An iterative method for solving the linear feasibility problem

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Abstract

Many optimization problems reduce to the solution of a system of linear inequalities (SLI). Some solution methods use relaxed, averaged projections. Others invoke surrogate constraints (typically stemming from aggregation). This paper proposed a blend of these two approaches. A novelty comes with introducing as surrogate constraint a halfspace defined by differences of algorithmic iterates. The first iteration is identical to surrogate constraints methods. In next iterations, for a given approximation $\bar{x}$, besides the violated constraints in $\bar{x}$, we also take into consideration the surrogate inequality, which we have obtained in the previous iteration.

The motivation for this research comes from the work of H. Scolnik et al. [1], who studied some projection methods for a system of linear equations.

1 Introduction

We consider the problem of finding a solution x^* of a consistent system of linear inequalities

$$G^\top x \leq b, \qquad (1)$$

where G is a matrix of size $n \times m$ with columns $g_i \in \mathbb{R}^n$, $i \in I = \{1, ..., m\}$, $x = (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n$, $b = (b_1, b_2, ..., b_m)^\top \in \mathbb{R}^m$.

The problem of finding a solution of a SLI is called also the *linear feasibility problem.*

Let $\bar{x}$ be the current approximation of a solution of this problem and let x^+ be the next approximation, and let $I(\bar{x}) = \{i \in I : g_i^\top \bar{x} > b_i\}$ denotes the subset of violated constraints in the point $\bar{x}$, $M = \{x \in \mathbb{R}^n : G^\top x \leq b\} \neq \varnothing$ denotes the solution set of (1) $M(\bar{x}) = \{x \in \mathbb{R}^n : g_i^\top x \leq b_i,\ i \in I(\bar{x})\}$ denotes the solution set of a subsystem of violated constraints for a given approximation $\bar{x}$ of a solution $x^* \in M$.

Let

$$s^i = P_i(\bar{x}) - \bar{x}, \qquad (2)$$

where P_i denotes the metric projection onto the halfspace $H_i = \{x \in \mathbb{R}^n : g_i^\top x \leq b_i\}$.

We assume, without loss of generality, that $\|g_i\| = 1$, $i \in I$, where $\|\cdot\|$ denotes the Euclidean norm.

Let $w = (w_1, ..., w_m)^\top \in \Delta_m = \{u \in \mathbb{R}^m : u \geq 0,\ e^\top u = 1\}$, where $e = (1, ..., 1)^\top \in \mathbb{R}^m$, be a *vector of weights.* We consider only vectors of weights with zero weights for nonviolated constraints, i.e., $w_i = 0$ for $i \in I \backslash I(\bar{x})$.

If we multiply the particular inequalities of (1) by coordinates of a vector of weights w and we add formed inequalities then we obtain a *surrogate inequality (surrogate constraint)*

$$w^\top G^\top x \leq w^\top b.$$

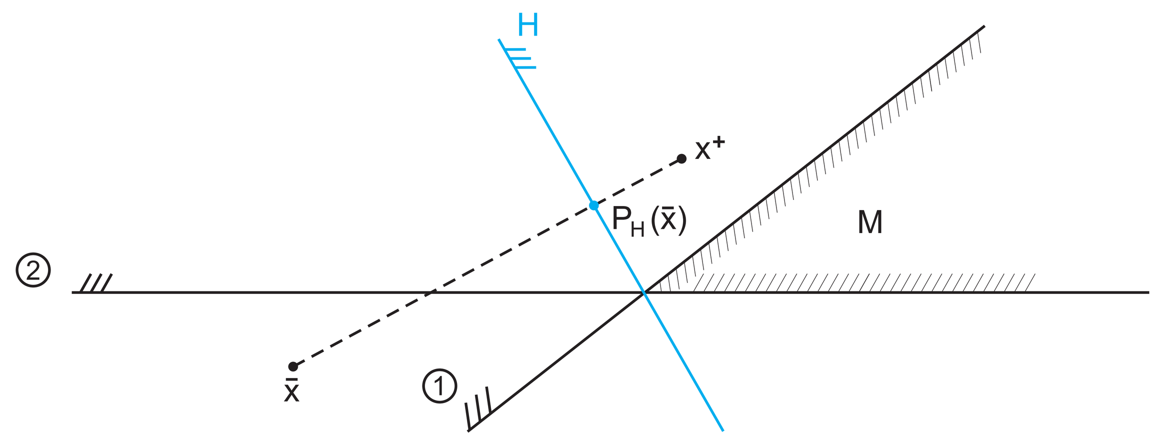
Of course $M \subset H_w = \{x \in \mathbb{R}^n : w^\top G^\top x \leq w^\top b\}$. In one iteration of the *surrogate constraints method we* calculate a vector of weights $w \in \Delta_m$ and a projection vector $t(w)$ onto the halfspace H_w, i.e.,

$$t(w) = P_{H_w}(\bar{x}) - \bar{x}.$$

Thus, the next point x^+ has the form

$$x^+ = \bar{x} + \lambda t(w), \qquad (3)$$

where $\lambda \in [0, 2]$ is a relaxation parameter.



2 Construction of a new surrogate constraint

Let $\bar{x} \notin M$ be an approximation of a solution $x^* \in M$ and let

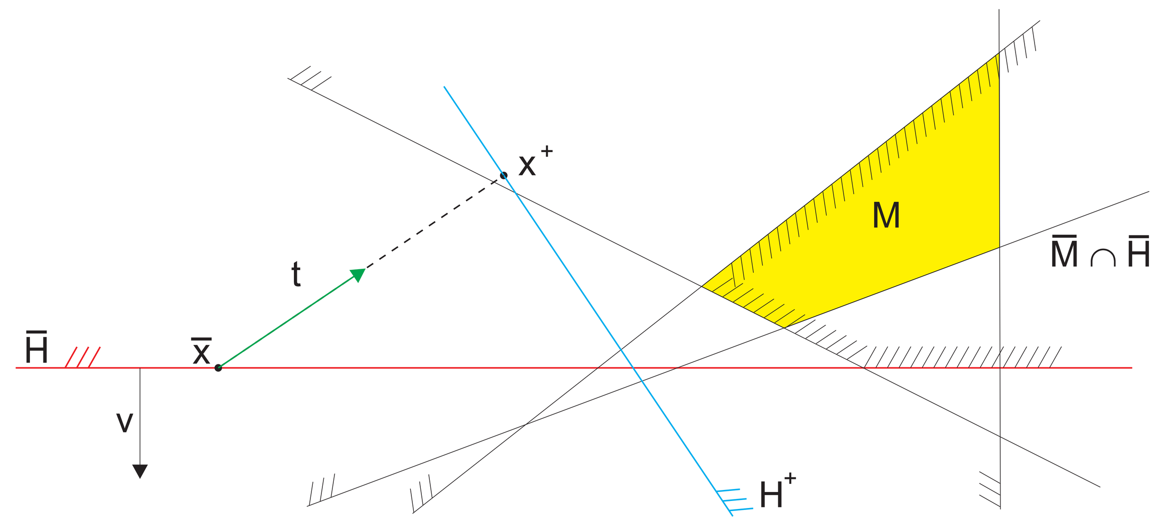
$$\bar{H} = H(\bar{x}) = \{x \in \mathbb{R}^n : v^\top (x - \bar{x}) \leq 0\}$$

be a halfspace with a normal vector $v \in \mathbb{R}$, such that $M \subset \bar{H}$.

Make a new point $x^+ \in \bar{H}$ which essentially better approximates a solution x^* will be our main aim. Let x^+ has the form

$$x^+ = \bar{x} + \lambda t,$$

where $\lambda \geq 0$ is a parameter and $t \in \mathbb{R}^n$ is a direction of search.



We propose the following manner of evaluation of the vector t :
(a) evaluate a vector of weights w,
(b) evaluate the vectors s^i for $i \in I(\bar{x})$, by (2),
(c) evaluate the projections $P_{\bar{H}}(\bar{x} + s^i)$ for $i \in I(\bar{x})$,
(d) set $t = P_{\bar{H}}(\bar{x} + s^i) - \bar{x}$.

In the point x^+ we create a new surrogate inequality H^+ with a normal vector $-t$. Thus, H^+ has a following form

$$H^+ = \{x \in \mathbb{R}^n : -t^\top (x - x^+) \leq 0\}.$$

In this Section we answer the question: how big may be the parameter λ to satisfy the condition $M \subset H^+$?

Denote for $i \in I(\bar{x})$

$$\hat{x}^i = \bar{x} + s^i, \qquad (4)$$
$$y^i = P_{\bar{H}}(\hat{x}^i), \qquad (5)$$
$$p^i = y^i - \hat{x}^i, \qquad (6)$$
$$t^i = y^i - \bar{x}. \qquad (7)$$

Let

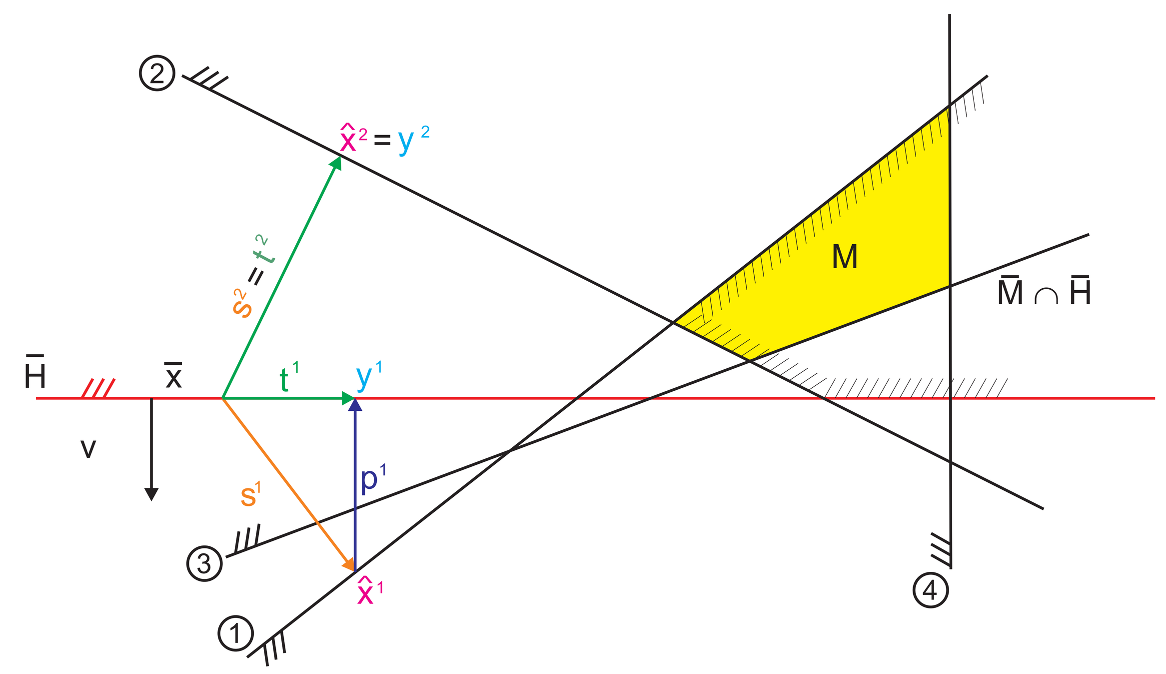
$$t = \sum_{i \in I(\bar{x})} w_i t^i$$

and

$$x^+ = \bar{x} + \lambda t,$$

where

$$\lambda = \mu \frac{\sum_{i \in I(\bar{x})} w_i \|s^i\|^2}{\|\sum_{i \in I(\bar{x})} w_i t^i\|^2}, \quad \mu \in (0, 2].$$



Lemma 2.1 *For each $z \in \bar{M} \cap \bar{H}$ the following inequality is satisfied*

$$\|x^+ - z\|^2 \leq \|\bar{x} - z\|^2 - \mu (2 - \mu) \frac{(\sum_{i \in I(\bar{x})} w_i \|s^i\|^2)^2}{\|\sum_{i \in I(\bar{x})} w_i t^i\|^2}. \qquad (8)$$

2.1 Iterative scheme

Step 0. (*Initialization*) Choose an arbitrary starting point $x^{(0)}$, an *optimality tolerance* $\varepsilon > 0$, a small positive quantity $\gamma > 0$ and a quantity $\delta \in (0, 1]$. Set $H^{(0)} = \mathbb{R}^n$ and $k = 0$ (*iteration's counter*).

Step 1. (*Stopping criterion*) If $x^{(k)}$ is an $\varepsilon-$ optimal solution, i.e., $\max\{g_i^\top x^{(k)} - b_i : i = 1, ..., m\} \leq \varepsilon$, terminate.

Step 2. (*Projection*) Evaluate for $i \in I(x^{(k)})$

(a) $s^{i(k)} = -(g_i^\top x^{(k)} - b_i) g_i$,

(b) $\hat{x}^{i(k)} = x^{(k)} + s^{i(k)}$,

(c) $y^{i(k)} = P_{H^k}(\hat{x}^{i(k)})$,

(d) $t^{i(k)} = y^{i(k)} - x^{(k)}$.

Step 3. (*Approximation's update*)

(a) Evaluate $x^{(k+1)} = x^{(k)} + \mu^{(k)} \frac{\sum_{i \in I(x^{(k)})} w_i^{(k)} \|s^{i(k)}\|^2}{\|\sum_{i \in I(x^{(k)})} w_i^{(k)} t^{i(k)}\|^2} \cdot t^{(k)}$, where

$t^{(k)} = \sum_{i \in I(x^{(k)})} w_i^{(k)} t^{i(k)}$ for a vector of weights $w^{(k)}$, where $w_i^{(k)} > \gamma$, $i \in I(x^{(k)})$ and $\mu^{(k)} \in (\delta, 1]$,

(b) Set $v^{(k)} = x^{(k)} - x^{(k+1)}$,

(c) Set $H^{(k+1)} = \{x \in \mathbb{R}^n : v^{(k)\top} (x - x^{(k+1)}) \leq 0\}$,

(d) Increase k by 1 and go to Step 1.

Theorem 1 *If $M \neq \emptyset$, then any infinite sequence $(x^{(k)})_{k=0}^{\infty}$ generated by Iterative scheme has the following properties:*
(a) $(x^{(k)})$ is strictly Fejér-monotone with respect to M,
(b) $\lim_{k \to \infty} \max_{i \in I(x^{(k)})} \|s^{i(k)}\| = 0$.
Consequently, $x^{(k)}$ converges to an $x^* \in M$.

3 Numerical results

The project can be summarized by the following scheme: We set $\mu = 1$, $\gamma = 10^{-3}$ and $\varepsilon = 10^{-6}$.

We have compared two methods: the surrogate constraints method of Yang-Murty [4] and the method presented in Section 2 (Iterative scheme).

For all presented methods we have performed the computation for two variants:
Variant I - equal weights for violated constraints, i.e., $w_i^{(k)} = \frac{1}{|I(x^{(k)})|}$,
Variant II - weights proportional to residuum of violated constraints

$$w_i^{(k)} = \max \left\{ \gamma, \frac{g_i^\top x^{(k)} - b_i}{\sum_{i \in I(x^{(k)})} g_i^\top x^{(k)} - b_i} \right\}, i \in I(x^{(k)}).$$

$n \times m$	l	Variant I		Variant II	
		Y-M Method	Iter. Schem. 2.1	Y-M Method	Iter. Schem. 2.1
20×20	12	23	9	25	14
	20	40	13	46	13
20×40	12	38	13	44	13
	24	71	18	76	19
20×80	12	55	15	65	17
	24	91	20	127	23
50×50	30	27	13	34	15
	50	38	16	53	18
50×100	30	49	18	62	20
	60	89	23	120	26
50×200	30	69	23	107	25
	60	178	32	235	34
100×100	60	29	14	40	17
	100	42	17	60	20
100×200	60	64	21	95	24
	120	116	28	171	30
200×200	120	30	16	43	18
	200	40	17	64	20
500×500	300	32	17	48	19
	500	36	18	55	20
500×1000	300	133	31	194	32
	500	234	40	348	52

References

[1] H. Scolnik, N. Echebest, M. T. Guardarucci, M. C. Vacchino, *A class of optimized row projection methods for solving large nonsymetric linear systems*, Applied Numerical Mathematics **41** (2002) 499-513.

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