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BAYESIAN ESTIMATION OF STOCHASTIC-TRANSITION MARKOV-SWITCHING MODELS FOR BUSINESS CYCLE ANALYSIS

By
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Bayesian Estimation of Stochastic-transition Markov-switching Models for Business Cycle Analysis

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Abstract

We propose a new class of Markov-switching (MS) models for business cycle analysis. As usually done in the literature, we assume that the MS latent factor is driving the dynamics of the business cycle but the transition probabilities can vary randomly over time. Transition probabilities are generated by random processes which may account for the stochastic duration of the regimes and for possible stochastic relations between the MS probabilities and some explanatory variables, such as autoregressive components and exogenous variables. The presence of latent factors and nonlinearities calls for the use of simulation-based inference methods. We propose a full Bayesian inference approach which can be naturally combined with Monte Carlo methods. We discuss the choice of the priors and a Markov-chain Monte Carlo (MCMC) algorithm for estimating the parameters and the latent variables. We provide an application of the model and of the MCMC procedure to data of Euro area. We also carry out a real-time comparison between different models by employing sequential Monte Carlo methods and some concordance statistics, which are widely used in business cycle analysis.

KEYWORDS: Markov-switching Models; Stochastic Duration Models; Bayesian Inference; Markov-Chain Monte Carlo; Sequential Monte Carlo.

AMS CLASSIFICATION: 62G07, 62M20, 62P20, 91B84.

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1 Introduction

Turning points detection and forecasting of the economic activity level are challenging problems in business cycle analysis. In this paper we consider a model-based framework and a Bayesian inference approach to deal with these issues.

Early contributions in the non-linear literature applied *Markov Switching* (MS) models (see for example Goldfeld and Quandt (1973) and Hamilton (1989)) and threshold autoregressive models (see Tong (1983) and Potter (1995)) to capture turning points and model the asymmetry in the business cycle dynamics.

These contributions have been extended in many directions. Kim (1994) applies MS to dynamic linear models. In this case the Hamilton's filter is useless and he proposes a Bayesian approach for inference. Kim and Nelson (1999b) provide a complete introduction to inference methods based on *Markov-chain Monte Carlo* (MCMC) for MS state space models.

In the basic MS models for business cycle the switching process indicates the phase of the economic cycle and may assume at least two regimes, which are usually interpreted as: positive growth trend and negative growth trend. Kim and Murray (2001) and Kim and Piger (2000) consider MS models with three-regimes (recession, high-growth and normal-growth).

Other extensions are in Sichel (1991), Watson (1994), Diebold and Rudebusch (1996), Durland and McCurdy (1994) and Filardo (1994), which assume that the MS transition probabilities depend on the duration of the current phase of the cycle and thus are time-varying. Moreover, Billio and Casarin (2009) consider stochastic transition probabilities. Finally some multivariate extensions to the Hamilton (1989) model can be found in Diebold and Rudebusch (1996) and Krolzig (1997, 2004).

In the present work we combine a state space approach to the business cycle with the possibility to consider a non homogeneous MS model. The first contribution is to assume that the transition probabilities of the Markov-switching process vary randomly over time. More specifically we assume that the probabilities are stochastic processes with beta distributed innovations. The main advantages of using a beta random variables is that it is naturally defined on a bounded interval and is a flexible probabilistic model. See Ferrari and Cribari-Neto (2004) for an introduction to beta regression models.

Models with stochastic transition have been already proposed in econometrics by Gagliardini and Gouriéroux (2005) in a continuous time setting. In that work the probability of transition of the credit quality from one rating class to another one is modelled as a Jacobi diffusion process, which is naturally defined on a bounded interval. The ergodic distribution of the Jacobi process is a beta. This probabilistic fact suggests us that a beta noise process could be a very good candidate process for modelling, in a discrete-time setting, random fluctuations on bounded intervals.

The use of random transition probabilities implies that the duration of the different regimes is stochastic and has thus a dynamics. The dynamics of the MS transition probabilities in the existing works is usually modelled by means of a deterministic (e.g. linear-logistic) relationship between the probabilities and a set of explanatory variables. We introduce a residual term in order to account for unexplained variations in duration. Moreover, the variations in the MS probabilities and in the duration can be explained thanks to an autoregressive structure.

In this work we introduce a very flexible parameterization which allows for a easy identification of the location and scale parameters involved in the model. In this sense the proposed model represents an extension of the stochastic transition model proposed in Billio and Casarin (2009). The proposed parameterization allow us to model directly the mean of the process (i.e. the transition probabilities) and make easier the inference procedure on the parameter.

Another relevant contribution of the present work is to propose a full Bayesian inference approach for random transition MS models. We refer the reader to Bauwens, Lubrano and Richard (1999) for an introduction to Bayesian inference for dynamic models. Kim and Nelson (1999b) provide a review on Bayesian inference method for MS models. Moreover, we follow an inference approach based on Monte Carlo simulation methods, see Billio, Casarin and Sartore (2007) for an updated review on the simulation-based inference methods for business cycle models and Billio, Monfort and Robert (1999) for MCMC-based inference for MS ARMA processes. We follow a data-augmentation approach and propose a MCMC algorithm for jointly estimating the latent MS autoregressive process and the stochastic transition probabilities. Finally, we provide an application to the business cycle of the Euro area and employ *Sequential Monte Carlo* (SMC) methods, also known as Particle Filters (Doucet, Freitas and Gordon (2001)), to obtain a sequential comparison between some competing models.

The work is structured as follows. Section 2 introduces a probabilistic state space representation of dynamic models and presents a new MS model with stochastic transition probabilities. Section 3 proposes a Bayesian inference approach for the stochastic transition models. Section 4 provides an application to the Business cycle of the Euro area. Section 5 concludes.

2 Stochastic-Transition Models

We consider a MS latent factor model to extract the phases of the business cycle. In this model, the observable y_t is a measure of the unobservable level x_t of the economic activity and the economic phases are represented through a Markov chain process, s_t. The transition probabilities of the MS models usually applied in the literature (see for example Kim and Nelson (1999b)) are constant over time or depend on a set of exogenous variables, which may include the lagged variables or a duration process.

Let y_t be the observable variable and x_t and s_t two latent variables. We propose a new *Stochastic-Transition MS* (ST-MS) model which has the following measure and transition densities

$$
\begin{aligned}
y_t|x_t &\sim \mathcal{N}(x_t, \sigma_y^2) && (1)\\
x_t|x_{t-1}, s_t &\sim \mathcal{N}(\mu_{s_t} + \rho_{s_t} x_t, \sigma^2_{x\, s_t}) && (2)\\
s_t|s_{t-1} &\sim \mathbb{P}(s_t = j|s_{t-1} = i) = p_{ij,t}, \quad \text{with } i, j \in \{0, \dots, K-1\}. && (3)
\end{aligned}
$$

with $K \in \mathbb{N}$ the maximum number of regimes. In this paper we will consider $K = 2$, that correspond to the recession, $s_t = 0$, and expansion, $s_t = 1$, phases of the business cycle.

Under a stochastic modelling point of view the assumption of a deterministic relation between the conditioning variables and the transition probabilities can be unsatisfactory. In fact, the empirical evidence is in favor of phases with different duration and this may be only partially explained by a set of exogenous variables. It seems more reasonable to assume that time variations in the duration may also depend on the intrinsic random nature of the adjustments in the economic activity. Billio and Casarin (2009) recently show that stochastic transition MS outperforms the constant and the time-varying transition MS models in terms of forecasting abilities. In this work we propose a more flexible parameterization.

A beta random variable takes values in the standard unit interval and its density can assume quite different shapes

$$
f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{I}_{[0,1]}(x) \qquad (4)
$$

with $\Gamma(z)$ the gamma function and $\alpha > 0$, $\beta > 0$. In particular, the mean and variance of a beta random variable are

$$
\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}, \qquad \mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
$$

This parameterization of the beta distribution is inconvenient because α and β are both shape parameters and they are difficult to interpret in terms of conditional expectations. If we let $\eta = \alpha/(\alpha+\beta)$ and $\phi = (\alpha+\beta)$, with $\eta \in]0, 1[$ and $\phi > 0$ we obtain a reparameterization of the model which allow us to model separately the mean and the scale of the variable.

We employ the above reparameterization and propose to capture the unexplained part of the transition probability $p_{ij,t}$ with a dynamic beta model (see for example Ferrari and Cribari-Neto (2004)). The conditional density of $p_{ii,t}$ is thus

$$p_{ii,t} \sim \frac{\Gamma(\phi)}{\Gamma(\eta_{it}\phi)\Gamma(\phi(1-\eta_{it}))} p_{ii,t}^{\phi\eta_{it}-1}(1-p_{ii,t})^{\phi(1-\eta_{it})-1}\mathbb{I}_{]0,1[}(p_{ii,t}) \tag{5}$$

with $\phi > 0$ and $\eta_{it} \in]0,1[$, $\forall i \in \{0,1\}$. The location parameter η_{it} represents the mean of the variable and ϕ can be interpreted as a precision parameter.

We consider a set of explanatory variables $\mathbf{v}_{it} \in \mathbb{R}^{n_{vi}}$ (boldface means that the quantity is a vector) and let $\mathcal{F}_t = \sigma(\{p_{ii,s}, s \leq t,\})$ and $\mathcal{G}_t = \sigma(\{(\mathbf{v}'_{1s}, \mathbf{v}'_{2s})', s \leq t-1\})$. Both the conditional mean and the conditional variance of the transition probabilities $\mathbb{E}(p_{ii,t}|\mathcal{F}_{t-1} \vee \mathcal{G}_t) = \eta_{it}$ and $\mathbb{V}(p_{ii,t}|\mathcal{F}_{t-1} \vee \mathcal{G}_t) = \eta_{it}(1-\eta_{it})/(1+\phi)$ depend on the set of variables and allow us to easily provide forecasts. Moreover, this model can account for heteroscedasticity. In particular, for a given value of η_{it}, the variance of the random variable increases with ϕ.

Let $\eta_{it} = h(\boldsymbol{\psi}'_i \mathbf{v}_{it})$, with with $\boldsymbol{\psi}_i \in \mathbb{R}^{n_{vi}}$ and $h(z) : \mathbb{R} \mapsto]0,1[$. There are several possible choices for h. For instance, we can use the probit or the log-log functions. In this paper we consider the logistic transform: $h(x) = 1/(1+\exp\{-x\})$. The set of variables $\mathbf{v}_{it}$ can be different for each transition probability and can include the lagged values of $p_{ii,t}$ and of s_t. The resulting class of models allows many useful specifications.

For example when the transition density $p_{ii,t}$ depends on its past values

$$\eta_{it} = h(\psi_{0i} + \psi_{1i}p_{ii,t-1} + \ldots + \psi_{pi}p_{ii,t-p}) \tag{6}$$

then we will have a pure beta autoregressive models of order p. Another special case is the duration dependent model

$$\eta_{it} = h(\psi_{0i} + \psi_{1i}d_{ii,t}) \tag{7}$$

with $d_t = (d_{t-1}+1)\mathbb{I}_{s_{t-1}}(s_{t-2}) + (1-\mathbb{I}_{s_{t-1}}(s_{t-2}))$ (see for example Durland and McCurdy (1994) and Filardo (1994)).

In this work we will focus on the beta autoregressive specification. We use some of the parameter estimates for the Euro area (see Section 3) to simulated a sample of transition probabilities (see Fig. 1) from the ST-MS model. In order to highlight the effect of the exogenous variable $\mathbf{v}_{it} = (1, u_t)'$ on the recession and expansion phases of the business cycle we assume for u_t the following artificial dynamics: $u_t = 0.2 + 0.4\mathbb{I}_{t\geq 150} + 0.4\mathbb{I}_{t\geq 300}$. Note that an increase in the exogenous produces and increase in the probability of staying in a recession and a decrease of the probability of staying in a recession phase. Moreover the simulation

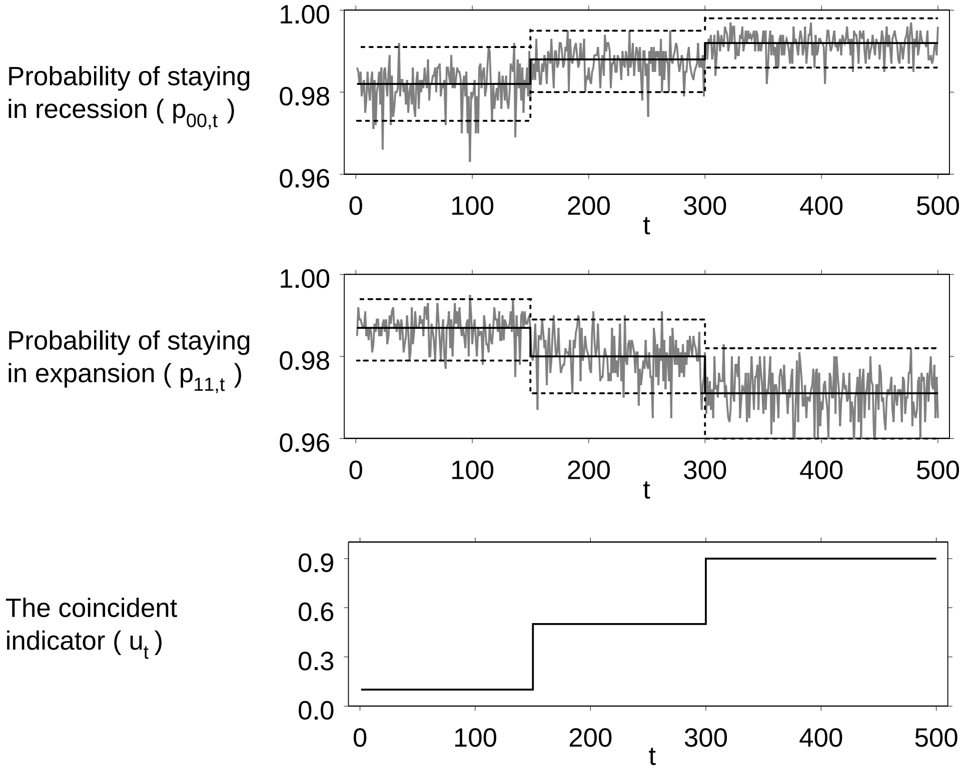

Figure 1: Up and middle: sample paths (continuous gray line) of the stochastic probabilities of staying in recession $p_{00,t}$ and in expansion $p_{11,t}$, simulated from model ST-MS with $\boldsymbol{\psi}_0 = (1.67, 0.71)'$, $\boldsymbol{\psi}_1 = (2.51, -1.03)'$, $\phi = 90$ and $\mathbf{v}_{it} = (1, u_t)'$, evolution of the conditional mean $\mathbb{E}(p_{ii,t}|\mathcal{F}_{t-1} \vee \mathcal{G}_t)$ (continuous black line) and of the 5% and 95% quantiles (dashed black lines) of the conditional density. Bottom: the exogenous process $u_t = 0.1 + 0.4\mathbb{I}_{t\geq 150} + 0.4\mathbb{I}_{t\geq 300}$.

experiment shows that the transition probabilities exhibit random fluctuations generated by the beta process.

Under a modelling point of view, the random variations in the transition probability make the ST-MS a flexible model, which may accounts for time-varying and stochastic duration of the MS regimes. It is not easy to find the analytical relationship between the parameters of the beta processes and the conditional and unconditional distributions of duration processes. Thus we carry out a Monte Carlo simulation study in order to estimate the effect of the autoregressive coefficient ψ_{1i}, with $i = 0, 1$, and of the precision parameter ϕ of the beta process on the unconditional distribution of the duration. The distribution of the duration has been estimated with 100 Monte Carlo

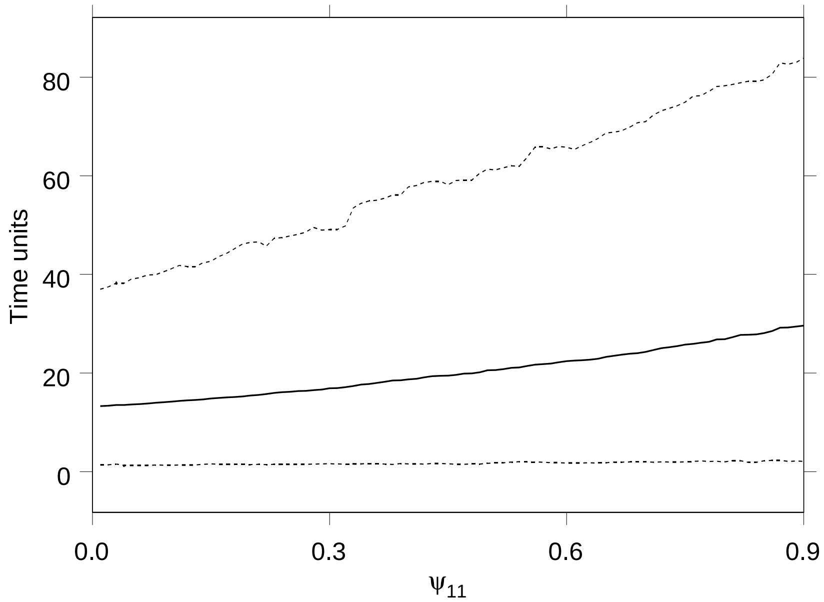

Figure 2: Effects of the autoregressive coefficient ψ_{1i} and of the scale ϕ of the beta process on the stochastic duration of the second regime ($i = 1$). The mean of the duration process has been estimated with 100 Monte Carlo experiments. In each experiments we simulated a path of 5,000 realizations from a ST-MS model with $\boldsymbol{\psi}_0 = (1.9, 0.01)'$, $\psi_{01} = 1.9$, varying $\psi_{11} \in [0, 1]$, $\phi = 90$ and explanatory variables $\mathbf{v}_{it} = (1, p_{ii,t-1})'$. Mean (continuous line) and 5% and 95% quantiles (dashed lines) of the duration distribution.

experiments. In each experiment we simulated a path of $5,000$ realizations from a ST-MS model with parameters $\boldsymbol{\psi}_0 = (1.9, 0.01)'$, $\psi_{01} = 1.9$, $\psi_{11} \in [0, 1]$, $\phi = 90$ and explanatory variables $\mathbf{v}_{it} = (1, p_{ii,t-1})'$. We observed that increasing the value of the parameter ϕ, with $\phi \in [0, 300]$, increases the dispersion level of the duration density and has a negligible effect on the duration mean. Increasing the persistence of the transition probability $p_{ii,t}$, i.e. the value of ψ_{1i}, produces a positive shift of the duration density. The graph on the left of Fig. 2 shows the Monte Carlo estimates of the relationship between ψ_{11} and the mean and the quantiles of the unconditional duration distribution for the second regime. In our application we will estimate the value of the persistence parameters, thus the duration mean will be implicitly determined. An alternative to the estimation of the persistence parameter is the calibration on the basis of the inverse relationship between persistence and duration mean and of exogenous information on the duration of the regimes.

3 Bayesian Inference

We introduce the following notation which will be useful for defining both the MCMC and SMC estimation procedures. Let $\mathcal{Z} \subset \mathbb{R}^{n_z}$, $\mathcal{Y} \subset \mathbb{R}^{n_y}$ and $\boldsymbol{\theta} \subset \mathbb{R}^{n_{\boldsymbol{\theta}}}$ be three measurable spaces, called state, observation and parameter spaces respectively. Denote with $\{\mathbf{z}_t;\, t \in \mathbb{N}\}$, $\mathbf{z}_t \in \mathcal{Z}$, the *hidden state* (or latent variable) vectors of a dynamic model, with $\{\mathbf{y}_t;\, t \in \mathbb{N}_0\}$, $\mathbf{y}_t \in \mathcal{Y}$, the *observable variable* vectors and with $\boldsymbol{\theta} \in \Theta$ the *parameter* vector of the model.

Let $\mathbf{z}_{r:t} \triangleq (\mathbf{z}_r, \ldots, \mathbf{z}_t)$ be the collection of state vectors from time r up to time t, with $r \leq t$ and $\mathbf{z}_{-t} \triangleq (\mathbf{z}_0, \ldots, \mathbf{z}_{t-1}, \mathbf{z}_{t+1}, \ldots, \mathbf{z}_T)$ the collection of all the state vectors up to time T, without the t-th element. We employ the same notation for the observable variables and the parameter vector.

For the proposed ST-MS model $n_y = 1$, with $\mathbf{y}_t = y_t$ and $n_z = 4$, with $\mathbf{z}_t = (x_t, s_t, p_{00,t}, p_{11,t})'$. For the estimation purposes we introduce the following reparameterization $\mu_{s_t} = \mu_0 + d_\mu s_t$ and $\rho_{s_t} = \rho_0 + d_\rho s_t$, with the subset, $\mathcal{A}_1 \subset \Theta$, of parameter values which are satisfying at some identifiability and stationarity constraints, which are $\delta_\mu > 0$ and $|\rho_{s_t}| < 1$ respectively. The dimension of the parameter space is $n_\theta = 8 + n_{v1} + n_{v2}$, and the parameter vector is $\boldsymbol{\theta} = (\sigma_y^2, \mu_0, d_\mu, \rho_0, d_\rho, \sigma_{x0}^2, \sigma_{x1}^2, \boldsymbol{\psi}_0', \boldsymbol{\psi}_1', \phi)'$

The complete-data likelihood of the model is

$$\mathcal{L}(\mathbf{y}_{1:T}, \mathbf{z}_{1:T}|\theta) = \prod_{t=1}^{T} \Big(f(y_t|x_t, \boldsymbol{\theta}) f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) f(s_t|s_{t-1}, p_{00t}, p_{11,t}, \boldsymbol{\theta})$$
$$f(p_{ii,t}|s_{t-r:t-1}, p_{ii,t-q:t-1}, \boldsymbol{\theta}) \Big) f(x_0|s_0, \boldsymbol{\theta}) f(s_0|p_{00,0}, p_{11,0}, \boldsymbol{\theta}) f(p_{00,0}) f(p_{11,0})$$

where

$$\begin{aligned}
f(y_t|x_t, \boldsymbol{\theta}) &= (2\pi\sigma_y^2)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_y^2}(y_t - x_t)^2 \right\} \mathbb{I}_{\mathbb{R}}(y_t) \\
f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) &= (2\pi\sigma_x^2)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\sigma_x^2}(x_t - \mu_{s_t} - \rho_{s_t} x_t)^2 \right\} \mathbb{I}_{\mathbb{R}}(x_t) \\
f(s_t|s_{t-1}, p_{00,t}, p_{11,t}, \boldsymbol{\theta}) &= (p_{00,t}^{1-s_t}(1 - p_{00,t})^{s_t})^{1-s_{t-1}} (p_{11,t}^{s_t}(1 - p_{11,t})^{1-s_t})^{s_{t-1}} \mathbb{I}_{\{0,1\}}(s_t) \\
f(p_{ii,t}|\mathbf{v}_{it}, \boldsymbol{\theta}) &= \frac{\Gamma(\phi)}{\Gamma(\eta_{it}\phi)\Gamma(\phi(1 - \eta_{it}))} p_{ii,t}^{\phi\eta_{it}-1} (1 - p_{ii,t})^{\phi(1-\eta_{it})-1} \mathbb{I}_{]0,1[}(p_{ii,t})
\end{aligned}$$

and $f(x_0|s_0, \boldsymbol{\theta})$, $f(s_0|p_{00,0}, p_{11,0}, \boldsymbol{\theta})$, $f(p_{00,0})$, $f(p_{11,0})$ represent the densities associated to the initial distributions of the latent variables. We assume the uniform distribution for the initial transition probabilities, the ergodic distribution associated to $p_{ii,0}$ for the initial regime and the stationary distribution for x_0 in each regimes. In the following we will assume that $\mathbf{v}_{it}$ may contain $p_{ii,t-1}$ and do not contain lagged values of s_t. Then we denote with $f(p_{ii,t}|p_{ii,t-1},\boldsymbol{\theta})$ the resulting transition density. The proposed inference approach can be extended to the case the transition depends on the past values of the chain, as in the duration dependent models.

3.1 Priors

Let us consider the following partition of the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2', \boldsymbol{\theta}_3', \boldsymbol{\theta}_4')'$, with $\boldsymbol{\theta}_1 = \sigma_y^2$, $\boldsymbol{\theta}_2 = (\mu_0, d_\mu, \rho_0, d_\rho)'$, $\boldsymbol{\theta}_3 = (\sigma_{x0}^2, \sigma_{x1}^2)'$, $\boldsymbol{\theta}_4 = (\boldsymbol{\psi}_0', \boldsymbol{\psi}_1')'$ and $\boldsymbol{\theta}_5 = \phi$.

Due to the linear and Gaussian dynamics of the observable variable we assume a conjugate prior for $\boldsymbol{\theta}_1$

$$\sigma_y^2 \sim \mathcal{IG}\left(\frac{\underline{\alpha}_0}{2}, \frac{\underline{\beta}_0}{2}\right) \tag{8}$$

which is an inverse gamma, with density $f(\boldsymbol{\theta}_1)$.

Given the conditionally linear and Gaussian dynamics of the latent factor, it is natural to consider a truncated multivariate normal prior for $\boldsymbol{\theta}_2$

$$\boldsymbol{\theta}_2 \sim \mathcal{N}_4\left(\underline{\boldsymbol{m}}_0, \underline{\Sigma}_0\right) \mathbb{I}_{\mathcal{A}_1}(\boldsymbol{\theta}_2) \tag{9}$$

with density $f(\boldsymbol{\theta}_2)$ and parameter constraints $\mathcal{A}_1$ as defined in the previous section. We assume the following independent priors for the elements of $\boldsymbol{\theta}_3$

$$\sigma_{x0}^2 \sim \mathcal{IG}\left(\frac{\underline{\alpha}_1}{2}, \frac{\underline{\beta}_1}{2}\right), \qquad \sigma_{x1}^2 \sim \mathcal{IG}\left(\frac{\underline{\alpha}_2}{2}, \frac{\underline{\beta}_2}{2}\right) \tag{10}$$

with densities $f(\sigma_{xk}^2)$, $k = 0, 1$.

For the elements of $\boldsymbol{\theta}_4$ we consider two independent priors

$$\boldsymbol{\psi}_k \sim \mathcal{N}_{n_{kv}}\left(\underline{\boldsymbol{m}}_{1+k}, \underline{\Sigma}_{1+k}\right) \tag{11}$$

with density $f(\boldsymbol{\psi}_k)$, $k = 0, 1$. Note that ϕ is a precision parameter, which should positive definite, thus we assume an inverse gamma distribution

$$\phi \sim \mathcal{IG}\left(\underline{\alpha}_3, \underline{\beta}_3\right) \tag{12}$$

as a prior for $\boldsymbol{\theta}_5$ and denote with $f(\boldsymbol{\theta}_5)$ its density.

The MCMC estimation procedure, which will be presented in the next section, requires proper prior distributions for the parameters. In the empirical application the hyper-parameters will be set to be nearly noninformative. The standard deviation of the prior will be chosen to have a range of variability of the prior which is greater than the range of variability in the actual parameters. This assumption allows us to have a flat prior in the regions of the parameter space where the likelihood have high values.

More specifically, for the scale parameter priors, the hyper-parameter values $\alpha_0 = 1$, $\beta_0 = 1$, $\underline{\alpha}_k = 1$, $\underline{\beta}_k = 1$, with $k = 1, 2$, are quite standard in business cycle analysis. We assume $\underline{\boldsymbol{m}}_0 = \mathbf{0}_4$ and $\underline{\Sigma}_0 = 100\, I_3$ for the prior on the latent factor parameters. Finally we set $\underline{\boldsymbol{m}}_{1+k} = \mathbf{0}_{n_{kv}}$, $\underline{\Sigma}_{1+k} = 100\, I_3$ for the priors on the coefficients of the stochastic transition and $\underline{\alpha}_3 = 1$ $\underline{\beta}_3 = 1$ for its scale parameter.

3.2 MCMC Algorithm

We follow the data augmentation approach (see Tanner and Wong (1987)) and apply MCMC in order to simulate from the joint posterior distribution of the parameters and latent variables. More specifically, we consider a Gibbs sampling algorithm. Some components of the Gibbs sampler can be simulated exactly while others will be simulated by a Metropolis-Hastings step. The resulting MCMC is an hybrid Gibbs sampler.

The iteration j, with $j = 1, \ldots, J$, of the hybrid Gibbs sampler includes two steps. First we simulate the parameter vector $\boldsymbol{\theta}^{(j)}$ from its full conditional distribution given the values of the latent variables $\mathbf{z}_{0:T}^{(j-1)}$ simulated at the previous step. In order to simulate from the full conditional of the parameter vector, we consider the following partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2', \boldsymbol{\theta}_3', \boldsymbol{\theta}_4', \boldsymbol{\theta}_5')'$, with the blocks of parameters defined in the previous section. Then we simulate from the full conditional distribution of $\boldsymbol{\theta}_i$ given the vector of remaining parameters denoted with $\boldsymbol{\theta}_{-i}$, for $i = 1, \ldots, 5$, i.e.

$$
\begin{aligned}
\boldsymbol{\theta}_1^{(j)} &\sim f(\boldsymbol{\theta}_1 | \boldsymbol{\theta}_2^{(j-1)}, \boldsymbol{\theta}_3^{(j-1)}, \ldots, \boldsymbol{\theta}_5^{(j-1)}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}^{(j-1)}) && (13)\\
\boldsymbol{\theta}_2^{(j)} &\sim f(\boldsymbol{\theta}_2 | \boldsymbol{\theta}_1^{(j)}, \boldsymbol{\theta}_3^{(j-1)}, \ldots, \boldsymbol{\theta}_5^{(j-1)}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}^{(j-1)}) && (14)\\
& \ldots && (15)\\
\boldsymbol{\theta}_5^{(j)} &\sim f(\boldsymbol{\theta}_5 | \boldsymbol{\theta}_1^{(j)}, \boldsymbol{\theta}_2^{(j)}, \ldots, \boldsymbol{\theta}_4^{(j)}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}^{(j-1)}) && (16)
\end{aligned}
$$

In the second step, we simulate the latent variables $\mathbf{z}_{0:T}^{(j)}$ given the updated parameter value $\boldsymbol{\theta}^{(j)}$ as follows

$$\begin{aligned}
\mathbf{x}_{0:T}^{(j)} &\sim f(\mathbf{x}_{0:T}|\mathbf{y}_{1:T},\mathbf{s}_{0:T}^{(j-1)},\mathbf{p}_{00,0:T}^{(j-1)},\mathbf{p}_{11,0:T}^{(j-1)},\boldsymbol{\theta}^{(j)}) && (17)\\
\mathbf{s}_{0:T}^{(j)} &\sim f(\mathbf{s}_{0:T}|\mathbf{y}_{1:T},\mathbf{x}_{0:T}^{(j)},\mathbf{p}_{00,0:T}^{(j-1)},\mathbf{p}_{11,0:T}^{(j-1)},\boldsymbol{\theta}^{(j)}) && (18)\\
\mathbf{p}_{00,0:T}^{(j)} &\sim f(\mathbf{p}_{00,0:T}|\mathbf{y}_{1:T},\mathbf{x}_{0:T}^{(j)},\mathbf{s}_{0:T}^{(j)},\mathbf{p}_{11,0:T}^{(j-1)},\boldsymbol{\theta}^{(j)}) && (19)\\
\mathbf{p}_{11,0:T}^{(j)} &\sim f(\mathbf{p}_{11,0:T}|\mathbf{y}_{1:T},\mathbf{x}_{0:T}^{(j)},\mathbf{s}_{0:T}^{(j)},\mathbf{p}_{00,0:T}^{(j)},\boldsymbol{\theta}^{(j)}) && (20)
\end{aligned}$$

We now present the full conditional distributions of the Gibbs sampler and discuss the sampling methods which will be used to generate values from these distributions.

Define $Y=(y_1,\ldots,y_T)'$ and $V=(x_1,\ldots,x_T)'$. The full conditional posterior distribution of σ_y^2 is

$$\begin{aligned}
&f(\sigma_y^2|\boldsymbol{\theta}_{-1},\mathbf{y}_{1:T},\mathbf{z}_{0:T})\propto\prod_{t=1}^{T}f(y_t|x_t,\boldsymbol{\theta})f(\boldsymbol{\theta}_2)\\
&\propto(\sigma_y^2)^{-T/2}\exp\left\{-\frac{1}{2\sigma_y^2}\left[(Y-V)'(Y-V)\right]\right\}(\sigma_y^2)^{-\underline{\alpha}_0/2-2}\exp\left\{-\frac{\underline{\beta}_0}{2\sigma_y^2}\right\}\\
&\propto(\sigma_y^2)^{-(\underline{\alpha}_0+T)/2-2}\exp\left\{-\frac{1}{2\sigma_y^2}\left[\underline{\beta}_0+(Y-V)'(Y-V)\right]\right\} \qquad (21)
\end{aligned}$$

which is proportional, up to a normalizing constant, to the density of the distribution $\mathcal{IG}\left(\bar{\alpha}_0/2,\bar{\beta}_0/2\right)$ with

$$\bar{\alpha}_0=\underline{\alpha}_0+T,\qquad \bar{\beta}_0=\underline{\beta}_0+\sum_{t=1}^{T}(y_t-x_t)^2 \qquad (22)$$

Thus we can be simulated exactly from the posterior of σ_y^2

Consider V introduced above and define the $(T\times 4)$ matrix $W=(\mathbf{w}_1,\ldots,\mathbf{w}_T)'$, with $\mathbf{w}_t=(1,s_t,x_{t-1},s_tx_{t-1})'$ and the T-dimensional diagonal matrix $\Sigma=\mathrm{diag}\{(\sigma_{x\,s_1}^2,\ldots,\sigma_{x\,s_T}^2)'\}$ then the full conditional of $\boldsymbol{\theta}_2$ can be written as

$$\begin{aligned}
&f(\boldsymbol{\theta}_2|\boldsymbol{\theta}_{-2},\mathbf{y}_{1:T},\mathbf{z}_{0:T})\propto\prod_{t=1}^{T}f(x_t|x_{t-1},s_t,\boldsymbol{\theta})f(x_0|s_0,\boldsymbol{\theta})f(\boldsymbol{\theta}_2)\\
&\propto\exp\left\{-\frac{1}{2}(V-W\boldsymbol{\theta}_2)'\Sigma^{-1}(V-W\boldsymbol{\theta}_2)-\frac{1}{2}(\boldsymbol{\theta}_2-\underline{\mathrm{m}}_2)'\underline{\Sigma}_0^{-1}(\boldsymbol{\theta}_2-\underline{\mathrm{m}}_2)\right\}g(\boldsymbol{\theta}_2)\\
&\propto\exp\left\{-\frac{1}{2}\left[\boldsymbol{\theta}_2'W'\Sigma^{-1}W\boldsymbol{\theta}_2-2\boldsymbol{\theta}_2'W'\Sigma^{-1}V+\boldsymbol{\theta}_2'\underline{\Sigma}_0^{-1}\boldsymbol{\theta}_2-2\boldsymbol{\theta}_2'\underline{\Sigma}_0^{-1}\underline{\boldsymbol{m}}_0\right]\right\}g(\boldsymbol{\theta}_2)\\
&\propto\exp\left\{-\frac{1}{2}\left[(\boldsymbol{\theta}_2-\bar{\boldsymbol{m}}_0)'\bar{\Sigma}_0^{-1}(\boldsymbol{\theta}_2-\bar{\boldsymbol{m}}_0)\right]\right\}g(\boldsymbol{\theta}_2) \qquad (23)
\end{aligned}$$

with $\bar{\mathbf{m}}_0 = \bar{\Sigma}_0(W'\Sigma^{-1}V + \underline{\Sigma}_0^{-1}\underline{\mathrm{m}}_0)$ and $\bar{\Sigma}_0 = (\underline{\Sigma}_0^{-1} + W'\Sigma^{-1}W)^{-1}$. Due to its diagonal structure the matrix Σ can be easily inverted. Note that the prior is not completely conjugate because the posterior is proportional to the density of a normal distribution with proportionality factor which depends on $\boldsymbol{\theta}_2$. Thus we adjust for this factor with a Metropolis-Hastings step. At the j-th step of the Gibbs, given $\boldsymbol{\theta}_2^{(j-1)}$, we simulate $\boldsymbol{\theta}_2^{(*)}$ form the proposal distribution $\mathcal{N}_4(\bar{\mathbf{m}}_0, \bar{\Sigma}_0)$, with and acceptance probability $\varrho = \min\{1, g(\boldsymbol{\theta}_2^{(*)})/g(\boldsymbol{\theta}_2^{(j-1)})\}$.

The full conditional of $\boldsymbol{\theta}_3$ is

$$
\begin{aligned}
f(\boldsymbol{\theta}_3|\boldsymbol{\theta}_{-3}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}) &\propto \prod_{t=1}^{T} f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) f(x_0|s_0, \boldsymbol{\theta}) f(\boldsymbol{\theta}_3) \\
&\propto \prod_{k=0}^{1} \left((\sigma_{x\,k}^2)^{-n_k/2} \exp\left\{ -\sigma_{x\,k}^{-2} \frac{1}{2} \sum_{t=1}^{T} (\mathbb{I}_{\{k\}}(s_t)(x_t - \mu_k - \rho_k x_{t-1})^2) \right\} \right) \\
&\quad \prod_{k=0}^{1} \left((\sigma_{x\,k}^2)^{-\underline{\alpha}_{1+k}/2} \exp\left\{ -\frac{\underline{\beta}_{1+k}}{2} \sigma_{x\,k}^{-2} \right\} \right) g(\boldsymbol{\theta}_3) \\
&\propto \prod_{k=0}^{1} (\sigma_{x\,k}^2)^{-\bar{\alpha}_{1+k}/2} \exp\left\{ -\frac{\bar{\beta}_{1+k}}{2} \sigma_{x\,k}^{-2} \right\} g(\boldsymbol{\theta}_3)
\end{aligned} \tag{24}
$$

with $n_k = \sum_{t=1}^{T} \mathbb{I}_{\{k\}}(s_t)$, $\bar{\alpha}_{1+k} = \underline{\alpha}_{1+k} + n_k$ and $\bar{\beta}_{1+k} = \underline{\beta}_{1+k} + \sum_{t=1}^{T} (\mathbb{I}_{\{k\}}(s_t)(x_t - \mu_k - \rho_k x_{t-1})^2)$. The posterior is proportional to the product of two inverse gamma, with a proportionality factor depending on $\boldsymbol{\theta}_3$. For simulating $\sigma_{x\,k}^{-2}$ we use a Metropolis-Hastings step with proposal $\mathcal{IG}(\bar{\alpha}_{1+k}/2, \bar{\beta}_{1+k}/2)$ and acceptance probability determined as we have done for $\boldsymbol{\theta}_2$.

The full conditional of the elements of $\boldsymbol{\theta}_4$ is

$$
\begin{aligned}
f(\boldsymbol{\psi}_i|\boldsymbol{\theta}_{-4}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}) &\propto \prod_{t=1}^{T} f(p_{ii,t}|p_{ii,t-1}, \boldsymbol{\theta}) f(\boldsymbol{\psi}_i) \\
&\propto \prod_{t=1}^{T} \left(\Gamma(\eta_{it}\phi)\Gamma(\phi(1-\eta_{it}))\right)^{-1} p_{ii,t}^{\phi\eta_{it}-1} (1-p_{ii,t})^{\phi(1-\eta_{it})-1} f(\boldsymbol{\psi}_i) \\
&\propto \prod_{t=1}^{T} \exp\left\{ A_{it} h(\boldsymbol{\psi}_i'\mathbf{v}_{it}) - \log\Gamma(\eta_{it}\phi) - \log\Gamma(\phi(1-\eta_{it})) \right\} f(\boldsymbol{\psi}_i)
\end{aligned} \tag{25}
$$

with $A_{it} = \log((p_{ii,t}/(1-p_{ii,t}))^{\phi})$. Values of $\boldsymbol{\psi}_i$ are generated with the aid of a M.-H. step. One possibility would be to employ a symmetric random walk, but this procedure does not use the local information of the posterior. Instead we obtain a proposal distribution by taking the logarithm of the full conditional,

$g(\boldsymbol{\psi}_i)$, and then calculating a second-order Taylor expansion centered around $\tilde{\boldsymbol{\psi}}_i$

$$g(\boldsymbol{\psi}_i) \approx g(\tilde{\boldsymbol{\psi}}_i) + (\boldsymbol{\psi}_i - \tilde{\boldsymbol{\psi}}_i)'\nabla^{(1)} g(\tilde{\boldsymbol{\psi}}_i) + \frac{1}{2}(\boldsymbol{\psi}_i - \tilde{\boldsymbol{\psi}}_i)'\nabla^{(2)} g(\tilde{\boldsymbol{\psi}}_i)(\boldsymbol{\psi}_i - \tilde{\boldsymbol{\psi}}_i)$$

with the gradient vector and Hessian matrix

$$\begin{aligned}
\nabla^{(1)} g(\boldsymbol{\psi}_i) = & \sum_{t=1}^{T} \left[A_{it} - \phi\Psi^{(0)}(\phi\eta_{it}) + \phi\Psi^{(0)}(\phi(1-\eta_{it}))\right] h^{(1)}(\mathbf{v}_{1t}'\boldsymbol{\psi}_i)\mathbf{v}_{1t} \\
& -\underline{\Sigma}_{1+i}^{-1}(\boldsymbol{\psi}_i - \underline{\mathrm{m}}_{1+i}) \\
\nabla^{(2)} g(\boldsymbol{\psi}_i) = & \sum_{t=1}^{T} \Big\{ \left[A_{it} - \phi\Psi^{(0)}(\phi\eta_{it}) + \phi\Psi^{(0)}(\phi(1-\eta_{it}))\right] h^{(2)}(\mathbf{v}_{1t}'\boldsymbol{\psi}_i) - \\
& \left[\phi^2\Psi^{(1)}(\phi\eta_{it}) + \phi^2\Psi^{(1)}(\phi(1-\eta_{it}))\right] \left(h^{(1)}(\mathbf{v}_{1t}'\boldsymbol{\psi}_i)\right)^2 \Big\} \mathbf{v}_{1t}\mathbf{v}_{1t}' - \underline{\Sigma}_{1+i}^{-1}
\end{aligned}$$

where $h^{(k)}$ is the k-th order derivative of h and $\Psi^{(0)}$ and $\Psi^{(1)}$ are the digamma and the trigamma functions respectively.

If $\tilde{\boldsymbol{\psi}}_i$ is the mode of the full conditional then $\nabla^{(1)} g(\tilde{\boldsymbol{\psi}}_i) = 0$. However, we do not know the mode, thus we evaluate $\tilde{\boldsymbol{\psi}}_i$ by a Newton-Rapson step. Suppose at the iteration j of the algorithm we have an estimate $\tilde{\boldsymbol{\psi}}_i^{(j-1)}$ of the mode, then it is updated as follows

$$\tilde{\boldsymbol{\psi}}_i^{(j)} = \tilde{\boldsymbol{\psi}}_i^{(j-1)} + \Sigma_i^{(j-1)} \nabla^{(1)} g(\tilde{\boldsymbol{\psi}}_i^{(j-1)})$$

where $\Sigma_i^{(j-1)} = -\left(\nabla^{(2)} g(\tilde{\boldsymbol{\psi}}_i^{(j-1)})\right)^{-1}$. On the j-th iteration, the M.-H. (Chib and Greenberg (1995) and Tanner(1993)) generates a candidate $\boldsymbol{\psi}_i^{(*)}$ from a normal distribution with mean $\tilde{\boldsymbol{\psi}}_i^{(j)}$ and variance $\Sigma_i^{(j-1)}$. Then the candidate is accepted with log-probability

$$\begin{aligned}
\min \Bigg\{ & 0, g(\boldsymbol{\psi}_i^{(*)}) - g(\boldsymbol{\psi}_i^{(j-1)}) - \frac{1}{2}(\boldsymbol{\psi}_i^{(j-1)} - \tilde{\boldsymbol{\psi}}_i^{(j)})' \left(\Sigma_i^{(j-1)}\right)^{-1} (\boldsymbol{\psi}_i^{(j-1)} - \tilde{\boldsymbol{\psi}}_i^{(j)}) + \\
& \frac{1}{2}(\boldsymbol{\psi}_i^{(*)} - \tilde{\boldsymbol{\psi}}_i^{(j)})' \left(\Sigma_i^{(j-1)}\right)^{-1} (\boldsymbol{\psi}_i^{(*)} - \tilde{\boldsymbol{\psi}}_i^{(j)}) \Bigg\}
\end{aligned}$$

After an initial, transitory period, the sequences $\tilde{\boldsymbol{\psi}}_i$ and $\Sigma_i^{(j-1)}$ stabilize and do not need to be updated. Thus the computational time for the M.-H. step decreases.

For the parameter ϕ the posterior is

$$
\begin{aligned}
f(\phi|\boldsymbol{\theta}_{-5}, \mathbf{y}_{1:T}, \mathbf{z}_{0:T}) &\propto \prod_{t=1}^{T} f(p_{ii,t}|p_{ii,t-1}, \boldsymbol{\theta}) f(\boldsymbol{\theta}_5) \\
&\propto \prod_{t=1}^{T}\prod_{i=0}^{1} \Gamma(\phi)\left(\Gamma(\eta_{it}\phi)\Gamma(\phi(1-\eta_{it}))\right)^{-1} p_{ii,t}^{\phi\eta_{it}-1}(1-p_{ii,t})^{\phi(1-\eta_{it})-1} f(\boldsymbol{\theta}_5) \\
&\propto \exp\left\{\phi\sum_{t=1}^{T} B_t + l(\phi) - \underline{\beta}_3/\phi - \log(\phi)(\underline{\alpha}_3+1)\right\} \qquad (26)
\end{aligned}
$$

with $B_t = \sum_{i=0}^{1}(\eta_{it}\log(p_{ii,t}) + (1-\eta_{it})\log(1-p_{ii,t}))$ and $l(\phi) = \sum_{t=1}^{T}\sum_{i=0}^{1}[\log\Gamma(\phi) - \log\Gamma(\eta_{it}\phi) - \log\Gamma(\phi(1-\eta_{it}))]$. We apply a M.-H. step with a Gaussian proposal for the transformed parameter $\varphi = \log\phi$ in order to guarantee the positive definiteness. We built the proposal by proceeding in the same fashion we did for the full conditional of $\boldsymbol{\psi}_i$. Let $g_\phi(\phi)$ be the log-posterior and consider the second-order approximation of $g(\varphi) = g_\phi(\exp(\varphi)) + \varphi$ about $\tilde{\varphi}$

$$
g(\varphi) \approx g(\tilde{\varphi}) + (\varphi-\tilde{\varphi})g^{(1)}(\tilde{\varphi}) + \frac{1}{2}g^{(2)}(\tilde{\varphi})(\varphi-\tilde{\varphi})^2 \qquad (27)
$$

The first and second derivatives with respect to ϕ are

$$
\begin{aligned}
g_\phi^{(1)}(\phi) &= \sum_{t=1}^{T} B_t + 2T\Psi^{(0)}(\phi) - \sum_{t=1}^{T}\sum_{i=0}^{1}\left[\Psi^{(0)}(\phi\eta_{it})\eta_{it}\right. \\
&\quad \left. +\Psi^{(0)}(\phi(1-\eta_{it}))(1-\eta_{it})\right] + \underline{\beta}_3/\phi^2 - (\underline{\alpha}_3+1)/\phi \\
g_\phi^{(2)}(\phi) &= 2T\Psi^{(1)}(\phi) - \sum_{t=1}^{T}\sum_{i=0}^{1}\left[\Psi^{(1)}(\phi\eta_{it})\eta_{it}^2\right. \\
&\quad \left. +\Psi^{(1)}(\phi(1-\eta_{it}))(1-\eta_{it})^2\right] - 2\underline{\beta}_3/\phi^3 + (\underline{\alpha}_3+1)/\phi^2
\end{aligned}
$$

thus $g^{(1)}(\varphi) = g_\phi^{(1)}(\exp(\varphi))\exp(\varphi) + 1$ and $g^{(2)}(\varphi) = g_\phi^{(2)}(\exp(\varphi))\exp(2\varphi) + g_\phi^{(1)}(\exp(\varphi))\exp(\varphi)$.

We update the value of the approximated mode $\tilde{\varphi}^{(j)}$ by the recursion

$$
\tilde{\varphi}^{(j)} = \tilde{\varphi}^{(j-1)} + \sigma^{(j-1)}g^{(1)}(\tilde{\varphi}^{(j-1)})
$$

with $\sigma^{(j-1)} = -(g^{(2)}(\tilde{\varphi}^{(j-1)}))^{-1}$. The proposal of the M.-H. steps is a Gaussian distribution with mean $\tilde{\varphi}^{(j)}$ and variance $\sigma^{(j-1)}$. The acceptance probability is determined as done for the sampler of $\boldsymbol{\psi}_i$.

We consider a single-move Gibbs sampler for the latent factor x_t. We found that in our application the sampler performs quite well in terms of mixing. More efficient multi-move Gibbs samplers can be applied (see for example Carter and Köhn (1994)). The full conditional of the latent factor x_t for $t = 1, \dots, T-1$, is

$$\begin{aligned}
& f(x_t|\mathbf{y}_{1:T}, \mathbf{x}_{-t}, \mathbf{s}_{1:T}, \mathbf{p}_{00,1:T}, \mathbf{p}_{11,1:T}, \boldsymbol{\theta}) \propto \\
& \propto f(y_t|x_t, \boldsymbol{\theta}) f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) f(x_{t+1}|x_t, s_{t+1}, \boldsymbol{\theta}) \\
& \propto \exp\Big\{-\frac{1}{2}\Big[x_t^2\left(\sigma_y^{-2} + \sigma_{x\,s_t}^{-2} + \rho_{s_{t+1}}^2 \sigma_{x\,s_{t+1}}^{-2}\right) - 2x_t\left(y_t\sigma_y^{-2} + (\mu_{s_t} + \rho_{s_t} x_{t-1})\sigma_{x\,s_t}^{-2}\right. \\
& \qquad \left.+\rho_{s_{t+1}}(x_{t+1} - \mu_{s_{t+1}})\sigma_{x\,s_{t+1}}^{-2}\right)\Big]\Big\} \qquad (28)
\end{aligned}$$

which is proportional to the density of the normal distribution $\mathcal{N}_1(m_{xt}, \Sigma_{xt})$ with

$$\begin{aligned}
\Sigma_{xt} &= \left(\frac{1}{\sigma_y^2} + \frac{1}{\sigma_{x\,s_t}^2} + \frac{\rho_{s_{t+1}}^2}{\sigma_{x\,s_{t+1}}^2}\right)^{-1} \\
m_{xt} &= \Sigma_{xt}\left(\frac{y_t}{\sigma_y^2} + \frac{\mu_{s_t} + \rho_{s_t} x_{t-1}}{\sigma_{x\,s_t}^2} + \frac{\rho_{s_{t+1}}(x_{t+1} - \mu_{s_{t+1}})}{\sigma_{x\,s_{t+1}}^2}\right)
\end{aligned}$$

The full conditional of x_T is proportional to a Gaussian density with mean m_{xT} and variance Σ_{xT} given by

$$\Sigma_{xT} = \left(\frac{\sigma_y^2 \sigma_{x\,s_T}^2}{\sigma_{x\,s_T}^2 + \sigma_y^2}\right), \qquad m_{xT} = \left(\frac{y_T \sigma_{x\,s_T}^2 + (\mu_{s_T} + \rho_{s_T} x_{T-1})\sigma_y^2}{\sigma_{x\,s_T}^2 + \sigma_y^2}\right)$$

For the initial value x_0 we simulate conditionally on x_1, s_0 and s_1 from $\mathcal{N}(m_0, \sigma_0)$ with $\sigma_0 = (a_{s_0}^{-1} + \sigma_{x\,s_1}^{-2}\rho_{s_1}^2)$ and $m_0 = \sigma_0(b_{s_0}^{-1} a_{s_0} + \sigma_{x\,s_1}^{-2}(x_1 - \mu_{s_1})\rho_{s_1})$ where $a_k = \mu_k/(1-\rho_k)$ and $b_k = \sigma_{x\,k}^2/(1-\rho_k^2)$ are the parameters of the initial distribution of x_0 conditionally on $s_0 = k$.

Due to its diagonal structure, Σ_X can be easily inverted. Note that for the application to the business cycle proposed in Section 4 the simulation method for x_t is computationally feasible. Note however that other sampling techniques such as a block-wise Gibbs sampling or a forward-filtering and backward-sampling algorithm, can be used. See for example Billio, Casarin and Sartore (2007) for a review with applications to business cycle models. These kind of algorithms allows for an efficient sequential sampling from the posterior when the number of observations is high and the hidden states cannot be updated simultaneously.

The full conditional density of $s_1, \dots, s_T$ is

$$f(\mathbf{s}_{1:T}|\mathbf{y}_{1:T}, \mathbf{x}_{0:T}, \mathbf{p}_{00,0:T}, \mathbf{p}_{11,0:T}, \boldsymbol{\theta}) \propto \prod_{t=1}^{T} \omega_t(s_t, s_{t-1}) \qquad (29)$$

with $\omega_t(s_t, s_{t-1}) = f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) f(s_t|s_{t-1}, p_{00,t}, p_{11,t}, \boldsymbol{\theta})$.

In order to generate s_t with $t = 1, \dots, T$ we do not a apply a single-move sampler (see Albert and Chib (1993), but a multi-move sampler as suggested by many authors (see Liu, Wong and Kong (1994) and Carter and Köhn (1994). More specifically we follow Billio, Monfort and Robert (1999) and built a global M.-H. algorithm. At the j-th steps of the Gibbs we simulate $s_t^{(*)}$ for $t = 1, \dots, T$ from the joint proposal

$$g(\mathbf{s}_{1:T}|\mathbf{s}_{1:T}^{(j-1)}) = \prod_{t=1}^{T-1} \frac{\nu_t(s_t, s_{t-1}, s_{t+1}^{(j-1)})}{\sum_{k=0}^{1} \nu_t(k, s_{t-1}, s_{t+1}^{(j-1)})} \frac{\nu_T(s_T, s_{T-1})}{\sum_{k=0}^{1} \nu_T(k, s_{T-1})} \tag{30}$$

with

$$\begin{aligned}\nu_t(s_t, s_{t-1}, s_{t+1}^{(j-1)}) &= f(x_t|x_{t-1}, s_t, \boldsymbol{\theta}) f(s_t|s_{t-1}, p_{00,t}, p_{11,t}, \boldsymbol{\theta}) \cdot \\ &\cdot f(s_{t+1}^{(j-1)}|s_t, p_{00,t+1}, p_{11,t+1}, \boldsymbol{\theta})\end{aligned}$$

and $\nu_T(s_T, s_{T-1}) = f(x_T|x_{T-1}, s_T, \boldsymbol{\theta}) f(s_T|s_{T-1}, p_{00,T}, p_{11,T}, \boldsymbol{\theta})$. Then we accept with probability $\min\{1, \varrho(\mathbf{s}_{1:T}^{(*)}, \mathbf{s}_{1:T}^{(j-1)})\}$ where

$$\begin{aligned}&\varrho(\mathbf{s}_{1:T}^{(*)}, \mathbf{s}_{1:T}^{(j-1)}) = \\ &= \prod_{t=1}^{T-1} \frac{\omega_t(s_t^{(*)}, s_{t-1}^{(*)})}{\omega_t(s_t^{(j-1)}, s_{t-1}^{(j-1)})} \frac{\nu_t(s_t^{(j-1)}, s_{t-1}^{(j-1)}, s_{t+1}^{(j-1)})}{\nu_t(s_t^{(*)}, s_{t-1}^{(*)}, s_{t+1}^{(j-1)})} \frac{\sum_{k=0}^{1} \nu_t(k, s_{t-1}^{(*)}, s_{t+1}^{(j-1)})}{\sum_{k=0}^{1} \nu_t(k, s_{t-1}^{(j-1)}, s_{t+1}^{(j-1)})} \\ &= \prod_{t=1}^{T-1} \frac{\sum_{k=0}^{1} \nu_t(k, s_{t-1}^{(*)}, s_{t+1}^{(j-1)})}{\sum_{k=0}^{1} \nu_t(k, s_{t-1}^{(j-1)}, s_{t+1}^{(j-1)})} \frac{f(s_{t+1}^{(j-1)}|s_t^{(j-1)}, p_{00,t}, p_{11,t}, \boldsymbol{\theta})}{f(s_{t+1}^{(j-1)}|s_t^{(*)}, p_{00,t}, p_{11,t}, \boldsymbol{\theta})}\end{aligned}$$

This M.-H. chain on the path $\mathbf{s}_{1:T}$ has smaller variance than the variance of the component-wise MCMC chain. In the implementation of the algorithm we choose to switch to a single-move Gibbs step when the acceptance rate of the global M.-H. is low. For the initial value s_0 we apply a specific M.-H. step.

The full conditional of $p_{00,t}$ is

$$\begin{aligned}&f(p_{00,t}|\mathbf{y}_{1:T}, \mathbf{x}_{0:T}, \mathbf{s}_{-t}, \mathbf{p}_{00,-t}, \mathbf{p}_{11,0:T}, \boldsymbol{\theta}) \propto \\ &\propto f(s_t|s_{t-1}, p_{00,t}, p_{11,t}, \boldsymbol{\theta}) f(p_{00,t}|p_{00,t-1}, s_t, \boldsymbol{\theta}) f(p_{00,t+1}|p_{00,t}, s_{t+1}, \boldsymbol{\theta}) \\ &\propto p_{00,t}^{\phi\eta_{it}+(1-s_t)(1-s_{t-1})-1} (1-p_{00,t})^{\phi(1-\eta_{it})+s_t(1-s_{t-1})-1} g(p_{00,t}) \mathbb{I}_{]0,1[}(p_{00,t})\end{aligned} \tag{31}$$

In order to simulate from the posterior we employ a M.-H. algorithm. At the j-th iteration, let $p_{00,t}^{(j-1)}$ be the previous value of the M.-H. chain, then we simulate $p_{00,t}^{(*)}$ from the proposal $\mathcal{B}e(\alpha_t, \beta_t)$, with $\alpha_t = \phi\eta_{it} + (1-s_t)(1-s_{t-1})$ and $\beta_t = \phi(1-\eta_{it}) + s_t(1-s_{t-1})$ and set $p_{00,t}^{(j)} = p_{00,t}^{(*)}$ with probability

$$\min\left\{1, g(p_{00,t}^{(*)})/g(p_{00,t}^{(j-1)})\right\}$$

where $g(p_{00,t}) = (p_{00,t}/(1-p_{00,t}))^{\phi\eta_{it}}$. We proceed in a similar way for simulating from the posterior of the transition probability $p_{11,t}$. We design a specific M.-H. steps for the initial values $p_{ii,0}$, $i = 0, 1$.

3.3 SMC Estimates

The MS models considered in this works have the following probabilistic *state-space representation* (Harrison and West (1997) and Doucet et al. (2001))

$$\begin{aligned} \mathbf{y}_t &\sim p(\mathbf{y}_t|\mathbf{z}_t, \mathbf{y}_{1:t-1}, \boldsymbol{\theta}) &(32)\\ \mathbf{z}_t &\sim p(\mathbf{z}_t|\mathbf{z}_{t-1}, \mathbf{y}_{1:t-1}, \boldsymbol{\theta}) &(33)\\ (\mathbf{z}_0, \boldsymbol{\theta}) &\sim p(\mathbf{z}_0|\boldsymbol{\theta})p(\boldsymbol{\theta}), &(34)\end{aligned}$$

with $t = 1, \ldots, T$. In this representation $p(\mathbf{y}_t|\mathbf{z}_t, \mathbf{y}_{1:t-1}, \boldsymbol{\theta})$ and $p(\mathbf{z}_t|\mathbf{z}_{t-1}, \mathbf{y}_{1:t-1}, \boldsymbol{\theta})$ are the *measurement* and *transition* densities respectively. The densities $p(\mathbf{z}_0|\boldsymbol{\theta})$ and $p(\boldsymbol{\theta})$ are the priors on initial state and parameters. The time index in the transition and measurement densities indicates that they could possibly depend on a set of exogenous variables.

In this work we deal with a sequential inference problem. We aim to estimate the parameters and the latent variables of the model (32)-(34) sequentially over time. Following Berzuini et al. (1997) we include the parameters $\boldsymbol{\theta}$ into the state vector and then apply a nonlinear filtering procedure defined on the augmented state space.

Let $\delta_x(y)$ be the Dirac's mass centered in x and let us introduce the following dynamics for the parameter vector: $\boldsymbol{\theta}_t \sim \delta_{\boldsymbol{\theta}_{t-1}}(\boldsymbol{\theta}_t)$, with initial condition $\boldsymbol{\theta}_0 = \boldsymbol{\theta}$ a.s. and include the parameter $\boldsymbol{\theta}_t$ into the hidden states. We define the augmented state vector $\boldsymbol{\xi}_t = (\mathbf{z}_t', \boldsymbol{\theta}_t')'$ and the augmented state space $\Xi = \mathcal{Z} \times \Theta$.

Assume that the density $p(\boldsymbol{\xi}_t|\mathbf{y}_{1:t})$ is known at time t. Note that if $t = 0$ the density $p(\boldsymbol{\xi}_0|\mathbf{y}_0) = p(\mathbf{z}_0|\boldsymbol{\theta}_0)p(\boldsymbol{\theta}_0)$ is the initial distribution of the model in Eq. (32)-(34). The following recursions

$$\begin{aligned} p(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t}) &= \int_{\Xi} p(\mathbf{z}_{t+1}|\mathbf{z}_t, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1})\delta_{\boldsymbol{\theta}_t}(\boldsymbol{\theta}_{t+1})p(\boldsymbol{\xi}_t|\mathbf{y}_{1:t})d\boldsymbol{\xi}_t &(35)\\ p(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t+1}) &= \frac{p(\mathbf{y}_{t+1}|\mathbf{z}_{t+1}, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1})p(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t})}{p(\mathbf{y}_{t+1}|\mathbf{y}_{1:t})} &(36)\end{aligned}$$

define the states *prediction* and *filtering* densities respectively.

When the model is nonlinear and non-Gaussian these recursions can be solved by employing Sequential Monte Carlo methods (see for example Arulampalam, Maskell, Gordon and Clapp (2001) and Doucet et al. (2000)). In this work we apply a special class of SMC algorithms called regularised particle filters, which have been introduced by Musso, Oudjane and LeGland (2001) and Liu and West (2001). Casarin and Marin (2009) compare different particle filters, within the class of the kernel-regularised filters, and find that the regularised APF outperforms regularised SIR and SIS when the unknown parameters of the model are estimated sequentially. Thus in what follows we will consider the regularised APF approach.

Assume that at the initial time step a weighted random sample (*particle set*) $\mathcal{S}^N_{0|0} = \{\boldsymbol{\xi}^i_0, w^i_0\}^N_{i=1}$ is approximating the prior density and that at time t a weighted sample $\mathcal{S}^N_{t|t} = \{\boldsymbol{\xi}^i_t, w^i_t\}^N_{i=1}$, is approximating the filtering density as follow

$$\hat{p}_N(\boldsymbol{\xi}_t|\mathbf{y}_{1:t}) = \sum_{i=1}^{N} \delta_{\boldsymbol{\xi}^i_t}(\boldsymbol{\xi}_t) w^i_t$$

The element $\boldsymbol{\xi}^i_t$ of the sample is called *particle* and the particles set, $\mathcal{S}^N_{t|t}$, can be viewed as a random discretisation of the state space Ξ at time t, with associated probability weights w^i_t. At the time step $t+1$, as a new observation $\mathbf{y}_{t+1}$ arrives, we can approximate Eq. (35)-(36) as follows

$$\hat{p}_N(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t}) = \sum_{i=1}^{N} p(\mathbf{z}_{t+1}|\mathbf{z}^i_t, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1}) \delta_{\boldsymbol{\theta}^i_t}(\boldsymbol{\theta}_{t+1}) w^i_t \tag{37}$$

$$\hat{p}_N(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t+1}) \propto \sum_{i=1}^{N} p(\mathbf{y}_{t+1}|\mathbf{z}_{t+1}, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1}) p(\mathbf{z}_{t+1}|\mathbf{z}^i_t, \mathbf{y}_{1:t}, \boldsymbol{\theta}^i_t) \delta_{\boldsymbol{\theta}^i_t}(\boldsymbol{\theta}_{t+1}) w^i_t \tag{38}$$

which are called state and observable *empirical prediction densities* and *empirical filtering density* respectively.

In the regularised APF algorithms the filtering density in Eq. (38) is approximated through a weighted kernel density estimator

$$\hat{p}_N(\boldsymbol{\xi}_{t+1}|\mathbf{y}_{1:t+1}) \approx \frac{1}{N} \sum_{i=1}^{N} \omega(\boldsymbol{\xi}_{t+1}) \mathcal{N}_{n_{\boldsymbol{\theta}}}(\boldsymbol{\theta}_{t+1}|\mathbf{m}^i_t, b^2 V_t), \tag{39}$$

where the weights are $\omega(\boldsymbol{\xi}_{t+1}) = w^i_t p(\mathbf{y}_{t+1}|\mathbf{z}_{t+1}, \mathbf{y}_{1:t}, \boldsymbol{\theta}^i_t) p(\mathbf{z}_{t+1}|\mathbf{z}^i_t, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1})$ and the parameters of the Gaussian distribution are $\mathbf{m}^i_t = a\boldsymbol{\theta}^i_t + (1-a)\bar{\boldsymbol{\theta}}_t$ and $V_t = \sum_{i=1}^N (\boldsymbol{\theta}^i_t - \bar{\boldsymbol{\theta}}_t)(\boldsymbol{\theta}^i_t - \bar{\boldsymbol{\theta}}_t)' w^i_t$, with $\bar{\boldsymbol{\theta}}_t = \sum_{i=1}^N \boldsymbol{\theta}^i_t w^i_t$, $a \in [0,1]$ and $b^2 = (1-a^2)$.

A new random set $\mathcal{S}^N_{t+1|t+1}$ from the regularised filtering density at time $t+1$, can be obtained by the following two steps. First jointly simulate the random index i^j (*selection step*) and the particle value $\boldsymbol{\xi}^j_{t+1}$ (*mutation step*), with $j = 1, \ldots, N$, from

$$q(\boldsymbol{\xi}^j_{t+1}, i^j|\mathbf{y}_{1:t+1}) \;\; \propto \;\; p(\mathbf{z}^j_{t+1}|\mathbf{z}^{i^j}_t, \mathbf{y}_{1:t}, \boldsymbol{\theta}^j_{t+1}) \mathcal{N}_{n_{\boldsymbol{\theta}}}(\boldsymbol{\theta}^j_{t+1}|\mathbf{m}^{i^j}_t, b^2 V_t) q(i^j|\mathbf{y}_{1:t+1})$$

with $q(i^j|\mathbf{y}_{1:t+1}) = p(\mathbf{y}_{t+1}|\mu_{t+1}^{ij}, \mathbf{y}_{1:t}, \mathbf{m}_t^{ij})w_t^{ij}$. Secondly apply an importance sampling argument to the kernel density estimator and evaluate the following weights

$$w_{t+1}^j \propto \frac{p(\mathbf{y}_{t+1}|\mathbf{z}_{t+1}^j, \mathbf{y}_{1:t}, \boldsymbol{\theta}_{t+1}^j)}{p(\mathbf{y}_{t+1}|\mu_{t+1}^{ij}, \mathbf{y}_{1:t}, \mathbf{m}_t^{ij})} \tag{40}$$

3.4 Hypothesis Testing

Assume that we are interested in a test for the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} \in \Theta_0$ against the alternative $\mathcal{H}_1 : \boldsymbol{\theta} \in \Theta_1$. In a Bayesian framework this corresponds to the evaluation of the Bayes factor (see (Robert 2001)), that is the ratio of the posterior probabilities of the null and the alternative hypotheses over the ratio of the prior of the null and the alternative hypotheses. Let

$$p(\mathbf{y}_{1:T}|\boldsymbol{\theta}) = \int_{\mathcal{X}^T} p(\mathbf{y}_{1:T}, \mathbf{z}_{1:T}|\boldsymbol{\theta})d\mathbf{z}_{1:T}$$

then the Bayes factor is,

$$B_{01}^{\pi,T} = \frac{\int_{\Theta_0} p(\mathbf{y}_{1:T}|\boldsymbol{\theta})\pi_0(\boldsymbol{\theta})d\boldsymbol{\theta}}{\int_{\Theta_1} p(\mathbf{y}_{1:T}|\boldsymbol{\theta})\pi_1(\boldsymbol{\theta})d\boldsymbol{\theta}} = \frac{m_0(\mathbf{y}_{1:T})}{m_1(\mathbf{y}_{1:T})} \tag{41}$$

with $m_0(\mathbf{y}_{1:T})$ and $m_1(\mathbf{y}_{1:T})$ the marginals under the null and the alternative hypotheses respectively. The test of hypothesis requires to run two MCMC chains, ones under the null hypothesis and the other under the alternative hypothesis. The two set of simulated values allow the approximation of all the integrals involved in the Bayes factor.

Finally we will apply the Jeffreys' scale to judge the evidence in favor of or against the null brought by the data: $\log_{10}(B_{10}^{\pi,T})$ between 0 and 0.5, then the evidence against the null is poor, if between 0.5 and 1, it is substantial, between 1 and 2, it is strong and above 2 is decisive.

4 An Application to the Euro-zone Business Cycle

4.1 Data

We consider monthly observations from January 1970 to May 2009 of the Industrial Production Index (IPI) of the Euro area. In order to get the IPI at the Euro zone level a back-recalculation has been performed (see Anas et al. (2007a, 2007b) and Caporin and Sartore (2006) for details). The ST-MS model

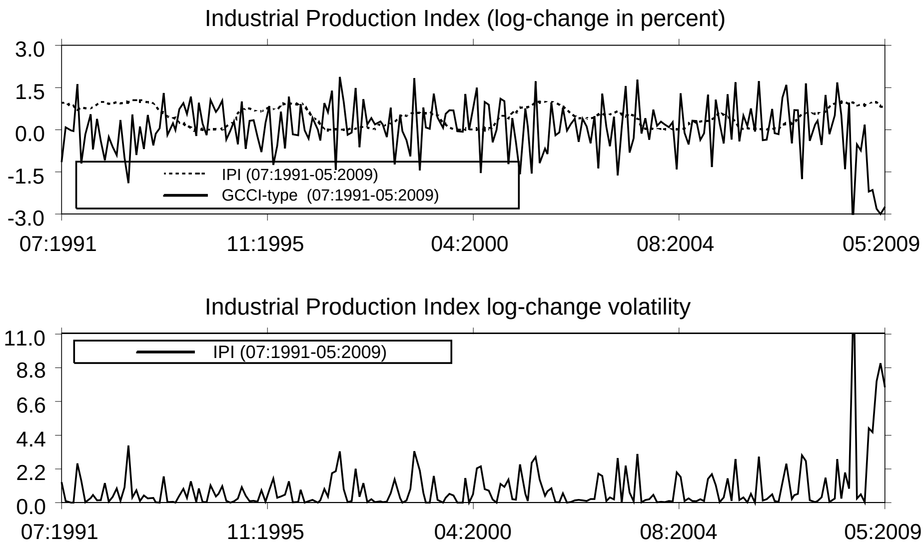

Figure 3: Up: log-change in percent of the European Industrial Production Index (IPI) and the GCCI-type index at the monthly frequency for the period: July 1991 to May 2009. Bottom: square of the IPI log-change.

has been applied to the log-change of the IPI index (upper chart of Fig. 3). The presence of time-varying volatility (bottom chart of Fig. 3) suggests that the model should account for different regimes in the volatility level.

The exogenous variables $\mathbf{v}_{it}$, $i = 1, 2$, which are driving the transition probabilities of the ST-MS model, are the constant term, an autoregressive component and a growth cycle coincident indicator (GCCI). For the construction of the coincident see Anas et al. (2008).

4.2 MCMC Estimation Results

Figures 4 and 5 graph the output of 5,000 MCMC iterations. For each parameter the graphs show the raw output of the MCMC chain (grey lines) and the ergodic averages (black lines). Consider for example the parameters $\boldsymbol{\psi}_i$, $i = 0, 1$ and ϕ of the stochastic transition, which are the most difficult to estimate due to the analytical form of the posterior density. From a graphical inspection of the M.-H. outputs for these parameters, it seems that the proposals, based on the second-order approximation of the log-posterior, are quite efficient allowing the M.-H. chains to explore the space and then to stabilize quickly after an initial burn-in period. The progressive average of the acceptance rate over the iterations of these two M.-H. steps is given in the last row of Fig. 5. At the last iteration the average acceptance rates are between 0.4 and 0.5.

The initial, transitory period, which is more evident in the ergodic averages for the different Gibbs components, is due to the initialization of the algorithm. After some iterations the MCMC chain is then converging to the stationary distribution. We try different starting points for the MCMC chain and verify that they result in similar final estimates and convergence behaviour. An initial sample of 1,000 values will be excluded from the MCMC sample when calculating the posterior means, the standard deviations and the quantiles (see Tab.1). We choose the size of the initial sample on the basis of a graphical inspection of the progressive averages over the MCMC iterations, but the a more rigorous method could be used. Even if convergence diagnostics for MCMC remains an open question (see Robert and Casella (1999)), the problems of the choice of the size of the initial sample and of the convergence detection of the Gibbs sampler may be assessed by using for example the convergence diagnostic (CD) statistics proposed in Geweke (1992). Let $n = 5,000$ be the MCMC sample size and $n_1 = 1000$, and $n_2 = 3000$ the sizes of two non-overlapping sub-samples. For the parameter θ, let

$$\hat{\theta}_1 = \frac{1}{n_1}\sum_{j=1}^{n_1} \theta^{(j)}, \quad \hat{\theta}_2 = \frac{1}{n_2}\sum_{j=n+1-n_2}^{n} \theta^{(j)}$$

be the MCMC sample mean and $\hat{\sigma}_i^2$ their variances estimated with the non-parametric estimator

$$\frac{\hat{\sigma}_i^2}{n_i} = \hat{\Gamma}(0) + \frac{2n_i}{n_i - 1}\sum_{j=1}^{h_i} K(j/h_i)\hat{\Gamma}(j),$$

$$\hat{\Gamma}(j) = \frac{1}{n_i}\sum_{k=j+1}^{n_i} (\theta^{(k)} - \hat{\theta}_i)(\theta^{(k-j)} - \hat{\theta}_i)$$

where we choose $K(x)$ to be the Parzen kernel (see Kim and Nelson (1999a)) and $h_1 = 100$ and $h_2 = 500$ are the bandwidths. Then the following statistics

$$CD = \frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\hat{\sigma}_1^2/n_1 + \hat{\sigma}_2^2/n_2}} \tag{42}$$

converges in distribution to a standard normal (see Geweke (1992)), under the null hypothesis that the MCMC chain has converged.

As indicated in Table 1 the means of the rate of log-change in the IPI index during the contraction and expansion phases are $\hat{\mu}_0 = -0.2425$ and $\hat{\mu}_1 = 0.2674$. The HPD region of the parameter d_ρ does not include zero, thus we conclude that the difference between the contraction and expansion rates of growth is significantly different from zero. The speed of reversion and the variance are $\hat{\rho}_0 = 0.1909$ and $\sigma^2_{x0} = 0.1179$ during the contraction phase and $\hat{\rho} = 0.1721$ and $\sigma^2_{x1} = 0.0885$ during the expansion phase. Note however that the HPD region of d_ρ contains zero thus the difference between the speed of recession and contraction is not significant. The HPD regions of σ^2_{x0} and σ^2_{x1} overlap, thus we should test the hypothesis that the two parameters are not significantly different. In particular we test the null hypothesis $\mathcal{H}_0 : \sigma^2_{x0} = \sigma^2_{x1}$ against the alternative $\mathcal{H}_1 : \sigma^2_{x0} \neq \sigma^2_{x1}$. The log-Bayes factor on the whole sample is $\log BF^{\pi,T}_{10} = 0.4201$, thus the evidence against the null is poor.

The estimates of the parameters of the stochastic-transition put on evidence that the constant component of the probability of the two regimes are $h(\hat{\psi}_{00}) = 0.8415$ and $h(\hat{\psi}_{01}) = 0.9255$ respectively, where h is the logistic function defined in Section 2. The estimates of the persistence parameters, $\hat{\psi}_{10} = 0.2609$ and $\hat{\psi}_{11} = 0.5073$, indicate that the MS transition probability has a significant autoregressive dynamics (the HPD regions of the two parameters do not include the zero). As expected the effect of the GCCI-type indicator is positive for the probability of staying in recession ($\hat{\psi}_{20} = 0.7144$) and negative for the probability of staying in expansion ($\hat{\psi}_{21} = -1.0384$).

4.3 Sequential Model Comparison

In this section we apply SMC to evaluate the ability of the model ST-MS to capture different features of the cycle. To this aim we compare the ST-MS with two competing models, i.e. the constant transition (CT-MS) model with transition probabilities

$$s_t|s_{t-1} \sim \mathbb{P}\left(s_t = j|s_{t-1} = i\right) = p_{ij}, \quad \text{with } i,j \in \{0,1\}. \tag{43}$$

and the dynamic transition (DT-MS) model with transition probabilities

$$s_t|s_{t-1} \sim \mathbb{P}_t\left(s_t = j|s_{t-1} = i\right) = p_{ij,t} \tag{44}$$

and then use the reference cycle in Anas et al. (2007b) as a benchmark.

In order to apply the PF to the ST-MS we consider the following monotonic transformation of the parameter vector: $\theta_t = (\ \log(\sigma_{yt}),\ \mu_{0t},\ \log(\mu_{1t}),\ \log((1+\rho_{0t})/(1-\rho_{0t})),\ \log((1+\rho_{1t})/(1-\rho_{1t})),\ \log(\sigma_{x0\,t}),\ \log(\sigma_{x1\,t}),\ \boldsymbol{\psi}'_{0t},\ \boldsymbol{\psi}'_{1t}\ \log(\phi_t),)$, which allows us to introduce some necessary constraints on the parameters space. Then we include θ_t into the state vector and apply the regularised APF given in Section 3 to filter the hidden states and sequentially estimate the parameters. In our applications we set $a = (3\delta-1)(2\delta)^{-1}$ and $b^2 = (1-a^2)$. From our simulation experiments we choose $\delta = 0.99$ and $N = 5,000$, in order to simultaneously

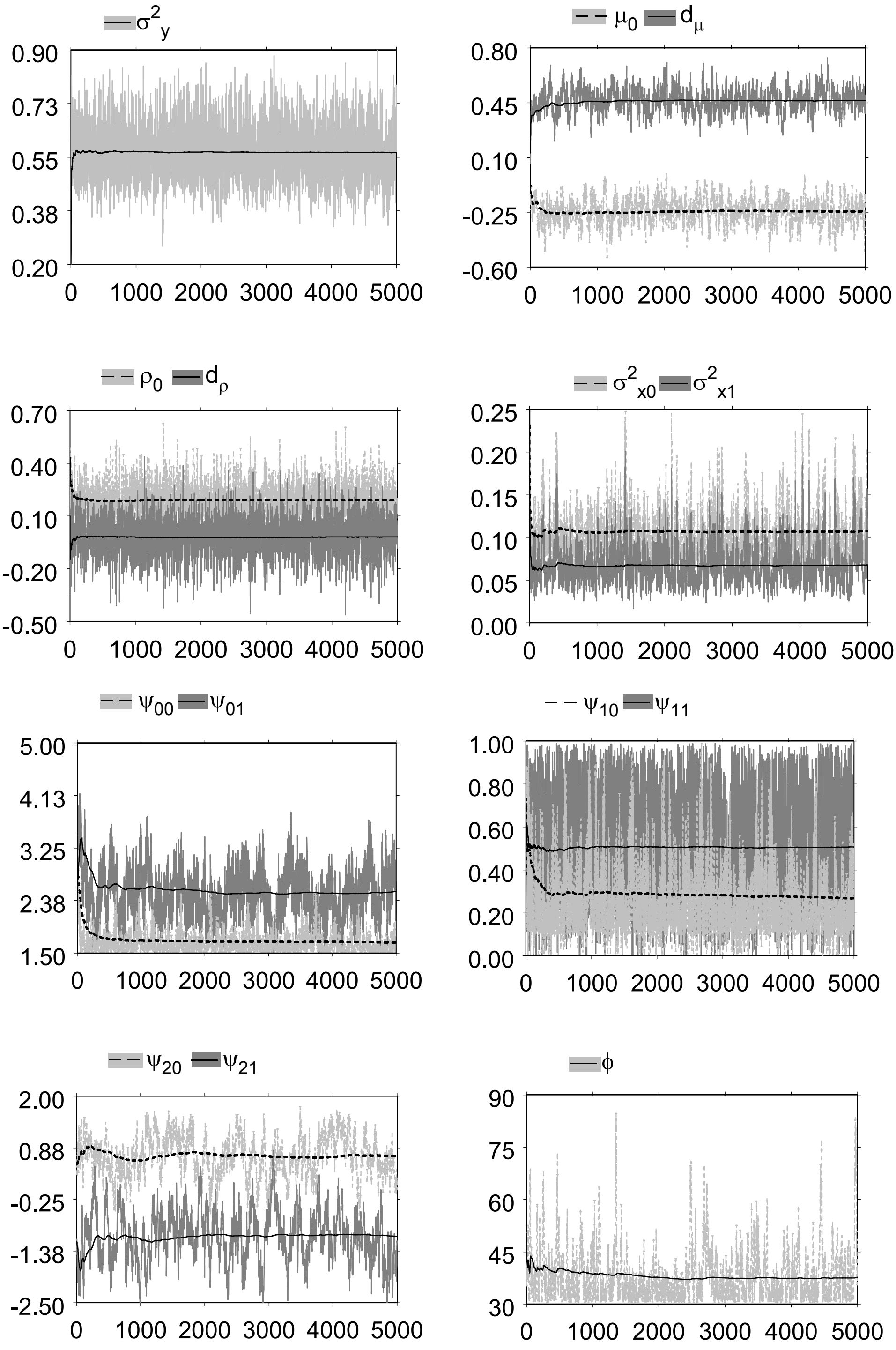

Figure 4: Raw MCMC output (grey lines) and ergodic averages (black lines) over the 5,000 MCMC iterations for the parameters σ_y^2, μ_0, d_μ, ρ_0, d_ρ, σ_{x0}^2, σ_{x1}^2, $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_1$ and ϕ of the stochastic transition model.

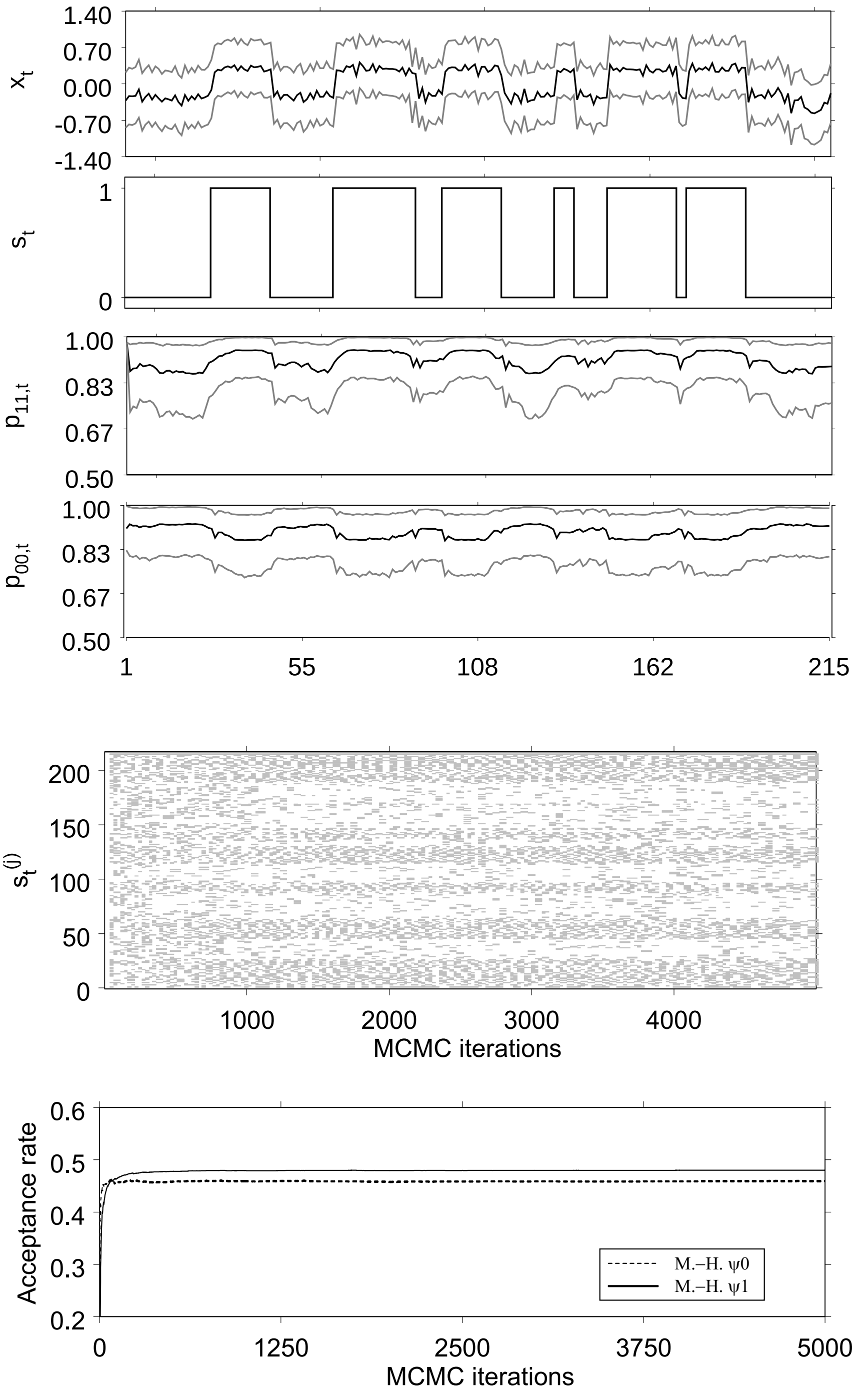

Figure 5: Ergodic averages (black lines) and quantiles (grey lines) over the 5,000 MCMC iterations for the latent processes x_t (first graph), s_t (second graph), p_{00t} and $p_{11,t}$ (third and fourth graphs). MCMC raw output for the regime switching process s_t (fifth graph) and progressive estimate of the M.-H. acceptance rate for the parameters $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_1$.

θ	$\hat{\theta}_T$	$q_{0.025}$	$q_{0.975}$	s.d.	CD
σ_y	0.5640	0.4065	0.7448	0.0866	0.0448
μ_0	-0.2425	-0.4077	-0.0864	0.0802	-0.0021
d_μ	0.5099	0.2949	0.7211	0.1084	-0.0458
ρ_0	0.1909	0.0031	0.3833	0.1779	-0.0160
d_ρ	-0.0188	-0.2373	0.1685	0.1002	-0.0047
σ^2_{x0}	0.1179	0.0396	0.1454	0.0268	-0.0564
σ^2_{x1}	0.0885	0.0349	0.1275	0.0241	-0.0653
ψ_{00}	1.6697	1.5053	2.1064	0.1588	0.0863
ψ_{10}	0.2609	0.1042	0.6636	0.1491	0.1164
ψ_{20}	0.7144	-0.1874	1.5079	0.4566	-0.0443
ψ_{01}	2.5190	1.6886	3.4467	0.4386	-0.0223
ψ_{11}	0.5073	0.1165	0.9582	0.2534	-0.0356
ψ_{21}	-1.0384	-2.1452	-0.0130	0.5441	0.0184
ϕ	36.9830	30.1593	58.3173	7.4694	-0.0037

Table 1: First column: estimated parameters for the ST-MS model for the log-change of the Euro Industrial Production Index. Other columns: parameter estimates, 0.025 and 0.975 quantiles, standard deviations (s.d.) and convergence diagnostic statistics (CD). The statistics have been obtained by iterating 5,000 times the Gibbs sampler and then discarding the first 1,000 iterations to have a MCMC sample from the stationary distribution.

minimize the parameter estimation bias, due to the regularization step, and avoid the degeneracy problem. We initialized the particle filter with a properly weighted sample (see Casarin and Marin (2009)) obtained by running the Gibbs sampler given in Section 3 on an initial set of observations.

In order to asses the degree of similarity between the three models we consider a set of indicators, which are widely used in the literature on business cycle analysis. First we employ the *concordance statistic* (C) for regular periodic behavior in the business cycles proposed by Harding and Pagan (2002). Let $\tilde{s}_{it} = \mathbb{I}_{]0.5,1]}(\hat{p}_t)$ be the filtered regime at time t, with $\tilde{p}_{it} = \sum_{j=1}^{N} w_t^j \mathbb{I}_{\{0\}}(s_t^j)$, for the three models: CT-MS ($i = 1$), DT-MS ($i = 2$) and ST-MS ($i = 3$). Then the concordance statistics measures the proportion of time during which two series $\tilde{s}_{it}$ and $\tilde{s}_{jt}$, are in the same state. The degree of concordance is then

$$C_T^{ij} = \frac{1}{T}\left\{\sum_{t=1}^{T}(\tilde{s}_{it}\tilde{s}_{jt}) + (1-\tilde{s}_{it})(1-\tilde{s}_{jt})\right\} \tag{45}$$

where T is the sample size. This measure ranges between 0 and 1, with 0 representing perfectly counter-cyclical switches, and 1 perfectly synchronous

	AF	CT-MS	DT-MS	ST-MS
AF	1	0.4261	**0.6150**	**0.6223**
CT-MS		1	0.3693	0.3681
DT-MS			1	**0.8750**
ST-MS				1

	$\beta = 0.5$		
Statistics	AF vs CT-MS	AF vs DT-MS	AF vs ST-MS
$\text{QPS}_T(\beta)$	0.0057	0.0057	0.0057
$\text{TPS}_T(\beta)$	0.0078	0.0071	0.0063
$\text{CGoF}_T(\beta)$	0.0057	0.0057	0.0057
$\text{RC}_T(\beta)$	0.0056	0.0057	0.0057

Table 2: Up: degree of concordance $\{C_T^{ij}\}$ between the filtered business cycle phases from our models: CT-MS, DT-MS and ST-MS and the cycle estimated in Anas et al. (2007b) (AF) for the sample period July 1991-February 2006. Bottom: concordance level measured with QPS, TPS, CGoF and RC statistics for the three models and for $\beta = 0.5$ with the reference cycle given in Anas et al. (2007b).

shifts. For two regimes described by random walks, the measure will be 0.5 in the limit.

We evaluate other criteria based on the following indicator

$$I_t = ((1 - \tilde{p}_{it}) - \tilde{p}_{it}) \tag{46}$$

The indicator I_t is in the $[-1, 1]$ interval. It is close to 1 when the economy is in a recession phase and close to 1 in a expansion phase. Given a threshold $\beta \in [0, 1]$, it is possible to define the following decision rule. We will say that the economy is a recession phase if $I_t \in [-1, -\beta[$ and in an expansion phase if $I_t \in [\beta, 1]$. The threshold β can be estimated empirically and take generally values in the $[0.3, 0.5]$ interval.

The first criteria is the the *Quadratic Probabilistic Score* (QPS) proposed in Brier (1950).

$$\text{QPS}_T(\beta) = \frac{1}{T} \sum_{t=1}^{T} \left(\mathbb{I}_{\{I_t < \beta\}} - r_t \right)^2 \tag{47}$$

with $\mathbb{I}_A$ the indicator function, which is 1 if $I_t < \beta$ and 0 otherwise and r_t the reference cycle given in Anas et al. (2007b). This criterion suffers from the drawbacks that two non-correlated variables may exhibit a high value of QPS if their persistence is strong (Harding and Pagan, 2006). Thus Darne and Ferrara

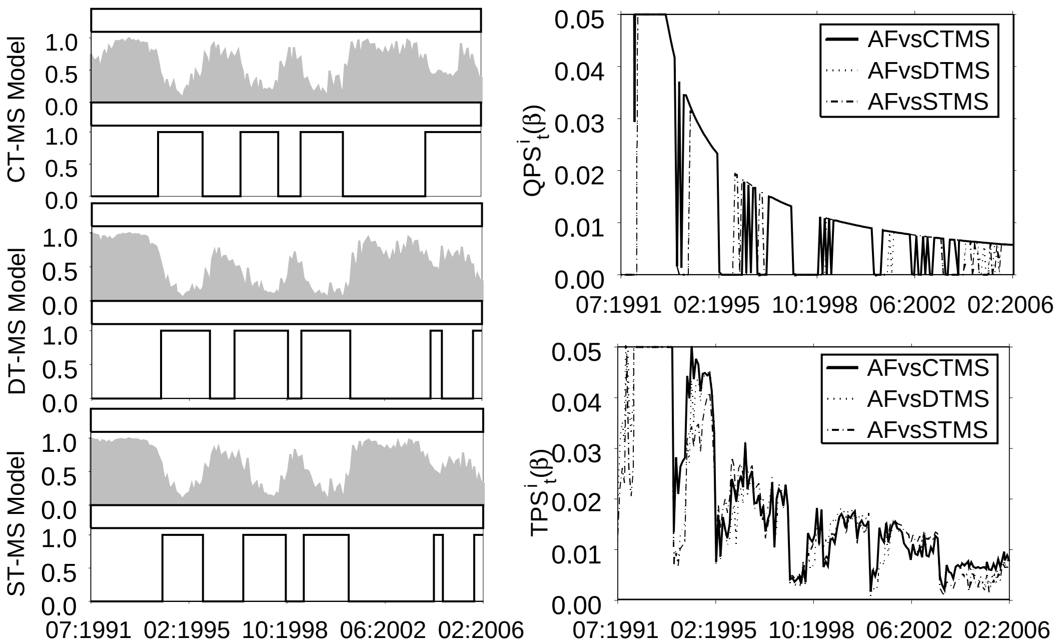

Figure 6: Left Column: Sequential estimate of the recession probability in the Euro area (*upper charts*) and sequentially filtered regimes (*bottom charts*, $\tilde{s}_t$) for model the three MS models. Right Column: Sequential evaluation of $QPS_t^i(\beta) = \frac{1}{t}\sum_{k=1}^{t}(\mathbb{I}_{\{I_{ik}<\beta\}} - r_k)^2$ and $TPS_t^i(\beta) = \frac{1}{t}\sum_{k=1}^{t}[1 + (2r_k - 1)(\arctan(I_{ik}\beta)/\arctan(\beta))]$ for $\beta = 0.5$, $t = 1,\ldots,T$, I_{ik} the recession indicator resulting from $\mathcal{M}_i$ and r_k the reference cycle in Anas et al. (2007b) (AF).

(2009) propose the *Cyclical Goodness of Fit* (CGoF) criterion, defined as

$$\mathrm{CGoF}_T(\beta) = \frac{1}{T}\sum_{t=1}^{T}\left[1 + (2r_t - 1)(\mathbb{I}_{\{I_t>\beta\}} - \mathbb{I}_{\{I_t<-\beta\}})\right]. \tag{48}$$

Another indicator is the *readability criterion* (RC)

$$\mathrm{RC}_T(\beta) = \frac{1}{T}\sum_{t=1}^{T}\mathbb{I}_{\{-\beta\leq I_t\leq\beta\}}. \tag{49}$$

In the regime between expansion and recession phases the signal is difficult to interpret. Therefore, a readable indicator counts how many times the signal stays in the intermediate zone. Note that QPS and RC are based on step-wise transform of the signal I_t which associate a loss equal to 2 to the large errors (out of the interval $[-\beta,\beta]$) and 1 for errors in the interval $[-\beta,\beta]$. It is also possible to introduce a continuously *transformed probabilistic score* (TPS)

$$\text{TPS}_T(\beta) = \frac{1}{T}\sum_{t=1}^{T}\left[1 + (2r_t - 1)\frac{\arctan(\beta I_t)}{\arctan(\beta)}\right], \tag{50}$$

with $\beta \in [0, +\infty[$. In this indicator a continuous nonlinear function associates a loss level between 2 and 0 to all the errors in the $[-1, 1]$ interval.

The concordance and QPS, TPS, CGoF and RC statistics allow us to conclude that each one of the three models captures different features of the recession phases, when compared to the AF's cycle. In particular the left column of Fig. 6 exhibits the sequentially filtered regimes. The outputs of the three models differ in terms of numbers of turning points detected in the business cycle and in terms of phases duration. The concordance and QPS statistics, over the period July 1991 - February 2006, indicates that the regime changes detected with the ST-MS are similar to the shifts in Anas et al. (2007b) (AF) and have a lower concordance, below 0.4, with the regimes changes detected with model CT-MS. The ST-MS and DT-MS have a high degree of concordance (above 0.6) with the reference cycle. The QPS and CGoF statistics (right column Fig. 6 and Tab. 2) bring us to conclude that the three models seem to be equivalent (CGoF=0.0057, QPS=0.0057), but the TPS, which considers a continuous weighting function for all the errors in the $[-1, 1]$ interval, indicates that the output of the ST-MS model (TPS=0.0063) is more similar to the reference cycle than the regime changes detected with the CT-MS (TPS=0.0078) and DT-MS (TPS=0.0071) models. Fig. 7 evidences the differences between the three models in detecting the beginning of the last recession period in the sample.

5 Conclusion

We propose a new class of Markov-switching latent factor models with stochastic transition probabilities. This class of models can account for time variation and randomness in the duration of the different regimes. The proposed parameterization has been employed in the context of inference on beta mixture and on beta regression modelling and allows a straightforward interpretation of the model parameters. We suggest an inference procedure based on the Bayesian paradigm and propose a MCMC estimation procedure. Finally, we apply the stochastic transition model and the MCMC estimation framework to the data of the Euro-zone business cycle and compare the results with exiting datations.

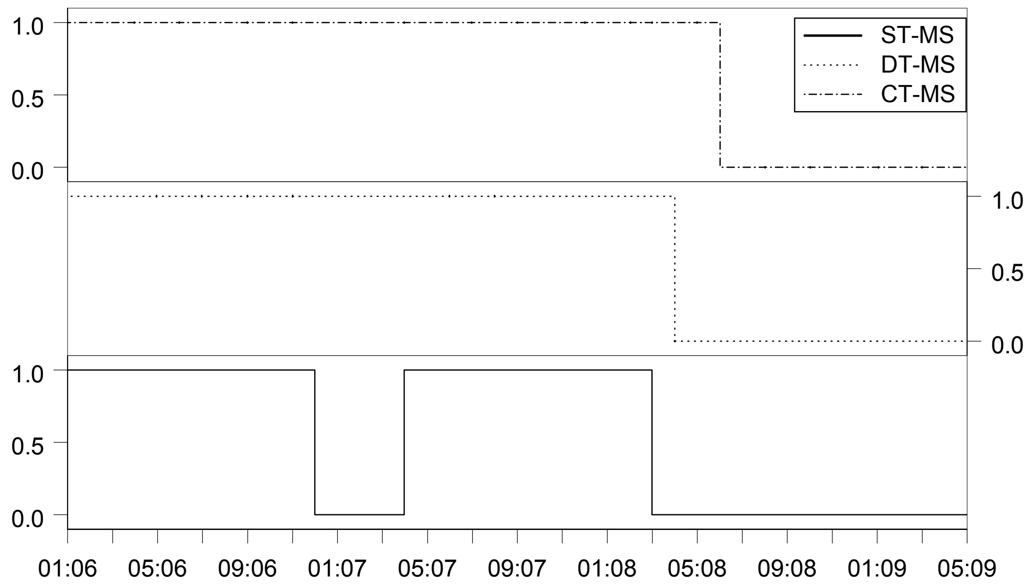

Figure 7: Sequential estimate of the recession periods (i.e. $s_t = 0$), with the CT-MS, DT-MS and ST-MS models, for the Euro area during the period January 2007 - May 2009.

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