} = \frac{L}{K}$ *if and only if* $N = L + K$.

(iii) $E(N_S) = (E(N))_S$; $(rad(N))_S \subseteq rad(N_S)$; $(wrad(N))_S \subseteq wrad(N_S)$.

(iv) *If $M = \bigoplus_{i\in I} M_i$ and $N = \bigoplus_{i\in I} N_i$, where $N_i \leq M_i \leq M$ for each $i \in I$, then $RE_M(N) = \bigoplus_{i\in I} RE_{M_i}(N_i)$.*

(v) *If $M_{\mathfrak{M}}$ s.t.r.f. (weakly s.t.r.f.) for all maximal ideals $\mathfrak{M}$ of R, then M s.t.r.f. (weakly s.t.r.f.).*

(vi) *The ring R s.t.r.f. (weakly s.t.r.f.) if and only if $rad_{M'}(0) = RE_{M'}(0)$ ($wrad_{M'}(0) = RE_{M'}(0)$) for every R-module M'.*

Proof. **(i)**, **(ii)** and **(iv)** Easy. **(iii)** See for example [5, Proposition 2.1] and the proof of [11, Proposition 1.6]. **(v)** follows from **(iii)** and the proof of **(vi)** follows from **(i)**. □

Note that in part **(ii)** of the above lemma, although $K \subseteq L$, but since L is not necessarily a submodule, the set $L + K$ need not be L.

Corollary 3.3. *If R is a one dimensional domain, then R weakly s.t.r.f. and $wrad(0) = \bigcap_{\mathfrak{M}\in\max(R)} (\mathfrak{M}T(M))_{\mathfrak{M}}{}^c$, where $\max(R)$ is the set of all maximal ideals of R.*

Proof. According to **(3.1)**, $\mathrm{wrad}(0) = \bigcap_{\mathfrak{M}\in\max(R)} \mathcal{C}_{\mathfrak{M}} - w_M(0)$, where $\mathcal{C}_{\mathfrak{M}} = \{0, \mathfrak{M}\}$. But if M_i's and N_i's are as in **(3.1)** with $\mathcal{C} = \mathcal{C}_{\mathfrak{M}}$, one can easily check that $M_1 = \mathrm{T}(M)$ and $M_2 = (\mathfrak{M}\mathrm{T}(M))_{\mathfrak{M}}{}^c$. To show that R weakly s.t.r.f., due to **(3.2)**, we can assume that $(R, \mathfrak{M})$ is local and show that $\mathfrak{M}\mathrm{T}(M) = \mathrm{wrad}(0) \subseteq R\mathrm{E}(0)$. Suppose that $r \in \mathfrak{M}$ and $m \in T(M)$. Then $\mathrm{Ann}(m) \neq 0$, and $r \in \sqrt{\mathrm{Ann}(m)} = \mathfrak{M}$, thus for some $n \in \mathbb{N}$, $r^n m = 0$. Hence $rm \in R\mathrm{E}(0)$, as required. □

Thus the class of rings which weakly s.t.r.f. is strictly larger than that of rings which s.t.r.f., as the following example illustrates.

Example 3.4. Let K be a field and $R = K[x^2, x^3]$, then since $S = K[x]$ is integral over R, we conclude that R has Krull dimension one. So according to the previous corollary, R weakly s.t.r.f. Now note that R is Noetherian, hence by [7, Theorem 1.1], if R s.t.r.f., then it must be a Dedekind domain. But R is not a Dedekind domain, since it is not integrally closed.

Lemma 3.5. *Let N, A and B be submodules of M, such that $A \subseteq B \cap N$ and $\mathcal{C}$ be a finite chain of prime ideals of R. Then $\mathcal{C} - w_B(A) \subseteq \mathcal{C} - w_M(N)$.*

Proof. By **(3.1)**, $\mathcal{C} - w_B(A)$ is the intersection of all weakly prime submodules W of B containing A with $\mathcal{C}(W) \subseteq \mathcal{C}$. But it is easy to check that if W is a $\mathcal{C}'$-weakly prime submodule of M, containing A, then $W' = W \cap B$ is either the whole B or a weakly prime submodule of B with $\mathcal{C}(W') \subseteq \mathcal{C}'$. Thus $\mathcal{C} - w_B(A) \subseteq \mathcal{C} - w_M(A) \subseteq \mathcal{C} - w_M(N)$. □

Proposition 3.6. *If R has finitely many prime ideals, then for every submodule N of M, we have $wrad_M(N) = \bigcup wrad_{M_f}(N_f)$, where the union is taken over all finitely generated submodules N_f of N and finitely generated submodules M_f of M with $N_f \subseteq M_f$.*

Proof. By **(3.5)**, $\bigcup \mathrm{wrad}_{M_f}(N_f) \subseteq \mathrm{wrad}_M(N)$. For the converse, first suppose that $\mathcal{C} = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_n\}$, where $\mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_n$ are prime ideals of R.

By induction on n, we will prove that $\mathcal{C} - w_M(N) \subseteq \bigcup \mathcal{C} - w_{M_f}(N_f)$, where the union is taken over all finitely generated submodules N_f of N and finitely generated submodules M_f of M with $N_f \subseteq M_f$. As the base step we take $n = 0$, where by $\emptyset - w_M(N)$ we mean M. So in this case the claim is clear.

Now suppose $n > 0$. Set $\mathcal{C}_i = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_i\}$, $C_0 = \emptyset$ and assume that $x \in \mathcal{C} - w_M(N)$. According to the Notation **3**, $N_n = N + \sum_{i=1}^n \mathfrak{P}_i\mathcal{C}_{i-1} - w_M(N)$ and $\mathcal{C} - w_M(N) = (N_n)_{\mathfrak{P}_n}{}^c$. Thus for some $s \in R \setminus \mathfrak{P}_n$ we have $sx \in N + \sum_{i=1}^n \mathfrak{P}_i\mathcal{C}_{i-1} - w_M(N)$. That is, there is an $a \in N$ and for each $1 \le i \le n$ there exist some $n_i \in \mathbb{N}$ and r_{ij}'s in $\mathfrak{P}_i$ and m_{ij}'s in $\mathcal{C}_{i-1} - w_M(N)$ such that $sx = a + \sum_{i=1}^n \sum_{j=1}^{n_i} r_{ij}m_{ij}$.

Now by the induction hypothesis for each $1 \le i \le n$ and $1 \le j \le n_i$ there are finitely generated submodules N_{ij} and M_{ij} of N and M, respectively, such that $N_{ij} \subseteq M_{ij}$ and $m_{ij} \in \mathcal{C}_i - w_{M_{ij}}(N_{ij})$. If $M' = \sum M_{ij} + Rx + Ra$ and $N' = \sum N_{ij} + Ra$, then by **(3.5)**, all m_{ij}'s are in $\mathcal{C}_i - w_{M'}(N')$ and consequently $x \in (N' + \sum_{i=1}^n \mathfrak{P}_i\mathcal{C}_{i-1} - w_{M'}(N'))_{\mathfrak{P}_n}{}^c = \mathcal{C} - w_{M'}(N')$ by the construction in **(3.1)**. Since x was arbitrary and M' and N' are finitely generated submodules of M and N, respectively, with $N' \subseteq M'$, therefore we have proved the induction statement.

Now because R has finitely many primes, it has finitely many chains of prime ideals, say $\mathcal{C}_1, \ldots, \mathcal{C}_n$. According to **(3.1)**, $\mathrm{wrad}_M(N) = \bigcap_{i=1}^n \mathcal{C}_i - w_M(N)$. Suppose $x \in \mathrm{wrad}_M(N)$. Then by the above argument, for all i's, there are finitely generated submodules M_i and N_i of M and N, respectively, with $N_i \subseteq M_i$ such that $x \in \mathcal{C}_i - w_{M_i}(N_i)$. Therefore if we set $M' = \sum M_i$ and $N' = \sum N_i$. Then by **(3.5)**, $x \in \bigcap_{i=1}^n \mathcal{C}_i - w_{M'}(N')$, and according to **(3.1)**, $\bigcap_{i=1}^n \mathcal{C}_i - w_{M'}(N') = \mathrm{wrad}_{M'}(N')$, so $x \in \mathrm{wrad}_{M'}(N')$, which completes the assertion. $\square$

Next we turn our attention to Noetherian rings. But first a lemma.

Lemma 3.7. *Suppose that every primary submodule of the R-module $M = R \oplus R$ weakly s.t.r.f. in M. Then*

- **(i)** *For every prime ideal $\mathfrak{P}$ of R and any $a \in R \setminus \mathfrak{P}$, we have $a\mathfrak{P} \subseteq a^2\mathfrak{P} + \mathfrak{P}^{(2)}$, where $\mathfrak{P}^{(2)} = (\mathfrak{P}^2_{\mathfrak{P}})^c$ is the second symbolic power of $\mathfrak{P}$.*
- **(ii)** *If $\mathfrak{P}$ is a non-maximal prime ideal of R containing a non-zero divisor element, then $\mathfrak{P}$ is not a principal ideal.*

Proof. **(i)** In [12, Lemma 2.3] the same result is proved, supposing that every primary submodule of M s.t.r.f. in M. One can easily check that the same proof works if we use weakly s.t.r.f. instead of s.t.r.f.

(ii) On the contrary suppose that $Rp = \mathfrak{P} \subset \mathfrak{M}$, for some $p \in R$ and a maximal ideal $\mathfrak{M}$ of R. Then it is easy to see that Rp^2 is primary and hence $\mathfrak{P}^{(2)} = Rp^2$. Now by **(i)**, if $a \in \mathfrak{M} \setminus \mathfrak{P}$, then $ap = ra^2p + r'p^2$ for some $r, r' \in R$.

Since $p \notin \mathrm{Z}(R)$, we get $a(1-ra) \in Rp$. But $a \notin Rp$, so $1-ra \in Rp \subseteq \mathfrak{M}$, which contradicts with $a \in \mathfrak{M}$. Consequently $\mathfrak{P}$ is not principal. □

In what follows $\mathrm{N}(R)$ is the nilradical of R.

Proposition 3.8. *If R is a Noetherian ring, the following are equivalent.*

- **(i)** *Every primary submodule of the R-module $M = R \oplus R$ weakly s.t.r.f. in M.*
- **(ii)** *Every primary submodule of every R-module M weakly s.t.r.f. in M.*
- **(iii)** *Every primary submodule of every R-module M s.t.r.f. in M.*
- **(iv)** *R is Artinian or R is one dimensional and $N(R)$ is an Artinian R-module.*
- **(v)** *Every non-maximal prime ideal $\mathfrak{P}$ of R is the only $\mathfrak{P}$-primary ideal of R.*
- **(vi)** *For every non-maximal prime ideal $\mathfrak{P}$ of R there exists $c \in R \setminus \mathfrak{P}$ such that $c\mathfrak{P} = 0$.*

Proof. The equivalence of **(iii)** and **(vi)** is proved in [12, Theorem 2.4] and that of **(iv)**, **(v)** and **(vi)** in [12, Theorem 1.9]. Also **(iii)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(i)** is trivial. The proof of **(i)** $\Rightarrow$ **(vi)** is quite the same as the proof of (ii) $\Rightarrow$ (iii) of [12, Theorem 2.4], just use **(3.7)(i)** instead of [12, Lemma 2.3]. □

Corollary 3.9. *If R is a Noetherian domain, then R weakly s.t.r.f. if and only if* $\dim R \leq 1$.

Proof. If $\dim R \leq 1$, then by **(3.3)**, R weakly s.t.r.f. Conversely if R weakly s.t.r.f., then by **(3.8)**, $\dim R \leq 1$. □

Theorem 3.10. *Suppose that R is a Noetherian ring. Then R weakly s.t.r.f. if and only if every finitely generated $\frac{R}{\mathrm{N}(R)}$-module weakly s.t.r.f.*

Proof. ($\Rightarrow$) It is easy to see that if the ring R weakly s.t.r.f., then the ring $\frac{R}{\mathrm{N}(R)}$ weakly s.t.r.f.

($\Leftarrow$) First assume that R is a reduced ring (that is, $\mathrm{N}(R) = 0$) and every finitely generated module over R weakly s.t.r.f. We will show that R weakly s.t.r.f. For this by **(3.2)(v)**, it suffices to show that $R_{\mathfrak{M}}$ weakly s.t.r.f., for every maximal ideal $\mathfrak{M}$ of R. Note that every finitely generated module M' over $R_{\mathfrak{M}}$ is of the form $N_{\mathfrak{M}}$ for some finitely generated R-module N (If $G = \{m_1, \ldots, m_k\}$ generates M' as an $\mathfrak{M}$-module, take N to be the R-submodule of M' generated by G). Since R is Noetherian we have $(\mathrm{wrad}_N(A))_{\mathfrak{M}} = \mathrm{wrad}_{N_{\mathfrak{M}}}(A_{\mathfrak{M}})$ for every submodule A of N by [5, Corollary 2.3]. Therefore by **(3.2)(iii)**, every finitely generated $R_{\mathfrak{M}}$-module weakly s.t.r.f., hence we can suppose that R is local.

Now by **(3.8)(iv)**, $\dim R \leq 1$. If R is zero dimensional, then according to [11, Theorem 2.8], R s.t.r.f. and so weakly s.t.r.f. Now assume that R is one dimensional. Since R is Noetherian it has finitely many minimal primes and since it is local it has exactly one maximal ideal, so R has finitely many prime ideals. Now by **(3.6)**, $\mathrm{wrad}_M(0) = \bigcup\{\mathrm{wrad}_{M_f}(0) \mid M_f$ is a finitely generated submodule of $M\} = \bigcup\{R\mathrm{E}_{M_f}(0) \mid M_f$ is a finitely generated submodule of $M\} \subseteq R\mathrm{E}_M(0)$. Therefore for every R-module M, $\mathrm{wrad}_M(0) = R\mathrm{E}_M(0)$. Consequently the result follows from **(3.2)(vi)**, in case R is a reduced ring.

Now for the general case, first note that by the above argument $\frac{R}{\mathrm{N}(R)}$ weakly s.t.r.f. Hence by **(3.8)(iv)**, dim R = dim $\frac{R}{\mathrm{N}(R)} \leq 1$ and again we can assume dim $R = 1$. Also by **(3.8)(v)**, every non-maximal prime ideal $\mathfrak{P}$ of R is the only $\mathfrak{P}$-primary ideal of R. So by [7, Proposition 2.6], there exist maximal ideals $\mathfrak{M}_1, \mathfrak{M}_2, \ldots, \mathfrak{M}_n$ and $k_1, k_2, \ldots, k_n \in \mathbb{N}$ such that

$$(*) \qquad \mathrm{N}(R) \cap \mathfrak{M}_1^{k_1} \cap \mathfrak{M}_2^{k_2} \cap \cdots \cap \mathfrak{M}_n^{k_n} = 0.$$

Similar to the proof of [7, Proposition 2.5], one can see that if $\frac{R}{\mathrm{N}(R)}$ weakly s.t.r.f. and $(*)$ holds, then R weakly s.t.r.f. $\square$

We end this paper with stating some conditions on local integral domains which s.t.r.f. or are semi-compatible.

Theorem 3.11. *Suppose that $(R, \mathfrak{M})$ is a local integral domain and $M = R \oplus R$.*

- **(i)** *If M is semi-compatible and $\mathfrak{P}$ is any prime ideal of R, then either $\mathfrak{P}$ is principal or $\mathfrak{P} = \mathfrak{M}\mathfrak{P}$.*
- **(ii)** *If M s.t.r.f., then $\mathfrak{P} = \mathfrak{M}\mathfrak{P}$ for every non-maximal prime ideal of R. Furthermore either $\mathfrak{M}$ is principal or $\mathfrak{M} = \mathfrak{M}^2$.*

Proof. **(i)** Suppose that $\mathfrak{P}$ is not principal. Choose arbitrary elements $a \in \mathfrak{P}$ and $b \in \mathfrak{P} \setminus Ra$ and set $N = \mathfrak{P}(a, b)$. If P is a prime submodule of M containing N, then either $\mathfrak{P}M \subseteq P$ or $(a, b) \in P$. Since $a, b \in \mathfrak{P}$, in both cases $(a, b) \in P$. Therefore $\mathrm{rad}(N) = \mathrm{rad}(R(a, b))$ and since M is semi-compatible $\mathrm{wrad}(N) = \mathrm{rad}(N) = \mathrm{rad}(R(a, b))$. In particular $(a, b) \in \mathrm{wrad}(N)$.

It is easy to check that $W_1 = \{(x, y) \mid xb = ya\}$ is a 0-prime submodule of M containing $R(a, b)$. Let $W_2 = (\mathfrak{P}W_1)_{\mathfrak{P}}{}^c$, then $N \subseteq W_2$ and either $W_2 = W_1$ or W_2 is a $\mathfrak{P}$-prime submodule of W_1 (for $\mathfrak{P}_{\mathfrak{P}} \subseteq ((W_2)_{\mathfrak{P}} : (W_1)_{\mathfrak{P}})$). Similarly for $W_3 = \mathfrak{M}W_2 + \mathfrak{P}W_1$, either $W_3 = W_2$ or W_3 is an $\mathfrak{M}$-prime submodule of W_2. Also clearly $N \subseteq W_3$ and one can easily check that if $W_i \neq W_{i-1}$, then $(W_i : W_{i-1}) \subseteq (W_3 : W_{i-1})$, where $1 \leq i \leq 3$ and $W_0 = M$. So if we delete the possible repeated terms of the chain $W_3 \leq W_2 \leq W_1 < W_0 = M$, it represents the standard basis of W_3 in M. Consequently, W_3 is a weakly prime submodule of M.

If $(x, y) \in W_1$, then by the definition of W_1, $xb \in Ra$ and since $b \notin Ra$, x is not a unit, thus $x \in \mathfrak{M}$. Whence $W_1 \subseteq \mathfrak{M} \oplus R$. Thus we deduce that $\mathfrak{P}W_1 \subseteq (\mathfrak{P}\mathfrak{M}) \oplus \mathfrak{P}$, $W_2 \subseteq ((\mathfrak{P}_{\mathfrak{P}}\mathfrak{M}_{\mathfrak{P}}) \oplus \mathfrak{P}_{\mathfrak{P}})^{\,c} \subseteq (\mathfrak{P} \oplus \mathfrak{P})$ and finally $W_3 \subseteq \big((\mathfrak{M}\mathfrak{P}) \oplus (\mathfrak{M}\mathfrak{P})\big) + \big((\mathfrak{M}\mathfrak{P}) \oplus \mathfrak{P}\big) = (\mathfrak{M}\mathfrak{P}) \oplus \mathfrak{P}$.

But $(a, b) \in \mathrm{wrad}(N) \subseteq W_3 \subseteq (\mathfrak{M}\mathfrak{P}) \oplus \mathfrak{P}$, thus $a \in \mathfrak{M}\mathfrak{P}$. Since a was an arbitrary element of $\mathfrak{P}$, we conclude $\mathfrak{P} \subseteq \mathfrak{M}\mathfrak{P}$.

(ii) The proof follows from **(i)** and **(3.7)(ii)**. $\square$

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