Journal of Mathematical Sciences, Vol. 140, No. 3, 2007

MASLOV DEQUANTIZATION, IDEMPOTENT AND TROPICAL MATHEMATICS: A BRIEF INTRODUCTION

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This paper is a brief introduction to idempotent and tropical mathematics. Tropical mathematics can be treated as the result of the so-called Maslov dequantization of the traditional mathematics over numerical fields as the Planck constant $\hbar$ tends to zero taking imaginary values. Bibliography: 187 *titles.*

To Anatoly Vershik with admiration and gratitude

This paper is a brief introduction, practically without exact theorems and proofs, to the Maslov dequantization and idempotent and tropical mathematics. Our list of references is not complete (not at all). Additional references can be found, e.g., in the electronic archive `http://arXiv.org` and in [9, 17, 21, 25, 28–31, 37, 38, 51, 62, 64–70, 84, 87, 89, 93, 96, 102, 104, 110, 115, 137, 187]. The present brief survey is an extended version of the paper [101].

The author thanks M. Akian, Ya. I. Belopolskaya, P. Butkovič, G. Cohen, S. Gaubert, R. A. Cuninghame-Green, P. Del Moral, W. H. Fleming, J. S. Golan, M. Gondran, I. Itenberg, Y. Katsov, V. N. Kolokoltsov, P. Loreti, Yu. I. Manin, G. Mascari, W. M. MacEneaney, G. Panina, E. Pap, M. Pedicini, H. Prade, A. A. Puhalskii, J.-P. Quadrat, M. A. Roytberg, G. B. Shpiz, I. Singer, and O. Viro for useful contacts and references. Special thanks are due to V. P. Maslov for his crucial help and support and to A. N. Sobolevskii for his great help including, but not limited to, three pictures presented below.

1. Some basic ideas

Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations (such as maximum or minimum), i.e., on replacing numerical fields by idempotent semirings and semifields. Typical examples are given by the so-called max-plus algebra $\mathbf{R}_{\max}$ and the min-plus algebra $\mathbf{R}_{\min}$. Let $\mathbf{R}$ be the field of real numbers. Then $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. Similarly, $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$. The new addition $\oplus$ is idempotent, i.e., $x \oplus x = x$ for all elements x.

Many authors (S. C. Kleene, S. N. N. Pandit, N. N. Vorobjev, B. A. Carré, R. A. Cuninghame-Green, K. Zimmermann, U. Zimmermann, M. Gondran, F. L. Baccelli, G. Cohen, S. Gaubert, G. J. Olsder, J.-P. Quadrat, and others) used idempotent semirings and matrices over these semirings for solving some applied problems in computer science and discrete mathematics, starting from the classical paper by S. C. Kleene [88]. The modern *idempotent analysis* (or *idempotent calculus*, or *idempotent mathematics*) was founded by V. P. Maslov and his collaborators in the 1980s in Moscow; see, e.g., [96, 119–124]. Some preliminary results are due to E. Hopf and G. Choquet, see [24, 71].

Idempotent mathematics can be treated as the result of a dequantization of the traditional mathematics over numerical fields as the Planck constant $\hbar$ tends to zero taking imaginary values. This point of view was presented by G. L. Litvinov and V. P. Maslov [102–104], see also [110, 111]. In other words, idempotent mathematics is an asymptotic version of the traditional mathematics over the fields of real and complex numbers.

The basic paradigm is expressed in terms of an *idempotent correspondence principle*. This principle is closely related to the well-known correspondence principle of N. Bohr in quantum theory. Actually, there exists a heuristic correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over fields, its "shadow," idempotent mathematics, appears. This "shadow" stands approximately in the same relation to traditional mathematics as classical physics does to quantum theory (see Fig. 1).

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 326, 2005, pp. 145–182. Original article submitted June 19, 2005.

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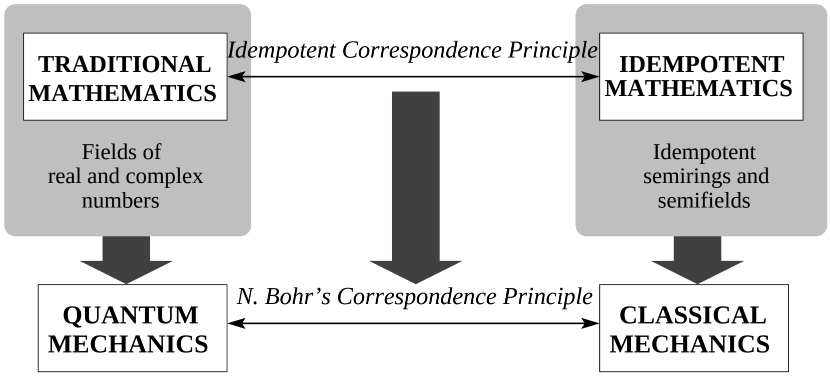

FIG. 1. Relations between idempotent and traditional mathematics.

In many respects, idempotent mathematics is simpler than the traditional one. However, the transition from traditional concepts and results to their idempotent analogs is often nontrivial.

2. Semirings, semifields, and dequantization

Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is said to be a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;
- the multiplication $\odot$ is distributive with respect to the addition $\oplus$:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z) \quad \text{and} \quad (x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$$

  for all $x, y, z \in S$.

A *unit* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral elements $\mathbf{0}$ and $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible. Note that dioïds in the sense of [9, 67, 68], quantales in the sense of [147, 148], and inclines in the sense of [84] are examples of idempotent semirings.

Let $\mathbf{R}$ be the field of real numbers and $\mathbf{R}_+$ be the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). The change of variables $x \mapsto u = h \ln x$, $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \to S = \mathbf{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbf{R}$ to S by Φ_h, i.e., let $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$, $\mathbf{0} = -\infty = \Phi_h(0)$, $\mathbf{1} = 0 = \Phi_h(1)$. Thus S obtains the structure of a semiring $\mathbf{R}^{(h)}$ isomorphic to $\mathbf{R}_+$ (see Fig. 2).

It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$ and that S forms a semiring with respect to the addition $u \oplus v = \max\{u, v\}$ and the multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by $\mathbf{R}_{\max}$; it is *idempotent*, i.e., $u \oplus u = u$ for all its elements. The semiring $\mathbf{R}_{\max}$ is actually a semifield. The analogy with quantization is obvious; the parameter h plays the role of the Planck constant, so $\mathbf{R}_+$ can be viewed as a "quantum object" and $\mathbf{R}_{\max}$ as the result of its "dequantization." A similar procedure (for $h < 0$) leads to the semiring $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$; in this case, $\mathbf{0} = +\infty$, $\mathbf{1} = 0$. The semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$ are isomorphic. This passage to $\mathbf{R}_{\max}$ or $\mathbf{R}_{\min}$ is called the *Maslov dequantization*. It is clear that the corresponding passage from $\mathbf{C}$ or $\mathbf{R}$ to $\mathbf{R}_{\max}$ is generated by the Maslov dequantization and the map $x \mapsto |x|$; *we will also call this passage the Maslov dequantization*. Connections with physics and the meaning of imaginary values of the Planck constant are discussed below (see Sec. 6) and in [110, 111]. The idempotent semiring $\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$, $\odot = \min$ can be obtained as the result of a "second dequantization" of $\mathbf{C}$, $\mathbf{R}$, or $\mathbf{R}_+$. Dozens of interesting examples of nonisomorphic idempotent semirings may be cited, as well as a number of standard methods of deriving new semirings from these (see, e.g., [26, 64, 66–70, 104, 110]). The so-called *idempotent dequantization* is a generalization of the Maslov dequantization; this is a passage from fields to idempotent semifields and semirings in mathematical constructions and results.

The Maslov dequantization is related to the well-known logarithmic transformation that was used, e.g., in the classical papers by E. Schrödinger [153] and E. Hopf [71]. The term "Cole–Hopf transformation" is also used.

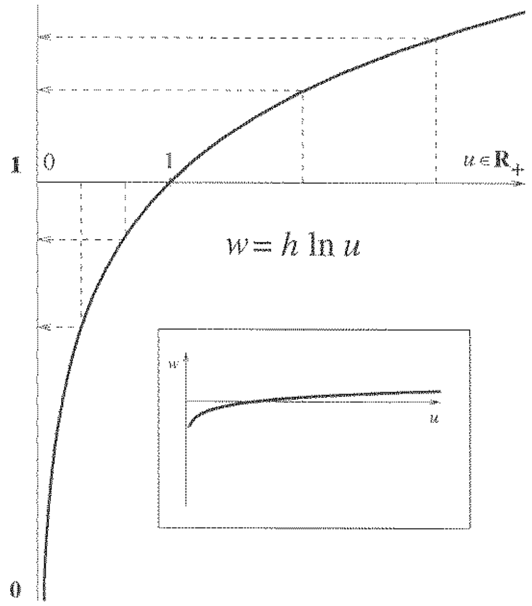

FIG. 2. Deformation of $\mathbf{R}_+$ to $\mathbf{R}^{(h)}$. Inset: same for a small value of h.

The subsequent progress of E. Hopf's ideas has culminated in the well-known vanishing viscosity method and the method of viscosity solutions, see, e.g., [10, 19, 54, 122, 125, 149, 167].

3. Terminology: Tropical semirings and tropical mathematics

The term "tropical semirings" was introduced in computer science to denote discrete versions of the max-plus algebra $\mathbf{R}_{\max}$ or min-plus algebra $\mathbf{R}_{\min}$ and their subalgebras; (discrete) semirings of this type were called tropical semirings by Dominic Perrin in honor of Imre Simon (who is a Brazilian mathematician and computer scientist) because of his pioneering activity in this area, see [140].

More recently, the situation and terminology have changed. For the most part of modern authors, "tropical" means "over $\mathbf{R}_{\max}$ (or $\mathbf{R}_{\min}$)" and tropical semirings are the idempotent semifields $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. The terms "max-plus" and "min-plus" are often used in the same sense. Now the term "tropical mathematics" usually means "mathematics over $\mathbf{R}_{\max}$ or $\mathbf{R}_{\min}$," see, e.g., [7, 11, 12, 14, 39, 41, 55, 56, 59, 60, 72–78, 85, 86, 126–129, 131, 132, 134, 135, 146, 155, 156, 161–165, 169–171]. The terms "tropicalization" and "tropification" (see, e.g., [87]) mean exactly dequantization and quantization in our sense. In any case, tropical mathematics is a natural and very important part of idempotent mathematics. Some well-known constructions and results of idempotent mathematics were repeated in the framework of tropical mathematics (and especially tropical linear algebra).

Note that in the papers [174–176], N. N. Vorobjev developed a version of idempotent linear algebra (with important applications, e.g., to mathematical economics) and predicted many aspects of the future extended theory. He used the terms "extremal algebras" and "extremal mathematics" for idempotent semirings and idempotent mathematics. Unfortunately, N. N. Vorobjev's papers and ideas were forgotten for a long period, so his remarkable terminology is not in use any more.

4. Idempotent algebra and linear algebra

The first known paper on idempotent linear algebra is due to S. Kleene [88]. Systems of linear algebraic equations over the exotic idempotent semiring of all formal languages over a fixed finite alphabet are examined in this work; however, S. Kleene's ideas are very general and universal. Since then, dozens of authors investigated matrices with coefficients belonging to an idempotent semiring and the corresponding applications to discrete mathematics, computer science, computer languages, linguistic problems, finite automata, optimization problems on graphs, discrete event systems and Petry nets, stochastic systems, evaluation of computer performance, computational problems, etc. This subject is very well known and well presented in the corresponding literature, see, e.g., [9, 17, 18, 21, 25, 28–30, 44, 62, 64–70, 84, 93, 96, 102, 104–107, 114, 115, 122, 174–177, 187].

Idempotent abstract algebra is not yet so well developed (on the other hand, from a formal point of view, lattice theory and the theory of ordered groups and semigroups are parts of idempotent algebra). However, there are many interesting results and applications, presented, e.g., in [29–31, 81, 147, 148, 154].

In particular, an idempotent version of the main theorem of algebra holds [31, 154] for radicable idempotent semifields (a semiring A is called *radicable* if the equation $x^n = a$ has a solution $x \in A$ for any $a \in A$ and any positive integer n). It is proved that $\mathbf{R}_{\max}$ and other radicable semifields are algebraically closed in a natural sense [154].

Over the last years, tropical algebraic geometry has attracted a lot of attention. This subject will be briefly discussed below (see Sec. 11).

5. Idempotent analysis

Idempotent analysis was initially constructed by V. P. Maslov and his collaborators and then developed by many authors. The subject is presented in the book by V. N. Kolokoltsov and V. P. Maslov [96] (a Russian version of this book [122] was published in 1994).

Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, zero $\mathbf{0}$, and unit $\mathbf{1}$. The set S is endowed with the *standard partial order* $\preceq$: by definition, $a \preceq b$ if and only if $a \oplus b = b$. Thus all elements of S are nonnegative: $\mathbf{0} \preceq a$ for all $a \in S$. Due to the existence of this order, idempotent analysis is closely related to lattice theory, the theory of vector lattices, and the theory of ordered spaces. Moreover, this partial order allows one to model a number of basic "topological" concepts and results of idempotent analysis at the purely algebraic level; this line of reasoning was examined systematically in [108–112] and [26].

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \to S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S pointwise.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $B(X, S)$ of all functions $X \to S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\, dx = \sup_{x \in X} \varphi(x), \tag{1}$$

where $\varphi \in B(X, S)$. Indeed, a Riemann sum of the form $\sum\limits_i \varphi(x_i) \cdot \sigma_i$ corresponds to the expression $\bigoplus\limits_i \varphi(x_i) \odot \sigma_i = \max\limits_i \{\varphi(x_i) + \sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \to 0$. Of course, this is a purely heuristic argument.

Formula (1) defines the *idempotent* (or *Maslov*) *integral* not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case, provided that any bounded (from above) subset of S has the least upper bound.

An *idempotent* (or *Maslov*) *measure* on X is defined by $m_\psi(Y) = \sup\limits_{x \in Y} \psi(x)$, where $\psi \in B(X, S)$, $Y \subseteq X$. The integral with respect to this measure is defined by the formula

$$I_\psi(\varphi) = \int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x \in X} (\varphi(x) \odot \psi(x)). \tag{2}$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order $\preceq$ is opposite to the conventional order $\leq$, so in this case Eq. (2) assumes the form

$$\int_X^{\oplus} \varphi(x)\, dm_\psi = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \inf_{x \in X} (\varphi(x) \odot \psi(x)),$$

where inf is understood in the sense of the conventional order $\leq$.

Note that the so-called pseudo-analysis (see, e.g., the survey paper by E. Pap [138]) is related to a special part of idempotent analysis; however, this pseudo-analysis is not a proper part of idempotent mathematics in the general case. Some generalizations of Maslov measures are discussed in [89, 137].

6. The superposition principle and linear problems

Basic equations of quantum theory are linear; this is the superposition principle in quantum mechanics. The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However, it is linear over the semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$. Similarly, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings; this is V. P. Maslov's idempotent superposition principle, see [119–123]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H \odot X \oplus F$, where X, H, and F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F [29]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. In [20], it was shown that principal optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc.

The linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$, which is the result of the Maslov dequantization of the Schrödinger equation, is closely related to the (conventional) linearity of the Schrödinger equation and can be deduced from this linearity. Thus it is possible to borrow standard ideas and methods of linear analysis and apply them to a new area.

Consider a classical dynamical system, specified by the Hamiltonian

$$H = H(p, x) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(x),$$

where $x = (x_1, \dots, x_N)$ are generalized coordinates, $p = (p_1, \dots, p_N)$ are generalized momenta, m_i are masses, and $V(x)$ is a potential. In this case, the Lagrangian $L(x, \dot{x}, t)$ has the form

$$L(x, \dot{x}, t) = \sum_{i=1}^{N} m_i \frac{\dot{x}_i^2}{2} - V(x),$$

where $\dot{x} = (\dot{x}_1, \dots, \dot{x}_N)$, $\dot{x}_i = dx_i/dt$. The value function $S(x, t)$ of the action functional has the form

$$S = \int_{t_0}^{t} L(x(t), \dot{x}(t), t)\, dt,$$

where the integration is performed along the factual trajectory of the system. The classical equations of motion are derived as the stationarity conditions for the action functional (the Hamilton principle or the least action principle).

For fixed values of t and t_0 and arbitrary trajectories $x(t)$, the action functional $S = S(x(t))$ can be regarded as a function taking the set of curves (trajectories) to the set of real numbers, which can be treated as elements of $\mathbf{R}_{\min}$. In this case, the minimum of the action functional can be viewed as the Maslov integral of this function over the set of trajectories, or an idempotent analog of the Euclidean version of the Feynman path integral. The minimum of the action functional corresponds to the maximum of e^{-S}, i.e., the idempotent integral $\int^{\oplus}_{\{\text{paths}\}} e^{-S(x(t))} D\{x(t)\}$ with respect to the max-plus algebra $\mathbf{R}_{\max}$. Thus the least action principle can be regarded as an idempotent version of the well-known Feynman approach to quantum mechanics. The representation of a solution to the Schrödinger equation in terms of the Feynman integral corresponds to the Lax–Oleinik solution formula for the Hamilton–Jacobi equation.

Since $\partial S/\partial x_i = p_i$, $\partial S/\partial t = -H(p, x)$, the following Hamilton–Jacobi equation holds:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x_i}, x_i\right) = 0. \tag{3}$$

Quantization (see, e.g., [48]) leads to the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \widehat{H}\psi = H(\hat{p}_i, \hat{x}_i)\psi, \tag{4}$$

where $\psi = \psi(x,t)$ is the wave function, i.e., a time-dependent element of the Hilbert space $L^2(\mathbf{R}^N)$, and $\widehat{H}$ is the energy operator obtained by the substitution of the momentum operators $\widehat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$ and the coordinate operators $\widehat{x}_i\colon \psi \mapsto x_i\psi$ for the variables p_i and x_i in the Hamiltonian function, respectively. This equation is linear in the conventional sense (the quantum superposition principle). The standard procedure of limit transition from the Schrödinger equation to the Hamilton–Jacobi equation is to use the following ansatz for the wave function: $\psi(x,t) = a(x,t)e^{iS(x,t)/\hbar}$, and to keep only the leading-order term as $\hbar \to 0$ (the "semiclassical" limit).

Instead of doing this, we switch to imaginary values of the Planck constant $\hbar$ by the substitution $h = i\hbar$, assuming $h > 0$. Thus the Schrödinger equation (4) turns into the following generalized heat equation:

$$h\frac{\partial u}{\partial t} = H\left(-h\frac{\partial}{\partial x_i}, \hat{x}_i\right)u, \tag{5}$$

where the real-valued function u corresponds to the wave function ψ (or rather to $|\psi|$). A similar idea (the switch to imaginary time) is used in Euclidean quantum field theory (see, e.g., [130]); recall that time and energy are dual quantities.

The linearity of Eq. (4) implies the linearity of Eq. (5). Thus if u_1 and u_2 are solutions of (5), then so is their linear combination

$$u = \lambda_1 u_1 + \lambda_2 u_2. \tag{6}$$

Let $S = h\ln u$ or $u = e^{S/h}$ as in Sec. 2 above. It can easily be checked that Eq. (5) then turns into

$$\frac{\partial S}{\partial t} = V(x) + \sum_{i=1}^{N}\frac{1}{2m_i}\left(\frac{\partial S}{\partial x_i}\right)^2 + h\sum_{i=1}^{n}\frac{1}{2m_i}\frac{\partial^2 S}{\partial x_i^2}. \tag{7}$$

Thus we have passed from (4) to (7) by means of the change of variables $\psi = e^{S/h}$. Note that $|\psi| = e^{\mathrm{Re}S/h}$, where $\mathrm{Re}S$ is the real part of S. Now let us regard S as a real variable. Equation (7) is nonlinear in the conventional sense. However, if S_1 and S_2 are its solutions, then so is the function

$$S = \lambda_1 \odot S_1 \oplus_h \lambda_2 \odot S_2$$

obtained from (6) by means of our substitution $S = h\ln u$. Here the generalized multiplication $\odot$ coincides with the ordinary addition, and the generalized addition $\oplus_h$ is the image of the conventional addition under the above change of variables. As $h \to 0$, we obtain the operations of the idempotent semiring $\mathbf{R}_{\max}$, i.e., $\oplus = \max$ and $\odot = +$, and Eq. (7) turns into the Hamilton–Jacobi equation (3), since the third term in the right-hand side of (7) vanishes.

Thus it is natural to regard the limit function $S = \lambda_1 \odot S_1 \oplus \lambda_2 \odot S_2$ as a solution of the Hamilton–Jacobi equation and to expect that this equation can be treated as linear over $\mathbf{R}_{\max}$. This argument (clearly, a heuristic one) can be extended to equations of a more general form. For a rigorous treatment of (semiring) linearity for these equations, see [52, 96, 122, 123] and also [120]. Note that if h is replaced by $-h$, then the resulting Hamilton–Jacobi equation becomes linear over $\mathbf{R}_{\min}$.

The idempotent superposition principle indicates that there exist important nonlinear (in the traditional sense) problems that are linear over idempotent semirings. Linear idempotent functional analysis (see below) is a natural tool for the investigation of those nonlinear infinite-dimensional problems that possess this property.

7. Convolution and the Fourier–Legendre transform

Let G be a group. Then the space $\mathcal{B}(X, \mathbf{R}_{\max})$ of all bounded functions $G \to \mathbf{R}_{\max}$ (see above) is an idempotent semiring with respect to the following analog $\circledast$ of the usual convolution:

$$(\varphi(x) \circledast \psi)(g) = \int_G^{\oplus} \varphi(x) \odot \psi(x^{-1}\cdot g)\,dx = \sup_{x\in G}(\varphi(x) + \psi(x^{-1}\cdot g)).$$

Of course, it is possible to consider other "function spaces" (and other basic semirings instead of $\mathbf{R}_{\max}$). In [96, 122], "group semirings" of this type are referred to as *convolution semirings*.

Let $G = \mathbf{R}^n$, where $\mathbf{R}^n$ is regarded as a topological group with respect to the vector addition. The conventional Fourier–Laplace transform is defined as

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G e^{i\xi\cdot x}\varphi(x)\,dx, \tag{8}$$

where $e^{i\xi\cdot x}$ is a character of the group G, i.e., a solution of the following functional equation:

$$f(x+y) = f(x)f(y).$$

The idempotent analog of this equation is

$$f(x+y) = f(x) \odot f(y) = f(x) + f(y),$$

so "continuous idempotent characters" are linear functions of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$. As a result, the transform in (8) assumes the form

$$\varphi(x) \mapsto \tilde{\varphi}(\xi) = \int_G^{\oplus} \xi \cdot x \odot \varphi(x)\, dx = \sup_{x\in G} (\xi \cdot x + \varphi(x)). \tag{9}$$

The transform in (9) is nothing but the *Legendre transform* (up to notation) [121]; transforms of this kind establish the correspondence between the Lagrangian and the Hamiltonian formulations of classical mechanics. The Legendre transform generates an idempotent version of harmonic analysis for the space of convex functions, see, e.g., [118].

Of course, this construction can be generalized to different classes of groups and semirings. Transformations of this type convert the generalized convolution $\circledast$ to the pointwise (generalized) multiplication and possess analogs of some important properties of the usual Fourier transform. For the case of semirings of Pareto sets, the corresponding version of the Fourier transform reduces the multicriterial optimization problem to a family of monocriterial problems [152].

The examples discussed in this section can be treated as fragments of an idempotent version of representation theory, see, e.g., [111]. In particular, "idempotent" representations of groups can be treated as representations of the corresponding convolution semirings (i.e., idempotent group semirings) in semimodules.

8. Correspondence to stochastics and duality between probability and optimization

Maslov measures are nonnegative (in the sense of the standard order), just as probability measures. The analogy between idempotent and probability measures leads to important interplay between optimization theory and probability theory. By now, idempotent analogs of many objects of stochastic calculus have been constructed and investigated, such as max-plus martingales, max-plus stochastic differential equations, and others. These results make it possible, for example, to transfer powerful stochastic methods to optimization theory. This was noticed and examined by many authors (G. Salut, P. Del Moral, M. Akian, J.-P. Quadrat, V. P. Maslov, V. N. Kolokoltsov, P. Bernhard, W. A. Fleming, W. M. McEneaney, A. A. Puhalskii, and others), see the survey paper by W. A. Fleming and W. M. McEneaney [53] and [1, 6, 13, 28, 35–38, 50–52, 69, 122, 141, 143, 144]. For relations and applications to large deviations, see [1, 35–38, 142] and especially the book by A. A. Puhalskii [141].

9. Idempotent functional analysis

Many other idempotent analogs may be given, in particular, for basic constructions and theorems of functional analysis. Idempotent functional analysis is an abstract version of idempotent analysis. For the sake of simplicity, take $S = \mathbf{R}_{\max}$ and let X be an arbitrary set. The idempotent integration can be defined by formula (1), see above. The functional $I(\varphi)$ is linear over S, and its values correspond to limiting values of the corresponding analogs of Lebesgue (or Riemann) sums. An idempotent scalar product of functions φ and ψ is defined by the formula

$$\langle \varphi, \psi \rangle = \int_X^{\oplus} \varphi(x) \odot \psi(x)\, dx = \sup_{x\in X} (\varphi(x) \odot \psi(x)).$$

Thus it is natural to suggest idempotent analogs of integral operators in the form

$$\varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x,y) \odot \varphi(y)\, dy = \sup_{y\in Y} \{K(x,y) + \varphi(y)\}, \tag{10}$$

where $\varphi(y)$ is an element of a space of functions defined on a set Y, and $K(x,y)$ is an S-valued function on $X \times Y$. Expressions of this type are known to be standard in optimization problems.

Recall that the definitions and constructions described above can be extended to the case of idempotent semirings that are conditionally complete in the sense of the standard order. Using the Maslov integration, one can construct various function spaces as well as idempotent versions of the theory of generalized functions (distributions). For some concrete idempotent function spaces, it was proved that every "good" linear operator (in the idempotent sense) can be presented in the form (10); this is an idempotent version of the kernel theorem of L. Schwartz; results of this type were proved by V. N. Kolokoltsov, P. S. Dudnikov and S. N. Samborskii, I. Singer, M. A. Shubin, and others, see, e.g., [44, 96, 122, 123, 158]. Thus a "good" linear functional can be presented in the form $\varphi \mapsto \langle \varphi, \psi \rangle$, where $\langle \cdot, \cdot \rangle$ is an idempotent scalar product.

In the framework of idempotent functional analysis, results of this type can be proved in a very general situation. In [108–112], an algebraic version of idempotent functional analysis is developed; this means that basic (topological) notions and results are simulated in purely algebraic terms. This treatment covers the subject from basic concepts and results (e.g., idempotent analogs of the well-known Hahn–Banach, Riesz, and Riesz–Fisher theorems) to idempotent analogs of A. Grothendieck's concepts and results on topological tensor products, nuclear spaces and operators. An abstract version of the kernel theorem is formulated. Note that the passage from the usual theory to idempotent functional analysis may be very nontrivial; for example, there are many nonisomorphic idempotent Hilbert spaces. Important results on idempotent functional analysis (duality and separation theorems) are recently published by G. Cohen, S. Gaubert, and J.-P. Quadrat [26]; see also a finite-dimensional version of the separation theorem in [183]. Idempotent functional analysis has received much attention in the last years, see, e.g., [2–5, 27, 68, 105, 113, 149, 158, 159, 178] and the works cited above.

10. The dequantization transform and a generalization of the Newton polytopes

In this section, we briefly discuss the results proved in [113]. For functions defined on $\mathbf{C}^n$ almost everywhere (a.e.), it is possible to construct a dequantization transform $f \to \hat{f}$ generated by the Maslov dequantization. If f is a polynomial, then the subdifferential $\partial \hat{f}$ of $\hat{f}$ at the origin coincides with the Newton polytope of f. For the semiring of polynomials with nonnegative coefficients, the dequantization transform is a homomorphism of this semiring to the idempotent semiring of convex polytopes with respect to the well-known Minkowski operations. These results can be generalized to a wide class of functions and convex sets.

10.1. The dequantization transform. Let X be a topological space. For functions $f(x)$ defined on X, we say that a certain assertion is valid *almost everywhere* (a.e.) if it is valid for all elements x of an open dense subset of X. Assume that X is $\mathbf{C}^n$ or $\mathbf{R}^n$; denote by $\mathbf{R}^n_+$ the set $x = \{ (x_1, \dots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. For $x = (x_1, \dots, x_n) \in X$, we set $\exp(x) = (\exp(x_1), \dots, \exp(x_n))$; so if $x \in \mathbf{R}^n$, then $\exp(x) \in \mathbf{R}^n_+$.

Denote by $\mathcal{F}(\mathbf{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbf{C}^n$ such that $U \supset \mathbf{R}^n_+$. In all the examples below we consider even more regular functions, which are holomorphic in U. It is clear that $\mathcal{F}(\mathbf{C}^n)$ is a ring (and an algebra over $\mathbf{C}$) with respect to the usual addition and multiplication of functions.

For $f \in \mathcal{F}(\mathbf{C}^n)$, let us define a function $\hat{f}_h$ by the following formula:

$$\hat{f}_h(x) = h \log |f(\exp(x/h))|, \tag{11}$$

where h is a (small) real parameter and $x \in \mathbf{R}^n$. Set

$$\hat{f}(x) = \lim_{h \to 0} \hat{f}_h(x) \tag{12}$$

if the right-hand side of (12) exists almost everywhere. We say that the function $\hat{f}(x)$ is the *dequantization* of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is the *dequantization transform*. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbf{R}_{\max}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend on the restriction of f to $\mathbf{R}^n_+$, so in fact the dequantization transform is constructed for functions defined on $\mathbf{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$. Of course, similar definitions can be given for functions defined on $\mathbf{R}^n$ and $\mathbf{R}^n_+$.

Denote by V the set $\mathbf{R}^n$ treated as a linear Euclidean space (with the scalar product $(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$) and set $V_+ = \mathbf{R}^n_+$. We say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *dequantizable* whenever its dequantization $\hat{f}(x)$ exists (and is defined on an open dense subset of V). Let $\mathcal{D}(\mathbf{C}^n)$ denote the set of all dequantizable functions and $\widehat{\mathcal{D}}(V)$ denote the set $\{ \hat{f} \mid f \in \mathcal{D}(\mathbf{C}^n) \}$. Recall that functions from $\mathcal{D}(\mathbf{C}^n)$ (and $\widehat{\mathcal{D}}(V)$) are defined almost everywhere and $f = g$ means that $f(x) = g(x)$ a.e., i.e., for x ranging over an open dense subset of $\mathbf{C}^n$ (respectively, of V). Denote by $\mathcal{D}_+(\mathbf{C}^n)$ the set of all functions $f \in \mathcal{D}(\mathbf{C}^n)$ such that $f(x_1, \ldots, x_n) \geq 0$ if $x_i \geq 0$ for $i = 1, \ldots, n$; so $f \in \mathcal{D}_+(\mathbf{C}^n)$ if the restriction of f to $V_+ = \mathbf{R}^n_+$ is a nonnegative function. Denote by $\widehat{\mathcal{D}}_+(V)$ the image of $\mathcal{D}_+(\mathbf{C}^n)$ under the dequantization transform. We say that functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ are *in general position* whenever $\hat{f}(x) \neq \widehat{g}(x)$ for x running through an open dense subset of V.

For functions $f, g \in \mathcal{D}(\mathbf{C}^n)$ and any nonzero constant c, the following equations are valid:

(1) $\widehat{fg} = \hat{f} + \widehat{g}$;

(2) $|\hat{f}| = \hat{f}$; $\widehat{cf} = f$; $\widehat{c} = 0$;

(3) $(\widehat{f + g})(x) = \max\{\hat{f}(x), \widehat{g}(x)\}$ a.e. if f and g are nonnegative on V_+ (i.e., $f, g \in \mathcal{D}_+(\mathbf{C}^n)$) or f and g are in general position.

The left-hand sides of these equations are well defined automatically.

The set $\mathcal{D}_+(\mathbf{C}^n)$ has a natural structure of a semiring with respect to the usual addition and multiplication of functions taking values in $\mathbf{C}$. The set $\widehat{\mathcal{D}}_+(V)$ has a natural structure of an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$; elements of $\widehat{\mathcal{D}}_+(V)$ can be naturally treated as functions taking values in $\mathbf{R}_{\max}$. The dequantization transform generates a homomorphism from $\mathcal{D}_+(\mathbf{C}^n)$ to $\widehat{\mathcal{D}}_+(V)$.

10.2. Simple functions. For any nonzero number $a \in \mathbf{C}$ and any vector $d = (d_1, \ldots, d_n) \in V = \mathbf{R}^n$, we set $m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$; we call functions of this kind *generalized monomials*. Generalized monomials are defined a.e. on $\mathbf{C}^n$ and on V_+, but not on V unless the numbers d_i take integer or suitable rational values. We say that a function f is a *generalized polynomial* whenever it is a finite sum of linearly independent generalized monomials. For instance, Laurent polynomials are examples of generalized polynomials.

As usual, for $x, y \in V$ we set $(x, y) = x_1 y_1 + \cdots + x_n y_n$. It is easy to prove that if f is a generalized monomial $m_{a,d}(x)$, then $\hat{f}$ is the linear function $x \mapsto (d, x)$. If f is a generalized polynomial, then $\hat{f}$ is a sublinear function.

Recall that a real function p defined on $V = \mathbf{R}^n$ is called *sublinear* if $p = \sup_\alpha p_\alpha$, where $\{p_\alpha\}$ is a collection of linear functions. Sublinear functions defined everywhere on $V = \mathbf{R}^n$ are convex, therefore continuous. We discuss sublinear functions of this kind only. Assume that p is a continuous function defined on V; then p is sublinear whenever

(1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$;

(2) $p(cx) = cp(x)$ for all $x \in V$, $c \in \mathbf{R}_+$.

So if p_1 and p_2 are sublinear functions, then $p_1 + p_2$ is a sublinear function.

We say that a function $f \in \mathcal{F}(\mathbf{C}^n)$ is *simple* if its dequantization $\hat{f}$ exists and coincides a.e. with a sublinear function; we denote this (uniquely defined everywhere on V) sublinear function by the same symbol $\hat{f}$.

Recall that simple functions f and g are *in general position* if $\hat{f}(x) \neq \widehat{g}(x)$ for all x belonging to an open dense subset of V. In particular, generalized monomials are in general position whenever they are linearly independent.

Denote by $\mathrm{Sim}(\mathbf{C}^n)$ the set of all simple functions defined on V and denote by $\mathrm{Sim}_+(\mathbf{C}^n)$ the set $\mathrm{Sim}(\mathbf{C}^n) \cap \mathcal{D}_+(\mathbf{C}^n)$. Let $\mathrm{Sbl}(V)$ denote the set of all (continuous) sublinear functions defined on $V = \mathbf{R}^n$ and $\mathrm{Sbl}_+(V)$ denote the image $\widehat{\mathrm{Sim}_+}(\mathbf{C}^n)$ of $\mathrm{Sim}_+(\mathbf{C}^n)$ under the dequantization transform.

The set $\mathrm{Sim}_+(\mathbf{C}^n)$ is a subsemiring of $\mathcal{D}_+(\mathbf{C}^n)$, and $\mathrm{Sbl}_+(V)$ is an idempotent subsemiring of $\widehat{\mathcal{D}_+}(V)$. The dequantization transform generates an epimorphism of $\mathrm{Sim}_+(\mathbf{C}^n)$ onto $\mathrm{Sbl}_+(V)$. The set $\mathrm{Sbl}(V)$ is an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$.

It is clear that polynomials and generalized polynomials are simple functions.

We say that functions $f, g \in \mathcal{D}(V)$ are *asymptotically equivalent* whenever $\hat{f} = \widehat{g}$; any simple function f is an *asymptotic monomial* whenever $\hat{f}$ is a linear function. A simple function f is called an *asymptotic polynomial* whenever $\hat{f}$ is the sum of a finite collection of nonequivalent asymptotic monomials. Every asymptotic polynomial is a simple function.

Example. Generalized polynomials, logarithmic functions of (generalized) polynomials, and products of polynomials and logarithmic functions are asymptotic polynomials. This follows from our definitions and formula (11).

10.3. Subdifferentials of sublinear functions and Newton sets of simple functions. It is well known that all convex compact subsets in $\mathbf{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $A, B \in \mathcal{S}$, the sum $A \oplus B$ is the convex hull of the union $A \cup B$, and the product $A \odot B$ is defined as $A \odot B = \{\, x \mid x = a + b, \text{ where } a \in A, b \in B \,\}$. In fact, $\mathcal{S}$ is an idempotent linear space over $\mathbf{R}_{\max}$ (see, e.g., [110]). Clearly, the Newton polytopes in V form a subsemiring $\mathcal{N}$ in $\mathcal{S}$.

We will use some elementary results from convex analysis. These results can be found, e.g., in [118].
For any function $p \in \mathrm{Sbl}(V)$, we set

$$\partial p = \{\, v \in V \mid (v, x) \leq p(x) \text{ for all } \ x \in V \,\}.$$

It is well known from convex analysis that for any sublinear function p the set ∂p is exactly the *subdifferential* of p at the origin. The following proposition is also known in convex analysis.

Proposition 1. *Assume that $p_1, p_2 \in \mathrm{Sbl}(V)$; then*

$$\partial(p_1 + p_2) = \partial p_1 \odot \partial p_2 = \{\, v \in V \mid v = v_1 + v_2, \quad \textit{where} \ \ v_1 \in \partial p_1, v_2 \in \partial p_2 \,\};$$

$$\partial(\max\{p_1(x), p_2(x)\}) = \partial p_1 \oplus \partial p_2.$$

Let $p \in \mathrm{Sbl}(V)$. Then ∂p is a nonempty convex compact subset of V.

Corollary 1. *The map $p \mapsto \partial p$ is a homomorphism of the idempotent semiring $\mathrm{Sbl}(V)$ to the idempotent semiring $\mathcal{S}$ of all convex compact subsets of V.*

For any simple function $f \in \mathrm{Sim}(\mathbf{C}^n)$, denote by $N(f)$ the set $\partial(\hat{f})$. We call $N(f)$ the *Newton set* of the function f. It follows from this proposition that for any simple function f, its Newton set $N(f)$ is a nonempty convex compact subset of V.

Theorem. *Assume that f and g are simple functions. Then*

(1) $N(fg) = N(f) \odot N(g) = \{\, v \in V \mid v = v_1 + v_2$ *with* $v_1 \in N(f), v_2 \in N(g)\}$;
(2) $N(f+g) = N(f) \oplus N(g)$ *if f_1 and f_2 are in general position or $f_1, f_2 \in \mathrm{Sim}_+(\mathbf{C}^n)$ (recall that $N(f) \oplus N(g)$ is the convex hull of $N(f) \cup N(g)$).*

Corollary 2. *The map $f \mapsto N(f)$ generates a homomorphism from $\mathrm{Sim}_+(\mathbf{C}^n)$ to $\mathcal{S}$.*

Proposition 2. *Let $f = m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$ be a generalized monomial, where $d = (d_1, \dots, d_n) \in V = \mathbf{R}^n$ and a is a nonzero complex number. Then $N(f) = \{d\}$.*

Corollary 3. *Let $f = \sum_{d \in D} m_{a_d, d}$ be a generalized polynomial. Then $N(f)$ is the polytope $\oplus_{d \in D}\{d\}$, i.e., the convex hull of the finite set D.*

This assertion follows from the theorem and Proposition 2. Thus in this case $N(f)$ is the well-known classical Newton polytope of the polynomial f.
The following corollary is obvious.

Corollary 4. *Let f be a generalized or asymptotic polynomial. Then its Newton set $N(f)$ is a convex polytope.*

Example. Consider the one-dimensional case (i.e., $V = \mathbf{R}$) and assume that $f_1 = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$ and $f_2 = b_m x^m + b_{m-1}x^{m-1} + \dots + b_0$ with nonzero a_n, b_m, a_0, b_0. Then $N(f_1)$ is the segment $[0, n]$ and $N(f_2)$ is the segment $[0, m]$. Thus the map $f \mapsto N(f)$ corresponds to the map $f \mapsto \deg(f)$, where $\deg(f)$ is the degree of a polynomial f. In this case, the theorem means that $\deg(fg) = \deg f + \deg g$ and $\deg(f+g) = \max\{\deg f, \deg g\} = \max\{n, m\}$ if $a_i \geq 0$, $b_i \geq 0$ or f and g are in general position.

11. Dequantization of geometry

An idempotent version of real algebraic geometry was discovered in the report of O. Viro for the Barcelona Congress [172]. Starting from the idempotent correspondence principle, O. Viro constructed a piecewise linear geometry of polyhedra of a special kind in finite-dimensional Euclidean spaces as the result of the Maslov dequantization of real algebraic geometry. He indicated important applications in real algebraic geometry (e.g., in the framework of Hilbert's 16th problem about constructing real algebraic varieties with prescribed properties and parameters) and relations to complex algebraic geometry and amoebas in the sense of I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky (see their book [61], and [173]). Later, complex algebraic geometry was dequantized by G. Mikhalkin, and the result turned out to be the same; this new "idempotent" (or asymptotic) geometry is now often called *tropical algebraic geometry*, see, e.g., [47, 72, 126–129, 146, 155, 164, 166].

There is a natural relation between the Maslov dequantization and amoebas. Assume that $(\mathbf{C}^*)^n$ is a complex torus, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ is the group of nonzero complex numbers under multiplication. For $z = (z_1, \ldots, z_n) \in (\mathbf{C}^*)^n$ and a positive real number h, denote by $\mathrm{Log}_h(z) = h\log(|z|)$ the element

$$(h\log|z_1|, h\log|z_2|, \ldots, h\log|z_n|) \in \mathbf{R}^n.$$

Let $V \subset (\mathbf{C}^*)^n$ be a complex algebraic variety; denote by $\mathcal{A}_h(V)$ the set $\mathrm{Log}_h(V)$. If $h = 1$, then the set $\mathcal{A}(V) = \mathcal{A}_1(V)$ is called the *amoeba* of V in the sense of [61], see also [7, 126, 128, 129, 139, 155, 161, 173]; the amoeba $\mathcal{A}(V)$ is a closed subset of $\mathbf{R}^n$ with a nonempty complement. Note that this construction depends on our coordinate system.

For the sake of simplicity, assume that V is a hypersurface in $(\mathbf{C}^*)^n$ determined by a polynomial f. Then there is a deformation $h \mapsto f_h$ of this polynomial generated by the Maslov dequantization, and $f_h = f$ for $h = 1$. Let $V_h \subset (\mathbf{C}^*)^n$ be the zero set of f_h and set $\mathcal{A}_h(V_h) = \mathrm{Log}_h(V_h)$. Then there exists a tropical variety $\mathrm{Tro}(V)$ such that the subsets $\mathcal{A}_h(V_h) \subset \mathbf{R}^n$ tend to $\mathrm{Tro}(V)$ in the Hausdorff metric as $h \to 0$, see [126, 151]. The tropical variety $\mathrm{Tro}(V)$ is the result of a deformation of the amoeba $\mathcal{A}(V)$ and the Maslov dequantization of the variety V. The set $\mathrm{Tro}(V)$ is called the *skeleton* of $\mathcal{A}(V)$.

Example [126]. For the line $V = \{\,(x, y) \in (\mathbf{C}^*)^2 \mid x+y+1 = 0\,\}$, the piecewise linear graph $\mathrm{Tro}(V)$ is a tropical line, see Fig. 3(a). The amoeba $\mathcal{A}(V)$ is represented in Fig. 3(b), while Fig. 3(c) demonstrates the corresponding deformation of the amoeba.

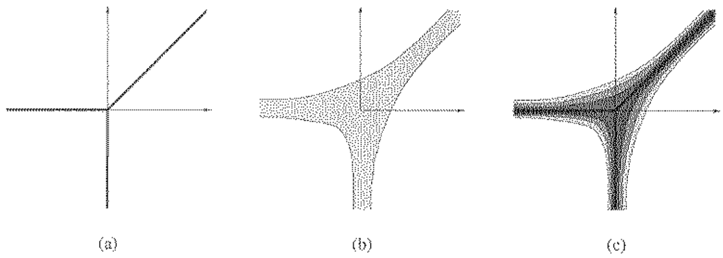

FIG. 3. A tropical line and amoebas.

In the important paper [80] (see also [47, 126, 128, 146]), tropical varieties appeared as amoebas over non-Archimedean fields. In 2000, M. Kontsevich noted that it is possible to use non-Archimedean amoebas in enumerative geometry, see [126, Sec. 2.4, Remark 4]. In fact, methods of tropical geometry lead to remarkable applications to algebraic enumerative geometry, Gromov–Witten and Welschinger invariants, see [59, 72–75, 126–129, 155, 156]. In particular, G. Mikhalkin presented and proved in [127, 129] a formula enumerating curves of arbitrary genus on toric surfaces. See also the papers [60, 72, 73, 131, 155].

Recently, many other papers on tropical algebraic geometry and its applications to the conventional (e.g., complex) algebraic geometry and other areas appeared, see, e.g., [7, 55, 56, 76, 131, 134, 135, 168, 169, 171]. The thing is that some difficult traditional problems can be reduced to their tropical versions which are hopefully not so difficult.

Note that tropical geometry is closely related to the well-known program of M. Kontsevich and Y. Soibelman, see, e.g., [98, 99].

There is an introductory paper [146] (see also [166]) on tropical algebraic geometry. However, on the whole, only first steps in idempotent/tropical geometry have been made, and the problem of systematic construction of idempotent versions of algebraic and analytic geometries is still open.

12. The correspondence principle for algorithms and their computer implementations

There are many important applied algorithms of idempotent mathematics, see, e.g., [9, 18, 21, 30, 31, 49, 52, 53, 67–69, 72, 84, 87, 93, 96, 102, 104, 106, 107, 114–116, 127, 129, 146, 150, 166, 174–176, 179, 186, 187].

The idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [102–104, 106, 107]. In particular, due to the superposition principle, analogs of linear algebra algorithms are especially important. It is well known that algorithms of linear algebra are convenient for parallel computations; their idempotent analogs also admit parallelization. This is a regular way to use parallel computations for many problems including basic optimization problems. It is convenient to use universal algorithms that do not depend on a specific semiring and its concrete computer model. Software implementations for universal semiring algorithms are based on object-oriented and generic programming; program modules can deal with abstract (and variable) operations and data types, see [102, 104, 106, 107, 116].

The most important and standard algorithms have many hardware implementations in the form of technical devices or special processors. These devices often can be used as prototypes for new hardware units generated by the replacement of the usual arithmetic operations with their semiring analogs, see [102, 104, 107]. Good and efficient technical ideas and decisions can be transposed from prototypes into new hardware units. Thus the correspondence principle generates a regular heuristic method for hardware design.

13. Idempotent interval analysis

An idempotent version of the traditional interval analysis is presented in [114, 115]. Let S be an idempotent semiring equipped with the standard partial order. A *closed interval* in S is a subset of the form $\mathbf{x} = [\underline{\mathbf{x}}, \bar{\mathbf{x}}] = \{x \in S \mid \underline{\mathbf{x}} \preceq x \preceq \bar{\mathbf{x}}\}$, where the elements $\underline{\mathbf{x}} \preceq \bar{\mathbf{x}}$ are called the *lower* and the *upper bounds* of the interval $\mathbf{x}$. The *weak interval extension* $I(S)$ of the semiring S is the set of all closed intervals in S endowed with the operations $\oplus$ and $\odot$ defined as $\mathbf{x} \oplus \mathbf{y} = [\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}, \bar{\mathbf{x}} \oplus \bar{\mathbf{y}}]$, $\mathbf{x} \odot \mathbf{y} = [\underline{\mathbf{x}} \odot \underline{\mathbf{y}}, \bar{\mathbf{x}} \odot \bar{\mathbf{y}}]$; the set $I(S)$ is a new idempotent semiring with respect to these operations. It is proved that basic interval problems of idempotent linear algebra are polynomial, whereas in the traditional interval analysis, problems of this kind are generally NP-hard. Exact interval solutions for the discrete stationary Bellman equation (see the matrix equation discussed in Sec. 8 above) and for the corresponding optimization problems are constructed and examined by G. L. Litvinov and A. N. Sobolevskii in [114, 115]. Similar results are presented by K. Cechlárová and R. A. Cuninghame-Green in [22].

14. Relations to the KAM theory and optimal transport

The subject of the Kolmogorov–Arnold–Moser (KAM) theory may be formulated as the study of invariant subsets in the phase spaces of nonintegrable Hamiltonian dynamical systems where the dynamics displays the same degree of regularity as in integrable systems (quasiperiodic behavior). Recently, a considerable progress has been made via a variational approach, where the dynamics is specified by the Lagrangian rather than Hamiltonian function. The corresponding theory was initiated by S. Aubry and J. N. Mather and recently dubbed *weak KAM theory* by A. Fathi (see his monograph *Weak KAM Theorems in Lagrangian Dynamics*, in preparation, and also [82, 83, 159, 160]). The minimization of a certain functional along trajectories of moving particles is a central feature of another subject, optimal transport theory, which also has undergone a rapid recent development. This theory dates back to G. Monge's work on cuts and fills (1781). A modern version of the theory is known now (after the work by L. Kantorovich [79]) as the *Monge–Ampère–Kantorovich (MAK) optimal transport theory.* There is a similarity between the two theories, and there are relations to problems of idempotent functional analysis (e.g., the problem of eigenfunctions for "idempotent" integral operators, see [159]). Applications of optimal transport to data processing in cosmology are presented in [15, 58].

15. Relations to logic, fuzzy sets, and possibility theory

Let S be an idempotent semiring with neutral elements $\mathbf{0}$ and $\mathbf{1}$ (recall that $\mathbf{0} \neq \mathbf{1}$, see Sec. 2 above). Then the Boolean algebra $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\}$ is a natural idempotent subsemiring of S. Thus S can be treated as a generalized (extended) logic with logical operations $\oplus$ (disjunction) and $\odot$ (conjunction). Ideas of this kind are discussed in many books and papers on generalized versions of logic and especially quantum logic, see, e.g., [42, 64, 90, 147, 148].

Let Ω be the so-called universe consisting of "elementary events." Denote by $\mathcal{F}(S)$ the set of functions defined on Ω and taking values in S; then $\mathcal{F}(S)$ is an idempotent semiring with respect to the pointwise addition and multiplication of functions. We say that elements of $\mathcal{F}(S)$ are *generalized fuzzy sets*, see [64, 100]. We obtain the well-known classical definition of fuzzy sets (L. A. Zadeh [180]) if $S = \mathbf{P}$, where $\mathbf{P}$ is the segment $[0, 1]$ with the semiring operations $\oplus = \max$ and $\odot = \min$. Certainly, functions from $\mathcal{F}(\mathbf{P})$ with values in the Boolean algebra $\mathbf{B} = \{0, 1\} \subset \mathbf{P}$ correspond to traditional sets from Ω, and the semiring operations correspond to the standard operations for sets. In the general case, functions from $\mathcal{F}(S)$ taking their values in $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\} \subset S$ can be treated as traditional subsets in Ω. If S is a lattice (i.e., $x \odot y = \inf\{x, y\}$ and $x \oplus y = \sup\{x, y\}$), then generalized fuzzy sets coincide with L-fuzzy sets in the sense of J. A. Goguen [63]. The set $I(S)$ of intervals is an idempotent semiring (see Sec. 11), so elements of $\mathcal{F}(I(S))$ can be treated as interval (generalized) fuzzy sets.

It is well known that the classical theory of fuzzy sets is a basis for the theory of possibility [43, 181]. It is also possible to develop a similar generalized theory of possibility starting from generalized fuzzy sets, see, e.g., [43, 90, 100]. Generalized theories can be noncommutative; they seem to be more qualitative and less quantitative compared to the classical theories presented in [180, 181]. We see that idempotent analysis and the theories of (generalized) fuzzy sets and possibility have the same objects, i.e., functions with values in semirings. However, basic problems and methods may be different for these theories (like for the measure theory and the probability theory).

16. Relations to other areas and miscellaneous applications

Many relations and applications of idempotent mathematics to various theoretical and applied areas of mathematical sciences are discussed above. Needless to say that optimization and optimal control problems form a very natural field for applications of ideas and methods of idempotent mathematics. There is a very good survey paper [93] by V. N. Kolokoltsov on the subject, see also [9, 18, 21, 25, 28–31, 35, 37, 38, 50–53, 67–69, 102, 104, 114, 115, 117, 119–124, 144, 174–176, 179, 186, 187].

There are many applications to differential equations and stochastic differential equations, see, e.g., [50–53, 69, 91, 92, 94, 96, 119–123, 138, 159, 160].

Applications to game theory are discussed, e.g., in [95, 96, 122]. There are interesting applications in biology (bioinformatics), see, e.g., [49, 135, 150]. Applications and relations to mathematical morphology are examined in the paper [38] by P. Del Moral and M. Doisy and especially in the extended preprint version of this article. There are many relations and applications to physics (quantum and classical physics, statistical physics, cosmology, etc.), see, e.g., Sec. 6 above and [23, 92, 96, 110, 111, 117, 133, 145].

There exist important relations and applications to purely mathematical areas. The so-called tropical combinatorics is discussed in the large survey paper [87] by A. N. Kirillov, see also [18, 187]. Interesting applications of tropical semirings to the traditional representation theory are presented in [11, 12, 87]. Tropical mathematics is closely related to the very attractive and popular theory of cluster algebras founded by S. Fomin and A. Zelevinsky, see their survey paper [57]. In both cases, there are relations with the traditional representation theory of Lie groups and related topics. For important relations with convex analysis and discrete convex analysis, see, e.g., [2, 27, 30, 32, 40, 113, 118, 123, 157, 183–185]. Some results on the complexity of idempotent/tropical calculations can be found, e.g., in [86, 114, 115, 170]. Interesting applications of tropical algebra to the theory of braids and the Yang–Baxter mappings (in the sense of [16]) can be found in [34, 45, 46].

Starting from the classical papers by N. N. Vorobjev [174–176], many authors examine, explicitly or not, relations and applications of idempotent mathematics to mathematical economics, see, e.g., [33, 95, 123, 182, 187].

This work has been supported by the RFBR grant 05–01–00824 and the Erwin Schrödinger International Institute for Mathematical Physics (ESI).

Translated by the author.

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