

ON GORENSTEIN WEAK INJECTIVE MODULES

TIWEI ZHAO AND YUNGE XU

ABSTRACT. In this paper, we introduce the notion of Gorenstein weak injective modules in terms of weak injective modules and characterize rings over which all modules are Gorenstein weak injective. Moreover, we discuss the relations between weak cosyzygy and Gorenstein weak cosyzygy of a module. In addition, we discuss the stability of Gorenstein weak injective modules.

1. INTRODUCTION

Throughout R is an associative ring with identity and all modules are unitary. Unless stated otherwise, an R -module will be understood to be a left R -module. As usual, $pd_R(M)$ and $id_R(M)$ will denote the projective and injective dimensions of an R -module M , respectively. For unexplained concepts and notations, we refer the readers to [7, 19].

In 1970, to generalize the homological properties from Noetherian rings to coherent rings, Stenström introduced the notion of FP-injective modules in [20]. In this process, finitely generated modules should in general be replaced by finitely presented modules. Recall that an R -module M is called FP-injective if $\text{Ext}_R^1(N, M) = 0$ for any finitely presented R -module N , and accordingly, the FP-injective dimension of M , denoted by $FP-id_R(M)$, is defined to be the smallest non-negative integer n such that $\text{Ext}_R^{n+1}(N, M) = 0$ for any finitely presented R -module N . Recently, as an extending work of Stenström's viewpoint, Gao and Wang introduced the notion of weak injective modules ([14]). In this process, finitely presented modules were replaced by super finitely presented modules (see [13] or Sec. 2 for the definition). It was shown that many results of a homological nature may be generalized from coherent rings to arbitrary rings (see [12, 14] for details).

In 1965, Eilenberg and Moore first introduced the viewpoint of relative homological algebra in [8]. Since then the relative homological algebra, especially the Gorenstein homological algebra, got a rapid development. Nowadays, it has been developed to an advanced level (e.g. [3, 4, 6, 7, 15, 16, 17, 21]). However, in the most results of Gorenstein homological algebra, the condition 'noetherian' is essential. In order to make the similar properties of Gorenstein homological algebra hold in a wider environment, Ding and his coauthors introduced the notions of Gorenstein FP-injective and strongly Gorenstein flat

2010 *Mathematics Subject Classification.* Primary 16E65; Secondary 16E10, 16E05.

Key words and phrases. weak injective module, Gorenstein weak injective module, Gorenstein weak cosyzygy, two-degree Gorenstein weak injective module.

modules (see [5, 18] for details). Later on, Gillespie renamed Gorenstein FP-injective modules as Ding injective, and strongly Gorenstein flat modules as Ding projective ([11]). However, under their definition these Gorenstein FP-injective modules are stronger than the Gorenstein injective modules (see [7, Def. 10.1.1] for the definition), and an FP-injective module is not necessarily Gorenstein FP-injective in general. To solve this question, Gao investigated another class of Gorenstein FP-injective modules in [9, 10] which extended the class of FP-injective modules, that is, an R -module M is called *Gorenstein FP-injective* if there exists an exact sequence of FP-injective R -modules

$$\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

such that $M = \text{Coker}(E_1 \rightarrow E_0)$ and the functor $\text{Hom}_R(P, -)$ leaves this sequence exact whenever P is a finitely presented R -module with $pd_R(P) < \infty$. Under Gao's definition these Gorenstein FP-injective modules are weaker than the Gorenstein injective modules, and every FP-injective module is Gorenstein FP-injective. However, the condition 'coherent' is essential in his paper. So, in order to make the properties of this class of Gorenstein FP-injective modules hold over any ring, it seems that we have to replace FP-injective modules by weak injective modules.

In Section 2, we first introduce the notion of Gorenstein weak injective modules in terms of weak injective modules and give some basic properties. Then we characterize rings over which all modules are Gorenstein weak injective and others. Finally, we discuss the relations between weak cosyzygy and Gorenstein weak cosyzygy of a module. In Section 3, we mainly discuss the stability of Gorenstein weak injective modules.

2. GORENSTEIN WEAK INJECTIVE MODULES

In this section, we give the definition of Gorenstein weak injective modules and discuss some of the properties of these modules. We first recall some terminologies and preliminaries. For more details, we refer the readers to [7, 12, 14].

Definition 2.1. ([7]) Let \mathcal{F} be a class of R -modules. By an \mathcal{F} -preenvelope of an R -module M , we mean a morphism $\varphi : M \rightarrow F$ where $F \in \mathcal{F}$ such that for any morphism $f : M \rightarrow F'$ with $F' \in \mathcal{F}$, there exists a morphism $g : F \rightarrow F'$ such that $g\varphi = f$, that is, there is the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & F \\ & \searrow f & \downarrow g \\ & & F' \end{array}$$

If furthermore, when $F' = F$ and $f = \varphi$, the only such g are automorphisms of F , then $\varphi : M \rightarrow F$ is called an \mathcal{F} -envelope of M .

Dually, one may give the notion of \mathcal{F} -(pre)cover of an R -module. Note that \mathcal{F} -envelopes and \mathcal{F} -covers may not exist in general, but if they exist, they are unique up to isomorphism.

In the process that some results of a homological nature may be generalized from coherent rings to arbitrary rings, the notion of super finitely presented modules plays a crucial role. Recall from [13] that an R -module M is called *super finitely presented* if there exists an exact sequence $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$, where each F_i is finitely generated and projective. Then Gao and Wang gave the definition of weak injective modules in terms of super finitely presented modules in [14], which is a generalization of the notion of FP-injective modules.

Definition 2.2. ([14]) An R -module M is called *weak injective* if $\text{Ext}_R^1(N, M) = 0$ for any super finitely presented R -module N .

We denote by $\mathcal{WI}(\mathcal{R})$ the class of all weak injective R -modules. By [12, Thm. 3.4], every R -module has a weak injective preenvelope. So for any R -module M , M has a right $\mathcal{WI}(\mathcal{R})$ -resolution, that is, there exists a $\text{Hom}_R(-, \mathcal{WI}(\mathcal{R}))$ -exact complex

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots,$$

where each E^i is weak injective. Moreover, since every injective R -module is weak injective, this complex is also exact.

Now we give the notion of Gorenstein weak injective modules in terms of weak injective modules as follows.

Definition 2.3. An R -module M is called *Gorenstein weak injective* if there exists an exact sequence of weak injective R -modules

$$\mathbf{W} = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

such that $M = \text{Coker}(W_1 \rightarrow W_0)$ and the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$.

Remark 2.4. (1) Every weak injective R -module is Gorenstein weak injective.

(2) Since every FP-injective R -module is weak injective, every Gorenstein FP-injective R -module (in sense of Gao's definition) is Gorenstein weak injective. If R is a left coherent ring, then the class of Gorenstein weak injective R -modules coincides with the class of Gorenstein FP-injective R -modules. Moreover, we have the following implications by [9, Prop. 2.5]:

$$\begin{aligned} \text{Gorenstein injective } R\text{-modules} &\Rightarrow \text{Gorenstein FP-injective } R\text{-modules} \\ &\Rightarrow \text{Gorenstein weak injective } R\text{-modules.} \end{aligned}$$

If R is an n -Gorenstein ring (i.e. a left and right Noetherian ring with self injective dimension at most n on both sides for some non-negative integer n), then these three kinds of R -modules coincide.

(3) The class of Gorenstein weak injective R -modules is closed under direct products.

(4) If $\mathbf{W} = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$ is an exact sequence of weak injective R -modules such that the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$, then by symmetry, all the images, the kernels and the cokernels of \mathbf{W} are Gorenstein weak injective.

Proposition 2.5. *Let M be a Gorenstein weak injective R -module. Then $\text{Ext}_R^i(N, M) = 0$ whenever N is a super finitely presented R -module with $pd_R(N) < \infty$ and $i \geq 1$.*

Proof. By Definition 2.3, there exists an exact sequence $0 \rightarrow M \rightarrow W^0 \rightarrow M^1 \rightarrow 0$ with W^0 weak injective and M^1 Gorenstein weak injective, such that the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$. Moreover, consider the following exact sequence

$$0 \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, W^0) \rightarrow \text{Hom}_R(N, M^1) \rightarrow \text{Ext}_R^1(N, M) \rightarrow \text{Ext}_R^1(N, W^0).$$

Since W^0 is weak injective, we have $\text{Ext}_R^1(N, W^0) = 0$, and hence $\text{Ext}_R^1(N, M) = 0$. Since M^1 is Gorenstein weak injective, we also have $\text{Ext}_R^1(N, M^1) = 0$. Consider the following exact sequence

$$0 = \text{Ext}_R^1(N, W^0) \rightarrow \text{Ext}_R^1(N, M^1) \rightarrow \text{Ext}_R^2(N, M) \rightarrow \text{Ext}_R^2(N, W^0).$$

Note that $\text{Ext}_R^2(N, W^0) = 0$ by [14, Prop. 3.1], and hence $\text{Ext}_R^2(N, M) \cong \text{Ext}_R^1(N, M^1) = 0$. We repeat the argument by replacing M^1 with M to get a Gorenstein weak injective R -module M^2 and the isomorphisms $\text{Ext}_R^3(N, M) \cong \text{Ext}_R^2(N, M^1) \cong \text{Ext}_R^1(N, M^2) = 0$. Continue this process, we may obtain a Gorenstein weak injective R -module M^{i-1} and the isomorphisms $\text{Ext}_R^i(N, M) \cong \text{Ext}_R^{i-1}(N, M^1) \cong \cdots \cong \text{Ext}_R^1(N, M^{i-1}) = 0$ for $i \geq 1$. \square

The following proposition shows that we may simplify the definition of Gorenstein weak injective R -modules.

Proposition 2.6. *An R -module M is Gorenstein weak injective if and only if there exists an exact sequence of weak injective R -modules*

$$\mathbf{W} = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

such that $M = \text{Coker}(W_1 \rightarrow W_0)$.

Proof. \Rightarrow It is trivial.

\Leftarrow By the definition of Gorenstein weak injective R -modules, it suffices to show that the complex $\text{Hom}_R(N, \mathbf{W})$ is exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$. We use induction on $n = pd_R(N) < \infty$. The case $n = 0$ is trivial. Let

$n \geq 1$, and assume that the result holds for the case $n - 1$. Consider an exact sequence $0 \rightarrow K \rightarrow P_0 \rightarrow N \rightarrow 0$, where P_0 is finitely generated projective and K is super finitely presented. Then $pd_R(K) = n - 1$. Since each term of \mathbf{W} is weak injective, we may get the following exact sequence of complexes

$$0 \rightarrow \text{Hom}_R(N, \mathbf{W}) \rightarrow \text{Hom}_R(P_0, \mathbf{W}) \rightarrow \text{Hom}_R(K, \mathbf{W}) \rightarrow 0.$$

Clearly, the complex $\text{Hom}_R(P_0, \mathbf{W})$ is exact. Moreover, the complex $\text{Hom}_R(K, \mathbf{W})$ is also exact by the induction hypothesis. So the complex $\text{Hom}_R(N, \mathbf{W})$ is exact, and hence M is Gorenstein weak injective. \square

Corollary 2.7. *Let M be an R -module. Then the following are equivalent:*

- (1) M is Gorenstein weak injective;
- (2) There is an exact sequence $\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0$, where each W_i is weak injective;
- (3) There is an exact sequence $0 \rightarrow L \rightarrow W \rightarrow M \rightarrow 0$, where W is weak injective and L is Gorenstein weak injective.

Proof. (1) \Rightarrow (3) It is trivial.

(3) \Rightarrow (2) Since L is Gorenstein weak injective, by Definition 2.3, there is an exact sequence $\cdots \rightarrow W_2 \rightarrow W_1 \rightarrow L \rightarrow 0$, where each W_i is weak injective. Assembling this sequence with the sequence given in (3), we have the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & \searrow & & \nearrow & & \\ & & & & & & & L & & & \\ & & & & \nearrow & & \searrow & & & & \\ & & & & 0 & & & & & & 0 \end{array}$$

Let $W_0 = W$, then (2) holds.

(2) \Rightarrow (1) Since every R -module has a weak injective preenvelope by [12, Thm. 3.4], we may easily get an exact sequence $0 \rightarrow M \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$, where each W^i is weak injective. Assembling this sequence with the sequence given in (2), we get the following exact sequence

$$\mathbf{W} = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

such that $M = \text{Coker}(W_1 \rightarrow W_0)$. By Proposition 2.6, M is Gorenstein weak injective. \square

Definition 2.8. The *Gorenstein weak injective dimension* of an R -module M , denoted by $Gwid_R(M)$, is defined as $\inf\{n \mid \text{there is an exact sequence } 0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow 0 \text{ with } G^i \text{ Gorenstein weak injective for any } 0 \leq i \leq n\}$. If no such n exists, set $Gwid_R(M) = \infty$.

Proposition 2.9. *Given an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with M weak injective. If L is Gorenstein weak injective, then so is N . Otherwise, $Gwid_R(L) \leq Gwid_R(N) + 1$.*

Proof. The first assertion follows from Corollary 2.7. Assume that $Gwid_R(N) = n < \infty$. Then, by Definition 2.8, there exists an exact sequence

$$0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow 0$$

with G^i Gorenstein weak injective for any $0 \leq i \leq n$. Assembling this sequence with the exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, we may easily get that $Gwid_R(L) \leq n + 1 = Gwid_R(N) + 1$. \square

Proposition 2.10. *Given an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with L weak injective. If N is Gorenstein weak injective, then so is M .*

Proof. Assume that N is Gorenstein weak injective. Then, by Corollary 2.7, there exists an exact sequence $0 \rightarrow K \rightarrow W \rightarrow N \rightarrow 0$, where K is Gorenstein weak injective and W is weak injective. Consider the following pull-back diagram:

$$(1) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & K & = & K & \\ & & & \downarrow & & \downarrow & \\ 0 & \rightarrow & L & \rightarrow & W' & \rightarrow & W \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & L & \rightarrow & M & \rightarrow & N \rightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since L and W are weak injective, it follows from the middle row in the diagram (1) that W' is weak injective. By the middle column in the diagram (1) and Corollary 2.7, we have that M is Gorenstein weak injective. \square

We now give a characterization for rings whose every module is Gorenstein weak injective as follows.

Proposition 2.11. *The following are equivalent:*

- (1) *Every R -module is Gorenstein weak injective;*
- (2) *Every projective R -module is weak injective.*

Proof. (1) \Rightarrow (2) Let P be a projective R -module. Then P is Gorenstein weak injective by hypothesis. So there exists an exact sequence $0 \rightarrow K \rightarrow W \rightarrow P \rightarrow 0$, where W is weak injective. It is obvious that this sequence is split, and hence P is weak injective as a direct summand of W .

(2) \Rightarrow (1) Let M be any R -module. If every projective R -module is weak injective, then by assembling a projective resolution of M with its weak injective resolution, we may get the following exact sequence of weak injective R -modules

$$\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

such that $M = \text{Coker}(W_1 \rightarrow W_0)$. Thus M is Gorenstein weak injective by Proposition 2.6. \square

Corollary 2.12. *Let R be a left noetherian ring. Then R is quasi-Frobenius if and only if every R -module is Gorenstein weak injective.*

Proof. If R is quasi-Frobenius, then every projective R -module is injective. Consequently, every projective R -module is weak injective, and the result holds by Proposition 2.11. The converse follows from Proposition 2.11 and the fact that the injective R -modules coincide with the weak injective R -modules over a left noetherian ring R . \square

Let M be an R -module. Recall from [14] that the *weak injective dimension* of M , denoted by $\text{wid}_R(M)$, is defined to be the smallest non-negative integer n such that $\text{Ext}_R^{n+1}(N, M) = 0$ for any super finitely presented R -module N . Accordingly, we define the *left global weak injective dimension* of a ring R as

$$\ell.\text{wiD}(R) = \sup\{\text{wid}_R(M) \mid M \text{ is any } R\text{-module}\}.$$

Proposition 2.13. *Every weak injective R -module is Gorenstein weak injective with equivalent if $\ell.\text{wiD}(R) < \infty$.*

Proof. Assume that $\ell.\text{wiD}(R) = n < \infty$ and let M be a Gorenstein weak injective R -module. The case $n = 0$ is trivial. Let $n \geq 1$. Since M is Gorenstein weak injective, there is an exact sequence

$$\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0$$

with each W_i weak injective. Let $K_n = \text{Ker}(W_{n-1} \rightarrow W_{n-2})$. Then we get an exact sequence

$$0 \rightarrow K_n \rightarrow W_{n-1} \rightarrow \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0.$$

By hypothesis, $\text{wid}_R(K_n) \leq n$, and hence M is weak injective by [14, Prop. 3.3]. \square

The next proposition gives a description of rings over which all Gorenstein weak injective R -modules are weak injective.

Proposition 2.14. *The following are equivalent:*

- (1) *Every Gorenstein weak injective R -module is weak injective;*
- (2) *For any R -module M , $G\text{wid}_R(M) = \text{wid}_R(M)$.*

Proof. (1) \Rightarrow (2) Let M be an R -module. Since every weak injective R -module is Gorenstein weak injective, it is obvious that $G\text{wid}_R(M) \leq \text{wid}_R(M)$. So it suffices to show that $\text{wid}_R(M) \leq G\text{wid}_R(M)$. Without loss of generality, we assume that $G\text{wid}_R(M) = n < \infty$ for some non-negative integer n . Then, by Definition 2.8, there is an exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow 0$ with G^i Gorenstein weak injective for

any $0 \leq i \leq n$. Note that each G^i is weak injective by hypothesis. Thus, $\text{wid}_R(M) \leq n = \text{Gwid}_R(M)$ by [14, Prop. 3.3], as desired.

(2) \Rightarrow (1) It is trivial. \square

Consider the following exact sequence

$$0 \longrightarrow M \xrightarrow{d^0} W^0 \xrightarrow{d^1} W^1 \longrightarrow \dots,$$

where each W^i is weak injective. Let $V^i = \text{Coker}d^{i-1}$ for any $i \geq 1$. Then we call V^i the i -th *weak cosyzygy* of M .

Similarly, if each W^i is Gorenstein weak injective in the above sequence, then we call V^i the i -th *Gorenstein weak cosyzygy* of M .

As what Huang and his coauthor have done in [17], we investigate the relations between weak cosyzygy and Gorenstein weak cosyzygy of an R -module as follows.

Since every weak injective R -module is Gorenstein weak injective, it is obvious that every i -th weak cosyzygy of an R -module M is the i -th Gorenstein weak cosyzygy of M . The following theorem shows that the converse holds to some extent.

Definition 2.15. A ring R is called *GW*-closed if the class of Gorenstein weak injective R -modules is closed under extensions.

Theorem 2.16. *Let R be a GW-closed ring, n a positive integer and V^n an n -th Gorenstein weak cosyzygy of an R -module M . Then V^n is an n -th weak cosyzygy of some R -module N , and there is an exact sequence $0 \rightarrow G \rightarrow N \rightarrow M \rightarrow 0$, where G is Gorenstein weak injective.*

Proof. We use induction on n . For the case $n = 1$, there is an exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow V^1 \rightarrow 0$ with G^0 Gorenstein weak injective. By the definition of Gorenstein weak injective R -modules, there is an exact sequence $0 \rightarrow G \rightarrow W^0 \rightarrow G^0 \rightarrow 0$ with W^0 weak injective and G Gorenstein weak injective.

Consider the following pull-back diagram:

$$(2) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G & \xlongequal{\quad} & G & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & W^0 & \longrightarrow & V^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & G^0 & \longrightarrow & V^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

It follows from the middle row in the diagram (2) that V^1 is the first weak cosyzygy of an R -module N . Moreover, we get the desired exact sequence $0 \rightarrow G \rightarrow N \rightarrow M \rightarrow 0$ from the second column in the diagram (2).

Now let $n \geq 2$ and suppose that the result holds for the case $n - 1$. Let V^n be an n -th Gorenstein weak cosyzygy of M . Then we have the following exact sequence

$$0 \longrightarrow M \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \dots \longrightarrow G^{n-1} \longrightarrow V^n \longrightarrow 0 ,$$

where each G^i is Gorenstein weak injective. Since G^{n-1} is Gorenstein weak injective, there is an exact $0 \rightarrow G' \rightarrow W^{n-1} \rightarrow G^{n-1} \rightarrow 0$ with G' Gorenstein weak injective and W^{n-1} weak injective.

Consider the following pull-back diagrams:

$$(3) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G' & \xlongequal{\quad} & G' & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V' & \longrightarrow & W^{n-1} & \longrightarrow & V^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & V^{n-1} & \longrightarrow & G^{n-1} & \longrightarrow & V^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

and

$$(4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G' & \xlongequal{\quad} & G' & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V^{n-2} & \longrightarrow & G'' & \longrightarrow & V' \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V^{n-2} & \longrightarrow & G^{n-2} & \longrightarrow & V^{n-1} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where $V^{n-i} = \text{Coker}(G^{n-i-2} \rightarrow G^{n-i-1})$ for $i = 1, 2$. For the exact sequence $0 \rightarrow G' \rightarrow G'' \rightarrow G^{n-2} \rightarrow 0$ in the diagram (4), since G' and G^{n-2} are Gorenstein weak injective, G'' is also Gorenstein weak injective by hypothesis. Moreover, by the middle row in the diagram (4), we have that V' is an $(n - 1)$ -st Gorenstein weak cosyzygy of M . Thus V' is the $(n - 1)$ -st weak cosyzygy of some R -module N by the induction hypothesis. In addition, by assembling the middle row in the diagram (3), we may get that V^n is the n -th weak cosyzygy of N , as desired. \square

3. THE STABILITY OF GORENSTEIN WEAK INJECTIVE R -MODULES

In this section, we mainly consider the stability of Gorenstein weak injective R -modules. We denote by $\mathcal{GWI}(\mathcal{R})$ the class of Gorenstein weak injective R -modules. The stability of Gorenstein categories was first considered by Sather-Wagstaff et al. in [21] and further

investigated by many authors, see [1, 2, 22, 23, 24, 25]. They considered a class of R -modules of the form $\text{Im}(G_0 \rightarrow G^0)$ for some exact sequence of Gorenstein projective R -modules (see [7, Def. 10.2.1] for the definition)

$$\mathbf{G} = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$$

such that the complexes $\text{Hom}_R(\mathbf{G}, H)$ and $\text{Hom}_R(H, \mathbf{G})$ (or, only $\text{Hom}_R(\mathbf{G}, H)$) are exact for any Gorenstein projective R -module H , and proved that these modules are nothing but Gorenstein projective R -modules.

Inspired by the above, we begin with the following question.

Question. Given an exact sequence of Gorenstein weak injective R -modules

$$\mathbf{G} = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$$

such that $M = \text{Coker}(G_1 \rightarrow G_0)$ and the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$, is M Gorenstein weak injective?

We call an R -module M defined as in the above question *two-degree Gorenstein weak injective*, and denote by $\mathcal{GWI}^2(\mathcal{R})$ the class of two-degree Gorenstein weak injective R -modules. It is obvious that there is containment $\mathcal{GWI}(\mathcal{R}) \subseteq \mathcal{GWI}^2(\mathcal{R})$.

In the following, we show that the answer to the above question is affirmative over GWI -closed rings.

Theorem 3.1. *Let R be a GWI -closed ring. Then $\mathcal{GWI}(\mathcal{R}) = \mathcal{GWI}^2(\mathcal{R})$.*

Before giving the proof of Theorem 3.1, we need the following preliminaries.

Definition 3.2. An R -module M is called *strongly two-degree Gorenstein weak injective* if there exists an exact sequence

$$\cdots \rightarrow G \xrightarrow{d} G \xrightarrow{d} G \xrightarrow{d} G \rightarrow \cdots,$$

where G is Gorenstein weak injective, such that $M = \text{Coker}d$ and the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$.

We denote by $\mathcal{SGWI}^2(\mathcal{R})$ the class of strongly two-degree Gorenstein weak injective R -modules. It is obvious that there is containment $\mathcal{SGWI}^2(\mathcal{R}) \subseteq \mathcal{GWI}^2(\mathcal{R})$.

Lemma 3.3. *Let M be an R -module. Then the following are equivalent:*

- (1) M is strongly two-degree Gorenstein weak injective;
- (2) There exists an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$, where G is Gorenstein weak injective, such that the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$;

(3) *There exists an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$, where G is Gorenstein weak injective, such that $\text{Ext}_R^1(N, M) = 0$ whenever N is a super finitely presented R -module with $\text{pd}_R(N) < \infty$.*

Proof. (1) \Rightarrow (2) It follows immediately from Definition 3.2.

(2) \Rightarrow (3) Let $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ be an exact sequence with G Gorenstein weak injective, applying the functor $\text{Hom}_R(N, -)$ with N super finitely presented to it, we have the following exact sequence

$$\cdots \rightarrow \text{Hom}_R(N, G) \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Ext}_R^1(N, M) \rightarrow \text{Ext}_R^1(N, G) \rightarrow \cdots$$

By Proposition 2.5, $\text{Ext}_R^1(N, G) = 0$. Moreover, the sequence $0 \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, G) \rightarrow \text{Hom}_R(N, M) \rightarrow 0$ is exact by hypothesis. Thus we have $\text{Ext}_R^1(N, M) = 0$.

(3) \Rightarrow (1) We first obtain the following commutative diagram from the exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ in (3):

$$\mathbf{G} = \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \swarrow & & \searrow & \\ & & & M & & & \\ & & & \swarrow & & \searrow & \\ \cdots & \longrightarrow & G & \longrightarrow & G & \longrightarrow & \cdots \\ & & \swarrow & & \searrow & & \\ & & M & & M & & \\ & & \swarrow & & \searrow & & \\ & & 0 & & 0 & & 0 \end{array}$$

Moreover, it is easy to get an exact sequence $0 \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, G) \rightarrow \text{Hom}_R(N, M) \rightarrow 0$ by hypothesis, and hence we have the following commutative diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \swarrow & & \searrow & \\ & & & \text{Hom}_R(N, M) & & \text{Hom}_R(N, M) & \\ & & & \swarrow & & \searrow & \\ \cdots & \longrightarrow & \text{Hom}_R(N, G) & \longrightarrow & \text{Hom}_R(N, G) & \longrightarrow & \cdots \\ & & \swarrow & & \searrow & & \\ & & \text{Hom}_R(N, M) & & \text{Hom}_R(N, M) & & \\ & & \swarrow & & \searrow & & \\ & & 0 & & 0 & & 0 \end{array}$$

It follows then that the complex $\text{Hom}_R(N, \mathbf{G})$ is exact whenever N is a super finitely presented R -module with $\text{pd}_R(N) < \infty$, and thus M is strongly two-degree Gorenstein weak injective. \square

Proposition 3.4. *Let M be an R -module. If M is two-degree Gorenstein weak injective, then it is a direct summand of some strongly two-degree Gorenstein weak injective R -module.*

Proof. Since M is two-degree Gorenstein weak injective, there exists an exact sequence of Gorenstein weak injective R -modules

$$\mathbf{G} = \cdots \rightarrow G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \rightarrow \cdots$$

such that $M = \text{Coker}d_1$ and the functor $\text{Hom}_R(N, -)$ leaves this sequence exact whenever N is a super finitely presented R -module with $pd_R(N) < \infty$. For all $m \in \mathbb{Z}$, we denote by $\Sigma^m \mathbf{G}$ the exact sequence obtained from \mathbf{G} by increasing all indexes by m : $(\Sigma^m \mathbf{G})_i = G_{i-m}$ and $d_i^{\Sigma^m \mathbf{G}} = d_{i-m}$ for all $i \in \mathbb{Z}$. Then we get the following exact sequence

$$\bigoplus_{m \in \mathbb{Z}} (\Sigma^m \mathbf{G}) = \cdots \rightarrow \bigoplus G_i \xrightarrow{\oplus d_i} \bigoplus G_i \xrightarrow{\oplus d_i} \bigoplus G_i \xrightarrow{\oplus d_i} \bigoplus G_i \rightarrow \cdots$$

Since $\text{Coker}(\oplus d_i) \cong \oplus \text{Coker}d_i$, M is a direct summand of $\text{Coker}(\oplus d_i)$. Moreover, from the isomorphism $\text{Hom}_R(N, \bigoplus_{m \in \mathbb{Z}} (\Sigma^m \mathbf{G})) \cong \prod_{m \in \mathbb{Z}} \text{Hom}_R(N, \Sigma^m \mathbf{G})$, we have that $\text{Coker}(\oplus d_i)$ is a strongly two-degree Gorenstein weak injective R -module, as desired. \square

Lemma 3.5. *Let R be a GWI-closed ring. Then the class of Gorenstein weak injective R -modules is closed under direct summands.*

Proof. Let $M \cong L \oplus N$ be a Gorenstein weak injective R -module. Then there exists an exact sequence $0 \rightarrow G_0 \rightarrow W_0 \rightarrow M \rightarrow 0$ such that W_0 is weak injective and G_0 is Gorenstein weak injective. Set $N_0 = \text{Ker}(W_0 \rightarrow N)$. Then we have two exact sequences: $0 \rightarrow N_0 \rightarrow W_0 \rightarrow N \rightarrow 0$ and $0 \rightarrow G_0 \rightarrow N_0 \rightarrow L \rightarrow 0$. By adding a trivial exact sequence $0 \rightarrow 0 \rightarrow N \rightarrow N \rightarrow 0$ to the second sequence, we get an exact sequence $0 \rightarrow G_0 \rightarrow N_0 \oplus N \rightarrow L \oplus N \rightarrow 0$. Note that G_0 and $L \oplus N$ are Gorenstein weak injective, so is $N_0 \oplus N$ by hypothesis. We repeat the argument by replacing N with N_0 to get N_1 and an exact sequence $0 \rightarrow N_1 \rightarrow W_1 \rightarrow N_0 \rightarrow 0$ with W_1 weak injective. Continue this process, we may obtain an exact sequence $\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow N \rightarrow 0$, where each W_i is weak injective, which shows that N is Gorenstein weak injective by Corollary 2.7. Similarly, L is Gorenstein weak injective. \square

Now we give the proof for our main theorem.

Proof of Theorem 3.1. Since $\mathcal{GWI}(\mathcal{R}) \subseteq \mathcal{GWI}^2(\mathcal{R})$, it suffices to show that $\mathcal{GWI}^2(\mathcal{R}) \subseteq \mathcal{GWI}(\mathcal{R})$. Since every two-degree Gorenstein weak injective R -module is a direct summand of some strongly two-degree Gorenstein weak injective R -module, and the class of Gorenstein weak injective R -modules is closed under direct summands by Lemma 3.5, so we only need to prove that every strongly two-degree Gorenstein weak injective R -module is Gorenstein weak injective R -module. Let M be a strongly two-degree Gorenstein weak injective R -module. Then, by Lemma 3.3, there is an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ with G Gorenstein weak injective. Moreover, by Corollary 2.7, there is an exact sequence $0 \rightarrow G_1 \rightarrow W \rightarrow G \rightarrow 0$ with W weak injective and G_1 Gorenstein weak injective.

Consider the following pull-back diagram:

$$(5) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G_1 & = & G_1 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & W & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the middle row in the diagram (5), we obtain an exact sequence $0 \rightarrow N \rightarrow W \rightarrow M \rightarrow 0$ with W weak injective. Thus, in order to show that M is Gorenstein weak injective, it suffices to prove that N is Gorenstein weak injective by Proposition 2.9.

Consider the following pull-back diagram:

$$(6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G_1 & = & G_1 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & G_2 & \longrightarrow & N \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since G and G_1 are Gorenstein weak injective, G_2 is also Gorenstein weak injective by the middle column in the diagram (6). Hence there exists an exact sequence $0 \rightarrow G_3 \rightarrow W_0 \rightarrow G_2 \rightarrow 0$ such that W_0 is weak injective and G_3 is Gorenstein weak injective.

Consider the following pull-back diagram:

$$(7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G_3 & = & G_3 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & W_0 & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & G_2 & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the middle row in the diagram (7), we obtain an exact sequence $0 \rightarrow N_1 \rightarrow W_0 \rightarrow N \rightarrow 0$ with W_0 weak injective. We repeat the argument by replacing N with N_1 to get N_2 and an exact sequence $0 \rightarrow N_2 \rightarrow W_1 \rightarrow N_1 \rightarrow 0$ with W_1 weak injective. Continue

this process, we may obtain an exact sequence $\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow N \rightarrow 0$, where each W_i is weak injective, which shows that N is Gorenstein weak injective by Corollary 2.7. We have completed the proof.

ACKNOWLEDGMENTS

The first author would like to thank Professor Zenghui Gao for sending his preprint [14].

REFERENCES

- [1] S. Bouchiba, Stability of Gorenstein classes of modules, *Algebra Colloq.*, 2013, 20(4): 623-636
- [2] S. Bouchiba, M. Khaloui, Stability of Gorenstein flat modules, *Glasgow Math. J.*, 2012, 54: 169-175
- [3] L. W. Christensen, Gorenstein Dimension, *Lecture Notes in Math.*, vol. 1747, Springer-Verlag, Berlin, 2000
- [4] L. W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions—a functorial description with applications, *J. Algebra*, 2006, 302: 231-279
- [5] N. Ding, Y. Li, L. Mao, Strongly Gorenstein flat modules, *J. Aust. Math. Soc.*, 2009, 86: 323-338
- [6] E. E. Enochs, O. M. G. Jenda, Gorenstein injective and projective modules, *Math. Z.* 1995, 220: 611-633
- [7] E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Exp. Math., vol. 30, Walter de Gruyter, Berlin, New York, 2000
- [8] S. Eilenberg, J. C. Moore, *Foundations of relative homological algebra*, Am. Math. Soc., 1965
- [9] Z. Gao, Coherent rings and Gorenstein FP-injective modules, *Comm. Algebra*, 2012, 40: 1669-1679
- [10] Z. Gao, On strongly Gorenstein FP-injective modules, *Comm. Algebra*, 2013, 41: 3035-3044
- [11] J. Gillespie, Model structures on modules over Ding-Chen rings, *Homology, Homotopy Appl.*, 2010, 12(1): 61-73
- [12] Z. Gao, Z. Huang, Weak injective covers and dimension of modules, available at <http://math.nju.edu.cn/~huangzy/papers/weakinj.pdf>
- [13] Z. Gao, F. Wang, All Gorenstein hereditary rings are coherent, *J. Algebra Appl.*, 2014, 13(4): 1350140 (5 pages)
- [14] Z. Gao, F. Wang, Weak injective and weak flat modules, *Comm. Algebra* (to appear)
- [15] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra*, 2004, 189: 167-193
- [16] Z. Huang, Proper resolutions and Gorenstein categories, *J. Algebra*, 2013, 393(1): 142-169
- [17] C. Huang, Z. Huang, Gorenstein syzygy modules, *J. Algebra*, 2010, 324(12): 3408-3419
- [18] L. Mao, N. Ding Gorenstein FP-injective and Gorenstein flat modules, *J. Algebra Appl.*, 2008, 7(4): 491-506
- [19] J. J. Rotman, *An Introduction to Homological Algebra*, New York: Springer, 2009
- [20] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.*, 1970, 2: 323-329
- [21] S. Sather-Wagstaff, T. Sharif, D. White, Stability of Gorenstein categories, *J. London Math. Soc.*, 2008, 77: 481-502
- [22] Z. Wang, Z. Liu, Stability of strongly Gorenstein flat modules, *Vietnam J. Math.* (to appear)
- [23] A. Xu, Notes on stability of Gorenstein categories, *J. Algebra Appl.*, 2013, 12(1): 1250209 (12 pages)
- [24] A. Xu, N. Ding, On stability of Gorenstein categories, *Comm. Algebra*, 2013, 41(10): 3793-3804
- [25] G. Yang, Z. Liu, On stability of Gorenstein flat categories, *Glasgow Math. J.*, 2012, 54(01): 177-191

SCHOOL OF MATHEMATICS AND STATISTICS, HUBEI UNIVERSITY
430062 WUHAN, P. R. CHINA

E-mail address: tiweizhao@gmail.com, xuy@hubu.edu.cn