LEARNING TO PROVE IN ORDER TO PROVE TO LEARN

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Abstract. Proving is a basic skill for mathematicians; however, this is a difficult skill for some students to learn. In fact, traditionally students have struggled with learning to prove in their junior level mathematics courses. Recently, many universities have instituted a transition course to help students make the transition from computational courses to more proof based courses. This paper is a survey of the current learning to prove literature. It will examine where students struggle, their notions of proof, and the proof strategies that students employ. Finally, I will examine some of the relevant literature regarding the teaching of proof.

Proof is a central form of discourse within the mathematical community. While there have been some discrepancies among scholars as to the definition of a proof or the standard of rigor required for a proof, it is agreed that proving is a necessary skill for mathematicians. The process has many uses: to verify, to explain, to communicate, to persuade, to construct new knowledge or even to synthesize knowledge into an axiomatic form [1]. Thus learning to read, write and understand proofs plays an essential part in becoming an active member in the community of mathematicians.

Throughout history the rigor and requirements of a proof have changed depending on the culture of mathematics at the time. Kleiner [19] provides an exemplary account of the historical development of these cultural rules. From the Babylonians' method of mathematical examples in geometry to the Greek's method of axiomatic proof in geometry to the crisis over the symbolic notation in calculus, each of these developmental stages in justification has brought a mathematically based change to perceptions about the required rigor of proof. Euclid recognized a need for generalized proof beyond the examples provided by Babylonian authors. As the mathematical community debated the notations to be used in calculus, they recognized the need for more rigorous proof. Although proof is commonly discussed in high school geometry courses; traditionally, college students don't encounter proof again until calculus or beyond.

Many universities include some type of transition course from calculus to the upper division courses in their mathematics major curriculum. This course is meant to prepare students for the upper division courses, which often consist of presenting students with a theorem and its proof as a form of teaching. The goal for the course is to introduce students to the rigor of proof writing. The curriculum for transition courses is not in any way standardized. Content, methods as well as expectations change from section to section and university to university. Most courses begin with some logic facts, perhaps including truth tables and basic logic vocabulary [23].

Date: April 17, 2005.

The goal of the transition course is to prepare students to handle the requirements of an upper division course where they are expected not only to produce proofs in homework, but also read and understand proofs presented in textbooks and lectures. It is expected that students be able to follow the logical implications of proofs in class and potentially fill in gaps which are left to the student. Finally students are asked to use theorems which have already been proved in order to prove further conjectures or calculate answers to a question.

There is a growing body of research focused on learning to prove at the undergraduate level, specifically in junior level proof writing or transition courses. In what follows I give a brief overview of this literature concerning students' experiences, abilities, difficulties, and notion of proof, as well as frameworks regarding student proof strategies and schemes. Finally some theoretical research with regards to the cognitive restructuring required of students learning to prove, and the knowledge students need to successfully prove conjectures and begin developing expertise is also discussed.

1. Student's abilities and difficulties

Producing a formal proof requires the use of several areas of knowledge. Students struggle with the content area involved in the proof as well as the laws of logic and deductive reasoning. Students may also be unaware of the logical reasoning and aspects of rigor which govern the proving process. These difficulties as well as the issues concerning the process of proof writing and the language issues are discussed in the research literature.

Ruthven and Coe [27] studied the practices of advanced mathematics students at the end of their first year in college. They found most of these students did not use formal justification even when prompted to do so. When given the opportunity to use a proof to justify a conjecture made, students did not take it. Those students who did provide a justification or explanation of their conjectures used empirical evidence, i.e. appealing to examples as a source of proof.

Part of becoming a member of the mathematical community is learning to align one's discourse with that of the larger community [34]. This ability is dependent on the student understanding socio-mathematical norms held by the community. Dreyfus found students are often unaware of the social norms developed within the mathematical community regarding proof [6]. This difficulty is reinforced by Knuth and Elliott [21], who found pre-service teachers to posses an inadequate understanding of proof as defined by the mathematical community. In a similar study with year 10 students in the UK, Almeida [2] found students held an inadequate understanding of the formalism required.

Other difficulties were noted by Selden and Selden [30], who found students focused on local issues within a proof, without seeing the global picture. Thus a proof could be proving something besides the conjecture, but if each of its local steps were good the students indicated the proof was acceptable.

In a separate study Selden and Selden [29] found college students in a proof transition course struggled to understand the logic required to validate a proof. Students could not reliably determine the logical structure of mathematical statements in order to formulate a proof framework. Students were also unable to unpack informal statements into their formal logical equivalent statements, which is a prerequisite to being able to prove the statement. Similarly, Finlow-Bates et al. [8]

found that first year math students at the university level failed to be able follow a chain of reasoning. Hoyles and Kűchemann [17] studied high school students' understanding of logical implications. They found if students were asked to assume a statement was true, then given the antecedent was true they could correctly induce the implication. However, when students had to pick an example, they struggled to understand the need to pick an example which satisfied the antecedent. Students tended to pick examples to confirm a conjecture not to prove it or test it.

Another facet of difficulties within the logic aspect of proof construction is the language used. Finlow-Bates et al. [8] discovered students misunderstood the language being used in proofs. This predominately occurred when the word carried a specific mathematical meaning, which differed from its everyday use. Likewise Dreyfus [6] concurs that university students lack the language to communicate effectively in proof writing. The professor in Moore's study [24] is concerned that students do not have the language or cultural understanding.

Language is a basis for several difficulties students have. Moore found language to be a difficulty for students along with their use of definitions and their abilities related to the specific concepts. He found students consistently exhibited the following seven difficulties:

- *D1: The students did not know the definitions. That is, they were unable to state the definitions.*
- *D2: The students had little intuitive understanding of the concepts.*
- *D3: The students' concept images were inadequate for doing the proofs.*
- *D4: The students were unable, or unwilling, to generate and use their own examples.*
- *D5: The students did not know how to use definitions to obtain the overall structure of proofs.*
- *D6: The students were unable to understand and use mathematical language and notation.*
- *D7: The students did not know how to begin proofs.* ([24] p.251-252).

These errors all relate to the students' conceptual understanding of the material at hand and not necessarily the logical implications required in a deductive proof. It is worthy of noting the importance of the students being able to generate examples, D4. Without the domain specific knowledge held by the professor, the students struggled to generate examples they could easily manage. This left them unable to explore the definitions and theorems they were trying to understand and prove.

Several other studies support students' difficulties with the conceptual understanding of the material. Dreyfus remarks, "In most cases, they still lack the conceptual clarity to actively use the relevant concepts in a mathematical argument,"([6] p. 91). Finlow-Bates, Lerman, and Morgan [8] determined misunderstanding of the mathematical concept as a source of difficulty for their students as well. Hart [13] found that in abstract algebra courses students' were confused by the operations involved, indicating some problems inherent in their concept image.

While it is clear that students struggle with several aspects of proving, these difficulties fit nicely into two categories. First students struggle with the logic, language and culture of the proof as determined by the community. Second students lack the domain specific knowledge, such as definitions, theorems, heuristics and the ability to generate examples. Difficulties in the first category are intertwined with student notions of what constitutes a proof and the values they hold, while the latter category is related to the strategic knowledge they need and is often content or domain specific.

2. Student's notion of proof

In learning to prove, the notion of a proof encompasses a wide range of issues and ideas. Students' notions of proof determine what they consider a valid proof, what convinces them of a fact, how they approach the task of proving, what they value in a proof, and how they determine what constitutes a proof. The research literature points to the fact that students at the undergraduate level do not have a robust notion of proof.

Students' perceptions of proof play an important role in their thinking and reasoning and hence in the types of proofs they produce [24]. Moore conducted a classroom research experiment in a transition course taught at the university level. He noticed the students perceived a proof to be procedural, a set of steps to perform. Gray, Pinto, Pitta and Tall [9] found that students with a procedural approach were less successful at advanced mathematical thinking. Moore also noted that students had a limited idea of the purpose of proof.

Some students see the purpose of proof in a very narrow or disconnected manner. Vinner [32] observed students distinguished between verifying a truth and proving. This may be due in part to the student's notion that proving is convincing. As Segal [28] notes, there is a difference in the intended audience between convincing and validating. One convinces an individual, but when putting forth an argument for the community, one is validating a truth. Segal noticed that students distinguished between these two aspects of proof when looking at an empirical justification, but they were unable to do so when analyzing a deductive argument. Raman [25] also distinguishes between the private audience and public audience. She found that students saw these two aspects of proof as disconnected, whereas experts in the field see them intertwined. Raman's students didn't see the explanatory key idea of the proof connecting the different aspects. Although students tend to look for meaning and explanation in proofs, they rate proofs by their form.

In several studies it was noted that students think a proof must be presented in a particular format [14, 20, 32]. Vinner evaluated students at the senior high school level. He found students spent time substituting values into a previously proved theorem to confirm the general formula. This hints that their notion of proof did not include the concept of generality. Students felt a particular case might still need to be confirmed even though a general proof is understood.

The students studied by Healy and Hoyles [14] also struggled with the generality of proof. Healy and Hoyles found that students simultaneously held two different proof conceptions. The proofs, which were given highest preference to the students personally, were explanatory, while the proofs to receive the highest mark from the teacher were algebraic, even when the algebraic proof was incorrect. However, the proofs they constructed were mainly empirical or narrative in nature. Knuth's [20] subjects also placed a higher value on algebraic proofs even if there were clear errors in the proof.

Knuth claims a robust notion of proof should include an understanding of the infallibility of a proof as well as the generality of a proof. He studied the conceptions of in-service high school teachers and found they did not have a robust notion of proof. The teachers used four criteria to determine if a particular response constituted a proof: valid method, sound mathematical reasoning, sufficient detail, and knowledge dependence. The proofs were judged by their mechanics. For instance if a proof used induction, it was likely to be judged as a proof because it used a valid method, even if the induction was done incorrectly. These criteria, however, were not the same criteria for what determined a convincing argument. The teachers were convinced by sound mathematical reasoning, empirical data (an example or visual reference), familiarity, generality and that it showed (in a visual way) why. Students interviewed by Finlow-Bates [7] were also convinced by examples and stated the role of a proof was to explain.

Almeida [1] surveyed first and second year university students. He found the students' professed ideas about proof were similar to what he considered ideal: they espoused formal views. However their private proof practices deviated from the ideals they had expressed. He suggests that even the best second year university student can understand the need for a formal proof but may be unable to live up to the demands of rigor in his/her own proof practices, ([1], p.847). The lower students lack the understanding to perceive a formal proof as anything more than symbolic manipulation.

The first year university students interviewed by Finlow-Bates [7] saw value in the informal arguments presented but placed more value on empirical evidence. In fact they valued an informal proof with examples higher than the informal proof alone. When validating proofs, students value being able to understand what has been written [30]. This is in contrast to when students were asked to rank a proof for a grade. There they were likely to pick a proof for its structure (algebraic) even if they could not understand its meaning. In fact when ranking proofs students consistently paid attention to "their clarity, usefulness, consistency, how convincing they were and how easy they were to understand before considering if they were logical and rigorous, "([8], p. 258). When constructing proofs, students also value understanding [1] and they see value in informal arguments.

Overall students' concept images of proof seem to include a notion of algebraic or symbolic manipulation. Although they are convinced by informal and explanatory methods they consistently mark formal proofs as a preferred method. Students value being able to understand arguments. They value explanations and are convinced by empirical evidence. Their own proof construction (concept usage) is not usually formal. Some of the students viewed the purpose of proofs to include explaining, and verifying, but for the most part their notion of the purpose of proof is rather narrow.

3. Student Proof Schemes

Aside from describing student difficulties and notions of proof, a significant portion of the proof literature seeks to categorize the arguments produced by students as proof. An argument that convinces a student or which a student would use to convince someone is classified as a type of proof scheme [12]. There has been significant research on proof to classify and characterize student proof schemes [4, 10–12, 15, 22]. Each of these frameworks defines student behavior differently. The Harel and Sowder framework is the most extensive of the frameworks.

Harel and Sowder give three classes of conviction: external, empirical, and analytical. Each of these classes is made up of several proof schemes. To the authors these schemes represent cognitive stages in the student's mathematical development. There are two issues that Harel and Sowder attempt to define in this framework. The first is the student's level of understanding and the second is the techniques of proving by which the student is convinced. These are certainly related issues, but there is some confusion in this framework as students are capable of using different schemes for different problems. There is certainly cognitive development involved in being able to use an analytic proof scheme. Harel and Sowder also comment that students need to go through the process of using empirical schemes to be able to use analytical schemes. They remark that external conviction is an unnecessary step in the process, which presents a potential stumbling block for students.

The class of external conviction consists of ritual, authoritarian and non –quantitative symbolic schemes. The ritual scheme is when students are convinced by the form of the proof. The authoritarian scheme represents students who are convinced by a textbook, teacher or some other authority. The non-quantitative symbolic scheme describes students who mindlessly manipulate symbols with little to no understanding of their meaning. Harel and Sowder comment that other than the authoritarian proof scheme these schemes are due to poor teaching.

Arguments which convinced students in the empirical proof schemes class fall into two categories: inductive and perceptual. Inductive arguments are based on specific cases or examples. The inductive scheme is a natural progression in learning to prove. When a student is convinced by a set of pictures which lack transformational reasoning, we define this to be the perceptual proof scheme. If transformational reasoning is present then the proof scheme is analytical. The analytical proof schemes category was greatly revised by Harel [11] and renamed as deductive proof schemes. There are two types of deductive proofs: transformational and modern axiomatic. Each type of proof requires the student use goal oriented mental operations and recognize the need for generality. The difference is that transformational proofs are characterized by the transformation of images that govern the deduction in the evidencing process.

Transformational proof schemes are divided into contextual, generic, and causal schemes. Each of these distinctions is due to a restriction a student places on the generality of the proof. Either they restrict the context of the argument (contextual), the generality of the justification (generic), or the mode of justification (causal). The last proof scheme under transformational reasoning is called constructive. This scheme describes students who construct the object in question rather than just prove its existence. Mathematicians also use this type of proof to show existence.

Modern axiomatic proofs are characterized by a set of (arbitrary) rules that govern the transformations of images in the evidencing process. The class of modern axiomatic schemes describes a progress of understanding. It begins with structural schemes. The structural proof scheme requires that students recognize that definitions and theorems belong in the structure created by a particular set of axioms. The progression of understanding proceeds to axiomatizing, which is the ability to analyze the implications of changing a set of axioms. Harel and Sowder see the class of deductive proofs as hierarchical. Transformational proofs are generally developed first by students, but the ultimate goal is for students to be able to produce modern axiomatic proofs. Likewise that structural conception must come before axiomatizing [10].

Balacheff's research [4] focuses on further developing the empirical proof scheme. He asserts that students use examples for a variety of reasons. He first defines two types of proofs: pragmatic and conceptual. Pragmatic proofs use actions to show that something is the case, while conceptual proofs rest on the ability to formulate properties or relations. Balacheff notes that students use examples for both pragmatic and conceptual proofs. He defines example usage in four categories: naïve empiricism, crucial experiment, generic example and thought experiment.

Students utilizing naïve empiricism believe a conjecture to be fact after verifying the results for a few cases. These students do not consider generalization at all. The crucial experiment is a move toward generic understanding. Students pick examples where the validity of the statement is not intuitive. The idea is that if it works for this very complicated or difficult example then it should work for everything else. The generic example has all of the reasoning of a proof, but the proof is done with one representative of the class of objects it represents. Finally, the thought experiment is when students are able to detach themselves from the example in question and recognize the key idea of the proof.

Several studies have used Balacheff's framework to analyze students uses of example. Knuth and Elliott [22] remark that the first two uses of examples are an inductive approach to proof. Generic examples and thought experiments are a move to deductive proofs. In fact a generic example for many students takes the place of an explanatory proof [26].

Other proof scheme frameworks have also been put forth [8, 14, 17, 31]. Finlow-Bates et al. classify students' modes of thinking as empirical, logical and aesthetic. Empirical thinking is indicated by students' reliance on examples to convince them. Students who preferred rational arguments (this is similar to a basic deductive proof scheme) are indicated as thinking logically. An aesthetic mode of thought describes students who prefer proofs which are visually or intuitively appealing. Similar distinctions are made by Healy and Hoyles [14], who classify students' proofs as empirical, narrative and formal. The empirical and formal proofs described are the same as Finlow-Bates et al. The narrative proofs are closely related to an aesthetic mode of thinking. They include all of the reasoning and mathematical relationships from the formal proof but they are written in everyday language or represented in pictures. This category is also similar to Tall's [31] iconic proofs.

Tall [31], drawing from Bruner's work, uses the cognitive development involved in communication to determine different representations of proofs: enactive, iconic and symbolic. Enactive communication is done with actions and gestures, hence an enactive proof needs physical movement to show something is true. Iconic communication uses pictures to explain. Tall's definition of symbolic communication refers to the language of logic. This is akin to Harel and Sowder's modern axiomatic proof scheme.

Each of these studies defined student behavior in a slightly different manner. There are, however, some clear areas of consistency. Students may appeal to outside authority and empirical evidence when proving. Students often focus on form and visual aspects of a proof rather than the analytical or deductive reasoning. It is important to understand that students may use several different schemes depending on the task before them. This is no different than mathematicians who may choose to use transformational reasoning to prove some conjectures and axiomatic reasoning for others. Harel and Sowder define the entire class of analytical or deductive proof schemes to be considered as mathematical proofs. From the student's perspective all of these schemes including the external proof schemes are convincing arguments of different levels.

4. What Student's Need to Know

Learning to prove is not a simple task. Students must learn strategic knowledge in the content areas in which they are proving. Students must also obtain knowledge specific to proving. Likewise students need a wealth of problem solving skills as well as socio-mathematical norms regarding proof. They must be able to manipulate this knowledge by reformulating ideas, introducing notation and generating examples. Finally, there is a need for students to develop some behaviors associated with experts in the field. These areas of cognitive development play a role in the student learning to prove.

Jones [18] found successful students were able to create a rich concept map of proof. A larger proof concept map does not however, guarantee success. Gray et al. [9] conclude students need to develop a level of sophistication which includes the ability to be flexible with definitions and procedures, as well as being capable of thinking about mathematics symbolically.

Weber [33] interviewed undergraduates and doctoral students in abstract algebra. He found students needed domain specific knowledge outside of a complete concept image. Students needed to know proof techniques that may be specifically suitable for that domain. They must be able to determine which proofs are important and when are they useful. Students must be able to determine when and when not to use syntactic strategies, defined as procedural or symbolic manipulation. For example when proving the intersection of two sets is empty the strategy first employed should be a proof by contradiction. This does not always work, but it is the best first guess. Weber found even students who had conceptual knowledge about the content of the conjecture, did not necessarily know how to proceed in the proof. Hart [13] found the advanced students in his study made use of these domain specific strategies in their proofs.

Students also find it difficult to know what constitutes a proof. This is related to an inherent lack of understanding about the socio-mathematical norms established for the community. Dreyfus [6] argues that the socio-mathematical norms of the community of practicing mathematicians are not taught in the college classroom. He implies that students do not know what is expected of them regarding the rigor required for a particular proof in a certain situation and therefore are unable to justify their claims in an acceptable manner.

Socio-mathematical norms may not be the only issue. Housman and Porter [15] suggest there may be a learning trajectory, which proceeds from external conviction through empirical conviction to analytical conviction or proof. As students gain experience in proof writing, they progress through this trajectory. Students with more proof experience are able to reformulate statements into logically equivalent statements or take definitions and reformulate them to informal statements. These students introduce notation and generate examples more often [13,15]. Being capable of reformulating statements is a necessary skill for proving.

Weber's description of the strategic knowledge students need to be successful at proving is exemplified in the tasks that Berliner ascribes to experts. Berliner [5] describes the characteristics of experts in the teaching field. The parallel skills in mathematics might include the following skills: experts know how to proceed when faced with the wording of a conjecture. Experts have access to a wide range of examples, which clarify the nuances of the conjecture. Experts can also generate new examples to fit a new situation. Experts can recognize patterns in new problems and are better able to find analogies. While these skills and behaviors do not guarantee a student will be successful at proving; students who are successful exhibit these abilities.

5. Implications for Teaching and Research

The question is how to provide students with experiences, which help them to develop the knowledge and skills necessary to prove a statement. Almeida [3] suggests this experience should be similar to experiences faced by "pioneering mathematicians."He argues that the natural sequence for developing mathematics is "intuition, trial, error, speculation, conjecture, proof."

Dreyfus [6] notes the issues involved in the current mathematics textbooks and the misconceptions they produce. For example he notes that textbooks often use examples as a way of justifying a theorem. Likewise textbooks often use informal language in justification and he notes that classroom discourse is often even more watered down. This gives students poor examples of proof which permeate their concept image of proof. Hoyles [16] argues that the influence the curriculum plays on student approaches and perceptions of proof is underestimated. The students in her study exhibited behavior determined by the curriculum in the UK, associating proof with numerical investigations. Their answers listed data and conjectured even when this method was not appropriate.

Due to the enormous importance of proof in the mathematical dialogue it is imperative that we as a community continue to improve students' proving. While the literature is not particularly positive about student's abilities at the undergraduate level, students are learning. Thus it would be helpful if further literature focused on the abilities of students rather than their difficulties. Finally, the research suggests that students learn to prove best when they are immersed in doing mathematics. Hence the transition courses would benefit from developing curriculum which promotes defining, conjecturing and proving in a similar manner to mathematicians.

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