Universal localization of triangular matrix rings

Desmond Sheiham

Abstract

If R is a triangular matrix ring, the columns, P and Q, are projective R-modules. We describe the universal localization of R which makes invertible an R-module morphism $\sigma : P \to Q$, generalizing a theorem of A.Schofield. We also describe universal localization of R-modules.

1 Introduction

Suppose R is an associative ring (with 1) and $\sigma : P \to Q$ is a morphism between finitely generated projective R-modules. There is a universal way to localize R in such a way that σ becomes an isomorphism. More precisely there is a ring morphism $R \to \sigma^{-1}R$ which is universal for the property that

$$\sigma^{-1}R \otimes_R P \xrightarrow{1\otimes\sigma} \sigma^{-1}R \otimes_R Q$$

is an isomorphism (Cohn [7, 9, 8, 6], Bergman [3, 5], Schofield [17]). Although it is often difficult to understand universal localizations when R is noncommutative[1] there are examples where elegant descriptions of $\sigma^{-1}R$ have been possible (e.g. Cohn and Dicks [10], Dicks and Sontag [11, Thm 24], Farber and Vogel [12] Ara, González-Barroso, Goodearl and Pardo [1, Example 2.5]). The purpose of this note is to describe and to generalize some particularly interesting examples due to A.Schofield [17, Thm 13.1] which have application in topology (e.g. Ranicki [16, Part 2]).

We consider a triangular matrix ring $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ where A and B are associative rings (with 1) and M is an (A, B)-bimodule. Multiplication in R is given by

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

for all $a, a' \in A$, $m, m' \in M$ and $b, b' \in B$. The columns $\begin{pmatrix} A \\ 0 \end{pmatrix}$ and $\begin{pmatrix} M \\ B \end{pmatrix}$ are projective left R-modules with

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \\ B \end{pmatrix} \cong R.$$

General theory of triangular matrix rings can found in Haghany and Varadarajan [13, 14].

April 16, 2008

[1]If R is commutative one obtains a ring of fractions; see Bergman [5, p68].

We shall describe in Theorem 2.4 the universal localization $R \to \sigma^{-1}R$ which makes invertible a morphism

$$\sigma : \begin{pmatrix} A \\ 0 \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix}$$

Such a morphism can be written $\sigma = \begin{pmatrix} j \\ 0 \end{pmatrix}$ where $j : A \to M$ is a morphism of left A-modules. Examples follow, in which restrictions are placed on A, B, M and σ. In particular Example 2.8 recovers Theorem 13.1 of Schofield [17]. We proceed to describe the universal localization $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ of an arbitrary left module N for the triangular matrix ring R (see Theorem 2.12).

The structure of this paper is as follows: Definitions, statements of results and examples are given in Section 2 and the proofs are collected in Section 3.

I am grateful to Andrew Ranicki, Aidan Schofield and Amnon Neeman for helpful conversations.

2 Statements and Examples

Let us first make more explicit the universal property of localization:

Definition 2.1. A ring morphism $R \to R'$ is called σ-inverting if

$$\mathrm{id} \otimes \sigma : R' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \to R' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

is an isomorphism. The universal localization $i_\sigma : R \to \sigma^{-1}R$ is the initial object in the category of σ-inverting ring morphisms $R \to R'$. In other words, every σ-inverting ring morphism $R \to R'$ factors uniquely as a composite $R \to \sigma^{-1}R \to R'$.

Definition 2.2. An (A, M, B)-ring (S, f_A, f_M, f_B) is a ring S together with ring morphisms $f_A : A \to S$ and $f_B : B \to S$ and an (A, B)-bimodule morphism $f_M : M \to S$.

$$\begin{array}{ccccc} A & \xrightarrow{f_A} & S & \xleftarrow{f_B} & B \\ & & {\scriptstyle f_M}\uparrow & & \\ & & M & & \end{array}$$

It is understood that the (A, B)-bimodule structure on S is induced by f_A and f_B, so that $f_A(a)f_M(m) = f_M(am)$ and $f_M(m)f_B(b) = f_M(mb)$ for all $a \in A$, $b \in B$ and $m \in M$.

A morphism $(S, f_A, f_M, f_B) \to (S', f'_A, f'_M, f'_B)$ of (A, M, B)-rings is a ring morphism $\theta : S \to S'$ such that i) $\theta f_A = f'_A$, ii) $\theta f_M = f'_M$ and iii) $\theta f_B = f'_B$.

Definition 2.3. Suppose $p \in M$. Let $(T(M, p), \rho_A, \rho_M, \rho_B)$ denote the initial object in the category of (A, M, B)-rings with the property $\rho_M(p) = 1$. For brevity we often write $T = T(M, p)$.

The ring T can be explicitly described in terms of generators and relations as follows. We have one generator x_m for each element $m \in M$ and relations:

(+) $x_m + x_{m'} = x_{m+m'}$

(a) $x_{ap} x_m = x_{am}$

(b) $x_m x_{pb} = x_{mb}$

(id) $x_p = 1$

for all $m, m' \in M$, $a \in A$ and $b \in B$. The morphisms ρ_A, ρ_M, ρ_B are

$$\begin{aligned} \rho_A &: A \to T; a \mapsto x_{ap} \\ \rho_B &: B \to T; b \mapsto x_{pb} \\ \rho_M &: M \to T; m \mapsto x_m \end{aligned}$$

Suppose $\sigma : \begin{pmatrix} A \\ 0 \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix}$ is a morphism of left R-modules. We may write $\sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$ for some $p \in M$. Let $T = T(M, p)$.

Theorem 2.4. *The universal localization $R \to \sigma^{-1}R$ is (isomorphic to)*

$$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \xrightarrow{\begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}} \begin{pmatrix} T & T \\ T & T \end{pmatrix}.$$

Example 2.5.

1. Suppose $A = B = M$ and multiplication in A defines the (A, A)-bimodule structure on M. If $p = 1$ then $T = A$ and $\rho_A = \rho_M = \rho_B = \mathrm{id}_A$.

2. Suppose $A = B$ and $M = A \oplus A$ with the obvious bimodule structure. If $p = (1, 0)$ then T is the polynomial ring $A[x]$ in a central indeterminate x. The map $\rho_A = \rho_B$ is inclusion of A in $A[x]$ while $\rho_M(1, 0) = 1$ and $\rho_M(0, 1) = x$.

The universal localizations corresponding to Example 2.5 are

1. $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \to \begin{pmatrix} A & A \\ A & A \end{pmatrix}$;

2. $\begin{pmatrix} A & A \oplus A \\ 0 & A \end{pmatrix} \to \begin{pmatrix} A[x] & A[x] \\ A[x] & A[x] \end{pmatrix}$.

Remark 2.6. One can regard the triangular matrix rings in these examples as path algebras over A for the quivers

1. $\bullet \longrightarrow \bullet$ 2. $\bullet \rightrightarrows \bullet$

The universal localizations $R \to \sigma^{-1}R$ are obtained by introducing an inverse to the arrow in 1. and by introducing an inverse to one of the arrows in 2. See for example Benson [2, p99] for an introduction to quivers.

The following examples subsume these:

Example 2.7. 1. (Amalgamated free product; Schofield [17, Thm 4.10]) Suppose $i_A : C \to A$ and $i_B : C \to B$ are ring morphisms and $M = A \otimes_C B$. If $p = 1 \otimes 1$ then T is the amalgamated free product $A \underset{C}{\sqcup} B$ and appears in the pushout square

$$\begin{array}{ccc} C & \xrightarrow{i_A} & A \\ {\scriptstyle i_B}\downarrow & & \downarrow{\scriptstyle \rho_A} \\ B & \xrightarrow[\rho_B]{} & T \end{array}$$

The map ρ_M is given by $\rho_M(a \otimes b) = \rho_A(a)\rho_B(b)$ for all $a \in A$ and $b \in B$. We recover part 1. of Example 2.5 by setting $A = B = C$ and $i_A = i_B = \mathrm{id}$.

2. (HNN extension) Suppose $A = B$ and $i_1, i_2 : C \to A$ are ring morphisms. Let $A \otimes_C A$ denote the tensor product with C acting via i_1 on the first copy of A and by i_2 on the second copy. Let $M = A \oplus (A \otimes_C A)$ and $p = (1, 0 \otimes 0)$. Now $T = A *_C \mathbb{Z}[x]$ is generated by the elements in A together with an indeterminate x and has the relations in A together with $i_1(c)x = xi_2(c)$ for each $c \in C$. We have $\rho_A(a) = \rho_B(a) = a$ for all $a \in A$ while $\rho_M(1, 0 \otimes 0) = 1$ and $\rho_M(0, a_1 \otimes a_2) = a_1 x a_2$. If $C = A$ and $i_1 = i_2 = \mathrm{id}_A$ we recover part 2. of Example 2.5.

The following example is Theorem 13.1 of Schofield [17] and generalizes Example 2.7.

Example 2.8. 1. Suppose p generates M as a bimodule, i.e. $M = ApB$. Now T is generated by the elements of A and the elements of B subject to the relation $\sum_{i=1}^n a_i b_i = 0$ if $\sum_{i=1}^n a_i p b_i = 0$ (with $a_i \in A$ and $b_i \in B$). This ring T is denoted $A \underset{(M,p)}{\sqcup} B$ in [17, Ch13]. The maps ρ_A and ρ_B are obvious and ρ_M sends $\sum_i a_i p b_i$ to $\sum_i a_i b_i$.

2. Suppose $M = ApB \oplus N$ for some (A, B)-bimodule N. Now T is the tensor ring over $A \underset{(M,p)}{\sqcup} B$ of

$$(A \underset{(M,p)}{\sqcup} B) \otimes_A N \otimes_B (A \underset{(M,p)}{\sqcup} B).$$

We may vary the choice of p as the following example illustrates:

Example 2.9. Suppose $A = B = M = \mathbb{Z}$ and $p = 2$. In this case $T = \mathbb{Z}\left[\frac{1}{2}\right]$ and $\rho_A = \rho_B$ is the inclusion of $\mathbb{Z}$ in $\mathbb{Z}\left[\frac{1}{2}\right]$ while $\rho_M(n) = n/2$ for all $n \in \mathbb{Z}$.

Example 2.9 can be verified by direct calculation using Theorem 2.4 or deduced from part 1. of Example 2.5 by setting $a_0 = b_0 = 2$ in the following more general proposition. Before stating it, let us remark that the universal property of $T = T(M, p)$ implies that $T(M, p)$ is functorial in (M, p). An (A, B)-bimodule morphism $\phi : M \to M'$ with $\phi(p) = p'$ induces a ring morphism $T(M, p) \to T(M', p')$.

Proposition 2.10. *Suppose A and B are rings, M is an (A, B)-bimodule and $p \in M$. If $a_0 \in A$ and $b_0 \in B$ satisfy $a_0 m = m b_0$ for all $m \in M$ then*

1. *The element $\rho_M(a_0 p) = x_{a_0 p} = x_{p b_0}$ is central in $T(M, p)$.*

2. *The ring morphism $\phi : T(M, p) \to T(M, a_0 p) = T(M, p b_0)$ induced by the bimodule morphism $\phi : M \to M; m \mapsto a_0 m = m b_0$ is the universal localization of $T(M, p)$ making invertible the element $x_{a_0 p}$.*

Since $x_{a_0} p$ is central each element in $T(M, a_0 p)$ can be written as a fraction α/β with numerator $\alpha \in T(M, p)$ and denominator $\beta = x^r_{a_0 p}$ for some non-negative integer r.

Having described universal localization of the ring R in Theorem 2.4 we may also describe universal localization $\sigma^{-1}R \otimes_R N$ of a left R-module N. For the convenience of the reader let us first recall the structure of modules over a triangular matrix ring.

Lemma 2.11. *Every left R-module N can be written canonically as a triple*

$$(N_A, N_B, f : M \otimes_B N_B \to N_A)$$

where N_A is a left A-module, N_B is a left B-module and f is a morphism of left A-modules.

A proof of this lemma is included in Section 3 below. Localization of modules can be expressed as follows:

Theorem 2.12. *If $N = (N_A, N_B, f)$ then the localization $\sigma^{-1}N = \sigma^{-1}R \otimes_R N$ is isomorphic to $\begin{pmatrix} T(M,p) \otimes_A N_A \\ T(M,p) \otimes_A N_A \end{pmatrix}$ with $\sigma^{-1}R = M_2(T(M, p))$ acting on the left by matrix multiplication.*

3 Proofs

The remainder of this paper is devoted to the proofs of Theorem 2.4, Proposition 2.10 and Theorem 2.12.

3.1 Localization as Pushout

Before proving Theorem 2.4 we show that there is a pushout diagram

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ {\scriptstyle \alpha}\downarrow & & \downarrow \\ R & \longrightarrow & \sigma^{-1}R \end{array}$$

where $\alpha\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $\alpha\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ and $\alpha\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&p\\0&0\end{pmatrix}$. Bergman observed [4, p69] that more generally, up to Morita equivalence every localization $R \to \sigma^{-1}R$ appears in such a pushout diagram.

It suffices to check that the lower horizontal arrow in the pushout

$$\begin{array}{ccc}\begin{pmatrix}\mathbb{Z}&\mathbb{Z}\\0&\mathbb{Z}\end{pmatrix} & \longrightarrow & \begin{pmatrix}\mathbb{Z}&\mathbb{Z}\\\mathbb{Z}&\mathbb{Z}\end{pmatrix}\\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\theta}\\ R & \xrightarrow[i]{} & S\end{array}$$

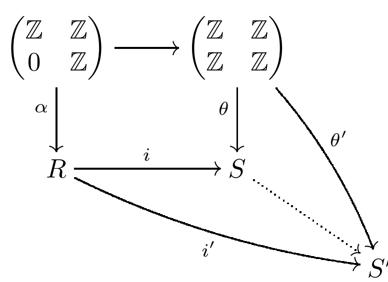
is i) σ-inverting and ii) Universal among σ-inverting ring morphisms.

i) The map $\mathrm{id}\otimes\sigma : S\otimes_R\begin{pmatrix}A\\0\end{pmatrix}\to S\otimes_R\begin{pmatrix}M\\B\end{pmatrix}$ has inverse given by the composite

$$S\otimes_R\begin{pmatrix}M\\B\end{pmatrix}\subset S\otimes_R R\cong S\xrightarrow{\ \gamma\ }S\cong S\otimes_R R\twoheadrightarrow S\otimes_R\begin{pmatrix}A\\0\end{pmatrix}$$

where γ multiplies on the right by $\theta\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

ii) If $i' : R\to S'$ is a σ-inverting ring morphism then there is an inverse $\psi : S'\otimes_R\begin{pmatrix}M\\B\end{pmatrix}\to S'\otimes_R\begin{pmatrix}A\\0\end{pmatrix}$ to $\mathrm{id}\otimes\sigma$. It is argued shortly below that there is a (unique) diagram



(1)

where θ' sends $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ to $\psi\left(1\otimes\begin{pmatrix}0\\1\end{pmatrix}\right)\in S'\otimes_R\begin{pmatrix}A\\0\end{pmatrix}\subset S'$. Since S is a pushout there is a unique morphism $S\to S'$ to complete the diagram and so i' factors uniquely through i.

To show uniqueness of (1), note that in S' multiplication on the right by $\theta'\begin{pmatrix}0&1\\0&0\end{pmatrix}$ must coincide with the morphism

$$\begin{pmatrix}0&0\\\mathrm{id}\otimes\sigma&0\end{pmatrix} : S'\otimes\begin{pmatrix}A\\0\end{pmatrix}\oplus S'\otimes\begin{pmatrix}M\\B\end{pmatrix}\longrightarrow S'\otimes\begin{pmatrix}A\\0\end{pmatrix}\oplus S'\otimes\begin{pmatrix}M\\B\end{pmatrix}$$

so multiplication on the right by $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ coincides with $\begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$. Now $1 \in S'$ may be written

$$\left(1 \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \in S' \otimes_R \begin{pmatrix} A \\ 0 \end{pmatrix} \oplus S' \otimes_R \begin{pmatrix} M \\ B \end{pmatrix}$$

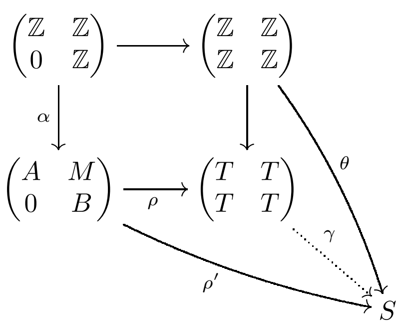
so $\theta' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \psi \left(1 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. The reader may verify that this formula demonstrates existence of a commutative diagram (1).

3.2 Identifying $\sigma^{-1}R$

Proof of Theorem 2.4. It suffices to show that the diagram of ring morphisms

$$\begin{array}{ccc} \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} & \longrightarrow & \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow \\ \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} & \xrightarrow[\rho]{} & \begin{pmatrix} T & T \\ T & T \end{pmatrix} \end{array}$$

is a pushout, where $T = T(M, p)$, $\rho = \begin{pmatrix} \rho_A & \rho_M \\ 0 & \rho_B \end{pmatrix}$ and α is defined as in Section 3.1. Given a diagram of ring morphisms



(2)

we must show that there is a unique morphism γ to complete the diagram. The map θ induces a decomposition of S as a matrix ring $M_2(S')$ for some ring S' and any morphism γ which makes the diagram commute must be of the form $\gamma = M_2(\gamma')$ for some ring morphism $\gamma' : T \to S'$ (e.g. Cohn [9, p1] or Lam [15, (17.7)]). Commutativity of the diagram implies that ρ' also respects the 2×2 matrix structure and we may write

$$\rho' = \begin{pmatrix} \rho'_A & \rho'_M \\ 0 & \rho'_B \end{pmatrix} : \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \longrightarrow \begin{pmatrix} S' & S' \\ S' & S' \end{pmatrix}$$

with $\rho'_M(p) = 1$. Since ρ' is a ring morphism, one finds

$$\begin{pmatrix} \rho'_A(aa') & \rho'_M(am' + mb') \\ 0 & \rho'_B(bb') \end{pmatrix} = \begin{pmatrix} \rho'_A(a)\rho'_A(a') & \rho'_A(a)\rho'_M(m') + \rho'_M(m)\rho'_B(b') \\ 0 & \rho'_B(b)\rho'_B(b') \end{pmatrix}$$

for all $a, a' \in A$, $b, b' \in B$ and $m, m' \in M$. Hence the maps $\rho'_A : A \to S'$ and $\rho'_B : B \to S'$ are ring morphisms and ρ'_M is a morphism of (A, B)-bimodules. By the universal property of T there exists a unique morphism $\gamma' : T \to S'$ such that $M_2(\gamma') : M_2(T) \to M_2(S') = S$ completes the diagram (2) above. □

Proof of Proposition 2.10. 1. In $T(M, p)$ we have $x_{a_0 p} x_m = x_{a_0 m} = x_{m b_0} = x_m x_{p b_0} = x_m x_{a_0 p}$ for all $m \in M$.
2. The map $\phi : M \to M; m \to a_0 m$ induces

$$\begin{aligned} \phi : T(M, p) &\to T(M, a_0 p) \\ x_m &\mapsto x_{a_0 m} \end{aligned} \tag{3}$$

In particular $\phi(x_{a_0 p}) = x_{a_0^2 p} \in T(M, a_0 p)$ and we have

$$x_{a_0^2 p} x_p = x_{a_0(a_0 p)} x_p = x_{a_0 p} = 1 = x_{p b_0} = x_p x_{p b_0^2} = x_p x_{a_0^2 p}$$

so $\phi(x_{a_0 p})$ is invertible.

We must check that (3) is universal. If $f : T(M, p) \to S$ is a ring morphism and $f(x_{a_0 p})$ is invertible, we claim that there exists unique $\widetilde{f} : T(M, a_0 p) \to S$ such that $\widetilde{f}\phi = f$.
Uniqueness: Suppose $\widetilde{f}\phi = f$. For each $m \in M$ we have

$$\widetilde{f}(x_{a_0 m}) = \widetilde{f}\phi(x_m) = f(x_m).$$

Now $f(x_{a_0 p})\widetilde{f}(x_m) = \widetilde{f}\phi(x_{a_0 p}) = \widetilde{f}(x_{a_0(a_0 p)} x_m) = \widetilde{f}(x_{a_0 m}) = f(x_m)$ so

$$\widetilde{f}(x_m) = (f(x_{a_0 p}))^{-1} f(x_m). \tag{4}$$

Existence: It is straightforward to check that equation (4) provides a definition of $\widetilde{f}$ which respects the relations (+),(a),(b) and (id) in $T(M, a_0 p)$. Relation (b), for example, is proved by the equations

$$\widetilde{f}(x_m)\widetilde{f}(x_{a_0 p b}) = f(x_{a_0 p})^{-1} f(x_m) f(x_{pb}) = f(x_{a_0 p})^{-1} f(x_{mb}) = \widetilde{f}(x_{mb})$$

and the other relations are left to the reader. □

3.3 Module Localization

We turn finally to the universal localization $\sigma^{-1}R \otimes_R N$ of an R-module N.

Proof of Lemma 2.11. If N is a left R-module, set $N_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} N$ and set $N_B = N/N_A$. If $m \in M$ and $n_B \in N_B$ choose a lift $x \in N$ and define the map $f : M \otimes N_B \to N_A$ by $f(m \otimes n_B) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} x$. Conversely, given a triple (N_A, N_B, f) one recovers a left R-module $\begin{pmatrix} N_A \\ N_B \end{pmatrix}$ with

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} an_A + f(m \otimes n_B) \\ bn_B \end{pmatrix}$$

for all $a \in A$, $b \in B$, $m \in M$, $n_A \in N_A$, $n_B \in N_B$. □

Proof of Theorem 2.12. Let $T = T(M, p)$. We shall establish an isomorphism of left T-modules

$$\begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \;\cong\; T \otimes_A N_A \tag{5}$$

and leave to the reader the straightforward deduction that there is an isomorphism of $\sigma^{-1}R$-modules

$$\sigma^{-1}R \otimes_R N = \begin{pmatrix} T & T \\ T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \cong \begin{pmatrix} T \otimes_A N_A \\ T \otimes_A N_A \end{pmatrix}.$$

Let $\alpha : T \otimes_A N_A \to \begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix}$ be given by $\alpha(t \otimes n) = \begin{pmatrix} t & 0 \end{pmatrix} \otimes_R \begin{pmatrix} n \\ 0 \end{pmatrix}$. Let $\beta : \begin{pmatrix} T & T \end{pmatrix} \otimes_R \begin{pmatrix} N_A \\ N_B \end{pmatrix} \to T \otimes_A N_A$ be given by

$$\beta\left(\begin{pmatrix} t & t' \end{pmatrix} \otimes_R \begin{pmatrix} n_A \\ n_B \end{pmatrix} \right) = t \otimes n_A + t' \otimes f(p \otimes n_B).$$

It is immediate that $\beta\alpha = \mathrm{id}$. To prove (5) we must check that $\alpha\beta = \mathrm{id}$ or in other words that

$$\begin{pmatrix} t & t' \end{pmatrix} \otimes \begin{pmatrix} n_A \\ n_B \end{pmatrix} = \begin{pmatrix} t & 0 \end{pmatrix} \otimes \begin{pmatrix} n_A \\ 0 \end{pmatrix} + \begin{pmatrix} t' & 0 \end{pmatrix} \otimes \begin{pmatrix} f(p \otimes n_B) \\ 0 \end{pmatrix}.$$

This equation follows from the following three calculations:

$$\begin{aligned} \begin{pmatrix} t & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ n_B \end{pmatrix} &= \begin{pmatrix} t & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ n_B \end{pmatrix} = 0; \\ \begin{pmatrix} 0 & t' \end{pmatrix} \otimes \begin{pmatrix} n_A \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & t' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} n_A \\ 0 \end{pmatrix} = 0; \\ \begin{pmatrix} 0 & t' \end{pmatrix} \otimes \begin{pmatrix} 0 \\ n_B \end{pmatrix} &= \begin{pmatrix} t' & 0 \end{pmatrix} \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ n_B \end{pmatrix} = \begin{pmatrix} t' & 0 \end{pmatrix} \otimes \begin{pmatrix} f(p \otimes n_B \\ 0 \end{pmatrix}. \end{aligned}$$ □

References

[1] P. Ara, M. A. González-Barroso, K. R. Goodearl, and E. Pardo. Fractional skew monoid rings. To appear in the Journal of Algebra.

[2] D. J. Benson. *Representations and cohomology.I. Basic Representation Theory of finite groups and associative algebras.* Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, 1995.

[3] G. M. Bergman. Coproducts and some universal ring constructions. *Transactions of the American Mathematical Society*, 200:33–88, 1974.

[4] G. M. Bergman. Modules over coproducts of rings. *Transactions of the American Mathematical Society*, 200:1–32, 1974.

[5] G. M. Bergman. Universal derivations and universal ring constructions. *Pacific Journal of Mathematics*, 79(2):293–337, 1978.

[6] P. M. Cohn. Localization in general rings, a historical survey. To appear in the Proceedings of the Conference on Noncommutative Localization in Algebra and Topology, ICMS, Edinburgh, 29-30 April, 2002.

[7] P. M. Cohn. *Free Rings and their Relations.* London Mathematical Society Monographs, 2. Academic Press, London, 1971.

[8] P. M. Cohn. Rings of fractions. *American Mathematical Monthly*, 78:596–615, 1971.

[9] P. M. Cohn. *Free Rings and their Relations.* London Mathematical Society Monographs, 19. Academic Press, London, 2nd edition, 1985.

[10] P.M. Cohn and W. Dicks. Localization in semifirs. II. *J.London Math.Soc. (2)*, 13(3):411–418, 1976.

[11] W. Dicks and E. Sontag. Sylvester domains. *J. Pure Appl. Algebra*, 13(3):243–275, 1978.

[12] M. Farber and P. Vogel. The Cohn localization of the free group ring. *Mathematical Proceedings of the Cambridge Philosophical Society*, 111(3):433–443, 1992.

[13] A. Haghany and K. Varadarajan. Study of formal triangular matrix rings. *Communications in Algebra*, 27(11):5507–5525, 1999.

[14] A. Haghany and K. Varadarajan. Study of modules over formal triangular matrix rings. *Journal of Pure and Applied Algebra*, 147(1):41–58, 2000.

[15] T. Y. Lam. *Lectures on Modules and Rings.* Number 189 in Graduate Texts in Mathematics. Springer, New York, 1999.

[16] A. A. Ranicki. Noncommutative localization in topology. To appear in the Proceedings of the Conference on Noncommutative Localization in Algebra and Topology, ICMS, Edinburgh, 29-30 April, 2002. arXiv:math.AT/0303046.

[17] A. H. Schofield. *Representations of rings over skew fields*, volume 92 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, 1985.

des@sheiham.com
International University Bremen, Bremen 28759, Germany.