The Scattering Support

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Abstract

We discuss inverse problems for the Helmholtz equation at fixed energy, specifically the inverse source problem and the inverse medium or obstacle problem. We introduce the *convex scattering support* of a far field, a set which will be a subset of the convex hull of the support of any source which can produce it. We give several theorems which explain how to compute the convex scattering support and how to relate it to the actual support of a source, medium, or obstacle.

1 Introduction

In inverse problems for the Helmholtz equation at fixed energy, the aim is to deduce properties of the source, the index of refraction, or the shape of the obstacle, from observations of scattered waves made at a distance. These waves are called *far fields*. The typical application involves a lot of far fields. For the inverse medium problem, the index of refraction is uniquely determined by the full scattering kernel, i.e. the observed scattered field for every possible incident wave.

Recent [7, 5] work has suggested that, in special cases, substantial information about the support of the scatterer can be obtained from the scattered field of a few, or even only one incident wave. The goal of this paper is to make a single precise definition and prove a few sharp theorems describing the extent to which one can find a scatterer from a single, or a few, observations. The first five sections discuss the inverse source problem, which is linear. All or our conclusions will have immediate corollaries for the (nonlinear) inverse

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medium and obstacle problems, which we include in section six. Although we believe that everything we say here extends to dimensions greater than or equal to two, we restrict ourselves to 2 dimensions in much of this paper.

There are two main aspects to this paper. In section 4, we prove the circular Paley-Wiener theorem. This theorem provides a test, which, when applied to a far field, determines whether that far field could have been produced by a source supported in a ball with center c and radius r. The key words here are *could have been produced*. Because the source is non-unique, it is not immediately clear that this potential source has anything to do with the true source of the far field.

To address this point, in section 3 we introduce the *convex scattering support*. The convex scattering support of a far field is the smallest convex set which can support that far field. We prove that any source which produces the far field must contain this set in the convex hull of its support, and that there is a source supported in any neighborhood of the convex scattering support which will produce that far field.
Our circular Paley-Wiener theorem locates this set, thus finding a subset of the convex hull of the support of the true source, in spite of the non-uniqueness inherent in the problem.
The *scattering support* is defined similarly but need not be convex; it must be contained in the support of any source which produces the far field. We don't yet know whether it is always possible to find a source, supported in a neighborhood of the scattering support, which produces the far field, or even that it will be nonempty for every far field.
In section 4, we prove the most technical theorem of the paper, which we call the *circular Paley-Wiener theorem*. For the source problem, the far field (at energy k^2) is exactly the Fourier transform of the source, restricted to the circle of radius k. We show that a far field has a source contained in a ball centered at c, of radius R, if an only if the (shifted) Fourier coefficients of the far field belong to a certain weighted l^2 space. If they do, we describe an explicit source, supported in that ball, which produces that far field. Furthermore, by introducing a more refined system of weighted l^2 spaces, we can detect how smooth the source is in a neighborhood of the boundary of the ball. Roughly speaking, we can tell the difference between a source that is a single layer on the boundary, one that increases linearly as we move in from the boundary, or one that increases like $(R-r)^L$ as r decreases from R.

In section 5, we define the (convex) scattering support of a source to be the (convex) scattering support of its far field. Some sources produce no far field, and hence have empty (convex) scattering supports. Both the scattering support and the convex scattering support of any radially symmetric source is just a point. Nevertheless, a priori knowledge about a source can be used to correlate its (convex) scattering support to its actual support. We show that any *convex corner* (see definition 7) in the support of a source must also be in the convex scattering support. As a corollary we see that a source that is a priori known to supported in a convex polygon, with a certain non-degeneracy condition at the corners, has equal support and convex scattering supports.

In section 6, we define the scattering support of a medium in terms of the far fields produced by a collection of one or more incident waves. We then draw corollaries from the previous sections. In particular, we conclude that, an inclusion (a jump in the index of refraction) supported on a convex polygonal domain has support equal to its convex scattering support for any single incident wave.

In section 7, we show that the preceding methods can also be applied to the obstacle problem.

Finally, we mention that the ideas we will discuss below are motivated by the *Linear Sampling Method* of Colton- Kirsch [2], and the subsequent factorization of the far field operator by Kirsch [6].

2 The Far Field of a Source

The unique outgoing solution to

$$\left(\Delta + k^2\right) u = f \in \mathcal{B} \tag{1}$$

can be computed by the limiting absorption principle (see e.g. [9],page 147).

$$u_f = \lim_{\epsilon \downarrow 0} \left(\Delta + (k - i\epsilon)^2\right)^{-1} f$$

Because u_f should be the Fourier transform of a solution $\tilde{u}$ to the wave equation which is zero in the past

$$u_f(k,x) = \int_0^\infty e^{-ikt}\tilde{u}(x,t)dt$$

extends to be holomorphic in $\{Im(k) \leq 0\}$ and is continuous up to the boundary.
We assume that $f \in \mathcal{B}$ [5] with norm

$$\|f\|_{\mathcal{B}} = \sum_{j=1}^{\infty}\left(R_j \int_{S^{n-1}}\int_{R_{j-1}}^{R_j} |f(r\Theta)|^2 r^{n-1} dr dS(\Theta)\right)^{\frac{1}{2}}$$

where $R_0 = 0$ and $R_j = 2^j$.
When $f \in \mathcal{B}$ then

$$u_f \in \mathcal{B}^*$$

where

$$\|f\|_{\mathcal{B}^*} = \sup_j \left(\frac{1}{R_j}\int_{S^{n-1}}\int_{R_{j-1}}^{R_j} |f(r\Theta)|^2 r^{n-1} dr\right)^{\frac{1}{2}}$$

and the linear map

$$f \xmapsto{W_k} u_f \in \mathcal{B}$$

is bounded ([4] pages 225-237).
As $r = |x| \to \infty$, u behaves asymptotically as

$$u_f \sim \quad \frac{e^{-ikr}}{r^{(n-1)/2}}\alpha_f(\theta) \quad = \quad \frac{e^{-ikr}}{r^{(n-1)/2}}\widehat{f}(k,\theta)$$

where we call the term α_f the far field of u_f, or just the far field f. Alternatively, we say that f *radiates* α. The previous equality points out that this far field is simply the restriction of the Fourier transform of f to the sphere of radius k.
We don't include a complete proof of this fact, but rather a brief sketch of one simple way to obtain it. Let $a(\theta) \in L^2(S^1)$ and

$$u_0(x) = \int_{S^{n-1}} a(\Theta)e^{ikx\cdot\Theta}dS(\Theta)$$

[5] We use $\mathcal{B}$, rather than the simpler weighted spaces L^2_δ because of theorem 1 below. The far field maps L^2_δ onto $H^{\delta-1/2}(S^{n-1})$, which is dense in, but not quite equal to $L^2(S^{n-1}_k)$.

where $\Theta \in S^{n-1}$ and $u_0(x)$ is a Herglotz wave function ([1]), which we note is a $\mathcal{B}^*$ solution to the free Helmholtz equation

$$(\Delta + k^2)u_0 = 0$$

having asymptotic behavior

$$u_0 \sim \frac{e^{-ikr}}{r^{(n-1)/2}}a(\Theta) + \frac{e^{ikr}}{r^{(n-1)/2}}a(-\Theta).$$

This can be seen from either a Bessel function expansion or a stationary phase calculation. If we multiply (1) by u_0, we obtain

$$\int_{B_R} u_0(\Delta + k^2)u_f = \int_{B_R} u_0 f$$

which means

$$\int_{S_R^{n-1}} u_0 \frac{\partial u_f}{\partial r} - u_f \frac{\partial u_0}{\partial r} = \int_{B_R} u_0 f$$

Letting $R \to \infty$, and making use of the asymptotic expansion of the far field u_f, shows that the left hand side is just

$$\int_{S^{n-1}} a(\Theta)\alpha(\Theta)dS(\Theta)$$

while inserting the definition of the Herglotz wave function u_0 tells us that the right hand side is

$$\int_{S^{n-1}} a(\Theta)\widehat{f}(k\Theta)dS(\Theta)$$

We summarize the above as a theorem.

Theorem 1. *The far field map*

$$\mathcal{F}_{\mathrm{k}}^{\infty} : f \to \widehat{f}(k\Theta)$$

is bounded and surjective, mapping

$$\mathcal{F}_{\mathrm{k}}^{\infty} : \mathcal{B} \to L^2(S_k^{n-1}).$$

Proof. $f \in \mathcal{B}$ implies that $\widehat{f}$ has a restriction to the sphere of radius k and the restriction map is continuous ([4] page 227) . □

We want to include the case where f is a distribution (in particular, a single layer) because this will allow us to apply our results to the obstacle problem as well as the inhomogeneous medium.
We define

$$
\begin{aligned}
f \in \mathcal{B}_s &\quad \text{if and only if} \quad (\Delta+1)^s f \in \mathcal{B} \\
f \in \mathcal{B}_s^* &\quad \text{if and only if} \quad (\Delta+1)^s f \in \mathcal{B}^*
\end{aligned}
$$

Theorem 2. *For all $s \in \mathbb{R}$ the maps*

$$W_k : \mathcal{B}_s \to \mathcal{B}_s^*$$

and

$$\mathcal{F}_{\mathrm{k}}^{\infty} : \mathcal{B}^s \to L^2(S_k^{n-1})$$

are bounded, with bounds depending only on δ.

Proof. Every distribution in H^s is the derivative of an L^2 function. Thus,

$$f = D^\mu F.$$

Since the Helmholtz equation has constant coefficients, W_k commutes with D^μ and

$$\mathcal{F}_{\mathrm{k}}^{\infty} D^\mu F = (k\Theta)^\mu \mathcal{F}_{\mathrm{k}}^{\infty} F.$$

□

3 The Scattering Support of a Far Field

Our goal is to *locate* f, using observations of its fixed frequency far field α_f. Since many f's produce a zero far field, e.g. any f whose Fourier transform vanishes on the sphere of radius k, we cannot hope to retrieve the support of f itself, but we can uniquely determine a set which must be part of the the convex hull of the support of any function which could have produced the far field α. We will call this set the convex scattering support of the far field α, and denote it by $\mathrm{cS}_k\mathrm{supp}\alpha$. We start with a few definitions. The first is just a reminder.

Definition 1. *A point x belongs to the support of a distribution, f, if there exists no open neighborhood $\mathcal{U}$ of x such that f restricted to $\mathcal{U}$ is zero.*

Definition 2. *The convex scattering support of the far field α is*

$$\mathrm{cS}_k\mathrm{supp}\alpha = \bigcap_{\mathcal{F}_\mathrm{k}^\infty f=\alpha} \mathrm{ch}(\mathrm{supp} f). \tag{2}$$

We also wish to define a notion of scattering support which doesn't require taking convex hulls. If we just mimic the definition from (2), taking the intersection of the supports instead of their convex hulls, the resulting intersection is always empty. To see this, let ϕ be a smooth cutoff equal to zero in a neighborhood of $\mathrm{supp} f$ and equal to one outside some ball. Then, $F = (\Delta + k^2)(\phi W_k f)$ radiates the same far field as f but has disjoint support. To avoid this difficulty , we define

Definition 3. *A point x belongs to the infinity-support of a distribution f if there exists no open unbounded neighborhood $\mathcal{U}$ of x such that f restricted $\mathcal{U}$ is zero. We denote this set by* $\mathrm{supp}_\infty f$.

We note that the $\mathrm{supp}_\infty f$ is the support of f plus the holes which can't be connected to infinity without crossing the support of f. It is the closure of the complement of the unbounded component of the complement of the support i.e. the support with any holes filled in. If $\mathrm{supp} f$ is an annulus, then $\mathrm{supp}_\infty f$ is the corresponding disk.

Definition 4. *The scattering support of the far field α is*

$$\mathrm{S}_k\mathrm{supp}\alpha = \bigcap_{\mathcal{F}_\mathrm{k}^\infty f=\alpha} \mathrm{supp}_\infty f.$$

We note that if $\alpha \in L^2(S^{n-1})$ we let the f's vary over $\mathcal{B}$, but we may chose $\mathcal{B}_s$ with $s < -\frac{1}{2}$, if we wish to look for single layers. Because we can approximate any distribution by smooth functions with support arbitrarily close to the support of the distribution, the scattering support doesn't depend on s (i.e. we don't need to define a $\mathrm{cS}_k\mathrm{supp}^s$).

In the next section, we will describe how to compute the convex scattering support of α in great detail. For the moment, we take note of some simple, but important properties. We begin with two lemmas. In the rest of this section, we will always assume that there is a compactly supported source which radiates α.

Lemma 3. *For any $\epsilon > 0$ and α with a compactly supported source ϕ, there exists an integer N and a sequence of sources f_n such that*

$$N_\epsilon(\mathrm{cS}_k\mathrm{supp}\alpha) \subset \bigcap_{n=1}^{N} \mathrm{chsupp}_\infty(f_n).$$

Similarly,

$$N_\epsilon(\mathrm{S}_k\mathrm{supp}\alpha) \subset \bigcap_{n=1}^{N} \mathrm{supp}_\infty(f_n).$$

Proof. Let $f_1 = f_\alpha$ be the compactly supported source. If $x_* \notin N_\epsilon(\mathrm{cS}_k\mathrm{supp}\alpha)$, there exists an f_* and an open set $\mathcal{O}(x_*)$ which does not intersect $\mathrm{chsupp}_\infty f_*$ (alternatively, $\mathrm{supp}_\infty f_*$) while $\mathcal{F}_\mathrm{k}^\infty f_* = \alpha$. Now, the complement of

$$[N_\epsilon(\mathrm{cS}_k\mathrm{supp}\alpha)] \cap \mathrm{supp}_\infty f_1$$

is compact, so finitely many $\mathcal{O}(x_*)$ cover this set. Numbering these x_*'s as $x_2, \cdots, x_N$ and the corresponding f's as $f_2, \cdots, f_N$, produces the conclusion. □

Lemma 4. *Suppose* supp $f_1 \subset \Omega_1$, supp $f_2 \subset \Omega_2$ *and that* $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$ *is connected and contains a neighborhood of* ∞*. If*

$$\mathcal{F}_\mathrm{k}^\infty f_1 = \mathcal{F}_\mathrm{k}^\infty f_2 = \alpha$$

then, for any $\delta > 0$, there exists an $f_3 \in C^\infty(\mathbb{R}^n)$ with

$$\mathrm{supp} f_3 \subset N_\delta(\Omega_1 \cap \Omega_2)$$

and

$$\mathcal{F}_\mathrm{k}^\infty f_3 = \alpha.$$

Proof. According to Rellich's lemma and unique continuation [1], $u_1 = W_k f_1$ and $u_2 = W_k f_2$ agree on the $\mathbb{R}^n \setminus (\Omega_1 \bigcup \Omega_2)$.
Let $\phi \in C^\infty(\mathbb{R}^n)$ satisfy

$$\phi = \begin{cases} 1, & x \in \mathbb{R}^n \backslash N_\delta(\Omega_1 \cap \Omega_2) \\ 0, & x \in N_{\frac{\delta}{2}}(\Omega_1 \cap \Omega_2) \end{cases}$$

then,

$$v = \begin{cases} \phi u_1, & x \in \mathbb{R}^n \backslash \Omega_1 \\ \phi u_2, & x \in \mathbb{R}^n \backslash \Omega_2 \\ 0, & x \in \Omega_1 \cap \Omega_2 \end{cases}$$

is a well-defined C^∞ function and $v = u_1 = u_2$ outside a compact set so that

$$f_3 = (\Delta + k^2)v$$

must also have far field α. □

Theorem 5. *Suppose that α has a compactly supported source f_α. Then, given any $\epsilon > 0$, there exists a C^∞ source f_ϵ such that $\mathcal{F}_{\mathrm{k}}^\infty f_\epsilon = \alpha$ and*

$$\mathrm{chsupp} f_\epsilon \subset N_\epsilon(\mathrm{cS}_k \mathrm{supp}\alpha).$$

Proof. Lemma 3 implies that $N_\epsilon(\mathrm{cS}_k\mathrm{supp}\alpha)$ is contained in the intersection of finitely many sources. We may take Ω_1 and Ω_2 in lemma 4 to be convex hulls of the supports of the sources, so that the hypothesis that $\mathbb{R}^n \backslash (\Omega_1 \bigcup \Omega_2)$ is connected is automatic. Thus we can produce a source supported on a neighborhood of the intersection of the convex hulls of the supports of any two sources, and complete the proof by induction. □

We suspect, but don't know for sure, that the the analog of theorem 5, with $\mathrm{cS}_k\mathrm{supp}\alpha$ replaced by $\mathrm{S}_k\mathrm{supp}\alpha$ is false. At present, we don't even know whether the convex hull of the scattering support is equal to the convex scattering support, or for that matter, whether the$\mathrm{S}_k\mathrm{supp}\alpha$ is nonempty for every α .

We do mention one alternative definition of the scattering support, which can be made in the case the source is compactly supported. In this case, Rellich's lemma guarantees that the far field extends uniquely to a solution u_α to the homogeneous free Helmholtz equation. This solution is real analytic outside some ball, and extends real analytically to larger open sets. The complement of these sets is also the scattering support. Specifically, let

$$M_\alpha := \left\{ x \middle| u_\alpha \textit{ can be analytically continued to an unbounded neighborhood } N_\epsilon(x) \right\}$$

Lemma 6.

$$\mathbb{R}^n \backslash \mathcal{M}_\alpha = \mathrm{S}_k \mathrm{supp}\alpha$$

Proof. Let u_{ext} be an extension of u_α to an unbounded open neighborhood $N_\epsilon(x)$. Let ϕ be smooth, supported in a slightly larger neighborhood, and equal to one on $N_\epsilon(x)$. Let

$$\tilde{f} = (\Delta + k^2)(1-\phi)u_{ext}$$

Then $\mathcal{F}_{\mathrm{k}}^{\infty}\tilde{f} = \alpha$, because $1-\phi = 1$ outside a ball, and $\tilde{f}$ is supported in an ϵ neighborhood of $\mathbb{R}^n \backslash \mathcal{M}_\alpha$. Thus

$$\mathrm{S}_k\mathrm{supp}\alpha \subset \mathbb{R}^n \backslash \mathcal{M}_\alpha$$

On the other hand, if $x \notin \mathrm{S}_k\mathrm{supp}\alpha$, there is an f with $x \notin \mathrm{supp}_\infty f$. Thus there is an unbounded neighborhood of x where the corresponding u_f satisfies the free equation $(\Delta + k^2)u_f = 0$. Hence u_f is real analytic there and $x \in \mathcal{M}_\alpha$. □

4 The Circular Paley Wiener Theorem

In this section and the following ones, we will begin to locate the convex scattering support. We now restrict ourselves to $\mathbb{R}^2$. We recall the classical Paley-Wiener theorem (see [3] page 181).

Theorem 7 (Paley-Wiener-Schwartz). *$F(\xi)$ extends to be a holomorphic function on $\mathbb{C}^n$ satisfying*

$$|F(\xi + i\eta)| \leq C|\xi + i\eta|^N \mathrm{e}^{R|\eta|}$$

if and only if F is the Fourier transform of a tempered distribution of order N supported in B_R, the ball of radius R.

We intend to extend far fields defined on the circle of radius k to all of $\mathbb{R}^2$, in such a way that the extensions are Fourier transforms of compactly supported functions. The following lemma provides us with our basic building blocks. J_n represents the Bessel function of order n.

Lemma 8. *Let $\rho^2 = \xi_1^2 + \xi_2^2$ and $\theta = \tan^{-1}\left(\frac{\xi_2}{\xi_1}\right)$ be polar coordinates on $\mathbb{R}^2$. Then*

$$F_{n,L}(\xi) = \frac{\mathrm{e}^{in\theta}J_{n+L}(\rho)}{\rho^L}$$

extends to a holomorphic function, $F_{n,L}(\xi+i\eta)$ on all of $\mathbb{C}^2$ and the estimates below hold.

$$\begin{aligned} |F_{n,L}(\xi+i\eta)| &\leq \mathrm{e}^{|\eta|} && (3)\\ &\leq \mathrm{e}^{(1+\varepsilon)|\eta|}\left(\frac{1}{1+\varepsilon}\right)^{n+L} && (4)\end{aligned}$$

Proof of the lemma. We first check that $F(\xi)$ is real analytic in $\mathbb{R}^2$ by writing the series expansion for $J_{n+L}(\rho)$, i.e.

$$\begin{aligned} F(\xi) &= \frac{\mathrm{e}^{in\theta}\left(\frac{\rho}{2}\right)^{n+L}\sum\frac{(-1)^k(\frac{\rho}{2})^{2k}}{k!(k+n)!}}{\rho^L}\\ &= \left(\frac{\mathrm{e}^{i\theta}\rho}{2}\right)^n\sum\frac{(-1)^k(\frac{\rho}{2})^{2k}}{k!(k+n)!}\\ &= \left(\frac{\xi_1+i\xi_2}{2}\right)^n\sum\frac{(-1)^k(\frac{\xi_1^2+\xi_2^2}{2})^{2k}}{k!(k+n)!}\end{aligned}$$

The last expression obviously extends to $\mathbb{C}^2$ as (with $\zeta=\xi+i\eta$)

$$\left(\frac{\zeta_1+i\zeta_2}{2}\right)^n\sum\frac{(-1)^k(\frac{\zeta_1^2+\zeta_2^2}{4})^{k}}{k!(k+n)!}$$

with the convergence of the series following from comparison with the series for $\mathrm{e}^{|\zeta_1^2+\zeta_2^2|}$.
To obtain (3), we will use *complex polar coordinates*, noting that every $\zeta=\xi+i\eta$ in $\mathbb{C}^2$ can be written as

$$\zeta=\xi+i\eta\ =\rho\mathrm{e}^{i\sigma}\begin{pmatrix}\cos(\theta+i\psi)\\ \sin(\theta+i\psi)\end{pmatrix}$$

and that

$$\eta=\frac{\rho}{2}\mathrm{e}^{-\psi}\begin{pmatrix}\cos(\sigma+\theta)\\ \sin(\sigma+\theta)\end{pmatrix}+\frac{\rho}{2}\mathrm{e}^{\psi}\begin{pmatrix}\cos(\sigma-\theta)\\ \sin(\sigma-\theta)\end{pmatrix}$$

For later use, we note also that $|\eta|$ increases as ψ moves away from zero and that

$$|\eta|\geq\rho\sin\sigma \tag{5}$$

Now

$$F(\xi)=\frac{\mathrm{e}^{in\theta}J_{n+L}(\rho)}{\mathrm{e}^{iL\theta}\rho^L}\qquad\text{extends to}\qquad\frac{\tilde{F}(\zeta)}{(\zeta_1+i\zeta_2)^L}$$

where $\tilde{F}$ extends $\mathrm{e}^{in\theta} J_{n+L}(\rho)$. Once we prove that

$$|\tilde{F}(\zeta)| \leq C\mathrm{e}^{|\eta|} \tag{6}$$

we will treat F by considering the function

$$h(z) = z^L(\zeta_1 + \frac{1}{z} + i\zeta_2)^2 F(\zeta_1 + z, \zeta_2)$$

$$F(\zeta_1 + i\zeta_2) \quad = \quad h(0)$$

so that, according to the maximal principle,

$$\begin{aligned} |F(\zeta_1 + i\zeta_2)| &\leq \max_{|z|=1} |h(z)| \\ &= \max_{|z|=1} \tilde{F}(\zeta_1 + z, \zeta_2) \\ &\leq C\mathrm{e}^{|z|+|\eta|} \leq \tilde{C}\mathrm{e}^{|\eta|} \end{aligned}$$

It remains to prove (6).

$$\begin{aligned} \tilde{F}(\zeta) &= \mathrm{e}^{im(\theta+i\psi)} J_{m+L}(\rho\mathrm{e}^{i\sigma}) \\ &= \int_0^{2\pi} \mathrm{e}^{i\rho\mathrm{e}^{i\sigma}\cos\tilde{\theta}} \mathrm{e}^{-im(\tilde{\theta}-\theta-i\psi)} \mathrm{e}^{-iL\tilde{\theta}} \frac{d\tilde{\theta}}{2\pi} \end{aligned}$$

If $m\psi \geq 0$ then the estimate is easy, namely

$$\begin{aligned} &\leq \mathrm{e}^{\mathrm{Im}(\rho\mathrm{e}^{i\sigma}\cos\tilde{\theta})} \\ &\leq \mathrm{e}^{\rho\sin\sigma} \leq \mathrm{e}^{|\eta|} \end{aligned}$$

by recalling (5).

If $m\psi \leq 0$, we compensate for the growing $\mathrm{e}^{-m\psi}$ by shifting the contour if integration into the complex plane, replacing $\tilde{\theta}$ by $\tilde{\theta} + i\psi$. The periodicity of the integral and the analyticity of the integrand justify the shift, giving

$$\tilde{F}(\zeta) = \int_0^{2\pi} \mathrm{e}^{i\rho\mathrm{e}^{i\sigma}\cos\tilde{\theta}+i\tilde{\psi}} \mathrm{e}^{-im(\tilde{\theta}-\theta+i(\tilde{\psi}-\psi))} \mathrm{e}^{-iL(\tilde{\theta}+i\tilde{\psi})} \frac{d\tilde{\theta}}{2\pi} \tag{7}$$

We may choose $\tilde{\psi}$ as we please. First, we choose $\tilde{\psi} = \psi$, so that

$$\begin{aligned} |\tilde{F}(\zeta)| &\leq \mathrm{e}^{-\mathrm{Im}(\rho\mathrm{e}^{i\sigma}\cos(\tilde{\theta}+i\psi))} \mathrm{e}^{L\psi} \\ &\leq \mathrm{e}^{|\eta|} \end{aligned}$$

The last step is justified because, since L has the same sign as m, $\mathrm{e}^{L\psi} \leq 1$, and because $\mathrm{Im}(\rho\mathrm{e}^{i\sigma}\cos(\tilde{\theta}+i\psi))$ is the imaginary part of the first component of a rotation of η, hence is less than $|\eta|$. This establishes (3). Finally, we return to (7) and choose $\tilde{\psi} = \psi - \log(1+\varepsilon)$ to establish (4). $\square$

In order to state our main theorem, we need a few definitions.

$$H_0^L(B_R) = \overline{\left\{ f \quad | \quad f \in C_0^\infty(B_R) \quad \text{and} \quad (\Delta+1)^{L/2} f \in L^2(\mathbb{R}^2) \right\}}.$$

Let $\alpha \in L^2(S^1)$ and $\{\alpha_n\}$ be its Fourier coefficients, i.e.

$$\alpha(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n \mathrm{e}^{in\theta}$$

We define

$$\sigma_n(R) = \left(\int_0^R J_n^2(s) s ds \right)^{\frac{1}{2}}$$

$$l_{L,R}^2 = \left\{ \alpha_n \quad | \quad (1+n^2)^{L/2} \frac{\alpha_n}{\sigma_n} \in l^2 \right\}$$

Theorem 9. *The following are equivalent:*

1. $\alpha_n \in l_{R,L}^2$
2. $\alpha = \mathcal{F}_{\mathrm{k}}^\infty f \ , \ f \in H_0^L(B_R)$
3. $\alpha = \mathcal{F}_{\mathrm{k}}^\infty f \ , \ (1-\Phi) f \in H_0^L(B_R)$ for $\Phi \in C_0^\infty$ *with* $\mathrm{supp}\Phi \subset B_{R-\epsilon}$

Remark 1. *Theorem 9 tells us that by examining the Fourier coefficients of the far field, we can determine first, the radius of the smallest ball which contains the scattering support of α, and then how smoothly the source may increase as we enter that ball. For example, with $\chi(r)$ defined to be the characteristic function of the ball of radius one.*

$$f = r\cos\theta \, \chi\left(\frac{r}{R}\right) \in H_0^L(B_R) \quad \textit{only for } L = 0$$

while

$$g = r\cos\theta \, (R-r)^M \chi\left(\frac{r}{R}\right) \in H_0^L(B_R) \quad \textit{for } L \leq M$$

The third item is in the theorem to emphasize that smoothness on the boundary of the ball is what we see in the far field, smoothness strictly inside the boundary is not relevant.

Before proceeding with the proof, we note a few properties of the weights σ_n, which follow from the asymptotic properties of the Bessel functions for fixed R as $n \to \infty$.

$$\sigma_n(R) \;\sim\; J_{n+\frac{1}{2}}(R) \;\sim\; \left(\frac{eR}{2(n+\frac{1}{2})}\right)^{n+\frac{1}{2}} \frac{1}{\sqrt{n}}$$

In particular,

$$\frac{\sigma_n(R_1)}{\sigma_n(R_2)} \;\sim\; \left(\frac{R_1}{R_2}\right)^n$$

so that, if $R_1 < R_2$, then

$$l^2_{R_1,L_1} \subset l^2_{R_2,L_2}$$

for any L_1 and L_2. We also note that

$$\frac{\sigma_{n+1}}{\sigma_n} \sim \frac{CR}{n} \tag{8}$$

with C a nonzero constant.

Proof of Theorem 9. By scaling variables, we may assume that $k = 1$. Suppose first that $f \in L^2(B_R)$, then write f as

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)\mathrm{e}^{in\phi}$$

Recalling that $\alpha = \hat{f}(\Theta)$, we have

$$\begin{aligned}
\alpha(\theta) &= \int_{B_R} \mathrm{e}^{ir\cos(\theta-\phi)} f(r,\phi) r dr d\phi \\
&= \sum \mathrm{e}^{in\theta} \int_0^R \int_0^{2\pi} \mathrm{e}^{ir\cos(\theta-\phi)} \mathrm{e}^{-in(\theta-\phi)} d\phi f_n(r) r dr \\
&= \sum \mathrm{e}^{in\theta} \int_0^R J_n(r) f_n(r) r dr
\end{aligned}$$

so that

$$\alpha_n = \int_0^R J_n(r) f_n(r) r dr.$$

Therefore,

$$\begin{aligned} |\alpha_n|^2 &\leq \int_0^R |J_n(r)|^2 r dr \int_0^R |f_n(r)|^2 r dr \\ \frac{|\alpha_n|^2}{\sigma_n^2(r)} &\leq \int_0^R |f_n(r)|^2 r dr \end{aligned}$$

so that

$$\begin{aligned} \sum \frac{|\alpha_n|^2}{\sigma_n^2(r)} &\leq \sum \int_0^R |f_n(r)|^2 r dr \\ &= ||f(r,\phi)||_{L^2(B_R)} \end{aligned}$$

Thus $\{\alpha_n\} \in l^2_{R,0}$. Now suppose $f \in H_0^L(B_R)$. If we write

$$f = \Phi f + (1-\Phi) f$$

with Φ smooth and supported in $B_{R-\epsilon}$, then

$$\tilde{\alpha} = \mathcal{F}_{\mathrm{k}}^\infty \Phi f \in l^2_{R-\epsilon,0} \subset l^2_{R,L} \quad \text{for all L},$$

so that we may assume that f is supported in $B_R \backslash B_{R-\epsilon}$. Now we use the identity

$$r J_n(r) = \left(r \frac{d}{dr} + n \right) J_{n+1}(r)$$

so that

$$\begin{aligned} \alpha_n &= \int_0^R r J_n(r) f_n(r) dr \\ &= \int_0^R \left(r \frac{d}{dr} + n \right) J_{n+1}(r) f_n(r) dr \\ &= -\int_0^R J_{n+1}(r) \left(r \frac{d}{dr} - n \right) f_n(r) dr \end{aligned}$$

and

$$|\alpha_n^2| \leq \int_0^R |J_{n+1}(r)|^2 r dr \int_0^R |g_n(r)|^2 r dr$$

so that

$$
\begin{aligned}
\sum \frac{|\alpha_n|^2}{\sigma^2_{n+1}(r)} &\leq \sum \int_0^R |g_n(r)|^2 r dr \\
&= ||g(r,\phi)||_{L^2(B_R)}
\end{aligned}
$$

where

$$
g_n(r) = \left(r\frac{d}{dr} - \frac{n}{r} \right) f_n(r)
$$

and therefore

$$
\begin{aligned}
g(r,\phi) &= \sum g_n(r) \mathrm{e}^{in\phi} \\
&= \left(\frac{d}{dr} - \frac{i}{r}\frac{\partial}{\partial \phi} \right) f(r,\phi) \qquad (9)
\end{aligned}
$$

so that

$$
||g||_{L^2(B(R))} \leq ||f||_{\mathrm{H}^1_0(B(R))}
$$

Take note that f is supported away from the origin, so that we needn't be concerned with the apparent difficulty associated with the $\frac{1}{r}$ in (9).
Repeating this procedure L times and noting that

$$
\begin{aligned}
\sum \frac{|\alpha_n|^2(1+n^2)^{L/2}}{\sigma^2_n(r)} &\leq \sum \frac{|\alpha_n|^2}{\sigma^2_{n+L}(r)} \\
&\leq ||f||^2_{\mathrm{H}^L_0(B(R))}
\end{aligned}
$$

finishes the proof that 2) implies 1), and also that 3) implies 1).
We next suppose 1) and define

$$
\hat{f} = \sum \frac{\alpha_n}{\sigma^2_{n+L}(R)} \mathrm{e}^{in\theta} \int_0^R J_{n+L}(s) \frac{J_{n+L}(\rho s)}{\rho^L} s ds \qquad (10)
$$

We intend to show that $\hat{f}$ belongs to $L^2_L(\mathbb{R}^2)$ (i.e. $(1+\rho^2)^{L/2}\hat{f} \in L^2$), extends to be holomorphic in $\mathbb{C}^2$, and satisfies the Paley-Wiener estimate

$$
|\hat{f}(\xi + i\eta)| \leq \mathrm{e}^{R|\eta|}
$$

This will allow us to conclude that the inverse Fourier transform, f, of $\hat{f}$

belongs to $\mathrm{H}_0^L(B(R))$. We start with a smooth g and compute

$$
\begin{aligned}
\int \quad \overline{g(\rho,\theta)} \quad & \rho^L \hat{f}(\rho,\theta)\rho d\rho d\theta \\
= \quad & \int \sum \frac{\alpha_n}{\sigma_{n+L}^2} \mathrm{e}^{in\theta} \int_0^R J_{n+L}(s) \frac{J_{n+L}(\rho s)}{\rho^L} s ds \sum \overline{g}_m(\rho) \mathrm{e}^{-in\theta} \rho d\rho d\theta \\
= \quad & \sum \frac{\alpha_n}{\sigma_{n+L}^2} \int_0^\infty \int_0^R J_{n+L}(s) J_{n+L}(\rho s) s ds \overline{g}_n(\rho) \rho d\rho \\
= \quad & \sum \frac{\alpha_n}{\sigma_{n+L}^2} \int_0^R \int_0^\infty J_{n+L}(\rho s) \overline{g}_n(\rho) \rho d\rho J_{n+L}(s) s ds \\
= \quad & \sum \frac{\alpha_n}{\sigma_{n+L}^2} \int_0^R \gamma_n(s) J_{n+L}(s) s ds \\
\leq \quad & \sum \frac{\alpha_n}{\sigma_{n+L}^2} \left(\int_0^R J_{n+L}^2(s) s ds \right)^{\frac{1}{2}} \cdot \left(\int_0^R |\gamma_n(s)|^2 s ds \right)^{\frac{1}{2}} \\
\leq \quad & \left(\sum \left(\frac{\alpha_n}{\sigma_{n+L}} \right)^2 \right)^{\frac{1}{2}} \cdot \left(\sum \int_0^R |\gamma_n(s)|^2 s ds \right)^{\frac{1}{2}}
\end{aligned}
$$

where

$$
\sum \int_0^R |\gamma_n(s)|^2 s ds = ||\mathrm{e}^{-iL\theta} g(\rho,\theta)||_{L^2}^2
$$

This allows us to conclude, via duality, that $\rho^L \hat{f} \in L^2$, but $\hat{f}$ is holomorphic, hence bounded, so $(1+\rho^2)^{L/2} \hat{f} \in L^2$, i.e. $\hat{f} \in L_L^2$ or $f \in \mathrm{H}^L(\mathbb{R}^2)$.

Checking the Paley-Wiener property is easier. We return to (10), extend to $\mathbb{C}^2$, and rewrite it as

$$
\begin{aligned}
\hat{f} &= \sum \frac{\alpha_n}{\sigma_{n+L}^2} \int_0^R J_{n+L}(s) \frac{\mathrm{e}^{in\theta+i\psi} J_{n+L}(\rho \mathrm{e}^{i\sigma} s)}{(\rho \mathrm{e}^{i\sigma})^L} s ds \\
|\hat{f}| &\leq \sum \frac{\alpha_n}{\sigma_{n+L}^2} \left(\int_0^R |J_{n+L}(s)|^2 s ds \right)^{\frac{1}{2}} \cdot \int_0^R \left| \frac{\mathrm{e}^{in\theta+i\psi} J_{n+L}(\rho \mathrm{e}^{i\sigma} s)}{(\rho \mathrm{e}^{i\sigma})^L} \right|^2 s ds \\
&\leq \left(\sum \left(\frac{\alpha_n}{\sigma_{n+L}} \right)^2 \right)^{\frac{1}{2}} \cdot R^2 \left(\sum \left| \frac{\mathrm{e}^{in\theta+i\psi} J_{n+L}(\rho \mathrm{e}^{i\sigma} s)}{(\rho \mathrm{e}^{i\sigma})^L} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum \left(\frac{\alpha_n}{\sigma_{n+L}} \right)^2 \right)^{\frac{1}{2}} \cdot R^2 \mathrm{e}^{(1+\epsilon)|\eta|} \left(\sum \frac{1}{(1+\epsilon)^{2n}} \right)^{\frac{1}{2}}
\end{aligned}
$$

the last step following from (4). Hence $\mathrm{supp} F \subset B_{R+\epsilon}$ for all ϵ, so $\mathrm{supp} F \subset B_R$. □

Now that we can locate the radius of the smallest sphere about the origin which contains the $\mathrm{cS}_k \mathrm{supp}\alpha$, we shift the center of the sphere, so that we can locate the the convex scattering support in the intersection of spheres.

Theorem 10. *Suppose that*

$$
\begin{aligned}
\mathcal{F}_{\mathrm{k}}^{\infty} f &= \alpha \\
&= \sum \alpha_n \mathrm{e}^{in\theta}
\end{aligned}
$$

then

$$
\begin{aligned}
\mathcal{F}_{\mathrm{k}}^{\infty} f(x+c) &= \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \alpha \\
&= \sum \alpha_n^c \mathrm{e}^{in\theta}
\end{aligned}
$$

where

$$
\alpha_m^c = \sum_n J_{m-n}(k|c|) e^{i\theta_c(n-m)} \alpha_n
$$

where $(|c|, \theta_c)$ are the polar coordinates of c.

Proof. Let $f_c(x) = f(x+c)$ denote the translate of f.

$$
\begin{aligned}
\hat{f}_c\big|_{|\xi|=k} &= \mathrm{e}^{ic\cdot\xi} \hat{f}(\xi)\big|_{|\xi|=k} \\
&= \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \hat{f}(k,\theta) \\
&= \sum \alpha_n \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \mathrm{e}^{in\theta}
\end{aligned}
$$

The α_n^c are the Fourier coefficients, i.e.

$$\begin{aligned}
\alpha_n^c &= \sum \alpha_n \mathrm{e}^{i(n-m)\theta_c} \int_0^{2\pi} \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \mathrm{e}^{i(n-m)(\theta-\theta_c)} d\theta \\
&= \sum \alpha_n \mathrm{e}^{i(n-m)\theta_c} \int_0^{2\pi} \mathrm{e}^{ik|c|\cos(\theta-\theta_c)} \mathrm{e}^{-i(m-n)(\theta-\theta_c)} d\theta \\
&= \sum \alpha_n \mathrm{e}^{i(n-m)\theta_c} J_{m-n}(k|c|)
\end{aligned}$$

□

Corollary 11.

$$\mathrm{cS}_k \mathrm{supp}\alpha \subset \bigcap_{j=1}^{n} B_{R_j}(c_j) \quad \textit{if and only if} \quad \alpha_n^{c_j} \in l^2_{R_j} \quad \forall j$$

Before finishing this section we would like to extend the circular Paley-Wiener theorem, theorem 9 to distributions, in particular single and multiple layer distributions on the boundary of a region. Note that all these distributions can be obtained as differential operators with smooth coefficients acting on the characteristic function of the region. We define:

Definition 5.

$$\mathcal{H}_0^{-L}(B_R) = \left\{ f \quad \middle| \quad f = P^L(x,D)g \middle| \quad g \in L^2(B_R) \right\}$$

where $P^L(x,D)$ is an L'th order differential operator with smooth coefficients. We have

Theorem 12 (Circular Paley-Wiener Theorem Extended). *The following are equivalent:*

1. $\alpha_n \in l^2_{R,-L}$
2. $\alpha = \mathcal{F}_{\mathrm{k}}^{\infty} f$, $f \in \mathcal{H}_0^{-L}(B_R)$
3. $\alpha = \mathcal{F}_{\mathrm{k}}^{\infty} f$, $(1-\Phi)f \in \mathcal{H}_0^{-L}(B_R)$ for $\Phi \in C_0^{\infty}$ *and* $\mathrm{supp}\Phi \subset B_{R-\epsilon}$

Proof. Without loss of generality, we may assume that $\mathrm{supp} f \subset B_R \backslash B_{R-\epsilon}$and that $k = 1$. We will carry out the proof below for $L = -1$. The general case then follows by repeating the argument L times. Suppose that $f \in L^2(B_R)$.We make the following simple observations:

$$\begin{aligned}
\{(\alpha_{\frac{\partial f}{\partial \phi}})_n\} &= \left\{\left(\frac{\partial \alpha_f}{\partial \theta}\right)_n\right\} \\
&= \{in\alpha_n\} \qquad \in l^2_{R,-1} \\
\{(\alpha_{\overline{\partial} f})_n\} &= \{(\mathrm{e}^{-i\theta}\alpha_f)_n\} \\
&= \{(\alpha_f)_{n-1}\} \qquad \in l^2_{R,-1}
\end{aligned}$$

because of (8). Combining these two with the polar coordinate representation d-bar operator

$$\overline{\partial} = (\partial_r + (i/r)\partial_\phi + 1/r)\,\mathrm{e}^{-i\phi}$$

yields

$$\{(\alpha_{\frac{\partial f}{\partial r}})_n\} = \{(\alpha_{\overline{\partial}(\mathrm{e}^{i\phi} f)})_n\} - \left\{\left(\alpha_{-(\frac{i\partial}{\partial \phi}+1)f/r}\right)_n\right\} \qquad \in l^2_{R,-1}$$

Now, if $f \in \mathcal{H}_0^{-1}(B_R)$ then we may write

$$f = \frac{\partial}{\partial r} a(r,\theta)g + \frac{\partial f}{\partial \theta} b(r,\theta)g + c(r,\theta)g$$

with smooth a, b, c. Thus $ag, bg, cg \ \in L^2(B_R)$ and , according to the observations above

$$\begin{gathered}
\{(\alpha_{\frac{\partial bg}{\partial \theta}})_n\} \in l^2_{R,-1} \\
\{(\alpha_{\frac{\partial ag}{\partial r}})_n\} \in l^2_{R,-1} \\
\{(\alpha_{cg})_n\} \in l^2_{R,0}
\end{gathered}$$

and therefore the sum of the three terms, $\{(\alpha_f)_n\} \in l^2_{R,-1}$.

Conversely, suppose that $\{\alpha_n\} \in l^2_{R,-1}$. We may assume that $\alpha_0 = 0$ because any finite sequence belongs to $l^2_{R,L}$ for every L. Consider

$$\{\beta_n\} = \left\{\frac{\alpha_n}{in}\right\} \in l^2_0$$

According to theorem 9,

$$\begin{aligned}\beta &= \mathcal{F}_{\mathrm{k}}^{\infty} g \quad \text{with } g \in L^2(B_R)\end{aligned}$$

so

$$\begin{aligned}f &= \frac{\partial g}{\partial \theta} \in \mathcal{H}_0^{-1}(B_R)\end{aligned}$$

and

$$\begin{aligned}\alpha_f &= \alpha\end{aligned}$$

□

5 The Scattering Support of a Source

We begin this section with a definition.

Definition 6. *By the (convex) scattering support of a source f we mean the (convex) scattering support of its far field.*

We note that the (convex) scattering support is a lower bound for the for the (convex hull of the) support of f, and in some instances it may be considerably smaller. We see this in the following example.

Example 1. *Let $f = (\Delta + k^2)\Phi$ for some $\Phi \in C_0^\infty(\mathbb{R}^2)$. Then $\mathcal{F}_{\mathrm{k}}^{\infty} f = 0$ and both the convex scattering support and the scattering support are empty. This is because Φ itself is the outgoing solution to the Helmholtz equation, and, having compact support, has far field zero.*

Example 2. *Let $f = \chi_{B_R(0)}$, where $B_R(0)$ is the ball of radius R centered at 0. Here, $\mathcal{F}_{\mathrm{k}}^{\infty} f$ is equal to the constant $\frac{2\pi}{k} R J_1(kR)$ and $\mathrm{cS}_k\mathrm{supp} f = \mathrm{S}_k\mathrm{supp} f = \{0\}$.*

Example 3. *Let $f = \chi_\Omega(\Delta + k^2)\Phi$, where again $\Phi \in C_0^\infty(\mathbb{R}^2)$. Then,*

$$\mathrm{S}_k\mathrm{supp} f \subset \partial\Omega \cap \mathrm{supp}_\infty \Phi.$$

Proof. This follows from the observation

$$\mathcal{F}_{\mathrm{k}}^{\infty}(\Delta + k^2)\Phi = 0$$

since Φ is compactly supported and hence the unique outgoing solution to $(\Delta + k^2)u = (\Delta + k^2)\Phi$. Therefore

$$\mathcal{F}_{\mathrm{k}}^{\infty} \chi_\Omega(\Delta + k^2)\Phi = \mathcal{F}_{\mathrm{k}}^{\infty} \left(1 - \chi_\Omega\right)(\Delta + k^2)\Phi.$$

We may apply lemma 4 with $\Omega_1 = \mathrm{supp}_\infty \Phi \cap \Omega$ and $\Omega_2 = \mathrm{supp}_\infty \Phi \cap (\mathbb{R}^2 \setminus \Omega)$, noting that the complement of the union is $\mathbb{R}^2 \setminus \mathrm{supp}_\infty \Phi$ which is connected. Therefore,

$$\mathrm{S}_k \mathrm{supp} \chi_\Omega (\Delta + k^2)\Phi \subset N_\epsilon \left(\mathrm{S}_k \mathrm{supp} \chi_\Omega (\Delta + k^2)\Phi \cap \mathrm{supp}\,(1 - \chi_\Omega)\,(\Delta + k^2)\Phi \right)$$

for any $\epsilon > 0$. Hence,

$$\mathrm{S}_k \mathrm{supp} \chi_\Omega (\Delta + k^2)\Phi \subset \partial\Omega \cap \mathrm{supp}(\Delta + k^2)\Phi \subset \partial\Omega \cap \mathrm{supp}\Phi.$$

$\square$

In some cases, we can show that $\mathrm{cS}_k \mathrm{supp} f$ is bigger than the small sets described in these examples. In particular, if $\mathrm{supp} f$ has a convex corner at some point p, and the leading term in the Taylor series for f satisfies a non-degeneracy condition at that point, then $p \in \mathrm{cS}_k \mathrm{supp} f$. That is, any source g that radiates α must have $p \in \mathrm{chsupp} g$. The remainder of this section is devoted to a proof of this fact.

Before stating the precise result, we define

$$s_- = \begin{cases} 0, & s > 0 \\ s, & s < 0 \end{cases} .$$

In this way, s_-^n has obvious meaning for any $n > 0$, and

$$s_-^0 = \begin{cases} 0, & s > 0 \\ 1, & s < 0 \end{cases} .$$

Next, we give the following definition.

Definition 7. *f has a corner at p if $f = s_-^0 t_-^0 f$ with*

$$\begin{aligned} s &= (x - p) \cdot v_1 \\ t &= (x - p) \cdot v_2 \end{aligned}$$

where $x, p, v_1, v_2 \in \mathbb{R}^2$ and $|v_1| = |v_2| = 1$. The corner is convex if p does not belong to the convex hull of $\mathrm{supp} f \setminus p$.

The angle, ω, of the corner located at p satisfies $\cos\omega = v_1 \cdot v_2$.

Lemma 13. *Suppose f has a convex corner at p. Then either*

i) $p \in \mathrm{cS}_k \mathrm{supp} f$

or

ii) There exists a neighborhood $N_\epsilon(x)$ and w_- solving

$$(\Delta + k^2) w_- = f_-$$

such that, in $N_\epsilon(x)$, $f_- = f$ and $\mathrm{supp} w_- \cap N_\epsilon(x) \subset \{s_- t_- < 0\}$*.*

Proof. Suppose $p \notin \mathrm{cS}_k \mathrm{supp} f$, then there exists a source g with $N_\epsilon(p) \bigcap \mathrm{chsupp} g = \emptyset$, and $\mathcal{F}_{\mathrm{k}}^\infty f = \mathcal{F}_{\mathrm{k}}^\infty g$. Thus the solution v to

$$(\Delta + k^2) v = g$$

solves the free Helmholtz equation to an unbounded neighborhood containing $N_\epsilon(x)$.

$$(\Delta + k^2) u = f = s_-^0 t_-^0 f \tag{11}$$

Because $\mathbb{R}^2 \setminus (\mathrm{ch\ supp} f \bigcap \mathrm{ch\ supp} g)$ is connected and unbounded, u is identically equal to v there (Rellich's lemmma and unique continuation, just as in lemma 4). We note that this set contains $\{s_- t_- \geq 0\} \bigcap N_\epsilon(x)$. Therefore, if we let $f - g = f_-$ and $u - v = w_-$,

$$(\Delta + k^2) w_- = f_-$$

with $w_- \equiv 0$ on $\{s_- t_- \geq 0\} \bigcap N_\epsilon(x)$ because $u = v$ there and $f_- = f$ in $N_\epsilon(x)$ because $g = 0$ there. □

A consequence of lemma 13 is that both w_- and $\partial w_- / \partial \nu$ vanish on $\{s_- = 0\}$ and on $\{t_- = 0\}$. Now, suppose f has the Taylor expansion at p, in the corner coordinates,

$$f = \sum_{n,m} f_{nm} s_-^n t_-^m, \tag{12}$$

then,

$$f_- = s_-^0 t_-^0 f = \sum_{n=0,m=0}^{N} f_{nm} s_-^n t_-^m + \tilde{f},$$

and, w_-, if it exists, will have an expansion

$$w_- = \sum_{n=2,m=2}^{N+2} w_{n,m} t_-^n s_-^m \tag{13}$$

The expansion for w_- must start at $n = 2$ and $m = 2$ because the Laplacian of s_-^0, s_-^1, t_-^0, or t_-^1 will contain a distribution supported on $t = 0$ or $s = 0$ and (11) guarantees that $\Delta w_- \in L^\infty$.
We can use this as a criterion for determining whether $p \in \mathrm{cS}_k\mathrm{supp} f$. We have the following theorem.

Theorem 14. *Let $T_p^m f$ denote the Taylor polynomial of order m of f at p and let $f^N(x, y)$ denote the non-vanishing homogeneous polynomial of lowest order which begins $T_p^m f$. Each of the conditions below guarantees that the corner $p \in \mathrm{cS}_k\mathrm{supp} f$.*

i. There exists no polynomial Q of degree $m - 2$ or less such that

$$T_p^m f = (\Delta + 1)s^2 t^2 Q \quad \textit{modulo terms of order } m{+}2$$

ii. There exists no homogeneous polynomial Q of degree $n - 2$ such that

$$f^N(x, y) = \Delta s^2 t^2 Q$$

iii. $f^N(x, y)$ is a spherical harmonic.

Remark 2. *The Laplacian referred to in the theorem is the standard one*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

We may use the s, t coordinates, but then

$$\Delta = \cos^2(\omega/2)\frac{\partial^2}{\partial t^2} + \sin^2(\omega/2)\frac{\partial^2}{\partial s^2}$$

Proof. If there exists a w_-, then it will have the expansion (13) as long as f has (12). Therefore, the first item implies that no such w_- exists. The second item implies the first condition, that is, if we seek to expand

$$T_p^m f = f^N + f^{N+1} + \cdots + f^m = \Delta t^2 s^2 (Q^{N-2} + Q^{N-3} + \cdots Q^{m-2})$$

the first step is to solve

$$f^N = \Delta t^2 s^2 Q^{N-2}$$

for the homogeneous polynomial Q^{N-2}, so if this fails so must the first condition. We claim that the third condition implies the second. That is, that it is impossible to solve

$$\Delta Q^{N+2} = H^N$$

with

$$Q^{N+2} = t^2 s^2 Q^{N-2}$$

and

$$\Delta H^N = 0.$$

To see this, we use the complex notation $z = x + iy$. First, we translate so that p sits at the origin, and then we rotate so that s and t become

$$\begin{aligned} t &= (\cos\omega/2, \sin\omega/2) \cdot (x, y) \\ s &= (\cos\omega/2, -\sin\omega/2) \cdot (x, y) \end{aligned}$$

Then, Q must solve

$$\Delta Q^{N+2} = H^N = 4(N+1)Az^N + (N+1)B\overline{z}^N$$

because z^N and $\overline{z}^N$ span the 2-dimensional space of spherical harmonics of order N. This means that

$$\partial_z \partial_{\overline{z}} Q^{N+2} = (N+1)Az^N + (N+1)B\overline{z}^N$$

and therefore,

$$Q^{N+2} = Az^{N+1}\overline{z} + Bz\overline{z}^{N+1} + Dz^{N+2} + E\overline{z}^{N+2}.$$

We show that no such Q^{N+2} can vanish, along with its first derivatives, on a both $\{t_- = 0\}$ and $\{s_- = 0\}$. If it does, then

$$\begin{aligned} \frac{\partial Q}{\partial z}|_{z=e^{\pm i\omega/2}} &= 0 \\ \frac{\partial Q}{\partial \overline{z}}|_{z=e^{\pm i\omega/2}} &= 0 \end{aligned}$$

so that

$$\begin{aligned}(-N+2)Be^{\mp i\omega} &= De^{\pm N\omega}+(N+1)E\\(-N+2)Ae^{\pm i\omega} &= (N+1)D+Ee^{\pm N\omega}.\end{aligned}$$

Taking linear combinations of the above equations yields

$$\begin{aligned}(-N+2)B\sin\omega &= D\sin N\omega\\(-N+2)B\sin(N-1)\omega &= (N+1)E\sin N\omega.\end{aligned}$$

A similar calculation for the previous simultaneous equations gives

$$\begin{aligned}(-N+2)A\sin\omega &= E\sin N\omega\\(-N+2)A\sin(N-1)\omega &= (N+1)D\sin N\omega.\end{aligned}$$

As long as $\omega \neq \pm\pi$, that is as long as we are dealing with a real corner, we divide and obtain

$$\begin{aligned}\frac{\sin(N-1)\omega}{\sin\omega} &= (N+1)\frac{E}{D}\\\frac{\sin(N-1)\omega}{\sin\omega} &= (N+1)\frac{D}{E}\end{aligned}$$

so that

$$\frac{\sin^2(N-1)\omega}{\sin^2\omega}=(N+1)^2,$$

but

$$\frac{\sin^2(N-1)\omega}{(N+1)^2\sin^2\omega}<1.$$

□

6 The Scattering support of an Inhomogeneous Medium

The results in the previous section, and the notion of scattering support, have immediate application to the inhomogeneous medium problem. If $n=1-m$ with $m\in L^\infty$ and compactly supported, there is a unique solution u to

$$(\Delta+k^2n(x))y=0 \quad \text{on} \quad \mathbb{R}^2 \tag{14}$$

which has the form

$$u = u_i + u_{sc}$$

with the incident wave u_i given by

$$u_i = \int_0^{2\pi} e^{ikx\cdot\Theta}\beta(\theta)d\theta$$

is a solution to the free Helmholtz equation (with Herglotz kernel β) and the outgoing scattered wave u_{sc}, i.e. one which satisfies the limiting absorption principle for some source $f \in \mathcal{B}$. The scattering operator $\mathcal{S}$ associated with the index of refraction $n(x)$ is defined to be the far field of the outgoing wave u_{sc}, that is

$$u_{sc} \sim \frac{e^{ikr}}{\sqrt{r}}\alpha(\theta), \quad r = |x|$$

so that

$$\mathcal{S}_{kn}\beta = \alpha$$

where

$$\mathcal{S}_{kn} : L^2(S^1) \to L^2(S^1).$$

We may write (14) as

$$(\Delta + k^2)u = k^2 mu.$$

We note the above source mu has the same support as m.

Lemma 15.

$$\mathrm{supp} mu = \mathrm{supp} m.$$

Proof. The weak unique continuation principle implies that u cannot vanish on an open set in $\mathbb{R}^2$. □

We conclude that

$$\mathrm{supp}_\infty m \supset \mathrm{S}_k \mathrm{supp}\mathcal{S}_{kn}\beta.$$

$$\mathrm{chsupp} m \supset \mathrm{cS}_k \mathrm{supp}\mathcal{S}_{kn}\beta.$$

If we have one or more far fields, resulting from an incident field generated by Herglotz kernel β_j at wavenumber k_j, then

$$\mathrm{supp}_\infty m \supset \bigcup_{j=1}^{N} \mathrm{S}_{k_j} \mathrm{supp}\mathcal{S}\beta_j.$$

$$\text{chsupp}m \supset \bigcup_{j=1}^{N} \text{cS}_{k_j}\text{supp}\mathcal{S}\beta_j.$$

We settle for one application of the previous section here (see [8] for an algorithm and a numerical method that finds this scattering support).

Theorem 16. *Suppose m has a convex corner at some point $p \in \mathbb{R}^2$ and $m(p) \neq 0$, then*

$$p \in \text{cS}_k\text{supp}m.$$

That is, if, for some $\tilde{n} = 1 + \tilde{m}$, the intersection of the ranges $\mathcal{R}(\mathcal{S}_{kn}) \cap \mathcal{R}(\mathcal{S}_{\tilde{n}k}) \neq \emptyset$, then

$$p \in \text{chsupp}(\tilde{m}).$$

Proof. Forget for the moment that $f = k^2 mu$ and write

$$(\Delta + k^2)u = f$$

If $p \notin \text{cS}_k\text{supp}f$ then u extends to a neighborhood of p as a solution to

$$(\Delta + k^2)u = 0$$

and therefore has a Taylor expansion at p. If the expansion starts with a homogeneous polynomial P^N of order N, then ΔP^N has order $N - 2$, and must be zero. Thus P^N is a harmonic polynomial.

Now remember that $f = k^2 mu$ and that $m(p) \neq 0$ and apply theorem 14 to conclude that p belongs to the convex scattering support.

□

7 The Scattering support of an Obstacle

The obstacle problem has a formulation parallel to that of the inhomogeneous medium problem i.e.

$$\begin{aligned} (\Delta + k^2)u &= 0 \qquad \text{in } \mathbb{R}^2 \backslash \Omega \\ u\big|_{\partial\Omega} &= 0 \\ u &= u_i + u_{sc} \end{aligned}$$

with the incident wave u_i given by

$$u_i = \int_0^{2\pi} e^{ikx\cdot\Theta}\beta(\theta)d\theta$$

a solution of the free Helmholtz equation (with Herglotz kernel β) and the scattered wave u_{sc} is required to be outgoing.
If we define

$$v = \begin{cases} 0 & \text{in} \quad \Omega \\ u & \text{in} \quad \mathbb{R}^2\backslash\Omega \end{cases}$$

then

$$\begin{aligned} (\Delta + k^2)v &= \delta_{\partial\Omega}\tfrac{\partial u}{\partial \nu}\big|_{\partial\Omega} \\ v &= u_i + v_{sc} \end{aligned}$$

so that v_{sc} is the unique outgoing solution to the source problem

$$(\Delta + k^2)v_{sc} = \delta_{\partial\Omega}\frac{\partial u}{\partial \nu}$$

and all the results of the previous sections apply.

The obstacle problem is formally determined. That is, we expect a single far field to determine the obstacle in most cases. For this reason, the concept of scattering support may not be as useful as in the previous cases, where the data was clearly insufficient to determine the all the properties of the scatterer. There already exist strong results due to Colton and Sleeman [1] regarding the unique determination of the obstacle from one or few incident waves.

One reason for extending the notion of scattering support to this case is that we can look for and find it without knowing *a priori* whether the scatterer is an obstacle or a penetrable medium, or for that matter, a source of any kind.

8 Conclusions and Summary

We have introduced the scattering support and the convex scattering support, and described in theorem 9, one way to compute the convex scattering support, by finding the intersection of all spheres which contain it. Another way to compute the scattering support is described, and implemented numerically, in [8].

We fully expect all of the two dimensional results to generalize to higher dimensions, and to other inverse problems, including the inverse gravitational problem and electrical impedance tomography, although the definition of the data will be slightly different.

There is much we do not yet understand about the scattering support. In the case of a penetrable medium, there is numerical evidence that the scattering support is much closer to the true support than we have been able to show here. A particularly pertinent question is to what extent, with or without a priori hypotheses about the medium, the two coincide.

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