

Routing versus Network Coding in Erasure Networks with Broadcast and Interference Constraints

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This paper determines upper and lower bounds on the order behavior (as the number of nodes increases) of throughput in an erasure network with transmit and receive-side constraints. This model is a useful abstraction of the network-layer behavior of a wireless network. An upper bound on throughput is determined for such a network, which is found to be achievable by routing if the parameter that determines the decay of channel quality with distance is greater than a critical value.

I. INTRODUCTION

In this paper, we study the asymptotic relationship between routing and coding as the number of nodes in a random, extended erasure network increases. Our network model is the wireless erasure network [1], where each edge in a graph represents an independent erasure channel with a (possibly unique) erasure probability. An erasure network is a useful model for studying packetized communication in a network. In addition, adding in transmit and receive-side constraints allows us to capture both the broadcast nature of the wireless medium as well as interference in a wireless network.

In our setting, we allow each transmitting node to broadcast one symbol across all of its outgoing edges (so-called broadcast constraint on each transmitting node). We consider two disparate settings on the receive-side: One with no interference among nodes, and the other with finite-field additive interference at the receivers. The reasons these two models are of great interest are as follows:

- An upper bound on throughput under the “no-interference model” provides an upper bound on throughput for a class of wireless channels with interference. If X_i , $1 \leq i \leq m$ represent the input to the channel, settings where the interference can be decomposed into *separate* erasures on each link producing Y_i , $1 \leq i \leq m$ followed by an

interference mapping $Z = g(Y_1, \dots, Y_m)$ with Z being the channel output, the no-interference case provides an outer bound on throughput. This is because the receiver in the no-interference case can “mimic” the interference function g , thus making the maximum throughput for the interference case less than that of the no-interference case.

- The finite field additive interference is intuitively a pessimistic interference model. This is because interference is traditionally thought to increase the total received power (or equivalently, for finite field inputs $X_i \in F$, the channel output Z belongs to a field with alphabet size larger than that of F). Thus, restricting the output to belong to the same (finite) field as each of the inputs represents a “stringent” interference requirement.

Thus, the no-interference and finite-field interference represent an optimistic and a pessimistic extreme respectively, with many other interference settings lying in between these two settings.

We have previously performed a similar analysis for networks where the probability of a successful transmission decays exponentially with distance between transmitter and receiver [2] (the case of absorption). This paper treats a power-law decay, where the probability of a successful reception decays polynomially with order α . For a network where n nodes are randomly placed in a square of area n (side length $\sqrt{n} \times \sqrt{n}$), each node has independent information which it desires to transmit to a unique randomly assigned destination node.

Our main results in this paper are as follows:

- We show that for all $\alpha > 3$, routing is order-optimal. That is, we establish an upper bound on the total throughput of the network that increases as $\Theta(\sqrt{n})$ within a poly-log factor, and that $\Theta(\sqrt{n})$ throughput growth is achievable using a

routing-only strategy. The upper-bounding technique that we use to do so is similar to that of [3], where we use the fact that nodes and source-destination are uniformly randomly located in order to upper-bound the number of nodes that are in a sense “close” to one another, and can therefore achieve a greater throughput than nodes that are situated further away from each other.

- In other words, we show that for $\alpha > 3$ gains from network coding in this model are sub-polynomial when studied from point of view of order behavior. For $2 \leq \alpha \leq 3$, further investigation is required to determine if network coding might potentially provide gains in throughput that increase polynomially with n .

A large body of work, on both multicast and multiple unicast systems (for example [4]) demonstrates that network coding can provide gains in many network settings. Our result is similar to that of [5] in that we claim that the gains of network coding in our models are order-wise negligible; however, we allow a larger selection of transmit strategies as our analysis is information-theoretic.

II. SYSTEM MODEL

In this paper, we study a class of wireless networks on directed graphs which all fall under the classification of *random broadcast erasure networks*:

- In a square of area n , n nodes are uniformly randomly distributed. Each node is a source of independent data, and is assigned a unique destination. We consider the total throughput achievable by all node pairs.
- The term *broadcast* specifies that in each time-slot, every node can choose only one symbol from its alphabet to transmit, and must transmit that identical signal along all outgoing edges.
- Each edge in the directed graph network acts as an independent *erasure channel*.
- A fairly general set of interference models are allowed.

Each of these characteristics are discussed in greater detail below.

Distribute n nodes on the square with $\sqrt{n} \times \sqrt{n}$ -length sides, and then consider the complete directed graph $G = (V, E)$ formed of the n nodes and $2\binom{n}{2}$ directed edges. For each edge (i, j) (directed from the node i to the node j) assign an erasure probability ϵ_{ij} . The erasure probability will describe the quality of the link between nodes i and j . We assume that the probability of a non-erasure event (i.e. a successful transmission) decays polynomially with distance. Specifically, if the physical distance between nodes i and j is d_{ij} , then

$$\epsilon_{ij} = 1 - \frac{1}{1 + d_{ij}^\alpha}$$

where the parameter α determines the attenuation law.

In each time-slot t , each node i chooses a single symbol $X_i(t)$ from the finite field alphabet F_q . Each edge acts as an independent erasure channel. That is, each edge (i, j) produces the output \tilde{X}_{ij} from the alphabet $\{F_q, \mathcal{E}\}$ where

$$\begin{aligned} p(\tilde{X}_{ij} = X_i) &= 1 - \epsilon_{ij} \\ p(\tilde{X}_{ij} = \mathcal{E}) &= \epsilon_{ij}. \end{aligned}$$

We call the \tilde{X}_{ij} the edge channel outputs.

At the end of each time-slot, each node j receives the channel output Y_j . To allow for a general set of interference models, we allow this output to be a probabilistic function of all the edge channel outputs. Specifically, we allow

$$Y_j(t) = f_j \left(\{\tilde{X}_{ij}(t) | (i, j) \in E\}, Q_j \right) \quad (1)$$

where Q_j are mutually independent random vectors.

We are most interested in two specific interference models:

- The case of *no interference*, where $Y_j(t)$ is the vector of all the incoming edges' \tilde{X}_{ij} . This model is most similar to the wireless erasure network model of [1].
- The case of *finite-field additive interference*, where

$$Y_j(t) = \sum_{\{i | (i, j) \in E \text{ and } \tilde{X}_{ij}(t) \neq \mathcal{E}\}} \tilde{X}_{ij}(t).$$

Here the output is the sum of all unerased symbols along incoming edges. We began the study of such a model in [2].

We also note that the *wireless broadcast additive interference network* model of [6] is a member of the set of interference models described by Equation (1).

Every node in the network has an independent source of data. We assign n source-destination pairs (s_l, t_l) randomly, such that each node is the unique destination for exactly one other node. We desire to determine asymptotic upper and lower bounds on the maximum common rate at which every source-destination pair can simultaneously communicate reliably.

III. UPPERBOUND

We prove the following converse result:

Theorem 1: For the set of random planar networks described in Section II, with high probability, the sum total throughput grow no faster than $O(\sqrt{n})$, within a polylog factor, when the decay parameter $\alpha > 3$.

This upperbound is valid for all of the broadcast erasure networks where the channel output available to each node is a composite function of the incoming edge channel outputs available to that node. The information-theoretic upper-bounding analysis allows

for the possibility of a wide variety of transmit and coding strategies, including traditional network coding and opportunistic network coding. [7]

Proof: We desire that all n source-destination pairs reliably communicate at some minimum rate R . That is, each source s_l , $l \in \{1, \dots, n\}$ chooses a message w_l independently and uniformly at random from the set $\{1, \dots, 2^{TR}\}$. We desire to determine the maximum R such that each node can, with arbitrarily small probability of error, identify which message was chosen using T network timeslots.

We will consider the case of a network with no interference, as described in Section II. The information available to each receiver j in all other models is a (possibly random) function of the vector of all edges' \tilde{X}_{ij} . Thus for each j , by the data processing inequality, the mutual information between the channel outputs and the set of all n source messages w_l for the no interference model –

$$I(\{\tilde{X}_{ij}(t) | (i, j) \in E\}; \{w_l | l \in 1..n\}) \geq I(Y_j(t); \{w_l | l \in 1..n\}) \quad (2)$$

is greater than that of any other model. The *no interference* model therefore provides an upper-bound on the maximum possible rate R .

Our primary tool for calculating an upperbound on R will be the cut-set bound, as described in, for example, [8]. In [9], we prove the following lemma for the wireless erasure network with no interference:

Lemma 1: For a wireless erasure network with *no receiver interference* divided into two sets of nodes, S and S^C , the cut-set bound on the feasible sum rate of data from S to S^C evaluates to

$$R_{cut} \leq I(X_S; Y_{S^C} | X_{S^C}) \leq \sum_{i \in S} \left(1 - \prod_{j \in S^C} \epsilon(d_{ij}) \right), \quad (3)$$

where d_{ij} is the distance between the i^{th} and j^{th} nodes.

(The difference between this lemma and the proofs in [1] is that we consider the possibility of cycles, while [1] limits their investigation to directed, acyclic graph networks.)

As in [3], we consider a cut that divides the area of the network in half, through the middle, into \mathcal{L} and $\mathcal{R} = \mathcal{L}^C$. For any positive δ , with high probability, between $(1 - \delta)/4$ and $(1 + \delta)/4$ of the source-destination pairs will have $s_l \in S$ and $t_l \in S^C$. Therefore, the sum-rate nR of the entire network is upperbounded by no more than 4 times the upperbound on sum-rate across the $\mathcal{L} - \mathcal{R}$ cut.

Take the total area of the network $n = r^2$ and divide it into n sub-blocks of size 1. Let \mathcal{V}_L be the set of nodes in the $r = \sqrt{n}$ squarelets directly to the left of the cut. We can then bound Equation (3), evaluated over the $\mathcal{L} - \mathcal{R}$ cut as

$$\begin{aligned} & \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{R}} (1 - \epsilon_{ij}) + |\mathcal{V}_L| \\ &= \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{R}} \frac{1}{1 + d_{ij}^\alpha} + |\mathcal{V}_L| \\ &\leq \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{R}} \frac{1}{d_{ij}^\alpha} + |\mathcal{V}_L|. \end{aligned} \quad (4)$$

The contribution to the sum from nodes in the $r = \sqrt{n}$ squarelets in \mathcal{L} on the boundary (i.e., nodes in \mathcal{V}_L) is no greater than 1 for each node. Every squarelet contains less than $\log n$ nodes with high probability, so this component of the rate sum is less than $O(\sqrt{n} \log n)$ w.h.p.

The left hand side of the sum in Equation (4) is bounded by $K\sqrt{n}$, where K is a constant dependent only upon α . The details of the mathematical proof is given in the Appendix.

Thus, we have shown that the total sum-rate across the cut, and therefore the total throughput, is bounded to within a polylog factor by $O(\sqrt{n})$. ■

IV. ACHIEVABILITY

We prove the following theorem:

Theorem 2: For the random *broadcast erasure network with finite-field additive interference*, $\Theta(\sqrt{n})$ sum total throughput is achievable, using a routing-only strategy, when $\alpha > 3$.

Proof: Our constructive strategy for achieving, within a polylog factor, $R = \Theta(1/\sqrt{n})$ throughput for each node pair is similar to the constructive strategy of Section IV of [10]. We will divide the network into square cells (different from the squarelets of the converse proof), this time of size $\rho \times \rho$, where ρ is a constant independent of n (but dependent on α) to be determined later. With high probability, each cell will have no more than $\rho^3 \log n$ nodes.

We will operate the network on a TDMA scheme, where each cell is allowed to have one node transmit in each of c^2 timeslots - either to a node in its own cell, or to a node in an adjacent cell. Under this scheme, the nearest simultaneously operating transmitter is at least $c - 1$ cells away from the intended receiver, or a distance of at least $\rho(c - 1)$ away. The 8 closest transmitters are all (at least) this distance from the receiver. The next 16 operating transmitters are all at least $2c - 1$ squares away, and so on so that there are $8k$ transmitters at least a distance of $(ck - 1)\rho$ from the intended receiver, for all positive integers k .

The union bound on the probability that the symbol from at least one of these transmitter is not erased, P_{int} , is thus

$$P_{int} \leq \sum_{k=1}^{\infty} 8k ((ck-1)\rho)^{-\alpha} \quad (5)$$

The sum converges for $\alpha > 3$, the range we are interested in, and by choosing an appropriate ρ (independent of n) the upperbound on the probability P_{int} can be made less than 1. The probability of a successful transmission between two nodes in adjacent cells (located no further apart than 2ρ , with no interfering symbols simultaneously received, is then better than

$$R_{neighbor} = \frac{1}{1 + (2\rho)^\alpha} (1 - P_{int}). \quad (6)$$

Each node in the cell gets at least a $1/\rho^2 \log n$ fraction of this rate; and the TDMA scheme allows the cell to operate at $1/c^2$ fraction of the time.

As is argued in [10], straight-line routing, or forwarding each transmission to the next cell closest to the destination, requires no more than $\sqrt{n \log n}$ paths share each node. Since each route must share a (within a log factor) constant $R_{neighbor}/(c^2 \rho^2 \log n)$ rate available in each cell, the rate available to each route is at least $\Theta(1/\sqrt{n})$, to a polylog factor. ■

Clearly, any order of throughput which is achievable in the network with interference is also achievable for the no-interference case.

Since each of the n source-destination node pairs can achieve a minimum of $\Theta(1/\sqrt{n})$ throughput, the total throughput for the system is at least $\Theta(\sqrt{n})$, to a polylog factor.

V. CONCLUSION

We have shown that for a class of random erasure networks that incorporate both broadcast constraints and receiver interference, an upper bound on total throughput grows as $O(\sqrt{n})$ in the total number of nodes if $\alpha > 3$. We also argue that finite-field additive interference is, in an intuitive sense, a pessimistic notion of interference for which $\Theta(\sqrt{n})$ total throughput is achievable. Thus for the setting under consideration routing is sufficient to provide the correct order of maximum throughput. That is, network coding in the case $\alpha > 3$ for an erasure network with broadcast constraints with or without receive side constraints can provide no more than a polylog factor improvement in performance. For the case of $2 < \alpha \leq 3$, we believe that further research might demonstrate the possibility of a larger increase in throughput with number of nodes using network coding.

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APPENDIX

Here we demonstrate that Equation (4) is bounded by $K\sqrt{n}$.

In every squarelet, there will be no more than $\log n$ nodes. The left side of Equation (4) is thus bounded by the expression

$$\sum_{i_l=2, i_r=1}^{r/2} \sum_{j_l, j_r=1}^r \frac{2(\log n)^2}{\left((i_l + i_r - 2)^2 + (j_l - j_r)^2\right)^{\alpha/2}} \quad (7)$$

by assuming that there are $\log n$ nodes in each squarelet, and that they are all located at the minimum source-destination distance possible for each pair of squarelets under consideration (Figure 7 on page 32 in [3]). We consider the rate achievable with just two nodes in each squarelet, and multiply it by the two $\log n$ terms.

We bound Equation (7) by

$$\sum_{i_l=0, i_r=1}^r \sum_{j_l, j_r=1}^r \frac{2(\log n)^2}{\left((i_l + i_r)^2 + (j_l - j_r)^2\right)^{\alpha/2}} \quad (8)$$

and break the summation into four cases:

- Case 1: $i_l = 0$ and $j_r = j_l = j$

$$\sum_{j=1}^r \sum_{i_r=1}^r \frac{1}{i_r^\alpha} \leq r \sum_{i_r=1}^{\infty} \frac{1}{i_r^\alpha} = \sqrt{n} K_1$$

when $\alpha > 1$ and the final sum converges.

- Case 2: $i_l \geq 1$ and $j_r = j_l = j$

$$\sum_{j=1}^r \sum_{i_l=1}^r \sum_{i_r=1}^r \frac{1}{(i_r + i_l)^\alpha} \leq r \sum_{i_l=1}^{\infty} \sum_{i_r=1}^{\infty} \frac{1}{(i_r + i_l)^\alpha} = \sqrt{n} K_2$$

when $\alpha > 2$ and the final sum converges.

- Case 3: $i_l = 0$ and $j_r \neq j_l$

$$\sum_{i_r=1}^r \sum_{j_r=1}^r \sum_{j_l=1, j_l \neq j_r}^r \frac{1}{\left(i_r^2 + (j_r - j_l)^2\right)^{\alpha/2}}$$

which we will show is bounded by $\sqrt{n} K_3$ when $\alpha > 2$.

- Case 4: $i_l \geq 1$ and $j_r \neq j_l$

$$\sum_{i_l=1}^r \sum_{i_r=1}^r \sum_{j_l=1}^r \sum_{j_r=1, j_l \neq j_r}^r \frac{1}{\left((i_l + i_r)^2 + (j_l - j_r)^2\right)^{\alpha/2}}$$

which we will show is bounded by $\sqrt{n}K_4$ when $\alpha > 3$.

A. Summary of Case 3

Bound

$$\sum_{i_r=1}^r \frac{1}{\left(i_r^2 + (j_r - j_l)^2\right)^{\alpha/2}}$$

by the integral

$$\int_0^\infty \frac{1}{(x^2 + a^2)^{\alpha/2}} dx = a^{1-\alpha} K_3'$$

using $a = |j_r - j_l|$ and the substitution $x = a \tan \theta$. Then

$$K_3' \sum_{j_l=1}^r \sum_{j_r=1, j_l \neq j_r}^\infty \frac{1}{|j_r - j_l|^{\alpha-1}} < \sqrt{n}K_3$$

for $\alpha > 2$.

B. Summary for Case 4

The procedure is similar to that in Case 3 except that we bound a double summation to get

$$K_4' \sum_{j_l=1}^r \sum_{j_r=1, j_l \neq j_r}^\infty \frac{1}{|j_r - j_l|^{\alpha-2}} < \sqrt{n}K_4$$

which converges only when $\alpha > 3$.

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