# USING THE REFINEMENT EQUATIONS FOR THE CONSTRUCTION OF PRE-WAVELETS II: POWERS OF TWO

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**Abstract**

*We study basic questions of wavelet decompositions associated with multiresolution analysis. A rather complete analysis of multiresolution associated with the solution of a refinement equation is presented. The notion of extensibility of a finite set of Laurent polynomials is shown to be central in the construction of wavelets by decomposition of spaces. Two examples of extensibility, first over the torus and then in complex space minus the coordinate axes are discussed. In each case we are led to a decomposition of the fine space in a multiresolution analysis as a sum of the adjacent coarse space plus an additional space spanned by the multiinteger translates of a finite number of pre-wavelets. Several examples are provided throughout to illustrate the general theory.*

## §1. Introduction

In this paper we record and refine some important new advances for multivariate wavelet decomposition. A resolution of this problem depends on certain matrix theory questions over the ring of multivariate Laurent series.

Let $s$ be a positive integer and let $\mathbb{R}^s$ be the $s$-dimensional real space equipped with the norm $|\cdot|$ given by

$$|x| := \Big(\sum_{j=1}^{s} |x_j|^2\Big)^{1/2} \quad \text{for } x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

By a function on $\mathbb{R}^s$ we mean a complex-valued Lebesgue measurable function on $\mathbb{R}^s$. By a sequence on $\mathbb{Z}^s$ we mean a mapping from $\mathbb{Z}^s$ to $\mathbb{C}$. An element of $\mathbb{Z}^s$ is called a multiinteger. For $1 \le p \le \infty$, we denote by $L^p = L^p(\mathbb{R}^s)$ the Banach space of all functions $f$ on $\mathbb{R}^s$ for which

$$\|f\|_p := \Big(\int_{\mathbb{R}^s} |f(x)|^p \, dx\Big)^{1/p} < \infty.$$

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We observe that $L^2$ is a Hilbert space with the usual inner product given by

$$\langle f, g\rangle := \int_{\mathbb{R}^s} f(x)\overline{g(x)}\,dx.$$

Analogously, denote by $\ell^p = \ell^p(\mathbb{Z}^s)$ the Banach space of all sequences $a$ on $\mathbb{Z}^s$ for which

$$\|a\|_p := \Big(\sum_{\alpha\in\mathbb{Z}^s} |a(\alpha)|^p\Big)^{1/p} < \infty.$$

For $\alpha \in \mathbb{Z}^s$ we let $\tau^\alpha$ be the shift operator given by $\tau^\alpha f = f(\cdot - \alpha)$, where $f$ can be a function on $\mathbb{R}^s$ or a sequence on $\mathbb{Z}^s$. For $\alpha \in \mathbb{Z}^s$, we call $f(\cdot - \alpha)$ *an integer translate* of $f$.

Following Mallat [26], we say that a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^p(\mathbb{R}^s)$ forms a multiresolution approximation of $L^p(\mathbb{R}^s)$, if it satisfies the following conditions:

(R1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.

(R2) $f \in V_j \Longrightarrow f(\cdot - 2^{-j}\alpha) \in V_j$ for all $j \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^s$.

(R3) $f \in V_j \Longleftrightarrow f(2\cdot) \in V_{j+1}$.

(R4) There is an isomorphism from $\ell^p$ onto $V_0$ which commutes with shift operators.

(R5) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(R6) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^p(\mathbb{R}^s)$.

The case $p = 2$ is particularly interesting, since $L^2(\mathbb{R}^s)$ is a Hilbert space. Let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$. Then $L^2(\mathbb{R}^s)$ can be decomposed as an orthogonal sum of $W_j$ ($j \in \mathbb{Z}$). It follows from (R3) that

$$f \in W_j \Longleftrightarrow f(2\cdot) \in W_{j+1}.$$

Hence any function $\psi \in W_0$ has the following property: For $j, k \in \mathbb{Z}$, $j \neq k$,

$$\langle \psi(2^j \cdot - \alpha), \psi(2^k \cdot - \beta)\rangle = 0 \quad \text{for all } \alpha, \beta \in \mathbb{Z}^s. \tag{1.1}$$

Following Battle [3], we call such a function a *pre-wavelet*. If, in addition, $\psi$ satisfies (1.1) for $j = k$ and $\alpha \neq \beta$, then $\psi$ is called a *wavelet*. A main problem in multiresolution analysis is to find pre-wavelets in $W_0$ such that their multiinteger translates form an unconditional basis for $W_0$. If they are wavelets, then their integer translates actually form an orthogonal basis for $W_0$. In such a case, $L^2(\mathbb{R}^s)$ has an orthogonal basis of wavelets.

The wavelet decomposition problem has been investigated by Meyer [27] and Mallat[26] for the case $s = 1$. Daubechies [17] has given a construction of smooth compactly supported wavelets. Dahmen and Micchelli [16] using results of [7] provided some improvement of [17] and gave an alternative derivation of Daubechies' theorem. Chui and Wang [8–9] considered a cardinal spline approach to wavelets in one variable. Chui and Wang [10], and independently Micchelli [28], provided a comprehensive study of univariate pre-wavelet decompositions. Moreover, in the multivariate case, Micchelli connected wavelet decompositions to the fundamental result of Quillen and Suslin in algebraic geometry which states that projective modules over polynomial rings are free.

An important example of a refinable function is a box spline, see [5] for its definition. This fact was discovered by Dahmen and Micchelli [15] and used as a basis for the line average subdivision scheme for computing box spline surfaces, see also Cohen, Lyche and Riesenfield [11]. Riemenschneider and Shen in a recent paper [29] provided wavelet decompositions based on box splines, see also [28].

This paper treats in detail the algebraic problems associated with wavelet decomposition of the spaces which comprise a multiresolution analysis. The central idea is to represent the space $V_1$ on the "fine scale" as a sum of $V_0$ on the "coarse scale" and $2^s-1$ additional spaces spanned by pre-wavelets. Before we undertake this task we provide very general conditions for the existence of a multiresolution analysis based on a solution to a refinement equation studied extensively in [7].

To this end, we introduce certain Banach spaces which are convenient and natural in the study of multiresolution analysis. Essential to our point of view about decompositions of spaces are algebraic and Fourier analytical issues concerning spaces generated by multiinteger translates of a fixed number of functions. For this purpose we draw upon results from our recent paper [21] on this subject. This allows use to develop criteria for change of bases and Gram-Schmidt orthogonalization in these spaces.

We couple these facts with the algebra of Laurent series associated with the refinement equation to solve the wavelet decomposition problem. As we shall demonstrate, the essential notion in this regard is what we call extensibility of a finite set of Laurent series. The extensibility of a set of Laurent series associated with the coarse space $V_0$ determines the existence of a wavelet decomposition of $V_1$.

There are two instances in which extensibility can be decided for Laurent polynomials. The first case, over the torus, depends on certain topological considerations, which will be described elsewhere, and the second case, over $(\mathbb{C}\backslash\{0\})^s$ requires the use of the celebrated Quillen-Suslin theorem. Numerous examples, of both a specific as well as a general nature, are provided throughout to highlight important aspects of the general theory.

Finally, we remark that much of what we develop here applies to multiresolution analysis based on general lattices. In a future publication this important extension will be addressed.

## §2. Multiresolution Analysis

Multiresolution analysis can be built on the multiinteger shifts of scales of a suitable function $\phi$. We will address this important technique in some generality next. Specifically, we discuss in this section the possible conditions on $\phi$ under which one can construct from $\phi$ a multiresolution approximation of $L^p(\mathbb{R}^s)$ $(1 \le p \le \infty)$.

We find it convenient to work in the following setting. Given a function $\phi$ on $\mathbb{R}^s$, set

$$\phi^\circ := \sum_{\alpha\in\mathbb{Z}^s} |\phi(\cdot - \alpha)|.$$

Then $\phi^\circ$ is a 1-periodic function. Define

$$|\phi|_p := \|\phi^\circ\|_{L^p([0,1)^s)}.$$

For $1 \le p \le \infty$, let $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^s)$ be the linear space of all functions $\phi$ for which $|\phi|_p < \infty$. Equipped with the norm $|\cdot|_p$, $\mathcal{L}^p$ becomes a Banach space. Clearly, $\|\phi\|_p \le |\phi|_p$, and $|\phi|_q \le |\phi|_p$ for $1 \le q \le p \le \infty$. This shows that

$$\mathcal{L}^p \subset L^p$$

and

$$\mathcal{L}^p \subset \mathcal{L}^q \quad \text{for } 1 \le q < p \le \infty.$$

The space $\mathcal{L}^\infty$ has already appeared in [28]. Also, note that $\mathcal{L}^1 = L^1$. There are several subspaces of $\mathcal{L}^p$ which are important in what follows. For instance, if $\phi \in L^p$ is compactly supported, then $\phi \in \mathcal{L}^p$ $(1 \le p \le \infty)$. Also, we say that a function $\phi \in L^p$ decays exponentially fast, if there are constants $C > 0$ and $q$, $0 < q < 1$, such that

$$\|\phi(\cdot + \alpha)\|_{L^p([0,1)^s)} \le Cq^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}^s.$$

If $\phi \in L^p$ decays exponentially fast, then $\phi \in \mathcal{L}^p$. We denote by $\mathcal{E}^p$ the subspace of $\mathcal{L}^p$ which consists of all exponentially decaying functions. Furthermore, we observe that if there are constants $C > 0$ and $\delta > 0$ such that

$$|\phi(x)| \le C(1 + |x|)^{-s-\delta} \quad \text{for all } x \in \mathbb{R}^s,$$

then $\phi \in \mathcal{L}^\infty$.

Given a function $\phi \in \mathcal{L}^p$ and a sequence $a \in \ell^\infty$, the semi-discrete convolution product $\phi *' a$ is, by definition, the sum $\sum_{\alpha\in\mathbb{Z}^s} \phi(\cdot - \alpha)a(\alpha)$. We also denote by $\phi *'$ the mapping $a \mapsto \phi *' a,\ a \in \ell^\infty$. The following theorem shows that $\phi *'$ maps $\ell^p$ to $L^p$ and maps $\ell^1$ to $\mathcal{L}^p$.

**Theorem 2.1.** *If $\phi \in \mathcal{L}^p(\mathbb{R}^s)$, then*

$$|\phi *' a|_p \le |\phi|_p \|a\|_1$$

*and*

$$\|\phi *' a\|_p \le |\phi|_p \|a\|_p.$$

**Proof:** The first inequality follows from the observation that

$$(\phi *' a)^\circ \le \|a\|_1 \phi^\circ.$$

The second inequality is obvious for $p = \infty$. In the case $1 \le p < \infty$, we let $I := \|\phi *' a\|_p$ and express $I^p$ as follows:

$$I^p = \sum_{\beta \in \mathbb{Z}^s} \int_{[0,1)^s+\beta} \left|(\phi *' a)(x)\right|^p dx = \int_{[0,1)^s} \sum_{\beta \in \mathbb{Z}^s} \left|(\phi *' a)(x+\beta)\right|^p dx.$$

Fix $x$ for the moment. Let $c$ be the sequence $\big(\phi(x+\beta)\big)_{\beta \in \mathbb{Z}^s}$. Then

$$(\phi *' a)(x+\beta) = (a * c)(\beta),$$

where $a*c$ denotes the discrete convolution product of $a$ and $c$. By a discrete version of Young's inequality , we have $\|a*c\|_p \le \|a\|_p \|c\|_1$ (see, *e.g.*, [18, Theorem 6.18]). It follows that

$$\sum_{\beta \in \mathbb{Z}^s} |(\phi *' a)(x+\beta)|^p = \|a*c\|_p^p \le \|a\|_p^p \|c\|_1^p = \|a\|_p^p (\phi^\circ(x))^p.$$

Consequently, we have

$$I^p \le \|a\|_p^p \int_{[0,1)^s} \big(\phi^\circ(x)\big)^p dx = \|a\|_p^p |\phi|_p^p,$$

as desired. ■

We denote by $S_p(\phi)$ the image of $\ell^p(\mathbb{Z}^s)$ under the mapping $\phi *'$. The integer translates of $\phi$ are said to be $\ell^p$*-stable* if there exists a constant $C_p > 0$ such that

$$\|\phi *' a\|_p \ge C_p \|a\|_p \quad \text{for all } a \in \ell^p.$$

In such a case, $\phi *'$ is an isomorphism from $\ell^p$ onto $S_p(\phi)$, and the integer translates of $\phi$ form an unconditional basis for $S_p(\phi)$.

A function $\phi \in \mathcal{L}^p$ is said to be *refinable*, if it satisfies a refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) \phi(2 \cdot - \alpha) \tag{2.1}$$

for some sequence $b \in \ell^1(\mathbb{Z}^s)$. The sequence $b$ is called the *mask* of the refinement equation.

For $j \in \mathbb{Z}$ we denote by $\sigma_j$ the scaling operator given by

$$\sigma_j f = f(2^j \cdot) \quad \text{for all functions } f \text{ on } \mathbb{R}^s.$$

We are now in a position to state the main result of this section.

**Theorem 2.2.** *Let $\phi \in \mathcal{L}^p(\mathbb{R}^s)$ $(1 \le p < \infty)$, $V_0 = S_p(\phi)$ and $V_j = \sigma_j(V_0)$. If $\phi$ is refinable and has $\ell^p$-stable integer translates, then $(V_j)_{j \in \mathbb{Z}}$ forms a multiresolution approximation of $L^p(\mathbb{R}^s)$.*

**Proof:** Using the discrete version of Young's inequality, we see that (R1) holds, because $\phi$ is refinable and $b \in \ell^1(\mathbb{Z}^s)$. (R2) and (R3) follow from the definition of $V_j$. (R4) is a consequence of $\ell^p$-stability. Indeed, $\phi *'$ is an isomorphism from $\ell^p$ onto $V_0$, which commutes with shift operators. (R5) will be proved in the following theorem, while (R6) is a consequence of Theorem 2.4 and Theorem 2.5, which will be proved later. ■

**Theorem 2.3.** *Let $\phi \in \mathcal{L}^p(\mathbb{R}^s)$ $(1 \le p < \infty)$, $V_0 = S_p(\phi)$ and $V_j = \sigma_j(V_0)$. If $\phi$ has $\ell^p$-stable integer translates, then $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.*

**Proof:** For $f \in \cap_{j\in\mathbb{Z}} V_j$, we have $f(2^j\cdot) \in V_0$; hence there is a sequence $a \in \ell^p$ ($a$ depends on $j$) such that $f(2^j x) = (\phi *' a)(x)$ $(x \in \mathbb{R}^s)$. But $\phi$ has $\ell^p$-stable integer translates, so there exists a constant $C_p > 0$ such that

$$\|a\|_p \le C_p^{-1}\Big(\int_{\mathbb{R}^s} |f(2^j x)|^p\,dx\Big)^{1/p} = C_p^{-1} 2^{-js/p}\|f\|_p. \tag{2.2}$$

It is easily seen that for all $x \in \mathbb{R}^s$,

$$|f(2^j x)| = |(\phi *' a)(x)| \le \|a\|_\infty \phi^\circ(x) \le \|a\|_p \phi^\circ(x).$$

It follows that

$$|f(x)|^p \le \|a\|_p^p |\phi^\circ(2^{-j}x)|^p.$$

Let $r > 0$. Integrating both sides of the above inequality on the ball

$$B_r := \{x \in \mathbb{R}^s : |x| \le r\},$$

we obtain

$$\int_{B_r} |f(x)|^p\,dx \le \|a\|_p^p \int_{B_r} |\phi^\circ(2^{-j}x)|^p\,dx \le 2^{js}\|a\|_p^p \int_{2^{-j}B_r} |\phi^\circ(x)|^p\,dx.$$

This together with the estimate for $\|a\|_p$ given in (2.2) yields

$$\int_{B_r} |f(x)|^p\,dx \le C_p^{-p}\|f\|_p^p \int_{2^{-j}B_r} |\phi^\circ(x)|^p\,dx.$$

Note that $\phi^\circ$ is 1-periodic and belongs to $L^p$ on $[0,1)^s$. Letting $j \to \infty$ in the above inequality, with $r$ fixed, we get $\int_{B_r} |f(x)|^p\,dx = 0$. This shows that $f = 0$, since $r$ can be any positive real number. ■

**Remark 2.1.** In the case $p = \infty$, Theorem 2.3 may fail to hold. For example, let $\phi$ be the characteristic function of $[0,1) \subset \mathbb{R}$ and let $V_0 = S_\infty(\phi)$, $V_j = \sigma_j(V_0)$. Then $1 \in V_j$ for all $j \in \mathbb{Z}$.

The Fourier-Laplace transform of a function $f$ is, by definition, the function given by

$$\widehat{f}(z) := \int_{\mathbb{R}^s} f(x) e^{-iz\cdot x}\,dx \quad (z \in \mathbb{C}^s),$$

where for $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$,

$$z \cdot x := \sum_{j=1}^{s} z_j x_j.$$

Restricted to $\mathbb{R}^s$, $\widehat{f}$ becomes the Fourier transform of $f$. For later use, we denote the $s$-torus $T^s$ as the set

$$\big\{(z_1, \ldots, z_s) \in \mathbb{C}^s : \ |z_1| = \cdots = |z_s| = 1\big\}.$$

The following theorem was first proved in [7] under the additional conditions that $\phi$ is a compactly supported continuous function.

**Theorem 2.4.** *If $\phi \in L^1$ is refinable, then*

$$\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = \widehat{\phi}(0). \tag{2.3}$$

**Proof:** Taking Fourier transforms on both sides of the refinement equation (2.1), we obtain

$$\widehat{\phi}(\xi) = 2^{-s} p(e^{-i\xi/2}) \widehat{\phi}(\xi/2) \quad (\xi \in \mathbb{R}^s) \tag{2.4}$$

where

$$p(z) = \sum_{\alpha \in \mathbb{Z}^s} b(\alpha) z^{\alpha} \quad (z \in T^s)$$

is the symbol of the sequence $b$. It follows from (2.4) that

$$\widehat{\phi}(\xi) = \prod_{j=1}^{k} \big(2^{-s} p(e^{-i\xi/2^j})\big) \widehat{\phi}(\xi/2^k). \tag{2.5}$$

If $|p(1)| < 2^s$, then choosing $\xi = 0$ in (2.4), we obtain $\widehat{\phi}(0) = 0$. Moreover, $|p(1)| < 2^s$ implies that for any fixed $\xi \in \mathbb{R}^s$ and sufficiently large $j$,

$$|2^{-s} p(e^{-i\xi/2^j})| < 1.$$

Thus, letting $k \to \infty$ in (2.5), we obtain $\widehat{\phi}(\xi) = 0$. This is true for any $\xi \in \mathbb{R}^s$, hence $\phi = 0$.

Now suppose $|p(1)| \geq 2^s$. Choosing $\xi = 2^{k+1}\beta\pi$ in (2.5), where $\beta \in \mathbb{Z}^s \backslash \{0\}$, we obtain

$$\widehat{\phi}(2^{k+1}\beta\pi) = (2^{-s} p(1))^k \widehat{\phi}(2\beta\pi).$$

It follows that

$$|\widehat{\phi}(2\beta\pi)| \leq |\widehat{\phi}(2^{k+1}\beta\pi)|.$$

Letting $k \to \infty$ in the above inequality and applying the Riemann-Lebesgue lemma, we obtain

$$\widehat{\phi}(2\beta\pi) = 0 \quad \text{for all } \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{2.6}$$

We claim that (2.6) implies (2.3). When $\phi \in L^1$ is compactly supported, this was proved by Strang and Fix [34] using the Poisson summation formula. In general, consider the Fourier series expansion of the 1-periodic function $\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha)$ in $L^1([0,1)^s)$:

$$\sum_{\alpha \in \mathbb{Z}^s} \phi(x - \alpha) \sim \sum_{\beta \in \mathbb{Z}^s} c(\beta) e^{i2\pi\beta \cdot x}, \quad x \in [0,1)^s.$$

The Fourier coefficients are

$$\begin{aligned} c(\beta) &= \int_{[0,1)^s} \sum_{\alpha \in \mathbb{Z}^s} \phi(x - \alpha) e^{-i2\pi\beta \cdot x} \, dx \\ &= \int_{\mathbb{R}^s} \phi(x) e^{-i2\pi\beta \cdot x} \, dx = \widehat{\phi}(2\pi\beta). \end{aligned}$$

Therefore (2.6) implies (2.3). ■

According to this theorem, if $\phi \in \mathcal{L}^p$ is refinable, then $\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha)$ is a constant. If, in addition, $\phi$ has $\ell^p$-stable integer translates, then this constant must be nonzero. This fact will be proved in Theorem 3.5. After normalization we may assume that $\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = 1$. Thus the property (R6) follows from the following theorem.

**Theorem 2.5.** *If $\phi \in \mathcal{L}^p$ $(1 \le p < \infty)$ and $\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = 1$, then for any $f \in L^p$,*

$$\|f - \sum_{\alpha \in \mathbb{Z}^s} a_h(\alpha) \phi(h^{-1} \cdot - \alpha)\|_p \to 0 \quad \text{as } h \to 0, \tag{2.7}$$

*where*

$$a_h(\alpha) = a_h(f, \alpha) := h^{-s} \int_{h\alpha + [0,h)^s} f(x) \, dx = \int_{[0,1)^s} f\big(h(x + \alpha)\big) \, dx. \tag{2.8}$$

**Proof:** In the proof of this theorem, we denote by $|x|$ the maximum norm of $x$, *i.e.*, $|x| = \max_{1 \le j \le s} |x_j|$ for $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$. Assume first that $f$ is a continuous function supported on a cube $\{x \in \mathbb{R}^s : |x| \le r\}$ for some $r > 0$. Let

$$\varepsilon_h(f) := \|f - \sum_{\alpha \in \mathbb{Z}^s} a_h(\alpha) \phi(h^{-1} \cdot - \alpha)\|_p. \tag{2.9}$$

Since $\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = 1$, we have

$$\varepsilon_h(f) = h^{s/p} \|g_h\|_p, \tag{2.10}$$

where

$$g_h(x) := \sum_{\alpha\in\mathbb{Z}^s} \big(f(hx) - a_h(\alpha)\big)\phi(x-\alpha) \quad (x\in\mathbb{R}^s).$$

To estimate $\|g_h\|_p$, we write

$$\|g_h\|_p^p = \int_{\mathbb{R}^s} |g_h(x)|^p\, dx = \sum_{\beta\in\mathbb{Z}^s}\int_{[0,1)^s} |g_h(x+\beta)|^p\, dx.$$

Let $N$ be a positive integer. From (2.8) we see that for $x\in[0,1)^s$ and $|\beta-\alpha| < N$,

$$\begin{aligned}&|f\big(h(x+\beta)\big) - a_h(\alpha)|\\ &\le \int_{[0,1)^s} \big|f\big(h(x+\beta)\big) - f\big(h(y+\alpha)\big)\big|dy \le \omega(f,Nh),\end{aligned}$$

where $\omega(f,\cdot)$ is the modulus of continuity of $f$:

$$\omega(f,t) := \sup_{|y|\le t} \|f - f(\cdot - y)\|_\infty.$$

In general, we have

$$|f\big(h(x+\beta)\big) - a_h(\alpha)| \le 2\|f\|_\infty \quad \text{for all } x\in[0,1)^s \text{ and } \alpha,\beta\in\mathbb{Z}^s.$$

It follows that for $x\in[0,1)^s$ and all $\beta\in\mathbb{Z}^s$,

$$\begin{aligned}|g_h(x+\beta)| &\le \sum_{|\alpha-\beta|<N} + \sum_{|\alpha-\beta|\ge N} |f(h(x+\beta)) - a_h(\alpha)||\phi(x+\beta-\alpha)|\\ &\le \omega(f,Nh)\phi^\circ(x) + 2\|f\|_\infty \phi_N^\circ(x),\end{aligned}$$

where

$$\phi_N^\circ := \sum_{|\alpha|\ge N} |\phi(\cdot - \alpha)|.$$

Let $M$ be the least integer bigger than $h^{-1}r+1$. Since the cardinality of the set $\{\beta\in\mathbb{Z}^s : |\beta| < 2M\}$ does not exceed a constant times $h^{-s}$, we obtain

$$\sum_{|\beta|<2M}\int_{[0,1)^s} |g_h(x+\beta)|^p\, dx \le C_1 h^{-s}\big[\omega(f,Nh)|\phi|_p + \|f\|_\infty \|\phi_N^\circ\|_{L^p([0,1)^s)}\big]^p \tag{2.12}$$

for some constnat $C_1 > 0$.

For $|\beta|\ge 2M$, we have

$$\begin{aligned}g_h(x+\beta) &= \sum_{\alpha\in\mathbb{Z}^s} (f(h(x+\beta)) - a_h(\alpha))\phi(x+\beta-\alpha)\\ &= -\sum_{\alpha\in\mathbb{Z}^s} a_h(\alpha)\phi(x+\beta-\alpha)\\ &= -\sum_{|\alpha|\le M} a_h(\alpha)\phi(x+\beta-\alpha),\end{aligned}$$

noting that $f(h(x+\beta))=0$ for $x\in[0,1)^s$ and $|\beta|\ge 2M$, and $a_h(\alpha)=0$ for $|\alpha|>M$. It follows that for $|\beta|\ge 2M$ and $x\in[0,1)^s$,

$$|g_h(x+\beta)|\le \|f\|_\infty \sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|.$$

Hence

$$\sum_{|\beta|\ge 2M}|g_h(x+\beta)|^p \le \|f\|_\infty^p \sum_{|\beta|\ge 2M}\Big[\sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|\Big]^p. \tag{2.13}$$

Observe that

$$\{\beta\in\mathbb{Z}^s : |\beta|\ge 2M\} = \bigcup_{|\gamma|\le 2M}\big(\gamma+4M(\mathbb{Z}^s\backslash\{0\})\big).$$

Let $\gamma$ be fixed, $|\gamma|\le 2M$. Since $1\le p<\infty$, we have that for $x\in[0,1)^s$,

$$\begin{aligned}&\sum_{\beta\in\gamma+4M(\mathbb{Z}^s\backslash\{0\})}\Big[\sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|\Big]^p\\ \le&\Big[\sum_{\beta\in\gamma+4M(\mathbb{Z}^s\backslash\{0\})}\sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|\Big]^p\le(\phi_M^\circ(x))^p,\end{aligned}$$

noting that $|\beta|\ge 2M$ and $|\alpha|\le M$ imply $|\beta-\alpha|\ge M$. Thus there exists a constant $C_2>0$ such that for all $x\in[0,1)^s$,

$$\begin{aligned}&\sum_{|\beta|\ge 2M}\Big[\sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|\Big]^p\\ \le&\sum_{|\gamma|\le 2M}\ \sum_{\beta\in\gamma+4M(\mathbb{Z}^s\backslash\{0\})}\Big[\sum_{|\alpha|\le M}|\phi(x+\beta-\alpha)|\Big]^p\\ \le& C_2h^{-s}(\phi_M^\circ(x))^p,\end{aligned}$$

where we have used again the fact that the cardinality of the set $\{\gamma\in\mathbb{Z}^s : |\gamma|\le 2M\}$ does not exceed a constant times $h^{-s}$. This together with (2.13) gives the following estimate:

$$\sum_{|\beta|\ge 2M}\int_{[0,1)^s}|g_h(x+\beta)|^p\,dx\le C_2\|f\|_\infty^p h^{-s}\|\phi_M^\circ\|_{L^p([0,1)^s)}^p. \tag{2.14}$$

We choose $N$ in such a way that $N\le M$. Then $\phi_M^\circ(x)\le\phi_N^\circ(x)$ for all $x\in[0,1)^s$.

Now (2.12) and (2.14) together tell us that there exists a constant $C>0$ such that

$$\begin{aligned}\|g_h\|_p^p&=\sum_{\beta\in\mathbb{Z}^s}\int_{[0,1)^s}|g_h(x+\beta)|^p\,dx\\ &\le C^p h^{-s}\big[\omega(f,Nh)|\phi|_p+\|f\|_\infty\|\phi_N^\circ\|_{L^p([0,1)^s)}\big]^p.\end{aligned}$$

This together with (2.10) yields

$$\varepsilon_h(f) \le C\big(\omega(f, Nh)|\phi|_p + \|f\|_\infty \|\phi_N^\circ\|_{L^p([0,1)^s)}\big). \tag{2.15}$$

Let us now choose $N$ to be the integer part of $1/\sqrt{h}$. When $h > 0$ is sufficiently small, we indeed have $M \ge h^{-1}r \ge N$. From (2.15) we conclude that

$$\lim_{h\to 0^+} \varepsilon_h(f) = 0.$$

This proves (2.7) for compactly supported continuous functions $f$.

Now let $f$ be an arbitrary function in $L^p$. By Theorem 2.1, it follows from the definition (2.9) of $\varepsilon_h(f)$ that

$$\varepsilon_h(f) \le \|f\|_p + h^{s/p}\|\phi *' a_h\|_p \le \|f\|_p + h^{s/p}|\phi|_p \|a_h\|_p.$$

From the definition (2.8) of $a_h(f, \alpha)$ and Hölder's inequality we see that

$$\|a_h\|_p \le h^{-s/p}\|f\|_p,$$

and so we obtain

$$\varepsilon_h(f) \le \|f\|_p(1 + |\phi|_p).$$

Hence for any $g \in C_0(\mathbb{R}^s)$, the space of compactly supported continuous functions on $\mathbb{R}^s$, we have

$$\varepsilon_h(f) \le \varepsilon_h(g) + \varepsilon_h(f - g) \le \varepsilon_h(g) + \|f - g\|_p(1 + |\phi|_p).$$

Since $C_0(\mathbb{R}^s)$ is dense in $L^p(\mathbb{R}^s)$ $(1 \le p < \infty)$, we conclude easily that $\varepsilon_h(f)$ converges to 0 as $h$ goes to 0 for any $f \in L^p$ $(1 \le p < \infty)$. ■

**Remark 2.2.** $L^\infty$-approximation by the integer translates of a function on $\mathbb{R}$ was first investigated by Schoenberg [33]. For the case $s > 1$, $L^2$-approximation was studied by Strang and Fix [34] using Fourier analysis. See also [13] and [6] for $L^p$-approximation by the integer translates of a compactly supported function on $\mathbb{R}^s$. When $\phi$ does not have compact support, $L^\infty$-approximation was recently studied by Light and Cheney [25], while $L^p$-approximation $(1 \le p \le \infty)$ was investigated by Jia and Lei [20].

## §3. Symbol Calculus

In this section we establish some basic properties of symbol calculus and apply them to the study of stability of the integer translates of a function.

Given a sequence $a$, let $\widetilde{a}(z)$ be its symbol, or its discrete Fourier transform:

$$\widetilde{a}(z) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)z^\alpha. \tag{3.1}$$

With some abuse of terminology, we call the series in (3.1) a Laurent series. If $a \in \ell^1(\mathbb{Z}^s)$, then the symbol $\widetilde{a}(z)$ is a continuous function of $z$ on the torus $T^s$. The sequence $a$ can be recovered from $\widetilde{a}$ through the inversion formula:

$$a(\alpha) = \int_{[0,1)^s} \widetilde{a}(e^{i2\pi\xi}) e^{-i2\pi\alpha\cdot\xi}\, d\xi, \quad \alpha \in \mathbb{Z}^s. \tag{3.2}$$

Let $\mathcal{B}$ be the set of all functions of the form (3.1) with $\|a\|_1 < \infty$. Normed by $\|\widetilde{a}\| := \|a\|_1$, $\mathcal{B}$ is a commutative Banach algebra, with pointwise multiplication. If $f \in \mathcal{B}$ and $f(z) \neq 0$ for every $z \in T^s$, then by Wiener's lemma (see *e.g.*, [32, p.266]) $1/f$ is also in $\mathcal{B}$.

If $a$ is a sequence decaying exponentially fast, *i.e.*, for some constants $C > 0$ and $q$ between 0 and 1,

$$|a(\alpha)| \le Cq^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}^s,$$

then $\widetilde{a}(z)$ is a holomorphic function of $z$ in a neighborhood of $T^s$. Conversely, if $f$ is a holomorphic function in a neighborhood of $T^s$, then $f = \widetilde{a}$ for some exponentially decaying sequence $a$. Let $\mathcal{H}$ be the set of all functions holomorphic in a neighborhood of $T^s$. Then $\mathcal{H}$ is a subalgebra of $\mathcal{B}$. If $f \in \mathcal{H}$ and $f(z) \neq 0$ for every $z \in T^s$, then $1/f$ is also in $\mathcal{H}$.

If $a$ is a finitely supported sequence, then $\widetilde{a}(z)$ is a Laurent polynomial, which is defined on $(\mathbb{C}\backslash\{0\})^s$. The set of all Laurent polynomials is denoted by $\mathcal{P}$. Then $\mathcal{P}$ is a subalgebra of $\mathcal{H}$. If $p$ is a Laurent polynomial which does not vanish on $(\mathbb{C}\backslash\{0\})^s$, then $1/p$ is also a Laurent polynomial. To see this, we observe that for some $\beta \in \mathbb{Z}^s$, $z^\beta p(z)$ is a polynomial. The polynomial $z_1 \cdots z_s$ vanishes on the zero set of $z^\beta p(z)$; hence by Hilbert Nullstellensatz, $z^\beta p(z)$ divides $(z_1 \cdots z_s)^n$ for some integer $n > 0$. This shows that $p(z)$ must have the form $\lambda z^\alpha$, where $\lambda \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{Z}^s$. Thus $1/p(z) = \lambda^{-1} z^{-\alpha}$ is also a Laurent polynomial.

Given $f \in L^p(\mathbb{R}^s)$ and $g \in L^q(\mathbb{R}^s)$ ($1 \le p \le \infty$, $1/p + 1/q = 1$), we denote by $c(f,g)$ the sequence on $\mathbb{Z}^s$ given by

$$c(f,g)(\alpha) := \int_{\mathbb{R}^s} f(x)\overline{g(x-\alpha)}\, dx = \int_{\mathbb{R}^s} f(x+\alpha)\overline{g(x)}\, dx. \tag{3.3}$$

**Theorem 3.1.** *For $1 \le p, q \le \infty$, $1/p + 1/q = 1$, the following inequalities hold:*

$$\|c(f,g)\|_1 \le |f|_p |g|_q \tag{3.4}$$

*and*

$$\|c(f,g)\|_p \le \|f\|_p |g|_q. \tag{3.5}$$

**Proof:** We have

$$\begin{aligned} \|c(f,g)\|_1 &\le \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} \int_{[0,1)^s+\beta} |f(\alpha+x)||g(x)|\, dx \\ &= \int_{[0,1)^s} \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} |f(\alpha+\beta+x)||g(x+\beta)|\, dx. \end{aligned}$$

By the definition of $f^\circ$ and $g^\circ$, the double sum under the above integral sign equals $f^\circ(x)g^\circ(x)$, hence it follows that

$$\|c(f,g)\|_1 \le \int_{[0,1)^s} f^\circ(x)g^\circ(x)\,dx \le |f|_p|g|_q.$$

This proves (3.4).

To prove (3.5) we shall use the converse of Hölder's inequality (see, *e.g.*, [18, p.181]). For two sequences $a$ and $b$ on $\mathbb{Z}^s$, let

$$\langle a,b\rangle := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\overline{b(\alpha)}.$$

For any finitely supported sequence $b$, we have

$$\langle c(f,g),b\rangle = \langle f, g*'b\rangle,$$

hence by Theorem 2.1 it follows that

$$|\langle c(f,g),b\rangle| \le \|f\|_p\|g*'b\|_q \le \|f\|_p|g|_q\|b\|_q.$$

This proves (3.5). ■

Denote by $[f,g](z)$ the symbol of the sequence $c(f,g)$:

$$[f,g](z) := \sum_{\alpha\in\mathbb{Z}^s} c(f,g)(\alpha)z^\alpha. \tag{3.6}$$

If $f,g\in\mathcal{L}^2$, then $c(f,g)\in\ell^1$ by Theorem 3.1, and hence $[f,g]\in\mathcal{B}$. If $f,g\in\mathcal{E}^2$, then one can easily see that $c(f,g)$ is an exponentially decaying sequence, so $[f,g]\in\mathcal{H}$. If $f,g\in L^2$ are compactly supported, then $c(f,g)$ is a finitely supported sequence, hence $[f,g](z)$ is a Laurent polynomial and $[f,g]\in\mathcal{P}$.

From the definition of $[f,g]$ we see that $\{\phi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^s}$ forms an orthonormal basis for $S_2(\phi)$ if and only if

$$[\phi,\phi](z) = 1 \quad \text{for all } z\in T^s.$$

Moreover, a function $f\in L^2$ is orthogonal to $S_2(\phi)$ if and only if

$$[f,\phi](z) = 0 \quad \text{for all } z\in T^s.$$

Furthermore, if $\phi,\psi\in\mathcal{L}^2$ and $a,b\in\ell^1$, then

$$[\phi*'a,\psi*'b](z) = \widetilde{a}(z)[\phi,\psi](z)\overline{\widetilde{b}(z)} \quad \text{for all } z\in T^s. \tag{3.7}$$

**Theorem 3.2.** *For $f,g\in\mathcal{L}^2$,*

$$[f,g](e^{-i\xi}) = \sum_{\alpha\in\mathbb{Z}^s} \widehat{f}(\xi+2\pi\alpha)\overline{\widehat{g}(\xi+2\pi\alpha)} \quad \text{for all } \xi\in\mathbb{R}^s.$$

**Proof:** This is a standard application of the Poisson summation formula. However, since $f$ and $g$ are in $\mathcal{L}^2$, we have to be careful about its proof. For $\xi \in \mathbb{R}^s$, using the dominated convergence theorem, we have

$$\begin{aligned}[f,g](e^{-i\xi}) &= \sum_{\alpha\in\mathbb{Z}^s} \int_{\mathbb{R}^s} f(x+\alpha)e^{-i\alpha\cdot\xi}\,\overline{g(x)}\,dx \\ &= \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} \int_{[0,1)^s} f(x+\alpha+\beta)e^{-i\alpha\cdot\xi}\,\overline{g(x+\beta)}\,dx \\ &= \int_{[0,1)^s} f_1(x)\overline{g_1(x)}\,dx,\end{aligned}$$

where

$$f_1(x) := \sum_{\alpha\in\mathbb{Z}^s} f(x+\alpha)e^{-i(x+\alpha)\cdot\xi}$$

and $g_1$ is defined in the same way. Since $f_1$ is a 1-periodic function and is square-integrable on $[0,1)^s$, it has a Fourier series expansion

$$f_1(x) \;\sim\; \sum_{\beta\in\mathbb{Z}^s} a(\beta)e^{i2\pi\beta\cdot x}.$$

where the Fourier coefficients $a(\beta)$ are given by

$$a(\beta) = \int_{[0,1)^s} f_1(x)e^{-i2\pi\beta\cdot x}\,dx = \widehat{f}(\xi+2\pi\beta).$$

In the same fashion the Fourier series expansion of $g_1$ is

$$g_1(x) \;\sim\; \sum_{\beta\in\mathbb{Z}^s} \widehat{g}(\xi+2\pi\beta)e^{i2\pi\beta\cdot x}.$$

Consequently, by Parseval's identity, we have

$$\int_{[0,1)^s} f_1(x)\overline{g_1(x)}\,dx = \sum_{\alpha\in\mathbb{Z}^s} \widehat{f}(\xi+2\pi\alpha)\overline{\widehat{g}(\xi+2\pi\alpha)},$$

as desired. ■

**Theorem 3.3.** *The integer translates of $\phi \in \mathcal{L}^2$ are $\ell^2$-stable if and only if one of the following conditions holds:*

*(i)* $\sum_{\alpha\in\mathbb{Z}^s} |\widehat{\phi}(\xi+2\pi\alpha)|^2 > 0$ *for all* $\xi \in \mathbb{R}^s$.

*(ii)* $[\phi,\phi](z) > 0$ *for all* $z \in T^s$.

*(iii) There exists* $g \in S_1(\phi)$ *such that*

$$\langle g, \phi(\cdot-\alpha)\rangle = \delta_{0\alpha} \quad \text{for all } \alpha \in \mathbb{Z}^s. \tag{3.8}$$

**Proof:** The function

$$\rho : \xi \mapsto \sum_{\alpha \in \mathbb{Z}^s} |\widehat{\phi}(\xi + 2\pi\alpha)|^2 \quad (\xi \in \mathbb{R}^s)$$

is a $2\pi$-periodic function. By Theorem 3.2, $\rho(\xi) = [\phi, \phi](e^{-i\xi})$, hence $\rho$ is continuous. Suppose that $\rho(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}^s$. Then for any given $\varepsilon > 0$ there exists $\delta$, $0 < \delta < \pi$, such that $|\xi - \xi_0| < \delta$ implies $|\rho(\xi)| < \varepsilon$. Let

$$h(\xi) := \begin{cases} 1, & \text{if } |\xi - \xi_0 - 2\pi\alpha| < \delta \text{ for some } \alpha \in \mathbb{Z}^s; \\ 0, & \text{elsewhere.} \end{cases}$$

Then $h$ is a $2\pi$-periodic function and is square-integrable on $[0, 2\pi)^s$. Hence $h(\xi) = \widetilde{a}(e^{-i\xi})$ for some $a \in \ell^2$. Let $f := \phi *' a$. The Fourier transform of $f$ is

$$\widehat{f}(\xi) = \widetilde{a}(e^{-i\xi})\widehat{\phi}(\xi) = h(\xi)\widehat{\phi}(\xi).$$

It follows that

$$\int_{\mathbb{R}^s} |\widehat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^s} |h(\xi)|^2 |\widehat{\phi}(\xi)|^2 \, d\xi = \int_{[0,2\pi)^s} |h(\xi)|^2 \rho(\xi) \, d\xi. \tag{3.9}$$

From the definition of $h$ we see that $h(\xi) \neq 0$ implies $\rho(\xi) < \varepsilon$. Hence it follows from (3.9) that

$$\|f\|_2^2 = (2\pi)^{-s} \|\widehat{f}\|_2^2 \le (2\pi)^{-s} \varepsilon^2 \int_{[0,2\pi)^s} |h(\xi)|^2 \, d\xi = \varepsilon^2 \|a\|_2^2.$$

This shows that the integer translates of $\phi$ are $\ell^2$-unstable, since $\varepsilon > 0$ can be arbitrary small. In other words, $\ell^2$-stability implies the condition (i).

By Theorem 3.2, (i) and (ii) are equivalent. Let us prove that (ii) implies (iii). If (ii) holds, then by Wiener's lemma, there exists $c \in \ell^1$ such that

$$\widetilde{c}(z) = 1/[\phi, \phi](z) \quad (z \in T^s). \tag{3.10}$$

Let $g = \phi *' c$. Then by (3.7),

$$[g, \phi](z) = \widetilde{c}(z)[\phi, \phi](z) = 1 \quad \text{for all } z \in T^s.$$

From (3.3) and (3.6) we see that this is equivalent to (3.8). Therefore (ii) implies (iii).

Finally, assume that $g \in S_1(\phi)$ satisfies (3.8). Since $\phi \in \mathcal{L}^2$, by Theorem 2.1 we have $g \in \mathcal{L}^2$. If $f = \phi *' a$ for some $a \in \ell^2(\mathbb{Z}^s)$, then by the dominated convergence theorem, we deduce from (3.8) that

$$a(\alpha) = \langle f, g(\cdot - \alpha) \rangle. \tag{3.11}$$

Applying Theorem 3.1 to $f$ and $g$, we obtain

$$\|a\|_2 \le \|f\|_2 |g|_2.$$

This shows that the integer translates of $\phi$ are $\ell^2$-stable. ■

The following theorem is an application of Theorem 3.3.

**Theorem 3.4.** *Let $\phi$ be a function in $\mathcal{E}^2$ having $\ell^2$-stable integer translates. If $a$ is a sequence in $\ell^2$ such that $f := \phi *' a$ is also in $\mathcal{E}^2$, then the sequence $a$ actually decays exponentially fast. In particular, if $\phi \in \mathcal{E}^2$ satisfies the refinement equation (2.1) with mask $b$ and has $\ell^2$-stable integer translates, then the mask $b$ decays exponentially fast.*

**Proof:** Since $\phi$ has $\ell^2$ stable integer translates, by Theorem 3.3, $[\phi, \phi](z) > 0$ for all $z \in T^s$. But $\phi \in \mathcal{E}^2$, hence $[\phi, \phi]$ is in $\mathcal{H}$, so is $1/[\phi, \phi]$. This shows that the sequence $c$ given by (3.10) decays exponentially fast. Let $g := \phi *' c$. Then $g \in \mathcal{E}^2$. By Theorem 3.3, this $g$ satisfies (3.11). Now that both $f$ and $g$ are in $\mathcal{E}^2$, hence by (3.11) we conclude that the sequence $a$ decays exponentially fast. ■

The following theorem extends Theorem 3.3 to the case in which $\phi \in \mathcal{L}^p$ $(1 \le p \le \infty)$.

**Theorem 3.5.** *Let $\phi \in \mathcal{L}^p$ $(1 \le p \le \infty)$. Then $\phi$ has $\ell^p$-stable integer translates if and only if*

$$\sup_{\alpha \in \mathbb{Z}^s} |\widehat{\phi}(\xi + 2\pi\alpha)| > 0 \quad \text{for all } \xi \in \mathbb{R}^s. \tag{3.12}$$

**Proof:** Suppose that for some $\xi \in \mathbb{R}^s$, $\widehat{\phi}(\xi + 2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$. We wish to prove that the integer translates of $\phi$ are $\ell^p$-unstable. By considering the function $x \mapsto e^{-i\xi\cdot x}\phi(x)$ $(x \in \mathbb{R}^s)$ if necessary, we may assume without loss of generality that $\widehat{\phi}(2\pi\alpha) = 0$ for all $\alpha \in \mathbb{R}^s$. With the help of the Poisson summation formula, this assumption implies that

$$\sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = 0 \tag{3.13}$$

(see the proof of Theorem 2.4). Thus the desired result for the case $p = \infty$ has been already established.

Let us now fix $p$, $1 \le p < \infty$. The sequence $e : \alpha \mapsto 1$ $(\alpha \in \mathbb{Z}^s)$ is not in $\ell^p(\mathbb{Z}^s)$, so we have to truncate $e$ as follows. For each integer $n > 0$, let $e_n$ be the sequence on $\mathbb{Z}^s$ given by

$$e_n(\alpha) = \begin{cases} 1, & \text{if } |\alpha| \le n; \\ 0, & \text{otherwise.} \end{cases}$$

To prove that the integer translates of $\phi$ are $\ell^p$-unstable, it suffices to show that

$$\|\phi *' e_n\|_p / \|e_n\|_p \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

To this end, we first truncate $\phi$ as follows. For each integer $N > 0$, let $\phi_N$ be the function on $\mathbb{R}^s$ given by

$$\phi_N(x) = \begin{cases} \phi(x), & \text{if } |x| \le N; \\ 0, & \text{otherwise.} \end{cases}$$

Then we set

$$\psi := \phi_N + \psi_N,$$

where $\psi_N$ is the function on $\mathbb{R}^s$ given by

$$\psi_N(x) := \begin{cases} \sum_{\alpha\in\mathbb{Z}^s}(\phi-\phi_N)(x-\alpha), & \text{if } x\in[0,1)^s; \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the construction of $\psi_N$ that

$$|\psi_N|_p \le |\phi-\phi_N|_p.$$

Hence we have

$$|\phi-\psi|_p = |\phi-\phi_N-\psi_N|_p \le |\phi-\phi_N|_p + |\psi_N|_p \le 2|\phi-\phi_N|_p. \qquad (3.15)$$

This together with Theorem 2.1 gives the following estimate:

$$\|(\phi-\psi)*'e_n\|_p \le |\phi-\psi|_p\|e_n\|_p \le 2|\phi-\phi_N|_p\|e_n\|_p.$$

Therefore we obtain

$$\|\phi*'e_n\|_p/\|e_n\|_p \le \|\psi*'e_n\|_p/\|e_n\|_p + 2|\phi-\phi_N|_p. \qquad (3.16)$$

By the dominated convergence theorem, $|\phi-\phi_N|_p \to 0$ as $N\to\infty$. Thus it remains to estimate $\|\psi*'e_n\|_p/\|e_n\|_p$. For this purpose, we first observe that $\psi$ is compactly supported:

$$\psi(x) = 0 \quad \text{for } |x| > N. \qquad (3.17)$$

Second, by (3.13), we see from the construction of $\psi$ that

$$\sum_{\alpha\in\mathbb{Z}^s}\psi(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}(\phi_N+\psi_N)(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\phi(\cdot-\alpha) = 0. \qquad (3.18)$$

Let $n > N$ and consider $\psi*'e_n$. By (3.17) and (3.18), we have

$$\sum_{\alpha\in\mathbb{Z}^s} e_n(\alpha)\psi(x-\alpha) = 0 \quad \text{for } |x| > n+N \text{ or } |x| < n-N.$$

It follows that

$$\begin{aligned} \|\psi*'e_n\|_p^p &= \int_{n-N\le|x|\le n+N}\Big|\sum_{\alpha\in\mathbb{Z}^s} e_n(\alpha)\psi(x-\alpha)\Big|^p\,dx \\ &\le \sum_{n-N\le|\beta|\le n+N}\int_{\beta+[0,1)^s}|\psi^\circ(x)|^p\,dx. \end{aligned}$$

The cardinality of the set $\{\beta \in \mathbb{Z}^s : n - N \le |\beta| \le n + N\}$ does not exceed $C_1 n^{s-1} N$, where $C_1 > 0$ is a constant. Thus the above estimate yields

$$\|\psi *' e_n\|_p^p \le C_1 n^{s-1} N |\psi|_p^p.$$

Furthermore, there exists a constant $C_2 > 0$ such that

$$\|e_n\|_p^p = \sum_{|\alpha| \le n} 1 \ge C_2 n^s.$$

Thus we get the following estimate:

$$\|\psi *' e_n\|_p / \|e_n\|_p \le \big((C_1 N)/(C_2 n)\big)^{1/p} |\psi|_p. \tag{3.19}$$

Note that by (3.15) we have

$$|\psi|_p \le |\phi|_p + 2|\phi - \phi_N|_p. \tag{3.20}$$

Choose $N$ to be the integer part of $\sqrt{n}$. Then we conclude from (3.19) and (3.20) that

$$\|\psi *' e_n\|_p / \|e_n\|_p \to 0 \quad \text{as } n \to \infty.$$

This verifies (3.14), so that the necessity part of the theorem has been proved.

Let us now assume that $\phi \in \mathcal{L}^p$ satisfies (3.12). To prove that $\phi$ has $\ell^p$-stable integer translates, we assume first that $\phi \in \mathcal{L}^\infty$. In this case, Theorem 3.3 is applicable; hence there exists $g \in S_1(\phi)$ satisfying (3.8). By Theorem 2.1, we have $g \in \mathcal{L}^\infty$. If $f = \phi *' a$ for some sequence $a \in \ell^p$, then (3.11) holds because of (3.8). Applying Theorem 3.1 to $f$ and $g$, we obtain

$$\|a\|_p \le \|f\|_p |g|_q \le \|f\|_p |g|_\infty.$$

Thus the integer translates of $\phi$ are $\ell^p$-stable.

To deal with the case $\phi \in \mathcal{L}^p$ $(1 \le p < \infty)$, we smooth $\phi$ by convolving it with the function $\chi$ given by

$$\chi(x) := e^{-\pi |x|^2}, \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s,$$

where $|x|^2 = \sum_{j=1}^s |x_j|^2$. Note that $\|\chi\|_1 = 1$ and

$$\widehat{\chi}(\xi) = e^{-|\xi|^2/(4\pi)}, \quad \xi \in \mathbb{R}^s,$$

(see, *e.g.*, [18, Proposition 8.24]). In particular, $\widehat{\chi}(\xi)$ is positive everywhere. Now, $\rho := \phi * \chi$ is in $\mathcal{L}^\infty$, because $|\rho|_\infty \le |\phi|_p |\chi|_q$, where $1/p + 1/q = 1$. Moreover, since $\widehat{\chi}$ never vanishes, $\rho$ also satisfies (3.12). By what has been proved, $\rho$ has $\ell^p$-stable integer translates. For any sequence $a \in \ell^p(\mathbb{Z}^s)$, by Young's inequality, we have

$$\|\rho *' a\|_p = \|\chi * (\phi *' a)\|_p \le \|\chi\|_1 \|\phi *' a\|_p = \|\phi *' a\|_p.$$

This shows that $\phi$ also has $\ell^p$-stable integer translates. ■

**Remark 3.1.** Since the condition (3.12) does not involve $p$, we may drop the qualifier $\ell^p$- henceforth. In what follows, we say that a function $\phi \in L^1$ has stable integer translates, if $\phi$ satisfies the conditions (3.12).

## §4. Stability

When $s > 1$, one has to find more than one functions to construct a pre-wavelet basis. Thus it is natural to consider the stability of the integer translates of several functions.

Let $\phi_1, \ldots, \phi_n$ be functions in $\mathcal{L}^p(\mathbb{R}^s)$. These functions give rise to a linear mapping

$$L_{\phi_1,\ldots,\phi_n} : (a_1, \ldots, a_n) \mapsto \sum_{j=1}^{n} \phi_j *' a_j, \quad a_1, \ldots, a_n \in \ell^p.$$

By Theorem 2.1, $L_{\phi_1,\ldots,\phi_n}$ is a bounded linear mapping from $(\ell^p)^n$ into $L^p$. The image of $(\ell^p)^n$ under the mapping $L_{\phi_1,\ldots,\phi_n}$ is denoted by $S_p(\phi_1, \ldots, \phi_n)$. The integer translates $\phi_j(\cdot - \alpha)$ ($\alpha \in \mathbb{Z}^s$; $j = 1, \ldots, n$) are said to be $\ell^p$-stable if there exists a positive constant $C_p$ such that

$$\|L_{\phi_1,\ldots,\phi_n}(a_1, \ldots, a_n)\|_p \geq C_p \sum_{j=1}^{n} \|a_j\|_p.$$

Recently, Jia and Micchelli [21] gave a necessary and sufficient condition for stability of integer translates of a finite number of compactly supported continuous functions. The following theorem extend their result to functions in $\mathcal{L}^2$.

**Theorem 4.1.** *Let $\phi_1, \ldots, \phi_n \in \mathcal{L}^2$. Then the integer translates of $\phi_1, \ldots, \phi_n$ are $\ell^2$-stable if and only if one of the following conditions holds:*

*(i) For any $\xi \in \mathbb{R}^s$, the sequences $\big(\widehat{\phi}_j(\xi + 2\pi\alpha)\big)_{\alpha \in \mathbb{Z}^s}$ ($j = 1, \ldots, n$) are linearly independent.*

*(ii) The matrix $\big([\phi_j, \phi_k](z)\big)_{1 \leq j,k \leq n}$ is positive definite for every $z \in T^s$.*

*(iii) There exist $g_1, \ldots, g_n \in S_1(\phi_1, \ldots, \phi_n)$ such that*

$$\langle g_j, \phi_k(\cdot - \alpha)\rangle = \delta_{jk}\delta_{0\alpha} \quad \text{for } 1 \leq j \leq k \leq n \text{ and } \alpha \in \mathbb{Z}^s. \tag{4.1}$$

*Moreover, if $\phi_1, \ldots, \phi_n \in \mathcal{E}^2$ have $\ell^2$-stable integer translates, $a_1, \ldots, a_n \in \ell^2$, and if $\sum_{j=1}^{n} \phi_j *' a_j \in \mathcal{E}^2$, then all the sequences $a_j$ actually decay exponentially fast.*

**Proof:** If, for some $\xi \in \mathbb{R}^s$, the sequences $\big(\widehat{\phi}_j(\xi + 2\pi\alpha)\big)_{\alpha \in \mathbb{Z}^s}$ ($j = 1, \ldots, n$) are linearly dependent, then there exist constants $r_j$ ($j = 1, \ldots, n$), not all zero, such that

$$\sum_{j=1}^{n} r_j \widehat{\phi}_j(\xi + 2\pi\alpha) = 0 \quad \text{for all } \alpha \in \mathbb{Z}^s.$$

Let $\phi := \sum_{j=1}^{n} r_j \phi_j$. Then by Theorem 3.3, the integer translates of $\phi$ are $\ell^2$-unstable. It follows that the integer translates of $\phi_1, \ldots, \phi_n$ are $\ell^2$-unstable. This proves that $\ell^2$-stability implies (i).

By Theorem 3.2, the matrix $\big([\phi_j,\phi_k](e^{-i\xi})\big)_{1\le j,k\le n}$ is the Gram matrix of the elements $\big(\widehat{\phi}_j(\xi+2\pi\alpha)\big)_{\alpha\in\mathbb{Z}^s}\in\ell^2$, hence (i) implies (ii).

If the matrix $\Phi(z):=\big([\phi_j,\phi_k](z)\big)_{1\le j,k\le n}$ is nonsingular for every $z\in T^s$, then its inverse matrix exists and has its all entries in $\mathcal{B}$. Choose $b_{jk}\in\ell^1$ $(j,k=1,\ldots,n)$ such that the matrix $\big(\widetilde{b}_{jk}(z)\big)_{1\le j,k\le n}$ is the inverse of $\Phi(z)$. Let

$$g_j:=\sum_{k=1}^n \phi_k *' b_{jk}.$$

Then $g_j\in S_1(\phi_1,\ldots,\phi_n)$ $(j=1,\ldots,n)$. By Theorem 2.1 we have $g_j\in\mathcal{L}^2$. We claim that $g_j$ satisfy (4.1). Indeed, for $1\le j,m\le n$, it follows from (3.7) and the above definition of $g_j$ that

$$[g_j,\phi_m](z)=\sum_{k=1}^n \widetilde{b}_{jk}(z)[\phi_k,\phi_m](z)=\delta_{jm}\quad\text{for all } z\in T^s.$$

Thus (ii) implies (iii).

Now let us prove that (iii) implies $\ell^2$-stability. Given $a_1,\ldots,a_n\in\ell^2$, let $f:=\sum_{j=1}^n a_j*'\phi_j$. Then $f\in L^2$. It follows from (4.1) that

$$a_j(\alpha)=\langle f, g_j(\cdot-\alpha)\rangle\quad\text{for all } \alpha\in\mathbb{Z}^s. \tag{4.2}$$

Consequently, by Theorem 3.1 we have

$$\|a_j\|_2\le|g_j|_2\|f\|_2\quad(j=1,\ldots,n).$$

In other words, $\phi_1,\ldots,\phi_n$ have $\ell^2$-stable integer translates.

Finally, if in addition, $\phi_1,\ldots,\phi_n\in\mathcal{E}^2$ and $f:=\sum_{j=1}^n\phi_j*'a_j\in\mathcal{E}^2$, then all the above sequences $b_{jk}$ decay exponentially fast. Hence all the functions $g_j$ are in $\mathcal{E}^2$. But $f$ is also in $\mathcal{E}^2$, therefore by Theorem 3.1 it follows from (4.2) that the sequences $a_j$ decay exponentially fast. ■

The following theorem gives a criterion for $\ell^p$-stability. Its proof is clear from those of Theorem 3.5 and Theorem 4.1.

**Theorem 4.2.** *Let $\phi_1,\ldots,\phi_n\in\mathcal{L}^p$ $(1\le p\le\infty)$. Then the integer translates of $\phi_1,\ldots,\phi_n$ are $\ell^p$-stable if and only if the conditions (i) of Theorem 4.1 holds.*

**Remark 4.1.** Again, by Theorem 4.2, we may drop the qualifier $\ell^p$- for stability of the integer translates of several functions. In what follows, we say that $\phi_1,\ldots,\phi_n\in L^1$ have stable integer translates, if they satisfy the condition (i) of Theorem 4.1.

Suppose that $\phi_1,\ldots,\phi_n\in\mathcal{L}^2$ have stable integer translates. Let

$$\psi_1,\ldots,\psi_m\in V:=S_1(\phi_1,\ldots,\phi_n).$$

We are interested in the conditions under which the integer translates of $\psi_1, \dots, \psi_m$ form an unconditional basis for $V$. Since $\psi_j \in S_1(\phi_1, \dots, \phi_n)$, there are sequences $a_{jk} \in \ell^1$ $(j = 1, \dots, m; k = 1, \dots, n)$ such that

$$\psi_j = \sum_{k=1}^{n} \phi_k *' a_{jk} \quad (j = 1, \dots, m). \tag{4.3}$$

Denote by $A(z)$ the $m \times n$ matrix $\big(\widetilde{a}_{jk}(z)\big)$. Taking Fourier transform of both sides of (4.3), we obtain

$$\widehat{\psi}_j(\xi) = \sum_{k=1}^{n} \widetilde{a}_{jk}(e^{-i\xi}) \widehat{\phi}_k(\xi). \tag{4.4}$$

Thus, if the matrix $A(z)$ has rank less than $m$ for some $z \in T^s$, or if $\phi_1, \dots, \phi_n$ have unstable integer translates, then for some $\xi \in \mathbb{R}^s$ the sequences

$$\big(\widehat{\psi}_j(\xi + 2\pi\alpha)\big)_{\alpha \in \mathbb{Z}^s} \quad (j = 1, \dots, m)$$

are linearly independent; hence, by Theorem 4.1, the integer translates of $\psi_1, \dots, \psi_m$ are unstable. Moreover, if $m < n$, then $V \neq S_1(\psi_1, \dots, \psi_m)$, for otherwise, by the same reason, the integer translates of $\phi_1, \dots, \phi_n$ would be unstable. Thus $m = n$ is a necessary condition for

$$\{\psi_j(\cdot - \alpha) : \ j = 1, \dots, m; \alpha \in \mathbb{Z}^s\}$$

to be an unconditional basis for $V$.

**Theorem 4.3.** *Suppose that $\phi_1, \dots, \phi_n \in \mathcal{L}^2$ have stable integer translates. Let $m = n$ and let $\psi_1, \dots, \psi_n$ be the functions given in (4.3) with all the sequences $a_{jk} \in \ell^1$. Then the following conditions are equivalent:*

*(i) $\psi_1, \dots, \psi_n$ have stable integer translates.*
*(ii) The matrix $A(z) := \big(a_{jk}(z)\big)$ is nonsingular for every $z \in T^s$.*
*(iii) $S_p(\phi_1, \dots, \phi_n) = S_p(\psi_1, \dots, \psi_n)$ for all $p$, $1 \le p \le 2$.*
*(iv) $S_p(\phi_1, \dots, \phi_n) = S_p(\psi_1, \dots, \psi_n)$ for some $p$, $1 \le p \le 2$.*

**Proof:** If the matrix $A(z)$ is singular for some $z = e^{-i\xi} \in T^s$, then from (4.4) we see that the sequences $\big(\widehat{\psi}_j(\xi + 2\pi\alpha)\big)_{\alpha \in \mathbb{Z}^s}$ $(j = 1, \dots, n)$ are linearly dependent. Hence by Theorem 4.1, the integer translates of $\psi_1, \dots, \psi_n$ are unstable. This shows that (i) implies (ii).

If (ii) holds, then $\big(A(z)\big)^{-1}$ exists for every $z \in T^s$ and has all its entries in $\mathcal{B}$. In other words, there exist $b_{jk} \in \ell^1(\mathbb{Z}^s)$ $(j, k = 1, \dots, n)$ such that $\big(\widetilde{b}_{jk}(z)\big) = \big(A(z)\big)^{-1}$. From (4.4) we deduce that

$$\widehat{\phi}_j(\xi) = \sum_{j=1}^{n} \widetilde{b}_{jk}(e^{-i\xi}) \widehat{\psi}_k(\xi) \quad \text{for all } \xi \in \mathbb{R}^s.$$

This shows that $\phi_j \in S_1(\psi_1, \dots, \psi_n)$ $(j = 1, \dots, n)$, from which (iii) follows. Evidently, (iii) implies (iv).

Finally, if (iv) holds, then $\psi_1, \dots, \psi_n$ must have stable integer translates, for otherwise $\phi_1, \dots, \phi_n$ would have unstable integer translates. ■

**Theorem 4.4.** *Suppose that $\phi_1, \dots, \phi_n \in \mathcal{L}^2$ have stable integer translates. Then there exist $\psi_1, \dots, \psi_n \in V := S_1(\phi_1, \dots, \phi_n)$ such that*

*(i) $\psi_1 = \phi_1$,*

*(ii) $S_1(\phi_1, \dots, \phi_j) = S_1(\psi_1, \dots, \psi_j)$ $(j = 2, \dots, n)$*

*and*

*(iii) The spaces $S_2(\psi_j)$ $(j = 1, \dots, n)$ are mutually orthogonal.*

*Thus $S_2(\phi_1, \dots, \phi_n)$ is the orthogonal sum of the spaces $S_2(\psi_j)$ $(j = 1, \dots, n)$. If $\phi_1, \dots, \phi_n \in \mathcal{E}^2$, then $\psi_1, \dots, \psi_n$ can be chosen to be functions in $\mathcal{E}^2$. Furthermore, if $\phi_1, \dots, \phi_n$ are compactly supported, then $\psi_1, \dots, \psi_n$ can be so chosen that they are also compactly supported.*

**Proof:** The proof proceeds by induction on $n$. The case $n = 1$ is trivial. Suppose that the theorem is true for $n$ and we wish to establish it for $n + 1$. Let $\phi_1, \dots, \phi_{n+1} \in \mathcal{L}^2$ have stable integer translates. By induction hypothesis, there exist $\psi_1, \dots, \psi_n$ satisfying (i), (ii) and (iii). By Theorem 4.3, it follows from (ii) that $\psi_1, \dots, \psi_n$ have stable integer translates. In particular,

$$[\psi_j, \psi_j](z) > 0 \quad \text{for every } z \in T^s.$$

Moreover, for $1 \le j, k \le n$, $j \ne k$,

$$[\psi_j, \psi_k](z) = 0 \quad \text{for every } z \in T^s. \tag{4.5}$$

We claim that there exist sequences $a_j \in \ell^1(\mathbb{Z}^s)$ $(j = 1, \dots, n)$ such that the function

$$\psi_{n+1} := \phi_{n+1} - \sum_{j=1}^{n} \psi_j *' a_j \tag{4.6}$$

is orthogonal to every $S_2(\psi_j)$. Using symbol calculus, we see that this is equivalent to

$$[\psi_{n+1}, \psi_j](z) = 0 \quad \text{for all } z \in T^s.$$

Substituting the expression (4.6) for $\psi_{n+1}$ in the above equation and taking (4.5) into account, we get

$$[\phi_{n+1}, \psi_j](z) = \widetilde{a}_j(z)[\psi_j, \psi_j](z). \tag{4.7}$$

Since $[\psi_j, \psi_j](z) > 0$ for every $z \in T^s$, by Wiener's lemma, there exists $a_j \in \ell^1$ such that (4.7) holds. With these $a_j$ $(j = 1, \dots, n)$, the function given in (4.6) is what we desired.

If all $\phi_1, \dots, \phi_{n+1} \in \mathcal{E}^2$, then the symbols $[\phi_{n+1}, \psi_j](z)$ and $[\psi_j, \psi_j](z)$ appearing in (4.7) are functions in $\mathcal{H}$. Since $[\psi_j, \psi_j](z) > 0$ for all $z \in T^s$,

$\widetilde{a}_j(z) = [\phi_{n+1}, \psi_j](z)/[\psi_j, \psi_j](z)$ is also a function in $\mathcal{H}$. Therefore the sequences $a_j$ $(j = 1, \dots, n)$ actually decay exponentially fast. Thus the function $\psi_{n+1}$ as given in (4.6) is in $\mathcal{E}^2$.

Suppose now that all $\phi_1, \dots, \phi_n, \phi_{n+1}$ are compactly supported. By induction hypothesis, there are compactly supported functions $\psi_1, \dots, \psi_n$ satisfying the condition (i), (ii) and (iii). Since $\psi_j$ are compactly supported $(j = 1, \dots, n)$, $[\psi_j, \psi_j](z)$ are Laurent polynomials in $z$. There exist finitely supported sequences $c_j$ such that $\widetilde{c}_j(z) = [\psi_j, \psi_j](z)$. Let $c = c_1 * c_2 * \cdots * c_n$. Then $c$ is also a finitely supported sequence and

$$\widetilde{c}(z) = \prod_{j=1}^{n} \widetilde{c}_j(z) > 0 \quad \text{for all } z \in T^s.$$

Let

$$\psi_{n+1} := \big(\phi_{n+1} - \sum_{j=1}^{n} \psi_j *' a_j\big) *' c,$$

where $a_j \in \ell^1$ are defined by (4.7). Then $\psi_{n+1}$ is orthogonal to $S_2(\phi_1, \dots, \phi_n)$. By the induction hypothesis we see from the construction of $\psi_{n+1}$ that

$$\psi_{n+1} \in S_1(\psi_1, \dots, \psi_n, \phi_{n+1}) = S_1(\phi_1, \dots, \phi_n, \phi_{n+1}).$$

Moreover, since $\widetilde{c}(z) > 0$ for all $z \in T^s$, by Wiener's Lemma we have

$$\phi_{n+1} \in S_1(\psi_1, \dots, \psi_{n+1}).$$

This shows that

$$S_1(\psi_1, \dots, \psi_{n+1}) = S_1(\phi_1, \dots, \phi_{n+1}).$$

We claim that $\psi_{n+1}$ is compactly supported. Indeed, $\psi_{n+1}$ can be rewritten as

$$\psi_{n+1} = \phi_{n+1} *' c - \sum_{j=1}^{n} \psi_j *' (a_j * c),$$

whereas

$$(\widetilde{a_j * c})(z) = \widetilde{a}_j(z) \prod_{k=1}^{n} \widetilde{c}_k(z) = [\phi_{n+1}, \psi_j](z) \prod_{k \neq j} [\psi_k, \psi_k](z)$$

are Laurent polynomials in $z$. This shows that $a_j * c$ are finitely supported sequences $(j = 1, \dots, n)$. It follows that $\psi_{n+1}$ is compactly supported. ■

**Remark 4.2.** Note that the above proof is essentially the Gram-Schmidt orthogonalization method.

## §5. Linear Independence

Linear independence of integer translates of functions is a concept closely related to stability. We shall see later that this concept plays an important role in constructing a pre-wavelet basis using box splines.

Denote by $\ell(\mathbb{Z}^s)$ the linear space of all sequences on $\mathbb{Z}^s$. Let $\phi_1, \ldots, \phi_n$ be compactly supported functions on $\mathbb{R}^s$. These functions give rise to a linear mapping $L_{\phi_1,\ldots,\phi_n}$ defined on $(\ell(\mathbb{Z}^s))^n$ as follows:

$$L_{\phi_1,\ldots,\phi_n} : (a_1, \ldots, a_n) \mapsto \sum_{j=1}^{n} \sum_{\alpha \in \mathbb{Z}^s} a_j(\alpha)\phi_j(\cdot - \alpha), \quad \text{for } a_1, \ldots, a_n \in \ell(\mathbb{Z}^s).$$

Denote by $S(\phi_1, \ldots, \phi_n)$ the image of $\bigl(\ell(\mathbb{Z}^s)\bigr)^n$ under the mapping $L_{\phi_1,\ldots,\phi_n}$. We say that the translates $\phi_j(\cdot - \alpha)$ ($\alpha \in \mathbb{Z}^s$, $j = 1, \ldots, n$) are *(algebraically) linearly independent* if the mapping $L_{\phi_1,\cdots,\phi_n}$ is injective.

Algebraic linear independence for integer translates of one function was investigated in [12]. It was later pointed out in [30] that this problem is related to the structure of closed shift invariant subspaces of the sequence space $\ell(\mathbb{Z}^s)$ equipped with the pointwise convergence topology, as was studied in [24]. Recently, Jia and Micchelli [21] proved the following theorem, which provides a necessary and sufficient condition for the algebraic linear independence of the integer translates of a finite number of functions.

**Theorem 5.1.** *Let $\phi_1, \ldots, \phi_n$ be compactly supported distributions. Then the integer translates of $\phi_1, \ldots, \phi_n$ are algebraically linearly independent if and only if for any $\xi \in \mathbb{C}^s$, the sequences*

$$\bigl(\widehat{\phi}_j(\xi + 2\pi\alpha)\bigr)_{\alpha \in \mathbb{Z}^s} \quad (j = 1, \ldots, n)$$

*are linearly independent.*

From this theorem and Theorem 4.2 we see that linear independence implies stability, but the converse is not true.

**Example 5.1**. Let

$$\phi(x) := \begin{cases} 1, & \text{if } 0 \le x < 1; \\ 1/2, & \text{if } 1 \le x < 2; \\ 0, & \text{elsewhere.} \end{cases}$$

Then the integer translates $\phi(\cdot - \alpha)$ ($\alpha \in \mathbb{Z}$) are stable but not linearly independent.

Note that $\phi$ satisfies the refinement equation

$$\phi = \sum_{j \in \mathbb{Z}} b(j)\phi(2 \cdot -j),$$

where the mask $b$ is given by

$$\widetilde{b}(z) = (z + 1)(z^2 + 2)/(z + 2).$$

Thus the mask $b$ is not finitely supported, even though $\phi$ is compactly supported. However, if $\phi$ has linearly independent integer translates and satisfies the refinement equation (2.1), then the mask $b$ must be finitely supported. This follows from the following theorem, which was proved in [4].

**Theorem 5.2.** *Let $\phi_1, \ldots, \phi_n$ be compactly supported functions having linearly independent integer translates, and let $a_1, \ldots, a_n$ be sequences on $\mathbb{Z}^s$. If the function $\sum_{j=1}^n \phi_j *' a_j$ is compactly supported, then all the sequences $a_1, \ldots, a_n$ must be finitely supported.*

Note that if the integer translates of $\phi$ are mutually orthogonal, then they are linearly independent. Hence a refinable function having orthogonal integer translates has a finitely supported mask. This special case of Theorem 5.2 was already observed in [17] and [16].

Suppose that $\phi_1, \ldots, \phi_n$ are compactly supported functions having linearly independent integer translates. Let $\psi_1, \ldots, \psi_m \in J := S(\phi_1, \ldots, \phi_n)$. Assume that they are also compactly supported. Since $\psi_j \in S(\phi_1, \ldots, \phi_n)$, there are sequences $a_{jk}$ $(j = 1, \ldots, m; k = 1, \ldots, n)$ such that

$$\psi_j = \sum_{k=1}^{n} \phi_k *' a_{jk} \quad (j = 1, \ldots, m). \tag{5.1}$$

By Theorem 5.2, all the sequences $a_{jk}$ are finitely supported. Denote by $A(z)$ the $m \times n$ matrix $\big(\widetilde{a}_{jk}(z)\big)$.Then the entries of $A(z)$ are Laurent polynomials. We see that if $m > n$, then the integer translates of $\psi_1, \ldots, \psi_n$ are linearly *dependent.* If $m < n$, then $J \neq S(\psi_1, \ldots, \psi_m)$. In the case $m = n$, the following theorem gives an analog of Theorem 4.3.

**Theorem 5.3.** *Suppose that $\phi_1, \ldots, \phi_n$ are compactly supported functions having linearly independent integer translates. Let $m = n$ and $\psi_1, \ldots, \psi_n$ be the compactly supported functions as given in (5.1). Then the following conditions are equivalent:*

*(i)* $\psi_1, \ldots, \psi_n$ *have linearly independent integer translates.*
*(ii) The matrix* $A(z) := \big(a_{jk}(z)\big)$ *is nonsingular for every* $z \in (\mathbb{C}\backslash\{0\})^s$.
*(iii)* $S(\phi_1, \ldots, \phi_n) = S(\psi_1, \ldots, \psi_n)$.

**Proof:** The proof is almost identical to that of Theorem 4.3. We only need to point out that if $A(z)$ satisfies the condition (ii), then $\det(A(z))$ is a Laurent polynomial which vanishes nowhere on $(\mathbb{C}\backslash\{0\})^s$. Hence by Hilbert Nullstellensatz, $\det(A(z))$ is of the form $rz^\alpha$, where $r \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{Z}^s$. This shows that all the entries of the inverse of $A(z)$ are Laurent polynomials. ■

## §6. Pre-Wavelet Decomposition

Let $\phi$ be a refinable function in $\mathcal{L}^2$ having stable integer translates. Let $V_0 := S_2(\phi)$, $V_j := \sigma_j(V_0)$ $(j \in \mathbb{Z})$ and let $W_j$ be the orthogonal complement of $V_j$ in $V_{j+1}$. Recall that a function in $W_0$ is a pre-wavelet. In this section we shall give a necessary and sufficient condition for $W_0$ to have an unconditional basis consisting of the integer translates of certain pre-wavelets.

Let $E = E_s$ be the set of all extreme points of the unit cube $[0,1]^s$, *i.e.*,

$$E = E_s := \{(\nu_1, \ldots, \nu_s) : \ \nu_j = 0 \text{ or } 1 \text{ for all } j\}.$$

This set affects a decomposition of the lattice $\mathbb{Z}^s$ into $2^s$ sublattices $2\mathbb{Z}^s + \mu$ ($\mu \in E$). It is convenient to use $E$ as an index set.

Suppose that $\phi$ satisfies the refinement equation (2.1) with $b \in \ell^1(\mathbb{Z}^s)$ as its mask. Let $p(z)$ be the symbol $\widetilde{b}(z)$. The sequence $b$ gives rise to $2^s$ sequences $b_\nu$ ($\nu \in E$) as follows:

$$b_\nu(\beta) := b(\nu + 2\beta), \quad \beta \in \mathbb{Z}^s. \tag{6.1}$$

Let

$$p_\nu(z) := \widetilde{b}_\nu(z) = \sum_{\beta \in \mathbb{Z}^s} b(\nu + 2\beta) z^\beta. \tag{6.2}$$

Then $p_\nu \in \mathcal{B}$. Correspondingly, we set

$$\phi_\nu := \phi(2 \cdot - \nu). \tag{6.3}$$

It follows from (6.1) and (6.3) that

$$\phi = \sum_{\nu \in E} \phi_\nu *' b_\nu. \tag{6.4}$$

Let $(p_1, \ldots, p_n)$ be an $n$-tuple of elements of $\mathcal{B}$. We say that $(p_1, \ldots, p_n)$ is *extensible* over $\mathcal{B}$, if there exist $n^2$ elements $p_{jk} \in \mathcal{B}$ ($j, k = 1, \ldots, n$) such that $p_{1k} = p_k$ for all $k$ and the matrix $P(z) := \big(p_{jk}(z)\big)$ is nonsingular for all $z \in T^s$. Note that if $(p_1, \ldots, p_n)$ is extensible, then $p_{jk}$ ($j = 2, \ldots, n; k = 1, \ldots, n$) can be chosen to be Laurent polynomials. Let us verify this fact. Given a Laurent series $p(z) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha$ and a positive integer $N$, we denote by $p^{(N)}(z)$ the partial sum $\sum_{|\alpha| \le N} a(\alpha) z^\alpha$. Since $a \in \ell^1(\mathbb{Z}^s)$, $p^{(N)}(z)$ converges to $p(z)$ uniformly on $T^s$ as $N$ goes to $\infty$. Let $P^{(N)}(z)$ be the matrix obtained from $P(z) = \big(b_{jk}(z)\big)$ through replacing $p_{jk}(z)$ by $p_{jk}^{(N)}(z)$ ($j = 2, \ldots, n; k = 1, \ldots, n$). Thus if $P(z)$ is nonsingular for every $z \in T^s$, then for sufficiently large $N$, $P^{(N)}(z)$ is nonsingular for every $z \in T^s$.

The following theorem shows that the existence of a pre-wavelet basis for $W_0$ is equivalent to the extensibility of the $2^s$-tuple $(p_\nu)_{\nu \in E}$ over $\mathcal{B}$.

**Theorem 6.1.** *There exist $2^s - 1$ functions $\psi_\mu \in V_1$ ($\mu \in E \backslash \{0\}$) such that their integer translates form an unconditional basis for $W_0$ if and only if the $2^s$-tuple $(p_\nu)_{\nu \in E}$ is extensible over $\mathcal{B}$, where $p_\nu$ ($\nu \in E$) are as given in (6.2). If $\phi \in \mathcal{E}^2$ and $(p_\nu)_{\nu \in E}$ is extensible over $\mathcal{B}$, then $\psi_\mu$ can be chosen to be functions in $\mathcal{E}^2$. Furthermore, if $\phi$ is compactly supported, then $\psi_\mu$ can be chosen to be compactly supported.*

**Proof:** Let $\phi_\nu$ be as given in (6.3). Since the integer translates of $\phi$ are stable, so are the integer translates of $\phi_\nu$ ($\nu \in E$). If one can find $2^s - 1$ functions $\psi_\mu \in V_1$ ($\mu \in E \backslash \{0\}$) such that their integer translates form an unconditional basis for $W_0$, then with $\psi_0 := \phi$, the integer translates of $\psi_\mu$ ($\mu \in E$) form an unconditional basis for $V_1$. Each $\psi_\mu$ is of the form

$$\psi_\mu = \sum_{\nu \in E} \phi_\nu *' b_{\mu\nu},$$

where $b_{\mu\nu}$ are sequences in $\ell^1(\mathbb{Z}^s)$. In particular, $b_{0\nu} = b_\nu$ for all $\nu \in E$. Let $p_{\mu\nu}(z) := \widetilde{b}_{\mu\nu}(z)$. Then

$$p_{0\nu}(z) = \widetilde{b}_{0\nu}(z) = \widetilde{b}_\nu(z) = p_\nu(z).$$

Moreover, by Theorem 4.3, the matrix $P(z) := \big(p_{\mu\nu}(z)\big)_{\mu,\nu\in E}$ is nonsingular for every $z \in T^s$. This proves the necessity.

Suppose conversely that $(p_\nu)_{\nu\in E}$ is extensible over $\mathcal{B}$. Then we can find Laurent polynomials $p_{\mu\nu}$ ($\mu \in E\backslash\{0\}, \nu \in E$) such that with $p_{0\nu} = p_\nu$ for all $\nu \in E$ the matrix $P(z) := \big(p_{\mu\nu}(z)\big)_{\mu,\nu\in E}$ is nonsingular for every $z \in T^s$. Each $p_{\mu\nu}(z)$ is the symbol of some sequence $b_{\mu\nu} \in \ell^1(\mathbb{Z}^s)$. Moreover, when $\mu \in E\backslash\{0\}$, the sequences $b_{\mu\nu}$ are finitely supported. Set

$$\rho_\mu := \sum_{\nu\in E} \phi_\nu *' b_{\mu\nu}.$$

Since $P(z)$ is nonsingular for every $z \in T^s$, by Theorem 4.3 the integer translates of $\rho_\mu$ ($\mu \in E$) are stable. Furthermore, by Theorem 4.4 we can find functions $\psi_\mu \in V_1$ ($\mu \in E$) such that (i)$\psi_0 = \rho_0$, (ii) the spaces $S_2(\psi_\mu)$ ($\mu \in E$) are mutually orthogonal and (iii) $V_1$ is the sum of the spaces $S_2(\psi_\mu)$ ($\mu \in E$). It follows that $W_0$ is the orthogonal sum of $S_2(\psi_\mu)$ ($\mu \in E\backslash\{0\}$). This proves the sufficiency.

If, in addition, $\phi \in \mathcal{E}^2$, then all $\rho_\mu \in \mathcal{E}^2$ ($\mu \in E$). Invoking Theorem 4.4, we conclude that $\psi_\mu$ can be chosen to bo functions in $\mathcal{E}^2$.

Finally, if $\phi$ is compactly supported, then all $\rho_\mu$ are compactly supported, so we can appeal to Theorem 4.4 again. ■

The following example illustrates Theorem 6.1.

**Example 6.1**. Let $\phi$ be the function on $\mathbb{R}^2$ which has value 1 at the origin, is piecewise linear and is zero outside the hexagon

$$\big\{(x_1, x_2) \in \mathbb{R}^2 : \ \max\{|x_1|, |x_2|, |x_1 - x_2|\} \le 1\big\}.$$

Then $\phi$ satisfies the refinement equation (2.1) with mask $b$ given by

$$\widetilde{b}(z) = 1 + (z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1})/2,$$

where $z = (z_1, z_2) \in (\mathbb{C}\backslash\{0\})^2$. Since $s = 2$, we have

$$E = E_2 = \{(0,0), (1,0), (0,1), (1,1)\}.$$

Let $b_\nu$ ($\nu \in E$) be the sequences as given in (6.1) and let $p_\nu(z)$ be the symbol $\widetilde{b}_\nu(z)$. Then

$$\begin{aligned} p_{0,0}(z) &= 1, & p_{1,0}(z) &= (1 + z_1^{-1})/2, \\ p_{0,1}(z) &= (1 + z_2^{-1})/2, & p_{1,1}(z) &= (1 + z_1^{-1} z_2^{-1})/2. \end{aligned}$$

The 4-tuple $\big(p_\nu(z)\big)_{\nu\in E}$ is extensible over the ring of Laurent polynomials in $z_1$ and $z_2$. Indeed, the matrix

$$\begin{bmatrix} 1 & (1+z_1^{-1})/2 & (1+z_2^{-1})/2 & (1+z_1^{-1}z_2^{-1})/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has $\big(p_{0,0}(z), p_{1,0}(z), p_{0,1}(z), p_{1,1}(z)\big)$ as its first row and has determinant 1. Let $\phi_\nu$ ($\nu \in E$) be as given in (6.3). Then the above discussion shows that the integer translates of $\phi$, $\phi_{1,0}$, $\phi_{0,1}$ and $\phi_{1,1}$ form an unconditional basis for $V_1$.

Now we can use the Gram-Schmidt orthogonalization as described in Theorem 4.4 to construct a pre-wavelet basis for $W_0$. Let $c$ and $a_\nu$ ($\nu \in E\backslash\{0\}$) be the sequences given by

$$\widetilde{c}(z) = [\phi, \phi](z)$$

and

$$\widetilde{a}_\nu(z) = [\phi_\nu, \phi](z).$$

Set

$$\psi_\nu := \phi_\nu *' c - \phi *' a_\nu, \quad \nu \in E\backslash\{0\}.$$

Then by Theorem 4.4, $\big\{\psi_\nu(\cdot - \alpha) : \ \nu \in E\backslash\{0\}, \alpha \in \mathbb{Z}^s\big\}$ is an unconditional basis for $W_0$. The sequences $c$ and $a_\nu$ can be easily found out:

$$\begin{aligned} [\phi, \phi](z) &= (6 + z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1})/12, \\ [\phi_{1,0}, \phi](z) &= (5 + 5z_1 + z_2^{-1} + z_1 z_2)/48, \\ [\phi_{0,1}, \phi](z) &= (5 + 5z_2 + z_1^{-1} + z_1 z_2)/48, \\ [\phi_{1,1}, \phi](z) &= (5 + 5z_1 z_2 + z_1 + z_2)/48. \end{aligned}$$

## §7. Wavelet Decomposition

Once a pre-wavelet basis is obtained, one can construct a wavelet basis from it by orthogonalization. However, if the integer translates of a given function $\phi$ already form an orthonormal basis for $V_0 = S_2(\phi)$, then it would be more convenient to construct wavelets directly from $\phi$.

Let $\phi$ be a function in $\mathcal{L}^2(\mathbb{R}^s)$ having orthonormal integer translates. Suppose that $\phi$ satisfies the refinement equation (2.1) with $b$ as its mask. For $\nu \in E$, let $b_\nu$ be the sequences given by (6.1), and let $\phi_\nu$ be the functions given by (6.3). We wish to find $2^s$ functions $\psi_\mu \in \mathcal{L}^2$ ($\mu \in E$) such that $\psi_0 = \phi$ and the integer translates of $\psi_\mu$ ($\mu \in E$) form an orthonormal basis for $V_1 := \sigma_1(V_0)$. Each $\psi_\mu$ is of the form

$$\psi_\mu = \sum_{\nu\in E} \phi_\nu *' b_{\mu\nu}, \tag{7.1}$$

where $b_{\mu\nu} \in \ell^1(\mathbb{Z}^s)$. Let $p_{\mu\nu}(z)$ be the symbol $\widetilde{b}_{\mu\nu}(z)$.

**Theorem 7.1.** *The set* $\{\psi_\mu(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, \mu \in E\}$ *forms an orthonormal basis for* $V_1$ *if and only if* $\big(2^{-s/2}p_{\mu\nu}(z)\big)_{\mu,\nu\in G}$ *is a unitary matrix for every* $z \in T^s$.

**Proof:** Let

$$\Phi(z) := \big([\phi_\mu, \phi_\nu](z)\big)_{\mu,\nu\in E},$$

$$\Psi(z) := \big([\psi_\mu, \psi_\nu](z)\big)_{\mu,\nu\in E},$$

and

$$B(z) := \big(\widetilde{b}_{\mu\nu}(z)\big)_{\mu,\nu\in E}.$$

Then (7.1) together with (3.7) yields

$$\Psi(z) = B(z)\Phi(z)B^*(z), \tag{7.2}$$

where $B^*(z)$ denotes the complex conjugate of the matrix $B(z)$. Since the integer translates of $\phi$ are orthonormal, we have

$$\Phi(z) = 2^{-s}I \quad \text{for all } z \in T^s, \tag{7.3}$$

where $I$ is the identity matrix. The integer translates of $\psi_\mu$ ($\mu \in E$) form an orthonormal basis for $V_1$ if and only if

$$\Psi(z) = I \quad \text{for all } z \in T^s.$$

By (7.2) and (7.3), this is equivalent to

$$2^{-s}B(z)B^*(z) = I,$$

that is, $\big(2^{-s/2}p_{\mu\nu}(z)\big)_{\mu,\nu\in E}$ is a unitary matrix for every $z \in T^s$. ■

Theorem 7.1 reduces the wavelet decomposition problem to the following matrix problem: Given $p_\nu \in \mathcal{B}$ ($\nu \in E$) satisfying

$$\sum_{\nu\in E} |p_\nu(z)|^2 = 1 \quad \text{for all } z \in T^s,$$

find $p_{\mu\nu} \in \mathcal{B}$ ($\mu, \nu \in E$) such that $p_{0\nu} = p_\nu$ for all $\nu \in E$ and $\big(p_{\mu\nu}(z)\big)_{\mu,\nu\in E}$ is a unitary matrix for every $z \in T^s$. This problem is solvable for the case in which $s \le 3$ and all $p_\nu$ are real-valued functions. Here we shall describe a solution given by Riemenschneider and Shen in [29]. Their solution is based on considering mappings $\eta$ on $E$ with the following property

$$\big(\eta(\mu) + \eta(\nu)\big) \cdot (\mu + \nu) \text{ is odd for all } \mu \ne \nu. \tag{7.4}$$

They essentially proved the following theorem, in which $E$ is viewed as the additive group $(\mathbb{Z}_2)^s$ with $\mathbb{Z}_2$ being the additive group of integers modulo 2.

**Theorem 7.2.** *If $\eta$ is a mapping on $E$ satisfying the condition (7.4), and if $x_\nu$ ($\nu \in E$) are real numbers such that $\sum_{\nu\in E} x_\nu^2 = 1$, then the matrix $\big((-1)^{\eta(\mu)\cdot\nu} x_{\nu-\mu}\big)_{\mu,\nu\in E}$ is an orthogonal matrix.*

**Proof:** For $\mu, \tau \in E$, let

$$I_{\mu\tau} := \sum_{\nu\in E} (-1)^{\eta(\mu)\cdot\nu} x_{\nu-\mu} (-1)^{\eta(\tau)\cdot\nu} x_{\nu-\tau}. \tag{7.5}$$

Then

$$I_{\mu\mu} = \sum_{\nu\in E} x_\nu^2 = 1 \quad \text{for all } \mu \in E.$$

It remains to prove $I_{\mu\tau} = 0$ for $\mu \neq \tau$. Since $2\mu = 0$ for all $\mu \in E$, we have

$$I_{\mu\tau} = \sum_{\nu\in E} (-1)^{(\eta(\mu)+\eta(\tau))\cdot\nu} x_{\nu+\mu} x_{\nu+\tau}.$$

Changing the indices from $\nu$ to $\nu - \mu - \tau$ in the above sum, we obtain

$$I_{\mu\tau} = \sum_{\nu\in E} (-1)^{(\eta(\mu)+\eta(\tau))\cdot(\nu-\mu-\tau)} x_{\nu-\mu} x_{\nu-\tau}. \tag{7.6}$$

It follows from (7.5) and (7.6) that

$$2I_{\mu\tau} = \sum_{\nu\in E} (-1)^{(\eta(\mu)+\eta(\tau))\cdot\nu} \big[1 + (-1)^{(\eta(\mu)+\eta(\tau))\cdot(\mu+\tau)}\big] x_{\nu-\mu} x_{\nu-\tau}.$$

Since the mapping $\eta$ satisfies the condition (7.4), we have $I_{\mu\tau} = 0$, as desired. ■

Riemenschneider and Shen have found the mappings $\eta$ satisfying the condition (7.4) for $s \le 3$. They already pointed out that when $s > 3$, there is no mapping $\eta$ satisfying the condition (7.4). For $s = 1$ one may choose $\eta$ to be the identity mapping. The mapping $\eta$ given in [29] is

$$\begin{matrix} (0,0) \mapsto (0,0) & \quad (1,0) \mapsto (1,1) \\ (0,1) \mapsto (0,1) & \quad (1,1) \mapsto (1,0) \end{matrix} \tag{7.7}$$

for $s = 2$ and in the case of $s = 3$ is

$$\begin{matrix} (0,0,0) \mapsto (0,0,0) & \quad (1,0,0) \mapsto (1,1,0) \\ (0,1,0) \mapsto (0,1,1) & \quad (1,1,0) \mapsto (1,0,0) \\ (0,0,1) \mapsto (1,0,1) & \quad (1,0,1) \mapsto (0,0,1) \\ (0,1,1) \mapsto (0,1,0) & \quad (1,1,1) \mapsto (1,1,1). \end{matrix}$$

Let us take a closer look into the case of $s = 3$. Number the elements of $E_3$ as follows: For $\mu = (\mu_1, \mu_2, \mu_3) \in E_3$ let $k(\mu) := \mu_1 + 2\mu_2 + 4\mu_3$. Write $x_{k(\mu)}$ for $x_\mu$. With the mapping $\eta$ as above the matrix $\big((-1)^{\eta(\mu)\cdot\nu} x_{\nu-\mu}\big)_{\mu,\nu\in E}$ becomes

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & -x_0 & -x_3 & x_2 & x_5 & -x_4 & -x_7 & x_6 \\ x_2 & x_3 & -x_0 & -x_1 & -x_6 & -x_7 & x_4 & x_5 \\ x_3 & -x_2 & x_1 & -x_0 & x_7 & -x_6 & x_5 & -x_4 \\ x_4 & -x_5 & x_6 & -x_7 & -x_0 & x_1 & -x_2 & x_3 \\ x_5 & x_4 & x_7 & x_6 & -x_1 & -x_0 & -x_3 & -x_2 \\ x_6 & x_7 & -x_4 & -x_5 & x_2 & x_3 & -x_0 & -x_1 \\ x_7 & -x_6 & -x_5 & x_4 & -x_3 & x_2 & x_1 & -x_0 \end{bmatrix}. \tag{7.8}$$

Similar orthogonal matrices have also appeared in [23, p.31] and [36], where the orthogonal matrices were derived from the multiplication table for the Cayley algebra (see, *e.g.*, [22, p.227]).

**Theorem 7.3.** *Let $\phi$ be a real and symmetric function in $\mathcal{L}^2(\mathbb{R}^s)$ satisfying the refinement equation (2.1). Suppose that $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ is an orthonormal basis for $V_0$. Let $\eta$ be a mapping on $E$ satisfying the condition (7.4) and $\eta(0) = 0$. Then with*

$$\psi_\mu := \sum_{\alpha\in\mathbb{Z}^s} (-1)^{\eta(\mu)\cdot\alpha} b(\alpha - \mu)\phi(2\cdot - \alpha)$$

*for each $\mu \in E$, the set $\{\psi_\mu(\cdot - \alpha) : \alpha \in \mathbb{Z}^s, \mu \in E\}$ is an orthonormal basis for $V_1$.*

**Proof:** Since $\phi$ is real and symmetric, the mask $b$ in the refinement equation (2.1) is real and symmetric. Let $b_\nu$, $p_\nu$ and $\phi_\nu$ ($\nu \in E$) be as given in (6.1), (6.2) and (6.3), respectively. Then $p_\nu(z) = \widetilde{b}_\nu(z)$ are real for every $z \in T^s$. Since $\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}$ is an orthonormal basis for $V_0$, the equation (6.4) implies that

$$1 = [\phi, \phi](z) = \sum_{\nu\in E} |p_\nu(z)|^2 [\phi_\nu, \phi_\nu](z) = 2^{-s} \sum_{\nu\in E} |p_\nu(z)|^2.$$

This shows that

$$\sum_{\nu\in E} |p_\nu(z)|^2 = 2^s \quad \text{for all } z \in T^s. \tag{7.9}$$

For $\mu \in E$, we rewrite $\psi_\mu$ as

$$\psi_\mu = \sum_{\nu\in E} \phi_\nu *' b_{\mu\nu},$$

where

$$b_{\mu\nu}(\beta) = (-1)^{\eta(\mu)\cdot\nu} b_{\nu-\mu}(\beta), \quad \beta \in \mathbb{Z}^s.$$

It follows that

$$p_{\mu\nu}(z) := \widetilde{b}_{\mu\nu}(z) = (-1)^{\eta(\mu)\cdot\nu} p_{\nu-\mu}(z). \tag{7.10}$$

By Theorem 7.2, (7.9) and (7.10) together imply that $\big(2^{-s/2} p_{\mu\nu}(z)\big)$ is a real orthogonal matrix for every $z \in T^s$. By Theorem 7.1, we conclude that the integer translates of $\psi_\mu$ ($\mu \in E$) form an orthonormal basis for $V_1$. ■

Let now $\phi$ be a function in $\mathcal{E}^2(\mathbb{R}^s)$ having stable integer translates. Then the function $P : z \mapsto [\phi, \phi](z)$ is in $\mathcal{H}$ and is positive on $T^s$. It follows that $1/\sqrt{P}$ is also in $\mathcal{H}$. Thus $1/\sqrt{P(z)}$ is the symbol of some exponentially decaying sequence $c$:

$$1/\sqrt{P(z)} = \widetilde{c}(z). \tag{7.11}$$

Set

$$\rho := \phi *' c. \tag{7.12}$$

Then by (3.7)

$$[\rho, \rho](z) = \widetilde{c}(z)[\phi, \phi](z)\widetilde{c}(z) = 1 \quad \text{for all } z \in T^s.$$

Hence the integer translates of $\rho$ form an orthonormal basis for $V_0$.

Suppose that $\phi$ satisfies the refinement equation (2.1) with $b$ as its mask. Then the Fourier transform of $\phi$ satisfies the following equation:

$$\widehat{\phi}(2\theta) = 2^{-s} Q_\phi(\theta)\widehat{\phi}(\theta), \tag{7.13}$$

where

$$Q_\phi(\theta) = \widetilde{b}(e^{-i\theta}) = \sum_{\alpha\in\mathbb{Z}^s} b(\alpha)e^{-i\alpha\cdot\theta}.$$

It follows from (7.11) and (7.12) that

$$\widehat{\rho}(\theta) = \widehat{\phi}(\theta)/\sqrt{P_\phi(\theta)}, \tag{7.14}$$

where

$$P_\phi(\theta) := [\phi, \phi](e^{-i\theta}).$$

Now (7.13) and (7.14) yield the following equation for the Fourier transform of $\rho$:

$$\widehat{\rho}(2\theta) = 2^{-s} Q_\phi(\theta)\sqrt{P_\phi(\theta)/P_\phi(2\theta)}\,\widehat{\rho}(\theta).$$

This shows that $\rho$ satisfies the refinement equation

$$\rho = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\rho(2\cdot - \alpha), \tag{7.15}$$

where

$$\widetilde{a}(e^{-i\theta}) = Q_\phi(\theta)\sqrt{P_\phi(\theta)/P_\phi(2\theta)}.$$

Thus the mask $a$ in (7.15) can be computed using the following formula:

$$a(\alpha) := \frac{1}{(2\pi)^s} \int_{[-\pi,\pi)^s} \sqrt{\frac{P_\phi(\theta)}{P_\phi(2\theta)}} Q_\phi(\theta) e^{i\alpha\cdot\theta}\, d\theta.$$

**Example 7.1**. Let $\phi$ be the same function as in Example 6.1. The functions $P_\phi$ and $Q_\phi$ were computed in [29]:

$$P_\phi(\theta) = 1/2 + \big(\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_1 + \theta_2)\big)/6, \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2.$$

$$Q_\phi(\theta) = 1 + \big(\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_1 + \theta_2)\big)/2, \quad \theta = (\theta_1, \theta_2) \in \mathbb{R}^2.$$

## §8. Extensibility

We have seen the importance of extensibility in wavelet analysis. In this section we shall elaborate this point in more details.

Let $R$ be a commutative ring with identity and let $R^n$ be the free $R$-module of rank $n$. We say that an element $(p_1, \dots, p_n) \in R^n$ is *extensible* over $R$ if $(p_1, \dots, p_n)$ is the first row of some $n \times n$ invertible matrix over $R$. An element $(p_1, \dots, p_n) \in R^n$ is called a unimodular row, if there exist $q_1, \dots, q_n \in R$ such that

$$\sum_{i=1}^{n} p_i q_i = 1.$$

Thus unimodularity is a necessary condition for $(p_1, \dots, p_n)$ to be extensible over $R$. However, in general, unimodularity is not sufficient for extensibility.

**Example 8.1**. For a positive integer $n$, let $S^{n-1}$ be the $(n-1)$-dimensional sphere

$$\Big\{(x_0, \dots, x_{n-1}) \in \mathbb{R}^n : \sum_{j=0}^{n-1} x_j^2 = 1\Big\}.$$

Let $R$ be the ring of all real-valued continuous functions on $S^{n-1}$. Consider the functions $p_k \in R$ given by

$$p_k(x) := x_k \quad (k = 0, 1, \dots, n-1).$$

Then $(p_0, \dots, p_{n-1})$ is a unimodular row. But $(p_0, \dots, p_{n-1})$ is extensible over $R$ only if $n = 1, 2, 4, 8$. This comes from Adam's Theorem (see [1]). However, it is shown in [23, p.38] that $(p_0, \dots, p_{n-1})$ is extensible over the ring of all complex-valued continuous functions on $S^{n-1}$.

A commutative ring with identity is said to have the *unimodular row property*, if for every $n$, every unimodular row in $R^n$ is extensible over $R$. Such a ring is called a Hermite ring in [23]. It is well known that if every finitely generated projective $R$-module is free, then $R$ has the unimodular row property (see [31, Theorem 4.51]).

The famous Quillen-Suslin theorem says that if $R$ is a polynomial ring over a field, then every finitely generated projective $R$-module is free. In particular, a polynomial ring over a field has the unimodular row property. The Quillen-Suslin theorem is also true for Laurent polynomial rings (see [35] and [23, Corollary 4.10]). Note that a row $(p_1(z), \ldots, p_n(z))$ of Laurent polynomials is unimodular if and only if $p_1(z), \ldots, p_n(z)$ have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. Thus we may state the Quillen-Suslin theorem in the following form:

**Theorem (Quillen-Suslin).** *Let $p_1(z), \ldots, p_n(z)$ be Laurent polynomials with complex coefficients which have no common zeros in $(\mathbb{C}\setminus\{0\})^s$. Then the $n$-tuple $(p_1, \ldots, p_n)$ is extensible over $\mathcal{P}$.*

The Quillen-Suslin theorem can be used to derive the following theorem about pre-wavelet decomposition.

**Theorem 8.1.** *Let $\phi$ be a compactly supported function in $L^2(\mathbb{R}^s)$. Let $W_0 = W_0(\phi)$ be the orthogonal complement of $V_0 := S_2(\phi)$ in $V_1 := \sigma_1(V_0)$. Suppose that $\phi$ is refinable and has linearly independent integer translates. Then there exists $2^s - 1$ compactly supported pre-wavelets $\psi_\mu$ $(\mu \in E\setminus\{0\})$ such that their integer translates form an unconditional basis for $W_0(\phi)$.*

**Proof:** Suppose that $\phi$ satisfies the refinement equation (2.1) with mask $b$. Then by Theorem 5.2 the sequence $b$ is finitely supported. Let $b_\nu$ and $p_\nu$ $(\nu \in E)$ be as defined in (6.1) and (6.2). Taking Fourier-Laplace transform of both sides of the refinement equation (2.1), we obtain

$$\widehat{\phi}(\xi) = 2^{-s}\widetilde{b}(e^{-i\xi/2})\widehat{\phi}(\xi/2) \quad (\xi \in \mathbb{C}^s).$$

By the definition of $p_\nu$, it follows from the above equation that

$$\widehat{\phi}(\xi + 2\pi\alpha) = 2^{-s}\Big(\sum_{\nu\in E} e^{-i\xi\cdot\nu/2}p_\nu(e^{-i\xi})\Big)\widehat{\phi}(\xi/2) \quad (\alpha \in \mathbb{Z}^s).$$

If $p_\nu$ $(\nu \in E)$ have a common zero in $(\mathbb{C}\setminus\{0\})^s$, say $z = e^{-i\xi}$ $(\xi \in \mathbb{C}^s)$, then $\widehat{\phi}(\xi + 2\pi\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$, which contradicts the hypothesis that $\phi$ has linearly independent integer translates. This shows that $(p_\nu)$ $(\nu \in E)$ do not have common zeros in $(\mathbb{C}\setminus\{0\})^s$. By the Quillen-Suslin theorem, $(p_\nu)_{\nu\in E}$ is extensible over $\mathcal{P}$. Therefore, by Theorem 6.1, we can find $2^s - 1$ compactly supported functions $\psi_\mu \in W_0$ $(\mu \in E\setminus\{0\})$ such that their integer translates form an unconditional basis for $W_0$. ■

Theorem 8.1 has a nice application to box splines. Let us first recall the definition of box splines. Given an $s \times n$ integer matrix of rank $s \leq n$, the box spline $B(\cdot|X)$ is defined by the equation

$$\int_{\mathbb{R}^s} f(x)B(x|X)\,dx = \int_{[0,1]^n} f(Xt)\,dt, \quad \text{for all } f \in C(\mathbb{R}^s).$$

For example, if

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

then $B(\cdot|X)$ is just a multiinteger shift of the hat function as given in Example 6.1. Any box spline is refinable (see [11], [15]). Let $x^1, \ldots, x^n$ denote the columns of $X$. Then $B(\cdot|X)$ satisfies the refinement equation (2.1) with mask $b$ given by

$$\widetilde{b}(z) = 2^{-n+s} \prod_{j=1}^{n} (1 + z^{x^j}). \tag{8.1}$$

Concerning the stability and linear independence, we have the following theorem, which was proved in [12], [14] and [19].

**Theorem 8.2.** *The following conditions are equivalent:*

*(i) The matrix $X$ is unimodular, i.e., every $s \times s$ submatrix of $X$ has determinant $-1$, $0$ or $1$.*
*(ii) The box spline $B(\cdot|X)$ has linearly independent integer translates.*
*(iii) The box spline $B(\cdot|X)$ has stable integer translates.*

Combining Theorem 8.1 and Theorem 8.2 together, we obtain the following result about the pre-wavelet decomposition of box splines.

**Theorem 8.3.** *If $\phi$ is a box spline $B(\cdot|X)$ with $X$ being unimodular, then there is a pre-wavelet basis for $W_0(\phi)$.*

Finally, we give an example to illustrate Theorem 8.3.

**Example 8.2**. Let

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then $B(\cdot|X)$ satisfies the refinement equation (1.1) with mask $b$ given by

$$\widetilde{b}(z) = 2^{-3}(1 + z_1)^2(1 + z_2)^2(1 + z_1 z_2).$$

The Laurent polynomials as defined in (6.2) are easily computed:

$$\begin{aligned} p_{0,0}(z) &= (1 + z_1 + z_2 + 5z_1 z_2)/8, \\ p_{1,0}(z) &= (2 + 4z_2 + 2z_1 z_2)/8, \\ p_{0,1}(z) &= (2 + 4z_1 + 2z_1 z_2)/8, \\ p_{1,1}(z) &= (5 + z_1 + z_2 + z_1 z_2)/8. \end{aligned}$$

It is easily seen that $(p_\nu(z))_{\nu \in E}$ is extensible over the Laurent polynomial ring. Indeed, the matrix

$$\begin{bmatrix} p_{0,0}(z) & p_{1,0}(z) & p_{0,1}(z) & p_{1,1}(z) \\ 1 & 0 & 0 & 0 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

has determinant $-8$ for all $z \in \mathbb{C}^s$. A pre-wavelet basis can be constructed as was done in Section 6.

We end the paper with the following comment, which is useful for explicit pre-wavelet decompositions. Suppose $\big(p_1(z), \ldots, p_n(z)\big)$, $z \in \mathbb{C}^s$ and $\big(q_1(\zeta), \ldots, q_m(\zeta)\big)$, $\zeta \in \mathbb{C}^t$ are extensible by means of the matrices $P(z)$ and $Q(\zeta)$, respectively. Then the tensor product element

$$\big(p_1(z), \ldots, p_n(z)\big) \times \big(q_1(\zeta), \ldots, q_m(\zeta)\big) := \big(p_1(z)q_1(\zeta), \ldots, p_n(z)q_m(\zeta)\big)$$

is extensible by means of the tensor product matrix

$$P(z) \times Q(\zeta) = \begin{pmatrix} p_{11}(z)Q(\zeta) & \ldots & p_{1n}(z)Q(\zeta) \\ \vdots & \ddots & \vdots \\ p_{n1}(z)Q(\zeta) & \ldots & p_{nn}(z)Q(\zeta) \end{pmatrix},$$

since

$$\det\big(P(z) \times Q(\zeta)\big) = \det P(z) \det Q(\zeta).$$

This observation was implicitly used in [2] to reduce multivariate decomposition to a tensor product construction based on univariate wavelets. Specifically, if $\phi(t)$, $\psi(t)$, $t \in \mathbb{R}$ is a pre-wavelet pair where $\phi$ satisfies a refinement equation with mask $b$, set

$$\psi^0(t) = \phi(t), \qquad \psi^1(t) = \psi(t), \quad t \in \mathbb{R}$$

and introduce the multivariate functions

$$\psi^e(x_1, \ldots, x_s) := \prod_{j=1}^{s} \psi^{e_j}(x_j), \quad e \in E_s;\ x = (x_1, \ldots, x_s) \in \mathbb{R}^s,$$

then $\phi(x) = \psi^0(x)$ satisfies a refinement equation with tensor product mask

$$b(\alpha) = b(\alpha_1) \cdots b(\alpha_s), \quad \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$$

and the remaining $2^s - 1$ functions $\big\{\psi^e : \ e \in E_s \backslash \{0\}\big\}$ form a pre-wavelet basis.

Coupled with Theorem 7.3 various explicit wavelet decompositions can be constructed.

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