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Integral equation methods for scattering by infinite rough surfaces

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SUMMARY

In this paper, we consider the Dirichlet and impedance boundary value problems for the Helmholtz equation in a non-locally perturbed half-plane. These boundary value problems arise in a study of time-harmonic acoustic scattering of an incident field by a sound-soft, infinite rough surface where the total field vanishes (the Dirichlet problem) or by an infinite, impedance rough surface where the total field satisfies a homogeneous impedance condition (the impedance problem). We propose a new boundary integral equation formulation for the Dirichlet problem, utilizing a combined double- and single-layer potential and a Dirichlet half-plane Green's function. For the impedance problem we propose two boundary integral equation formulations, both using a half-plane impedance Green's function, the first derived from Green's representation theorem, and the second arising from seeking the solution as a single-layer potential. We show that all the integral equations proposed are uniquely solvable in the space of bounded and continuous functions for all wavenumbers. As an important corollary we prove that, for a variety of incident fields including an incident plane wave, the impedance boundary value problem for the scattered field has a unique solution under certain constraints on the boundary impedance. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: scattering; integral equation; Helmholtz equation; rough surface; impedance problem; Dirichlet problem; outdoor sound propagation

1. INTRODUCTION

We consider the two-dimensional Dirichlet and impedance boundary value problems for the Helmholtz equation, $\Delta u+k^2u=0$, in a non-locally perturbed half-plane. The Dirichlet boundary

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value problem models time-harmonic ($e^{-i\omega t}$ time dependence) acoustic scattering by a sound-soft, infinite rough surface where the total field u_t (the sum of the incident field u^i and the scattered field u) vanishes; the same boundary value problem (in $\mathbb{R}^2$) also arises in electromagnetic scattering by one-dimensional rough, perfectly conducting surfaces in one of the polarization cases [1]. The impedance problem is of interest as a model of monofrequency outdoor sound propagation over inhomogeneous terrain. In this context u is the scattered or reflected part of the acoustic field (the total field u_t satisfies the homogeneous impedance boundary condition, $\partial u_t/\partial v - ik\beta u_t = 0$ on the boundary surface) [2]. The function β is the relative surface admittance of the ground surface, and is a function of the angular frequency ω and of local properties of the ground surface. The same boundary value problem arises, in both polarization cases, in electromagnetic scattering by a one-dimensional rough surface satisfying the Leontovich boundary condition [3].

The well-posedness of the Dirichlet problem has recently been studied in References [4–7] using integral equation methods. The boundary integral equation formulation derived in these papers utilizes the Green's function for an impedance half-plane in place of the more usual free-space Green's function. This choice of fundamental solution leads, notwithstanding the infinite boundary, to a well-posed integral equation formulation, and the unique solvability of this integral equation in the space of bounded and continuous functions is established, for all wavenumbers, in References [5,7]. Using this result, for a variety of incident fields including an incident plane wave, it can be shown that the boundary value problem for the scattered field has a unique solution [5,6]. Recently it has been shown that, once unique solvability of the boundary integral equation in the space of bounded and continuous functions has been established, unique solvability in a variety of other function spaces can be obtained, including the L^p spaces, for $1 \leqslant p \leqslant \infty$ [8], and these general results are applied in Reference [8] to the integral equation formulation proposed in References [4–7].

For the impedance problem existence and uniqueness have been established in Reference [9] for arbitrary L^∞ boundary data β, with $\Re\beta \geqslant \varepsilon > 0$, also by means of an integral equation method, but only in the case when the boundary is a flat surface. Integral equation methods have also been used to study scattering by infinite periodic structures (see e.g. References [10–14] and the references therein) and scattering by infinite, rough interfaces and inhomogeneous layers [15–17].

Integral equation methods are also currently widely used for practical computations of rough surface scattering (see e.g. References [18–25] and the references quoted therein). However, the integral equation formulations usually employed, which utilize the free field Green's function as fundamental solution, lack a theoretical basis, specifically an understanding of their uniqueness, existence, and stability properties, though see Reference [18] for steps in this direction.

In this paper, we propose a new boundary integral equation formulation for the Dirichlet problem, utilizing the combined double- and single-layer potential approach first proposed in Reference [26] for scattering by a bounded surface, but replacing the standard fundamental solution in our layer potentials by the Dirichlet half-plane Green's function. We prove that this integral equation is uniquely solvable in the space of bounded and continuous functions for all wavenumbers, this function space setting allowing for a variety of incident fields including an incident plane wave. We remark that our integral equation formulation is rather simpler than that, using the half-plane impedance Green's function, shown to be uniquely solvable in References [4–7] since the Dirichlet Green's function is considerably simpler than the impedance one. Thus, it is anticipated that this new formulation will be more straightforward and efficient for practical implementation.

For the impedance problem we derive a direct and an indirect boundary integral equation formulation using the Green's function for an impedance half-plane. The direct integral equation is obtained from Green's representation theorem: we show that it is equivalent to the impedance boundary value problem for arbitrary choices of the function β, including the important special case $\beta \equiv 0$ when the impedance boundary value problem reduces to the Neumann problem. The indirect integral equation is obtained by seeking the solution of the impedance problem in the form of a single-layer potential. We show, if certain conditions on β hold, that both integral equations have exactly one solution in the space of bounded and continuous functions for all wavenumbers. As a corollary we establish that, for a variety of incident fields, including an incident plane wave, the impedance boundary value problem for the scattered field has a unique solution under the same conditions on β.

Results related to those contained in this paper, including a numerical analysis of a novel Nyström discretization scheme suitable for all the integral equation formulations we propose, and a study of the stability and convergence of truncation to a finite section of the integrals over the infinite boundary which occur in each integral equation, are discussed in References [27,28]. For the special case of a flat surface, efficient boundary element techniques for the impedance problem have recently been proposed and analysed in Reference [29].

We conclude this section by introducing some notations used throughout. For $h \in \mathbb{R}$, define $\Gamma_h := \{x = (x_1, x_2) \in \mathbb{R}^2 | x_2 = h\}$ and $U_h := \{x \in \mathbb{R}^2 | x_2 > h\}$. For $V \subset \mathbb{R}^n$, $n = 1, 2$, we denote by $BC(V)$ the set of functions bounded and continuous on V, a Banach space under the norm $\|\psi\|_{\infty,V} := \sup_{x\in V} |\psi(x)|$, and by $BUC(V)$ the closed subspace of functions bounded and uniformly continuous on V. We abbreviate $\|\cdot\|_{\infty,\mathbb{R}^n}$ by $\|\cdot\|_\infty$ and $BC(\mathbb{R})$ by Y. For $0 < \alpha \leqslant 1$, we denote by $C^{0,\alpha}(V)$ the Banach space of functions $\phi \in BC(V)$ which are uniformly Hölder continuous with exponent α, with norm $\|\cdot\|_{C^{0,\alpha}(V)}$ defined by $\|\phi\|_{C^{0,\alpha}(V)} := \|\phi\|_\infty + \sup_{x,y\in V, x\neq y} [|\phi(x) - \phi(y)|/|x - y|^\alpha]$. We let $C^{1,\alpha}(\mathbb{R}) := \{\phi \in BC(\mathbb{R}) \cap C^1(\mathbb{R}) \mid \phi' \in C^{0,\alpha}(\mathbb{R})\}$, a Banach space under the norm $\|\phi\|_{C^{1,\alpha}(\mathbb{R})} := \|\phi\|_\infty + \|\phi'\|_{C^{0,\alpha}(\mathbb{R})}$. Given an open set $V \subset \mathbb{R}^2$ and $v \in L^\infty(V)$, denote by $\partial_j v$, $j = 1, 2$, the (distributional) derivative $\partial v(x)/\partial x_j$. Finally, for $A > 0$, $x \in \mathbb{R}^2$, $V \subset \mathbb{R}^2$, let $B_A(x) := \{y \in \mathbb{R}^2 \mid |y - x| < A\}$ and $V(A) := \{x \in V \mid |x_1| < A\}$.

2. THE SCATTERING PROBLEMS

Given $f \in C^{1,1}(\mathbb{R})$ with $f_- := \inf_{x_1\in\mathbb{R}} f(x_1) > 0$, define the two-dimensional region D by

$$D := \{x = (x_1, x_2) \in \mathbb{R}^2 | x_2 > f(x_1)\}$$

so that the boundary Γ of D is $\Gamma := \partial D = \{(x_1, f(x_1)) | x_1 \in \mathbb{R}\}$. Whenever we wish to denote explicitly the dependence of the region on the boundary function f we will write D_f for D and Γ_f for Γ.

We consider the problem of scattering of a time-harmonic wave u^{i}, a solution of the Helmholtz equation $\Delta u^{\mathrm{i}} + k^2 u^{\mathrm{i}} = 0$ in D, incident on the infinite boundary Γ. We assume that k is a complex constant with $\Im k \geqslant 0$, $\Re k > 0$ and restrict our attention to two cases: the case where the total field vanishes on the boundary, so that the scattered field u, also a solution of the Helmholtz equation in D, satisfies the Dirichlet boundary condition $u = -u^{\mathrm{i}}$ on Γ, and the case when the total field satisfies the homogeneous impedance boundary condition,

$\partial u_t/\partial \nu - ik\beta u_t = 0$ on Γ, where, and subsequently, $\nu(x)$ stands for the unit normal vector at $x \in \Gamma$ pointing out of D and $\partial/\partial\nu$ is the rate of change in this direction.

In order for the problem to have a unique solution, a radiation condition as x_2 tends to infinity has to be imposed on the scattered field u, that is, the scattered field u should behave as an outgoing wave as $x_2 \to +\infty$. We wish to consider incident fields including plane wave incidence for which the Sommerfeld radiation condition will not be satisfied. We adopt the so-called *upward propagating radiation condition*, proposed in [15,30], that the scattered field u is required to satisfy, for some $h > f_+ := \sup_{x_1 \in \mathbb{R}} f(x_1)$ and $\phi \in L^\infty(\Gamma_h)$,

$$u(x) = 2\int_{\Gamma_h} \frac{\partial \Phi(x,y)}{\partial y_2}\phi(y)\,\mathrm{d}s(y), \quad x \in U_h \tag{1}$$

where $\Phi(x,y) := (\mathrm{i}/4)H_0^{(1)}(k|x-y|)$, $x, y \in \mathbb{R}^2$, $x \neq y$, is the free-space Green's function for $\Delta + k^2$. This radiation condition is a generalization of the Rayleigh expansion condition for one-dimensional periodic gratings [30], and many of its properties are explored in Referene [15, Theorem 2.9].

The above problems of scattering of an incident field by an infinite rough surface can now be formulated as the following boundary value problems for the scattered field u. The function space specified for the Dirichlet and impedance boundary data g includes, when $k>0$, the usual incident fields of interest including the incident plane wave.

Dirichlet problem (*DP*): Given $g \in BC(\Gamma)$, determine $u \in C^2(D) \cap C(\bar{D})$ such that:

(i) u is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } D \tag{2}$$

(ii) $u = g$ on Γ;

(iii) For some $a \in \mathbb{R}$,

$$\sup_{x \in D} x_2^a |u(x)| < \infty \tag{3}$$

(iv) u satisfies the upward propagating radiation condition (1).

Let $\mathscr{R}(D)$ denote the set of functions $v \in C^2(D) \cap C(\bar{D})$ for which the normal derivative defined by $\partial v/\partial\nu(x) := \lim_{h\to 0+} \nu(x)\cdot\nabla v(x - h\nu(x))$ exists for $x \in \Gamma$, with the convergence uniform in x on every compact subset of Γ. For $h > f_+$ let $D_h := D \backslash \bar{U}_h$.

Impedance problem (*IP*): Given $g \in BC(\Gamma)$, $\beta \in BC(\Gamma)$, determine $u \in \mathscr{R}(D)$ such that:

(i) u is a solution of the Helmholtz equation (2) in D;
(ii) $\partial u/\partial\nu - ik\beta u = g$ on Γ;
(iii) u satisfies (3) for some $a \in \mathbb{R}$;
(iv) For some $\theta \in (0,1)$ and some constant $C_\theta > 0$,

$$|\nabla u(x)| \leqslant C_\theta [x_2 - f(x_1)]^{\theta - 1} \tag{4}$$

for $x \in D_b$, where $b = f_+ + 1$;

(v) u satisfies the upward propagating radiation condition (1).

Remark 2.1
If $u \in C^2(D)$ satisfies (2) and (3) then, by standard local elliptic regularity results [31], ∇u satisfies the same bound (3) as u in the interior of D, precisely

$$\sup_{x_1 \in \mathbb{R}, x_2 > f(x_1)+h} x_2^a |\nabla u(x)| < \infty \tag{5}$$

for all $h > 0$. For the impedance problem (IP) we need the additional bound (4) on ∇u in a neighbourhood of the boundary of D in order to carry out our proof of uniqueness.

3. THE DIRICHLET PROBLEM

For some $c_1, c_2 > 0$ let B be defined by

$$B = B(c_1, c_2) := \{ f \in C^{1,1}(\mathbb{R}) | f(s) \geqslant c_1, \; s \in \mathbb{R} \text{ and } \|f\|_{C^{1,1}(\mathbb{R})} \leqslant c_2 \} \tag{6}$$

The following result has been proved in Reference [5, Theorem 5.3] (see also Reference [7, Theorem 4.32]).

Theorem 3.1
The Dirichlet problem (DP) has exactly one solution. Moreover, for some constant $C > 0$ depending only on B and k,

$$|u(x)| \leqslant C x_2^{1/2} \|g\|_{\infty, \Gamma} \tag{7}$$

for all $f \in B$, $g \in BC(\Gamma)$.

In the remaining part of this section we are concerned with deriving a boundary integral equation formulation for the Dirichlet problem (DP) using a combined double- and single-layer potential based on the half-plane Dirichlet Green's function, which is uniquely solvable in the space of bounded and continuous functions for all wavenumbers.

3.1. *An integral equation formulation*

Let $G(x, y) := \Phi(x, y) - \Phi(x, y')$, where $y = (y_1, y_2)$, $y' = (y_1, -y_2)$, be the Dirichlet Green's function for $\Delta + k^2$ in the upper half-plane U_0. Then it is shown in Reference [4] that, for some constant $C > 0$ depending only on k,

$$|G(x, y)|, \; |\nabla_y G(x, y)| \leqslant C(1 + |x_2|)(1 + |y_2|)\{|x - y|^{-3/2} + |x - y'|^{-3/2}\} \tag{8}$$

for $x, y \in \mathbb{R}^2$ $x \notin \{y, y'\}$.

We seek a representation of the solution u to the Dirichlet problem (DP) in the form of a combined double- and single-layer potential

$$u(x) = \int_\Gamma \left[\frac{\partial G(x, y)}{\partial \nu(y)} + \mathrm{i}\eta G(x, y) \right] \psi(y) \, \mathrm{d}s(y), \quad x \in D \tag{9}$$

for some $\psi \in BC(\Gamma)$, where $\eta \neq 0$ is an arbitrary complex number to be fixed later, and $\nu(y)$ stands for the normal vector at $y \in \Gamma$ pointing out of D. In view of (8), (9) is well-defined.

For $\psi \in BC(\Gamma)$, the integrals

$$W(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial \nu(y)} \psi(y)\,\mathrm{d}s(y), \quad x \in \overline{U_0} \backslash \Gamma \tag{10}$$

$$V(x) = \int_\Gamma G(x,y)\psi(y)\,\mathrm{d}s(y), \quad x \in \overline{U_0} \backslash \Gamma \tag{11}$$

will be called, respectively, double-layer and single-layer potentials with density ψ. Properties of the double-layer potential (10) and the single-layer potential (11) can be established similarly as in References [4,5,7], and are summarized in Appendix A.

It can be shown, using the argument of Chandler-Wilde and Zhang [6], that (for $\psi \in BC(\Gamma)$) the double- and single-layer potentials (10) and (11), satisfy the upward propagating radiation condition (1). Combining this fact with Lemmas A.1 and A.2 in Appendix A, the following result can be obtained.

Theorem 3.2
The combined double- and single-layer potential (9) satisfies the Dirichlet problem (DP) with $a = -\frac{1}{2}$, provided $\psi \in BC(\Gamma)$ satisfies the boundary integral equation

$$\psi(x) = 2\int_\Gamma \left[\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right] \psi(y)\,\mathrm{d}s(y) - 2g(x), \quad x \in \Gamma \tag{12}$$

Defining $\tilde{\psi}, \tilde{g} \in Y = BC(\mathbb{R})$ by

$$\tilde{\psi}(s) := \psi((s, f(s))), \quad \tilde{g}(s) := g((s, f(s))), \quad s \in \mathbb{R} \tag{13}$$

and parametrizing the integral in (12) in the obvious way we obtain the following integral equation problem: find $\tilde{\psi} \in Y$ such that

$$\tilde{\psi}(s) - 2\int_{\mathbb{R}} \left[\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right] \tilde{\psi}(t)\sqrt{1 + [f'(t)]^2}\,\mathrm{d}t = -2\tilde{g}(s), \quad s \in \mathbb{R} \tag{14}$$

where $x = (s, f(s))$, $y = (t, f(t))$. Define the kernel κ by

$$\kappa(s,t) := 2\left[\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right] \sqrt{1 + [f'(t)]^2}, \quad s,t \in \mathbb{R}, \ s \neq t \tag{15}$$

with $x = (s, f(s))$, $y = (t, f(t))$. Using this kernel, define the integral operator K for $\phi \in Y$ by

$$(K\phi)(s) := \int_{\mathbb{R}} \kappa(s,t)\phi(t)\,\mathrm{d}t, \quad s \in \mathbb{R} \tag{16}$$

noting that whenever we wish to denote explicitly the dependence of the kernel and operator on the boundary function f we will write κ_f and K_f for κ and K, respectively. Then the integral equation (14) can be written in terms of the operator K and I, the identity operator, as

$$(I - K)\tilde{\psi} = -2\tilde{g} \tag{17}$$

3.2. Solvability of the boundary integral equation

In this section, we show firstly that the boundary integral equation (12) has at most one solution $\psi \in BC(\Gamma)$ for every $g \in BC(\Gamma)$. This uniqueness result, in common with uniqueness proofs for integral equation formulations for scattering by bounded obstacles (see e.g. Reference [3]), depends not only on the uniqueness result for the original boundary value problem (Theorem 3.1 in this case) but also on the unique solvability of an associated homogeneous boundary value problem on the other side of the scattering surface (the interior of the scatterer in the bounded obstacle case). Since we are using a combined double- and single-layer potential with the half-plane Dirichlet Green's function in our integral equation formulation it follows that our homogeneous problem consists of the Helmholtz equation in that part $D_- := U_0 \backslash \bar{D}$ of the upper half-plane which lies below the scattering surface Γ, together with a homogeneous Dirichlet condition on Γ_0 and an impedance condition on Γ.

Theorem 3.3
If $\Re(\bar{k}\eta) > 0$, then the integral equation (12) has at most one solution in $BC(\Gamma)$.

Proof
Clearly, in view of the comments before (17), we need only show that, if $\tilde{\psi} \in Y$ and

$$\tilde{\psi} - K_f \tilde{\psi} = 0 \tag{18}$$

then $\tilde{\psi} = 0$.

So suppose $\tilde{\psi} \in Y$ satisfies (18), define $\psi \in BC(\Gamma)$ by $\psi((t, f(t))) = \tilde{\psi}(t)$, $t \in \mathbb{R}$, and define u in $D \cup D_- \cup \Gamma_0 = \bar{U}_0 \backslash \Gamma$ to be the combined double- and single-layer potential

$$u(x) := \int_\Gamma \left[\frac{\partial G(x, y)}{\partial \nu(y)} + \mathrm{i}\eta G(x, y) \right] \psi(y)\, \mathrm{d}s(y), \quad x \in D \cup D_- \cup \Gamma_0$$

Then ψ satisfies (12) with $g = 0$, and it follows from Theorem 3.2 that u satisfies the Dirichlet problem (DP) with $g = 0$, so that, by Theorem 3.1, $u = 0$ in D. Further, by Lemma A.1(ii), (iv) and Lemma A.2(iii), where $u_\pm$ and $\partial u_\pm / \partial \nu$ are defined as in (A2) and (A7), respectively, it follows that $u_+ = \partial u_+ / \partial \nu = 0$ and further that u_- and $\partial u_- / \partial \nu$ exist and satisfy $u_- - u_+ = \psi$ and $\partial u_- / \partial \nu - \partial u_+ / \partial \nu = -\mathrm{i}\eta\psi$, so that $\psi = u_-$ and $\partial u_- / \partial \nu = -\mathrm{i}\eta u$. Also, by Lemmas A.1 and A.2, $u \in C^2(D_- \cup \Gamma_0)$ and is bounded and satisfies the Helmholtz equation in D_- and, from the definition of G, $u = 0$ on Γ_0. Furthermore, from (18) and Lemma A.3 we deduce that, for every $\theta \in (0, 1)$, $\psi \in C^{0,\theta}(\Gamma)$, so that Lemmas A.4 and A.5 can be applied to give that u satisfies (C2). Thus u is a solution of the mixed problem (MP) in Appendix C with the boundary condition (iii)(a), so, by Theorem C.1, $u \equiv 0$ in D_- and hence $\psi = 0$ on Γ. □

We now apply Theorem B.1 to establish the solvability of the integral equation (14) and use the notation of Appendix B. To this end, let $W = \{\kappa_f \mid f \in B\}$, where B is given by (6), for some $c_1, c_2 > 0$. We then have immediately from the above theorem that $I - \mathscr{K}_l$ is injective for all $l \in W$. Also, $T_a(W) = W$ for all $a \in \mathbb{R}$ and, by Lemma B.2(i), $W \subset BC(\mathbb{R}, L^1(\mathbb{R})) \subset \mathbf{K}$ and W satisfies (B4). It is also easy to see, by Lemma B.3, that W is σ-sequentially compact in $\mathbf{K}$. To apply Theorem B.1 we need also to show that, for every $l \in W$, there exists a sequence $(l_n) \subset W$ such that $l_n \xrightarrow{\sigma} l$ and (B5) holds. Let $l \in W$ and let $f \in B$ be such that $l = \kappa_f$. For each $n \in \mathbf{N}$ choose $f_n \in B$ so that f_n is periodic and $f_n(x_1) = f(x_1)$, $-n \leqslant x_1 \leqslant n$. Then $l_n := \kappa_{f_n} \in W$

and $f_n \xrightarrow{s} f$, $f_n' \xrightarrow{s} f'$, so that, by Lemma B.3(ii), $l_n \xrightarrow{\sigma} l$. Since $T_{a_n} l_n = l_n$, where $a_n > 0$ is the period of f_n, and $l_n \in BC(\mathbb{R}, L^1(\mathbb{R}))$, it follows from Theorem 2.10 in [32] that (B5) holds. Thus, W satisfies all the conditions of Theorem B.1. Applying this theorem we obtain the following result.

Theorem 3.4
Let $\Re(\bar{k}\eta) > 0$. Then, for all $f \in B$ the integral operator $I - K_f : Y \to Y$ is bijective (and so boundedly invertible) with

$$\sup_{f \in B} \|(I - K_f)^{-1}\| < \infty$$

Thus the integral equations (14) and (12) have exactly one solution for every $f \in B$ and $g \in BC(\Gamma)$, with

$$\|\psi\|_{\infty,\Gamma} = \|\tilde{\psi}\|_\infty \leqslant C\|\tilde{g}\|_\infty = C\|g\|_{\infty,\Gamma}$$

for some constant $C > 0$ depending only on B and k.

4. THE IMPEDANCE PROBLEM

In this section, we derive novel integral equation formulations for the impedance problem (IP) and use these to establish existence and uniqueness of solution to the impedance boundary value problem. To this end let

$$G_i(x, y) := \Phi(x, y) + \Phi(x, y') + P(x - y') \tag{19}$$

for $x, y \in \mathbb{R}^2$ with $x - y' \in \overline{U_0}$, $x \neq y$, where $y = (y_1, y_2)$, $y' = (y_1, -y_2)$, and

$$P(x) := \frac{e^{ik|x|}}{\pi} \int_0^\infty \frac{t^{-1/2} e^{-k|x|t}(1 + \gamma(1 + it))}{\sqrt{t - 2i}(t - i(1 + \gamma))^2}\, dt, \quad x \in \overline{U_0} \tag{20}$$

with $\gamma = x_2/|x|$ and the square root in (20) taken with $-\pi/2 < \arg\sqrt{t - 2i} < 0$. Then (see References [9,33]), $P \in C(\overline{U_0}) \cap C^\infty(\overline{U_0} \backslash \{0\})$, P satisfies the Helmholtz equation in U_0 and the Sommerfeld radiation and boundedness conditions,

$$\left.\begin{aligned} &P(x) = O(r^{-1/2}), \\ &\frac{\partial P(x)}{\partial r} - ikP(x) = o(r^{-1/2}), \end{aligned}\right\} r := |x| \to \infty \tag{21}$$

and

$$\frac{\partial P(x)}{\partial x_2} = -ik(P(x) + \Phi(x, 0)), \quad x \in \overline{U_0} \backslash \{0\}$$

From these properties it follows that $G_i(x, y)$ is the Green's function for the operator $\Delta + k^2$ in the upper half-plane U_0 which satisfies the impedance boundary condition

$$\frac{\partial G_i(x, y)}{\partial x_2} + ikG_i(x, y) = 0, \quad x \in \Gamma_0, \quad y \in \overline{U_0}, \ x \neq y \tag{22}$$

Define $G_R(x) := 2\Phi(x, 0) + P(x)$, $x \in \bar{U}_0 \setminus \{0\}$. Then

$$G_i(x, y) = G(x, y) + G_R(x - y') \tag{23}$$

where G is the half-plane Dirichlet Green's function given in Section 3, and it follows from bounds in Reference [4] and the asymptotic behaviour of the Hankel function for small argument that, for some constant $C > 0$ depending only on k,

$$|G_R(x)|, |\nabla G_R(x)| \leqslant C(1 + x_2)|x|^{-3/2}, \quad x \in \bar{U}_0 \setminus \{0\} \tag{24}$$

We remark that the Green's function G_i has been extensively studied (see Reference [33] and the references therein). In particular, very efficient calculation methods for the function P are developed in Reference [33].

In the following sections, we derive a direct and an indirect integral equation formulations for the impedance problem (IP). We further prove that the impedance problem (IP) has exactly one solution, employing integral equation methods.

4.1. Green's representation theorem

In this section, we derive a form of Green's representation theorem for the solution u of the impedance problem (IP), using Green's theorem combined with the radiation condition (1).

Theorem 4.1
Let u be a solution of the impedance problem (IP). Then

$$u(x) = -\int_\Gamma \left[\frac{\partial G_i(x, y)}{\partial \nu(y)} - \mathrm{i}k\beta(y) G_i(x, y) \right] u(y)\, \mathrm{d}s(y) + \int_\Gamma G_i(x, y) g(y)\, \mathrm{d}s(y), \quad x \in D \tag{25}$$

Proof
Take $x \in D$, choose $b > \max(x_2, f_+)$, $A > |x_1|$, δ in the range $0 < \delta < x_2 - f(x_1)$, and set $D_b^\delta(A) = \{x \in D(A) \mid f(x_1) + \delta < x_2 < b\}$. Apply Green's second theorem to $G_i(x, \cdot)$ and u in the bounded region $D_b^\delta(A) \setminus \overline{B_\varepsilon(x)}$, for $\varepsilon > 0$ sufficiently small, and then let $\varepsilon \to 0$ to obtain that

$$u(x) = -\int_{\partial D_b^\delta(A)} \left[u(y) \frac{\partial G_i(x, y)}{\partial n(y)} - G_i(x, y) \frac{\partial u}{\partial n}(y) \right] \mathrm{d}s(y) \tag{26}$$

where $\partial/\partial n$ denotes the partial derivative in the normal direction directed out of $D_b^\delta(A)$. Let $T_A := \partial D_b(A) \setminus (\Gamma \cup \Gamma_b)$. Then, from (4) and (5), it follows that $\int_{T_A} |\partial u/\partial n|\, \mathrm{d}s$ exists for all $A > 0$ and is bounded uniformly in A Thus, and in view of (8) and (24),

$$\left| \int_{T_A} G_i(x, y) \frac{\partial u}{\partial n}(y)\, \mathrm{d}s(y) \right| \leqslant \sup_{y \in T_A} |G_i(x, y)| \int_{T_A} \left| \frac{\partial u}{\partial n} \right| \mathrm{d}s \to 0$$

as $A \to \infty$. Further, by using (4), condition (ii) of the impedance problem (IP) and the fact that $f \in C^{1,1}(\mathbb{R})$ it is easy to prove that, where $x^* = (x_1, f(x_1))$, $\partial u/\partial n(x) \to \mathrm{i}k\beta(x^*)u(x^*) + g(x^*)$ as $\delta \to 0$, uniformly in $x \in \Gamma_A^\delta := \{(x_1, f(x_1) + \delta) \mid |x_1| \leqslant A\}$ (cf. the argument leading to (C4) in Appendix C). Thus, letting $\delta \to 0$ and then $A \to \infty$ in (26) we obtain that

$$u(x) = -\int_\Gamma \left[\frac{\partial G_i(x, y)}{\partial \nu(y)} - \mathrm{i}k\beta(y) G_i(x, y) \right] u(y)\, \mathrm{d}s(y) + \int_\Gamma G_i(x, y) g(y)\, \mathrm{d}s(y) + R_b \tag{27}$$

where

$$R_b = -\int_{\Gamma_b}\left[u(y)\,\frac{\partial G_i(x,y)}{\partial n(y)} - G_i(x,y)\,\frac{\partial u}{\partial n}(y)\right]\,\mathrm{d}s(y)$$

Since u satisfies the upward propagating radiation condition (1) and $G_i(x,\cdot)$ satisfies the Sommerfeld radiation and boundedness conditions (21) in U_h for $h>x_2$, it follows, from the equivalence of (i) and (v) in Theorem 2.9 of Chandler–Wilde and Zhang [15] and bounds (8) and (24), that $R_b=0$. □

4.2. A direct integral equation formulation

By making use of the Green's representation theorem, Theorem 4.1, we can reformulate the problem (IP) as a direct boundary integral equation formulation. From (25), Lemma A.1(ii) and Lemma A.2(iii) it follows that

$$u(x) + 2\int_{\Gamma}\left[\frac{\partial G_i(x,y)}{\partial \nu(y)} - \mathrm{i}k\beta(y)G_i(x,y)\right]u(y)\,\mathrm{d}s(y) = 2\int_{\Gamma} G_i(x,y)g(y)\,\mathrm{d}s(y), \quad x\in\Gamma \tag{28}$$

If u satisfies (25) and (28) and $u|_{\Gamma}\in BC(\Gamma)$, where $u|_{\Gamma}$ denotes the restriction of u to Γ, then, using Lemmas A.1 and A.2, it can be shown, similarly to the proof of Theorem 4.1 of Chandler-Wilde and Zhang [6], that $u\in C^2(D)$ and satisfies conditions (i), (iii), and (v) of the impedance problem (IP), with $a=-\frac{1}{2}$ in (iii). From (28) and Lemma A.3, together with the fact that $\beta, g, u\in BC(\Gamma)$, it follows that $u\in C^{0,\theta}(\Gamma)$, $0<\theta<1$. Then, applying Lemmas A.4 and A.5, it is seen that condition (iv) of problem (IP) is satisfied for every $\theta\in(0,1)$. To see that $u\in\mathscr{R}(D)$ and satisfies the impedance boundary condition (ii), define v in D_- by

$$v(x) = -\int_{\Gamma}\left[\frac{\partial G_i(x,y)}{\partial \nu(y)} - \mathrm{i}k\beta(y)G_i(x,y)\right]u(y)\,\mathrm{d}s(y)$$

$$+\int_{\Gamma} G_i(x,y)g(y)\,\mathrm{d}s(y), \quad x\in D_-\cup\Gamma_0 \tag{29}$$

Then, by Lemmas A.1, A.2, A.4, and A.5, Equations (22) and (28), and noting Remark C.1, v is a solution of the mixed problem (MP) of Appendix C with the boundary condition (iii)(b), so, by Theorem C.1, $v\equiv 0$ in D_-. Further, from (25), (29), the jump relations for the layer potentials (see Lemma A.1(ii) and (iv) and Lemma A.2(iii)), and since $\partial v/\partial \nu$ exists, it follows that $u\in\mathscr{R}(D)$ with $\partial u/\partial\nu - \mathrm{i}k\beta u = \partial v/\partial \nu + g$ on Γ, and hence that u satisfies the impedance boundary condition (ii). Thus, we have proved the following result which together with Theorem 4.1 establishes the equivalence of the impedance problem (IP) and the boundary integral equation (28).

Theorem 4.2

If u satisfies (27) and (28) and $u|_{\Gamma}\in BC(\Gamma)$ then u satisfies the impedance problem (IP) with $a=-1/2$ in (3) and with every $\theta\in(0,1)$ in (4).

4.3. Uniqueness results

Suppose that u_1 and u_2 are solutions of the impedance problem (IP). Then $u=u_1-u_2$ satisfies the problem (IP) with $g=0$. Therefore, in order to prove that the impedance problem has at most one solution, it is enough to show that the problem (IP) with $g=0$ has only the trivial solution. Throughout this section, we set $g\equiv 0$ and are concerned with showing that the problem (IP) then has only the trivial solution. We consider two cases: (a) $\Im k>0$, $\Re\beta\geqslant 0$; (b) $k>0$, $\Re\beta\geqslant d$, for some $d>0$. Note that, in physical terms, the conditions $\Im k\geqslant 0$ and $\Re\beta\geqslant 0$ ensure that energy is not generated in the medium and on the boundary, respectively, while if $\Im k>0$ ($\Re\beta>0$) then the medium (boundary) absorbs energy.

The following result deals with the simpler case $\Im k>0$. It can be proved in a similar manner to the proof of Theorem 4.5 in Reference [9], starting with an application of Green's theorem similar to the argument leading up to (21) in the proof of Theorem 4.1. The proof does not require the radiation condition (1) which is superfluous when $\Im k>0$ (see Reference [9, Remark 4.3]).

Theorem 4.3

If u satisfies the impedance problem (IP) with $g=0$, $\Im k>0$, and $\Re\beta\geqslant 0$, then $u\equiv 0$.

To establish the uniqueness result for the more subtle case $k>0$, we require an equality satisfied by u and contained in the next lemma. To this end, for $b>f_+$, $t\in\mathbb{R}$, let $\gamma_b(t)=\{(t,x_2)|f(t)\leqslant x_2\leqslant b\}$.

Lemma 4.4

Let $u\in\mathscr{R}(D)$ satisfy conditions (i)–(iv) of the impedance problem (IP) (with $g=0, k>0$). Then

$$k\int_{\Gamma(A)}\Re(\beta)|u|^2\,\mathrm{d}s+J_A=R_1(A),\quad A>0 \tag{30}$$

where

$$J_A:=\Im\int_{\Gamma_b(A)}\bar{u}\partial_2 u\,\mathrm{d}s,\quad R_1(A):=\Im\left[\int_{\gamma_b(-A)}-\int_{\gamma_b(A)}\right]\bar{u}\partial_1 u\,\mathrm{d}s$$

and $f_+<b<f_++1$.

Proof

Apply Green's first theorem to u and $\bar{u}$ in $S_\varepsilon:=\{x\in D_b(A)|x_2>f(x_1)+\varepsilon\}$, with $0<\varepsilon<1$, to obtain that

$$\int_{S_\varepsilon}[|\nabla u|^2-k^2|u|^2]\,\mathrm{d}x=\int_{\partial S_\varepsilon}\bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}s \tag{31}$$

on noting that $\Delta u=-k^2u$ in D, where $\partial/\partial n$ is the partial derivative in the normal direction directed out of S_ε. It has been shown in the proof of Theorem 4.1 that, for $x=(x_1,f(x_1)+\varepsilon)\in\partial S_\varepsilon$, $\partial u/\partial n(x)\to \mathrm{i}k\beta(x^*)u(x^*)+g(x^*)$ as $\varepsilon\to 0$, uniformly in x_1, where $x^*=(x_1,f(x_1))$. Thus, taking the imaginary part of Equation (31) and then the limit $\varepsilon\to 0$ we obtain Equation (30). □

To use equality (30) we need the following two lemmas. The first of these is a consequence of the UPRC and was proved in Reference [15, Lemma 6.1] and the second is an immediate consequence of Lemma A in Reference [6].

Lemma 4.5
If $\phi \in L^2(\Gamma_h) \cap L^\infty(\Gamma_h)$ and v is defined by (1) with u replaced with v, then the restrictions of v and $\partial_2 v$ to Γ_b are in $L^2(\Gamma_b) \cap BC(\Gamma_b)$ for all $b > h$ and $\Im \int_{\Gamma_b} \bar{v} \partial_2 v \, \mathrm{d}s \geqslant 0$.

Lemma 4.6
Suppose that $F \in L^\infty(\mathbb{R})$ and that, for some non-negative constants c, ε, and A_0,

$$\int_{-A}^{A} |F(t)|^2 \, \mathrm{d}t \leqslant c \int_{\mathbb{R} \setminus [-A,A]} G_A^2(t) \, \mathrm{d}t + c \int_{-A}^{A} (G_\infty(t) - G_A(t)) G_\infty(t) + \varepsilon, \quad A > A_0$$

where, for $A_0 < A \leqslant +\infty$,

$$G_A(s) := \int_{-A}^{A} (1 + |s - t|)^{-3/2} |F(t)| \, \mathrm{d}t, \quad s \in \mathbb{R}$$

Then $F \in L^2(\mathbb{R})$ and $\int_{-\infty}^{+\infty} |F(t)|^2 \, \mathrm{d}t \leqslant \varepsilon$.

To make use of Lemma 4.5 we define

$$v(x) = -\int_{\Gamma(A)} \left[\frac{\partial G_i(x, y)}{\partial \nu(y)} - \mathrm{i}k\beta(y) G_i(x, y) \right] u(y) \, \mathrm{d}s(y), \quad x \in D \tag{32}$$

Then, by (8), (23), (24), and condition (iii) of the impedance problem (IP) combined with the Cauchy–Schwarz inequality, we have that $v|_{\Gamma_a}$, the restriction of v to Γ_a, is in $L^2(\Gamma_a) \cap BC(\Gamma_a)$ for all $a > f_+$. On the other hand, utilizing (21), we see that v satisfies the Sommerfeld radiation condition in U_a for $a > f_+$, so that, by Theorem 2.9 in Reference [15], v satisfies (1) with $h = a$ and $\phi = v|_{\Gamma_a}$, for every $a > f_+$. Set, for some $b > f_+ + 1$,

$$J'_A := \Im \int_{\Gamma_b(A)} \bar{v} \partial_2 v \, \mathrm{d}s, \quad J''_A := \Im \int_{\Gamma_b} \bar{v} \partial_2 v \, \mathrm{d}s$$

for $A > 0$. Then, by Lemma 4.5, $J''_A \geqslant 0$, so that, if $\Re(\beta) \geqslant d > 0$, then, by (30),

$$K_A := \int_{\Gamma(A)} |u|^2 \, \mathrm{d}s \leq (kd)^{-1} [-J_A + R_1(A)] \leqslant (kd)^{-1} [(J''_A - J_A) + R_1(A)]$$

Now set $w(x_1) = u((x_1, f(x_1)))$, $x_1 \in \mathbb{R}$. Then, for some constant $C_0 > 0$ and all $A > 0$,

$$\int_{-A}^{A} |w(x_1)|^2 \, \mathrm{d}x_1 \leqslant K_A \leqslant C_0 \int_{-A}^{A} |w(x_1)|^2 \, \mathrm{d}x_1 \tag{33}$$

By (8), (23)–(25) with $g = 0$, and (32),

$$|v(x)| \leqslant C_1 W_A(x_1), \quad x \in U_{b-1} \setminus U_{b+1} \tag{34}$$

$$|u(x) - v(x)| \leqslant C_1 (W_\infty(x_1) - W_A(x_1)), \quad x \in U_{b-1} \setminus U_{b+1} \tag{35}$$

for some constant $C_1 > 0$, where, for $0 \leqslant A \leqslant +\infty$,

$$W_A(x_1) := \int_{-A}^{A} (1 + |x_1 - y_1|)^{-3/2} |w(y_1)| \, dy_1, \quad x_1 \in \mathbb{R}$$

Moreover, since u and v satisfy the Helmholtz equation in D, it follows by local regularity estimates [31], that $|\nabla u(x)|$ and $|\nabla u(x) - \nabla v(x)|$ can be bounded by multiples of maximum values of $|u|$ and $|u - v|$, respectively, in neighbourhoods of x. Hence, for $x \in \Gamma_b$, $|\nabla u(x)|$ and $|\nabla u(x) - \nabla v(x)|$ also satisfy (34) and (35), respectively, for some constant $C_1 > 0$. It follows that, for some constant $C_2 > 0$,

$$|J'_A - J''_A| \leqslant C_2 \int_{\mathbb{R} \setminus [-A, A]} (W_A(x_1))^2 \, dx_1$$

and

$$|J_A - J'_A| \leqslant C_2 \int_{-A}^{A} (W_\infty(x_1) - W_A(x_1)) W_\infty(x_1) \, dx_1$$

so that, for some constant $C > 0$ and all $A > 0$,

$$\begin{aligned} \int_{-A}^{A} |w(x_1)|^2 \, dx_1 &\leqslant (kd)^{-1} [(J''_A - J_A) + R_1(A)] \\ &\leqslant C \left\{ \int_{\mathbb{R} \setminus [-A, A]} W_A^2(x_1) \, dx_1 + \int_{-A}^{A} (W_\infty(x_1) - W_A(x_1)) W_\infty(x_1) \, dx_1 + |R_1(A)| \right\}. \end{aligned} \quad (36)$$

Applying Lemma 4.6 to (36) we obtain that $w \in L^2(\mathbb{R})$ (equivalently, by (33), $u \in L^2(\Gamma)$) and, for all $A_0 > 0$,

$$C_0^{-1} \int_\Gamma |u|^2 \, ds \leqslant \int_{-\infty}^{\infty} |w(x_1)|^2 \, dx_1 \leqslant C \sup_{A > A_0} |R_1(A)| \quad (37)$$

Since $\beta \in BC(\Gamma)$ and, from condition (iii) of the impedance problem (IP), $u \in BC(\Gamma)$, we obtain, in view of (28) with $g = 0$ and by Lemma A.3, that, for every $\lambda \in (0, 1)$, $u \in C^{0,\lambda}(\Gamma)$. Thus $u \in BUC(\Gamma) \cap L^2(\Gamma)$, from which it follows that $u(x) \to 0$ as $|x| \to \infty$ with $x \in \Gamma$.

For $A > 0$ choose $\psi_A \in BC(\Gamma)$ with $\|\psi_A\|_\infty = 1$ such that $\psi_A(x) = 1$ for $|x_1| \leqslant A/3$ and $\psi_A(x) = 0$ for $|x_1| \geqslant 2A/3$. Let u_1 and u_2 be defined by (25) with $g = 0$ and with the density u replaced with $u(1 - \psi_A)$ and $u\psi_A$, respectively, so that $u(x) = u_1(x) + u_2(x)$ for $x \in D$. Then, by Lemmas A.1(iii) and A.2(ii), for some constant $C > 0$ and all $x \in \gamma_b(-A) \cup \gamma_b(A)$, $|u_1(x)| \leqslant C \|u(1 - \psi_A)\|_{\infty, \Gamma} \to 0$ as $A \to \infty$. Also, by (8) and (24),

$$\sup_{x \in \gamma_b(-A) \cup \gamma_b(A)} |u_2(x)| \leqslant C \|u\|_{\infty, \Gamma} \int_{A/3}^{\infty} t^{-3/2} \, dt \to 0$$

as $A \to \infty$. In view also of the bound (4) it follows that $R_1(A) \to 0$ as $A \to \infty$, and thus, from (37), that $u = 0$ on Γ and hence, from (25) with $g = 0$, that $u \equiv 0$ in D. We have shown the following result on noting the remark made at the beginning of this section.

Theorem 4.7
Assume that $k>0$. If, for some $d>0$, $\Re(\beta(x))\geqslant d$ for all $x\in\Gamma$, then the impedance problem (IP) has at most one solution.

We remark that an example in Reference [9, Remark 4.4] for the case of a flat boundary shows that Theorem 4.7 no longer holds if we require only that $\Re\beta\geqslant 0$.

From Theorems 4.1 and 4.2, the impedance problem (IP) and the direct boundary integral equation (28) are equivalent, so by Theorems 4.3 and 4.7 we have the following uniqueness result for the integral equation (28).

Theorem 4.8
Suppose that, for some $d\geqslant 0$, $\Re(\beta(x))\geqslant d$ for all $x\in\Gamma$ and that either $\Im k>0$ or $d>0$. Then the boundary integral equation (28) has at most one solution in $BC(\Gamma)$.

4.4. Existence results

To prove existence of solution we note that from Theorems 4.1 and 4.2, the impedance problem (IP) and the integral equation (28) are equivalent. Thus it is enough to show existence of solution for the integral equation (28), which will be done by utilizing Theorem B.1. To this end define

$$\phi(x):=\int_\Gamma G_i(x,y)g(y)\,\mathrm{d}s(y),\quad x\in\Gamma$$

Then, by Lemma A.2(ii), $\phi\in BC(\Gamma)$ and

$$\|\phi\|_{\infty,\Gamma}\leqslant C^*\|g\|_{\infty,\Gamma}\tag{38}$$

for some constant $C^*>0$. Define $\tilde{u},\tilde{\phi},\tilde{\beta}\in Y$ by

$$\tilde{u}(s):=u((s,f(s))),\quad \tilde{\phi}(s):=\phi((s,f(s))),\quad \tilde{\beta}(s):=\beta((s,f(s))),\quad s\in\mathbb{R}\tag{39}$$

Again parameterizing integral (28) in the obvious way we obtain the following integral equation problem: find $\psi\in Y$ such that

$$\psi(s)-\int_{\mathbb{R}}\kappa_{\tilde{\beta},f}(s,t)\psi(t)\,\mathrm{d}t=2\tilde{\phi}(s),\quad s\in\mathbb{R}\tag{40}$$

where

$$\kappa_{\tilde{\beta},f}(s,t):=2\left[\mathrm{i}k\beta(y)G(x,y)-\frac{\partial G(x,y)}{\partial\nu(y)}\right]\sqrt{1+[f'(t)]^2},\quad s,t\in\mathbb{R},\quad s\neq t\tag{41}$$

with $x=(s,f(s))$, $y=(t,f(t))$. The integral equation (40) can be written as

$$(I-K_{\tilde{\beta},f})\tilde{u}=2\tilde{\phi}\tag{42}$$

where $K_{\tilde{\beta},f}$ is defined by (16) but with κ replaced by $\kappa_{\tilde{\beta},f}$.

For some $d_2 > 0$, $d_1 \geqslant 0$ and some $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(s) \to 0$ as $s \to 0$, let E be defined by

$$E = E(d_1, d_2, \omega) := \{\tilde{\beta} \in Y \,|\, \Re(\tilde{\beta}(s)) \geqslant d_1,\ s \in \mathbb{R},\ \|\tilde{\beta}\|_\infty \leqslant d_2, \text{ and } |\tilde{\beta}(s) - \tilde{\beta}(t)| \leqslant \omega(|s-t|),\ s,t \in \mathbb{R}\} \tag{43}$$

Note that $E \subset BUC(\mathbb{R})$. Conversely, given $\tilde{\beta} \in BUC(\mathbb{R})$, it holds that $\tilde{\beta} \in E$ provided $d_1 \leqslant \inf \Re(\tilde{\beta})$, $d_2 \geqslant \|\tilde{\beta}\|_\infty$ and $\omega \geqslant \omega_\beta$, where $\omega_\beta(h) := \sup_{s \in \mathbb{R}, |\eta| \leqslant h} |\tilde{\beta}(s+\eta) - \tilde{\beta}(s)|$, $h \geqslant 0$, is the modulus of continuity of $\tilde{\beta}$. We have the following existence result for the integral equations (28) and (42).

Theorem 4.9
If either $\Im k > 0$ or $d_1 > 0$, then, for all $f \in B$ and all $\tilde{\beta} \in E$, the integral operator $I - K_{\tilde{\beta}, f} : Y \to Y$ is bijective (and so boundedly invertible) with

$$\sup_{f \in B, \tilde{\beta} \in E} \|(I - K_{\tilde{\beta}, f})^{-1}\| < \infty$$

Thus the integral equations (28) and (42) have exactly one solution for every $f \in B$, $\tilde{\beta} \in E$, and $g \in BC(\Gamma)$, with

$$\|u\|_{\infty,\Gamma} = \|\tilde{u}\|_\infty \leqslant C\|\tilde{\phi}\|_\infty = C\|\phi\|_{\infty,\Gamma} \leqslant CC^*\|g\|_{\infty,\Gamma}$$

for some constants $C > 0$ and $C^* > 0$ depending only on B, E, and k.

Proof
Let $W := \{\kappa_{\tilde{\beta}, f} \,|\, f \in B,\ \tilde{\beta} \in E\}$. We then have immediately from Theorem 4.8 that $I - \mathscr{K}_l$ is injective for all $l \in W$. Also, $T_a(W) = W$ for all $a \in \mathbb{R}$ and, by Lemma B.2(ii), $W \subset BC(\mathbb{R}, L^1(\mathbb{R})) \subset \mathbf{K}$ and W satisfies (B4). It also follows from Lemmas B.3 and B.4 that W is σ-sequentially compact in $\mathbf{K}$. To apply Theorem B.1 we need also to show that, for every $l \in W$, there exists a sequence $(l_n) \subset W$ such that $l_n \xrightarrow{\sigma} l$ and (B5) holds. Let $l \in W$ and let $f \in B$ and $\tilde{\beta} \in E$ be such that $l = \kappa_{\tilde{\beta}, f}$. For each $n \in \mathbf{N}$ choose $f_n \in B$ and $\tilde{\beta}_n \in E$ so that f_n and $\tilde{\beta}_n$ are periodic with the same period and $f_n(x_1) = f(x_1)$, $\tilde{\beta}_n(x_1) = \tilde{\beta}(x_1)$, $-n \leqslant x_1 \leqslant n$. Then $l_n := \kappa_{\tilde{\beta}_n, f_n} \in W$ and $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$, $\tilde{\beta}_n \xrightarrow{s} \tilde{\beta}$, so that, by Lemma B.4(ii), $l_n \xrightarrow{\sigma} l$. Since $T_{a_n} l_n = l_n$, where $a_n > 0$ is the period of f_n and $\tilde{\beta}_n$, and $l_n \in BC(\mathbb{R}, L^1(\mathbb{R}))$, it follows from Theorem 2.10 in Reference [32] that (B5) holds. Thus W satisfies all the conditions of Theorem B.1 and hence Theorem 4.9 follows from Theorem B.1. □

Combining Theorems 4.2, 4.3, 4.7 and 4.9 we obtain a unique solvability result for the impedance problem (IP). Specifically, these theorems give that unique solvability holds whenever $f \in B$, defined by (6), and $\tilde{\beta} \in E$, defined by (43), provided that either $\Im k > 0$ or $d_1 > 0$. Since, in view of the remark after (43), every $\tilde{\beta} \in BUC(\mathbb{R})$ is in $E = E(d_1, d_2, \omega)$ for the choices $d_1 := \inf \Re\beta$, $d_2 := \|\beta\|_\infty$, and $\omega := \omega_\beta$, where ω_β is the modulus of continuity of $\tilde{\beta}$, we have the following solvability result.

Theorem 4.10
Suppose that $\beta \in BUC(\Gamma)$ and that, for some $d \geqslant 0$, $\Re(\beta(x)) \geqslant d$ for all $x \in \Gamma$. If either $\Im k > 0$ or $d > 0$, then the impedance problem (IP) has exactly one solution. Moreover, if either $\Im k > 0$ or $d_1 > 0$, then, for some constant $C > 0$ depending only on B, E, and k,

$$|u(x)| \leqslant C x_2^{1/2} \|g\|_{\infty,\Gamma}$$

for all $f \in B$, $g \in BC(\Gamma)$, and $\tilde{\beta} \in E$, with β defined in terms of $\tilde{\beta}$ by (39).

4.5. *An indirect boundary integral equation*

In this section, we derive an indirect boundary integral equation for the impedance problem (IP) by seeking a solution in the form of the single-layer potential,

$$u(x) = \int_\Gamma G_i(x, y)\psi(y)\, \mathrm{d}s(y), \quad x \in D \tag{44}$$

for some $\psi \in BC(\Gamma)$. It can be proved similarly to the proof of Theorem 4.2 (utilizing Lemmas A.2, A.3, and A.5) that the single-layer potential (44) satisfies all the conditions of problem (IP) (with $a = -1/2$ in (iii) and every $\theta \in (0, 1)$ in (iv)) except possibly the boundary condition (ii). From Lemma A.2(iii) the single-layer potential (44) satisfies the boundary condition (ii) provided ψ satisfies

$$\psi(x) + 2\int_\Gamma \left[\frac{\partial G_i(x, y)}{\partial \nu(x)} - \mathrm{i}k\beta(x)G_i(x, y) \right] \psi(y)\, \mathrm{d}s(y) = 2g(x), \quad x \in \Gamma \tag{45}$$

Thus we obtain the following result providing an indirect integral equation formulation of the impedance problem (IP).

Theorem 4.11
The single-layer potential (44) satisfies the problem (IP), with $a = -1/2$ in (iii) and every $\theta \in (0, 1)$ in (iv), provided $\psi \in BC(\Gamma)$ satisfies the boundary integral equation (45).

Using the uniqueness Theorems 4.3 and 4.7 for the impedance problem (IP) we can establish the following uniqueness result for the integral equation (45).

Theorem 4.12
Suppose that, for some $d \geqslant 0$, $\Re(\beta(x)) \geqslant d$ for all $x \in \Gamma$ and that either $\Im k > 0$ or $d > 0$. Then the boundary integral equation (45) has at most one solution in $BC(\Gamma)$.

Proof
It is enough to show that, if $\psi \in BC(\Gamma)$ satisfies (45) with $g = 0$, then $\psi = 0$.

Suppose that $\psi \in BC(\Gamma)$ satisfies (45) with $g = 0$ and define u in $\overline{U_0}$ to be the single-layer potential

$$u(x) := \int_\Gamma G_i(x, y)\psi(y)\, \mathrm{d}s(y), \quad x \in \overline{U_0}$$

Then it follows from Theorem 4.11 that u satisfies the problem (IP) with $g = 0$, so that, by Theorems 4.3 and 4.7, $u = 0$ in D. Further, by Lemma A.2(iii), where $u_\pm$ and $\partial u_\pm/\partial \nu$ are defined as in (A1) and (A6) respectively, $u_- - u_+ = 0$ and $\partial u_+/\partial \nu - \partial u_-/\partial \nu = \psi$, so $u_- = 0$ and $\partial u_-/\partial \nu = -\psi$. Also, by Lemma A.2, $u \in \mathscr{R}(D_-) \cap BC(\bar{D}_-)$, where $\mathscr{R}(D_-)$ is defined as in Appendix C, and u satisfies the Helmholtz equation in D_- and, from (22), $\partial_2 u = -\mathrm{i}ku$ on Γ_0. Furthermore, by Lemma A.5, u satisfies (C2). Thus u is a solution of the mixed problem (MP) with the boundary condition (iii)(b), and so it follows by Theorem C.1 that $u \equiv 0$ in D_- and hence that $\psi = -\partial u_-/\partial \nu = 0$. □

By making use of Theorem 4.12 and exactly the same argument as in the proof of Theorem 4.9 we are able to obtain the following existence result for the integral equation (45).

Theorem 4.13
Suppose that, for some $d \geqslant 0$, $\Re(\beta(x)) \geqslant d$ for all $x \in \Gamma$. If either $\Im k > 0$ or $d > 0$, then the integral equation (45) has exactly one solution for every $f \in B$, $g \in BC(\Gamma)$, and $\beta \in BUC(\Gamma)$. Moreover, if either $\Im k > 0$ or $d_1 > 0$, then, for some constant $C > 0$ depending only on B, E, and k, $\|\psi\|_{\infty,\Gamma} \leqslant C\|g\|_{\infty,\Gamma}$ for all $f \in B$, $g \in BC(\Gamma)$, $\tilde{\beta} \in E$, with β defined in terms of $\tilde{\beta}$ by (39).

APPENDIX A. PROPERTIES OF THE LAYER POTENTIALS

In this appendix, we establish properties of double- and single-layer potentials. Note that these properties, Lemmas A.1–A.5, are also true if the Dirichlet Green's function G is replaced by the impedance Green's function G_i. Indeed, Equation (23) and the fact that $G_R \in C^\infty(U_0)$ and satisfies the Helmholtz equation, and the bound (24) in U_0, make it clear that it is sufficient to prove these results for one or other of G and G_i for them to hold for both. We assume throughout that $f \in B$, where B is defined by (6), for some $c_1, c_2 > 0$. We remind the reader that $\nu(x)$ denotes the unit normal at $x \in \Gamma$, directed out of D.

Lemma A.1
Let W be the double-layer potential (10) with density $\psi \in BC(\Gamma)$. Then the following results hold.

(i) Define $S := \{x \in \mathbb{R}^2 \mid -c_1 < x_2 < f(x_1)\}$. Then $W \in C^2(S \cup D)$ and $\Delta W + k^2 W = 0$ in $S \cup D$.

(ii) W can be continuously extended from D to $\bar{D}$ and from $U_0 \backslash \bar{D}$ to $U_0 \backslash D$ with limiting values

$$W_\pm(x) = \int_\Gamma \frac{\partial G}{\partial \nu(y)}(x, y)\psi(y)\,\mathrm{d}s(y) \mp \frac{1}{2}\psi(x), \quad x \in \Gamma \tag{A1}$$

where

$$W_\pm(x) := \lim_{h \to 0, h > 0} W(x \mp h\nu(x)) \tag{A2}$$

The integral exists as an improper integral.

(iii) For every $\varepsilon < c_1$ there exists $C_\varepsilon > 0$ such that, for all $f \in B$ and $\psi \in BC(\Gamma_f)$,

$$|(1 + \varepsilon + x_2)^{-1/2} W(x)| \leqslant C_\varepsilon \|\psi\|_\infty, \quad x \in U_{-\varepsilon} \backslash \Gamma_f \tag{A3}$$

(iv) There holds

$$(\nabla W(x + h\nu(x)) - \nabla W(x - h\nu(x))) \cdot \nu(x) \to 0$$

as $h \to 0$, uniformly for x in compact subsets of Γ.

For a proof of these properties (with G replaced by G_i) see Reference [5, Lemmas A.1 and A.2. 4,7].

Similarly to the proof of Lemma A.1, using the corresponding results for single-layer potentials supported on bounded surfaces [3], we obtain the following properties of the single-layer potential.

Lemma A.2

The following results hold for the single-layer potential V, defined by (11), with density $\psi \in BC(\Gamma)$.

(i) With S defined as in Lemma A.1, $V \in C^2(S \cup D)$ and $\Delta V + k^2 V = 0$ in $S \cup D$.

(ii) For every $\varepsilon < c_1$, V is continuous in $U_{-\varepsilon}$ and there exists $C_\varepsilon > 0$ such that, for all $f \in B$ and $\psi \in BC(\Gamma_f)$,

$$|(1+\varepsilon+x_2)^{-1/2} V(x)| \leqslant C_\varepsilon \|\psi\|_\infty, \quad x \in U_{-\varepsilon} \tag{A4}$$

(iii) On the boundary Γ we have

$$V(x) = \int_\Gamma G(x,y)\psi(y)\,\mathrm{d}s(y), \quad x \in \Gamma \tag{A5}$$

$$\frac{\partial V_\pm}{\partial \nu}(x) = \int_\Gamma \frac{\partial G}{\partial \nu(x)}(x,y)\psi(y)\,\mathrm{d}s(y) \pm \frac{1}{2}\psi(x), \quad x \in \Gamma \tag{A6}$$

where

$$\frac{\partial V_\pm}{\partial \nu}(x) := \lim_{h \to 0, h > 0} \nu(x) \cdot \nabla V(x \mp h\nu(x)) \tag{A7}$$

and the convergence in (A7) is uniform on compact subsets of Γ. The integrals (A5) and (A6) exist as improper integrals.

The following lemmas A.3 and A.4 were proved, as they relate to the double-layer potential, in Reference [5, Lemmas A.2 and A.3]. The results in Lemma A.3 for the single-layer potential can be shown similarly. Lemma A.5 is a generalization of Lemma 4.4 in Reference [9] and can be proved similarly.

Lemma A.3

Let $\psi \in BC(\Gamma)$. Then the direct values of the double-layer potential

$$W(x) := \int_\Gamma \frac{\partial G}{\partial \nu(y)}(x,y)\psi(y)\,\mathrm{d}s(y), \quad x \in \Gamma$$

and the single-layer potential

$$V(x) := \int_\Gamma G(x,y)\psi(y)\,\mathrm{d}s(y), \quad x \in \Gamma$$

represent uniformly Hölder continuous functions on Γ with

$$\|W\|_{C^{0,\lambda}(\Gamma)}, \ \|V\|_{C^{0,\lambda}(\Gamma)} \leqslant C\|\psi\|_{\infty,\Gamma} \tag{A8}$$

for some constant $C > 0$ depending only on B and k.

Lemma A.4
If $\psi\in C^{0,\lambda}(\Gamma)$ with $0<\lambda<1$, then

$$|\nabla W(x)|\leqslant C|f(x_1)-x_2|^{\lambda-1},\quad x\in U_0\backslash(\bar{U}_b\cup\Gamma)$$

where C is a positive constant and $b=f_+ + 1$.

Lemma A.5
If $\psi\in BC(\Gamma)$ then, for $0<\lambda<1$, there exists a positive constant C such that

$$|\nabla V(x)|\leqslant C|f(x_1)-x_2|^{\lambda-1},\quad x\in U_0\backslash(\bar{U}_b\cup\Gamma)$$

where $b=f_+ + 1$.

APPENDIX B. INTEGRAL OPERATORS ON THE REAL LINE

B.1. Invertibility of integral operators

Consider the integral operator $\mathscr{K}_l$ with kernel $l:\mathbb{R}^2\to\mathbf{C}$, defined by

$$\mathscr{K}_l\psi(s)=\int_{\mathbb{R}} l(s,t)\psi(t)\,\mathrm{d}t,\quad s\in\mathbb{R} \tag{B1}$$

It is easy to see that integral (B1) exists in a Lebesgue sense for every $\psi\in X:=L^\infty(\mathbb{R})$ and $s\in\mathbb{R}$ iff $l(s,\cdot)\in L^1(\mathbb{R})$, $s\in\mathbb{R}$, and that $\mathscr{K}_l:X\to Y:=BC(\mathbb{R})$ and is bounded iff, in addition,

$$|||l|||:=\operatorname{ess\,sup}_{s\in\mathbb{R}}\|l(s,\cdot)\|_1<\infty \tag{B2}$$

and $\mathscr{K}_l\psi\in C(\mathbb{R})$ for every $\psi\in X$. If $\mathscr{K}_l:X\to Y$ and is bounded, then $\|\mathscr{K}_l\|=|||l|||$.

In the case that (B2) holds it is convenient to identify $l:\mathbb{R}^2\to\mathbf{C}$ with the mapping $s\to l(s,\cdot)$ in $\mathbf{Z}:=L^\infty(\mathbb{R},L^1(\mathbb{R}))$, which mapping is essentially bounded with norm $|||l|||$. Let $\mathbf{K}$ denote the set of those functions $l\in\mathbf{Z}$ having the property that $\mathscr{K}_l\psi\in C(\mathbb{R})$ for every $\psi\in X$, where $\mathscr{K}_l$ is the integral operator (B1). Then $\mathbf{Z}$ is a Banach space with the norm $|||\cdot|||$ and $\mathbf{K}$ is a closed subspace of $\mathbf{Z}$. Further, in view of the above comments, $\mathscr{K}_l:X\to Y$ and is bounded iff $l\in\mathbf{K}$. Also, note that certainly $BC(\mathbb{R},L^1(\mathbb{R}))\subset\mathbf{K}$, i.e. $l\in\mathbf{K}$ if $l\in\mathbf{Z}$ and if, additionally, for all $s\in\mathbb{R}$,

$$\|l(s,\cdot)-l(s',\cdot)\|_1\to 0\quad\text{as } s'\to s \tag{B3}$$

For $(\phi_n)\subset Y$, $\phi\in Y$, we say that (ϕ_n) *converges strictly* to ϕ and write $\phi_n\stackrel{s}{\to}\phi$ if $\sup_{n\in\mathbf{N}}\|\phi_n\|_\infty<\infty$ and $\phi_n(t)\to\phi(t)$ uniformly on every compact subset of $\mathbb{R}$. Similarly (see Reference [5,34]), for $(l_n)\subset\mathbf{K}$, $l\in\mathbf{K}$, we say that (l_n) is *σ-convergent* to l and write $l_n\stackrel{\sigma}{\to}l$ if $\sup_{n\in\mathbf{N}}|||l_n|||<\infty$ and, for all $\psi\in X$,

$$\int_{\mathbb{R}} l_n(s,t)\psi(t)\,\mathrm{d}t\to\int_{\mathbb{R}} l(s,t)\psi(t)\,\mathrm{d}t$$

as $n\to\infty$, uniformly on every compact subset of $\mathbb{R}$.

We also need the following definitions. For $a\in\mathbb{R}$, define the translation operator $T_a:\mathbf{Z}\to\mathbf{Z}$ by

$$T_a l(s,t)=l(s-a,t-a),\quad s,t\in\mathbb{R}$$

A subset $W\subset\mathbf{K}$ is said to be *σ-sequentially compact* in $\mathbf{K}$ if each sequence in W has a σ-convergent subsequence with limit in W. Let $B(Y)$ denote the Banach space of bounded linear operators on Y and I the identity operator on Y. Then the following result is proved in Reference [34] on the invertibility of the integral operator $I-\mathscr{K}_l$ (see also Reference [5]).

Theorem B.1
Suppose that $W\subset\mathbf{K}$ is σ-sequentially compact and satisfies that, for all $s\in\mathbb{R}$,

$$\sup_{l\in W}\int_{\mathbb{R}}|l(s,t)-l(s',t)|\,\mathrm{d}t\to 0,\quad \text{as } s'\to s \tag{B4}$$

that $T_a(W)=W$ for some $a\in\mathbb{R}$, and that $I-\mathscr{K}_l$ is injective for all $l\in W$. Then $(I-\mathscr{K}_l)^{-1}$ exists as an operator on the range space $(I-\mathscr{K}_l)(Y)$ for all $l\in W$ and

$$\sup_{l\in W}\|(I-\mathscr{K}_l)^{-1}\|<\infty$$

If also, for every $l\in W$, there exists a sequence $(l_n)\subset W$ such that $l_n\xrightarrow{\sigma} l$ and, for each n, it holds that

$$I-\mathscr{K}_{l_n}\quad \text{injective}\ \Rightarrow\ I-\mathscr{K}_{l_n}\ \text{surjective} \tag{B5}$$

then also $I-\mathscr{K}_l$ is surjective for each $l\in W$ so that $(I-\mathscr{K}_l)^{-1}\in B(Y)$.

B.2. Properties of the integral operators K_f and $K_{\tilde{\beta},f}$

We summarize some properties of the integral operators K_f in Section 3 and $K_{\tilde{\beta},f}$ in Section 4, for $f\in B$ and $\tilde{\beta}\in E$, where B and E are defined by (6) and (43).

Lemma B.2
Let $\kappa\in L^1(\mathbb{R})$ be defined by

$$\kappa(s):=\begin{cases}1-\log|s|, & 0<|s|\leqslant 1\\ |s|^{-3/2}, & |s|>1.\end{cases}$$

(i) For all $f\in B$,

$$|\kappa_f(s,t)|\leqslant C|\kappa(s-t)|,\quad s,t\in\mathbb{R},\ s\neq t \tag{B6}$$

for some constant $C>0$ depending only on c_1, c_2, η, and k, and

$$\sup_{|s_1-s_2|\leqslant h,\, f\in B}\int_{\mathbb{R}}|\kappa_f(s_1,t)-\kappa_f(s_2,t)|\,\mathrm{d}t\to 0$$

as $h\to 0$.

(ii) For all $f \in B$, $\tilde{\beta} \in E$,

$$|\kappa_{\tilde{\beta},f}(s,t)| \leqslant C|\kappa(s-t)|, \quad s,t \in \mathbb{R}, \ s \neq t \tag{B7}$$

for some constant $C > 0$ depending only on c_1, c_2, d_2, and k, and

$$\sup_{|s_1-s_2| \leqslant h, f \in B, \tilde{\beta} \in E} \int_{\mathbb{R}} |\kappa_{\tilde{\beta},f}(s_1,t) - \kappa_{\tilde{\beta},f}(s_2,t)| \, \mathrm{d}t \to 0$$

as $h \to 0$.

Part (i) was proved in Reference [4, Lemmas 5.1 and 5.2] for the case when $\eta = 0$; the case with $\eta \neq 0$ can be shown similarly. Part (ii) was proved in Reference [28, Lemmas 3.10 and 3.13]. Note that this lemma implies that κ_f, $\kappa_{\tilde{\beta},f} \in BC(\mathbb{R}, L^1(\mathbb{R})) \subset \mathbf{K}$, so that K_f, $K_{\tilde{\beta},f} : X \to Y$ and are bounded.

To apply Theorem B.1 we also need the following lemma which was proved as [5, Lemma 4.6].

Lemma B.3

(i) Every sequence $(f_n) \subset B$ has a subsequence (f_{n_m}) such that $f_{n_m} \xrightarrow{s} f$, $f'_{n_m} \xrightarrow{s} f'$, with $f \in B$.

(ii) Suppose that $(f_n) \subset B$ and that $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$, with $f \in B$. Then $\kappa_{f_n} \xrightarrow{\sigma} \kappa_f$.

Lemma B.4

(i) Every sequence $(\tilde{\beta}_n) \subset E$ has a subsequence $(\tilde{\beta}_{n_m})$ such that $\tilde{\beta}_{n_m} \xrightarrow{s} \tilde{\beta}$ with $\tilde{\beta} \in E$.

(ii) If $(f_n) \subset B$, $(\tilde{\beta}_n) \subset E$ and $f_n \xrightarrow{s} f$, $f'_n \xrightarrow{s} f'$, $\tilde{\beta}_{n_m} \xrightarrow{s} \tilde{\beta}$, with $f \in B$ and $\tilde{\beta} \in E$, then $\kappa_{\tilde{\beta}_n, f_n} \xrightarrow{\sigma} \kappa_{\tilde{\beta},f}$.

Proof

(i) Using the Arzela–Ascoli theorem and the definition of strict convergence together with the observation that E is bounded and equicontinuous it is easy to see that we can find a subsequence such that $\tilde{\beta}_{n_m} \xrightarrow{s} \tilde{\beta}$. That $\tilde{\beta} \in E$ follows since $(\tilde{\beta}_n) \subset E$.

(ii) It is not difficult to see that $\kappa_{\tilde{\beta}_n, f_n}(s,t) \to \kappa_{\tilde{\beta},f}(s,t)$ uniformly on compact subsets of $\{(s,t) \in \mathbb{R}^2 | s \neq t\}$. From this observation and the bound (B7) the result easily follows. □

APPENDIX C. MIXED BOUNDARY VALUE PROBLEMS IN A FINITE STRIP

Uniqueness proofs for integral equation formulations of scattering by unbounded surfaces, in common with those for scattering by bounded obstacles (see, for example, Reference [3]), depend not only on the uniqueness result for the original scattering problem but also on the unique solvability of an associated homogeneous boundary value problem on the other side of the scattering surface (the interior of the scatterer in the bounded obstacle case). Motivated by this we establish in this appendix a uniqueness result for a class of homogeneous mixed boundary value problems in the finite strip $D_- := U_0 \backslash \bar{D}$ of the upper half-plane, which lies below the scattering surface Γ.

Let $\mathscr{R}(D_-)$ denote the set of functions $v \in C^2(D_-) \cap C^1(D_- \cup \Gamma_0) \cap C(\bar{D}_-)$ for which the normal derivative defined by $\partial v/\partial v(x) := \lim_{h \to 0+} v(x) \cdot \nabla v(x + hv(x))$ exists uniformly for x in any compact subset of Γ. We consider in this appendix the following homogeneous mixed problem (MP): find $u \in \mathscr{R}(D_-) \cap BC(\bar{D}_-)$ such that:

(i) u is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } D_- \tag{C1}$$

(ii) For some $\theta \in (0, 1)$ and some constant $C_\theta > 0$,

$$|\nabla u(x)| \leqslant C_\theta [f(x_1) - x_2]^{\theta - 1} \tag{C2}$$

for $x \in D_-$;

(iii) u satisfies one of the following sets of boundary conditions:

(a) $\partial u/\partial v + \mathrm{i}\eta u = 0$ on Γ and $u = 0$ on Γ_0;
(b) $u = 0$ on Γ and $\partial u/\partial x_2 + \mathrm{i}ku = 0$ on Γ_0;
(c) $\partial u/\partial v = 0$ on Γ and $\partial u/\partial x_2 + \mathrm{i}ku = 0$ on Γ_0;
(d) $\partial u/\partial v + \mathrm{i}\eta u = 0$ on Γ and $\partial u/\partial x_2 + \mathrm{i}ku = 0$ on Γ_0;

where $\eta \in \mathbf{C}$.

Remark C.1
If $u \in C^2(D_-) \cap C^1(D_- \cup \Gamma_0) \cap BC(\bar{D}_-)$ satisfies (C1) and the boundary condition $u = 0$ on Γ then, by standard local elliptic regularity results [31], it follows that $u \in C^1(\bar{D}_-)$, so that automatically $u \in \mathscr{R}(D_-)$ holds. Further, by arguing exactly as in the proof of [6, Theorem 3.1], we can show that (C2) holds with $\theta > 1/2$. Thus (C2) is superfluous in this case and the assumption that $u \in \mathscr{R}(D_-) \cap BC(\bar{D}_-)$ can be replaced by the weaker assumption that $u \in C^2(D_-) \cap C^1(D_- \cup \Gamma_0) \cap BC(\bar{D}_-)$.

The main result of this appendix is the following theorem.

Theorem C.1
Let $\Re(\bar{k}\eta) > 0$. Then the mixed boundary value problem (MP) has only the trivial solution.

The following lemma is a first step in the proof of Theorem C.1, and indeed establishes this theorem in the case $\Im k > 0$.

Lemma C.2
Let $\Re(\bar{k}\eta) > 0$. Suppose u is a solution of the mixed problem (MP). Then $u \in L^2(\Gamma)$ if condition (iii)(a) is satisfied, $u \in L^2(\Gamma_0)$ if either condition (iii)(b) or condition (iii)(c) is satisfied, and $u \in L^2(\Gamma) \cap L^2(\Gamma_0)$ if condition (iii)(d) is satisfied. Furthermore, if $\Im(k) > 0$ then $u = 0$ in D_- in each case.

Proof
Consider first the case when condition (iii)(a) is satisfied.
Let, for $A > 0$ and δ in the range $0 < \delta < f_-$,

$$D_\delta(A) := \{x \in D(A) | 0 < x_2 < f(x_1) - \delta\}$$

where $D(A) := \{x \in D_- \mid |x_1| < A\}$. Apply Green's first Theorem to u and $\bar{u}$ in $D_\delta(A)$ to obtain that

$$\int_{D_\delta(A)} [|\nabla u|^2 - k^2 |u|^2]\,\mathrm{d}x = \int_{\partial D_\delta(A)} \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}s \tag{C3}$$

where $\partial/\partial n$ is the partial derivative in the normal direction directed out of $D_\delta(A)$. Let $T_A := \partial D(A) \backslash (\Gamma \cup \Gamma_0)$. Then, from (C2) and since $u \in BC(D_-)$, we deduce that $\int_{T_A} |u \partial \bar{u}/\partial n|\,\mathrm{d}s$ exists for all $A > 0$ and is bounded uniformly in A. Further, it is easy to see that there exists a constant $C_1 > 0$ such that, for all $\delta \in (0, f_-)$ and $x \in \Gamma_A^\delta := \{(x_1, f(x_1) - \delta) \mid |x_1| \leqslant A\}$, there exists $x_* \in \Gamma$ with $x = x_* + h\nu(x_*)$ and with $0 < h \leqslant C_1 \delta$. Furthermore, since $f \in C^{1,1}(\mathbb{R})$, $|\nu(x^*) - \nu(x_*)| \leqslant C_2 \delta$ for some constant $C_2 > 0$ independent of x, where $x^* = (x_1, f(x_1))$. Thus, for $x \in \Gamma_A^\delta$, we have, noting the definition of $\partial u/\partial \nu$ and the fact that $\partial u/\partial \nu = -\mathrm{i}\eta u$ on Γ and since $|\nabla u(x) \cdot (\nu(x^*) - \nu(x_*))| \leqslant C_1 C_2 \delta^\theta$, that

$$-\frac{\partial u}{\partial n}(x) = \nabla u(x_* + h\nu(x_*)) \cdot \nu(x_*) + \nabla u(x) \cdot (\nu(x^*) - \nu(x_*)) \to -\mathrm{i}\eta u(x^*)$$

as $\delta \to 0$, uniformly in $x \in \Gamma_A^\delta$. Thus, multiplying (C3) by $-\bar{k}$ and taking the imaginary part and then the limit as $\delta \to 0$, noting that $u = 0$ on Γ_0, we obtain that

$$\begin{aligned}\Im(k) \int_{D(A)} [|\nabla u|^2 + |k|^2 |u|^2]\,\mathrm{d}x + \Re(\bar{k}\eta) \int_{\Gamma(A)} |u|^2\,\mathrm{d}s &= \Im\left(-\bar{k} \int_{T_A} \bar{u}\,\frac{\partial u}{\partial n}\,\mathrm{d}s\right) \\ &\leqslant |k| \sup_{x \in T_A} |u(x)| \int_{T_A} \left|\frac{\partial u}{\partial n}\right|\,\mathrm{d}s \end{aligned} \tag{C4}$$

which is bounded as $A \to \infty$. Thus $u \in L^2(\Gamma)$. Further, if $\Im(k) > 0$ then, by (C4), $\nabla u \in L^2(D_-)$, which implies that there exists a positive sequence (A_n) with $A_n \to \infty$ as $n \to \infty$, such that $\int_{T_{A_n}} |\partial u/\partial n|\,\mathrm{d}s \to 0$ as $n \to \infty$. Letting $A \to \infty$ through this sequence in (C4), we see that $u = 0$ in D_-. The proof for case (iii)(a) is thus complete.

The other cases can be proved similarly, with (C4) replaced by

$$\Im(k) \int_{D(A)} [|\nabla u|^2 + |k|^2 |u|^2]\,\mathrm{d}x + |k|^2 \int_{\Gamma_0(A)} |u|^2\,\mathrm{d}s \leqslant |k| \sup_{x \in T_A} |u(x)| \int_{T_A} \left|\frac{\partial u}{\partial n}\right|\,\mathrm{d}s \tag{C5}$$

in the cases (iii)(b) and (iii)(c). □

To prove Theorem C.1 we also need the following integral representations of solutions to problem (MP) which can be obtained similarly to the derivation of (C4) and (27).

Lemma C.3

Let u be a solution of the mixed problem (MP). Then

$$u(x) = -\int_\Gamma \left[\frac{\partial G(x, y)}{\partial \nu(y)} + \mathrm{i}\eta G(x, y)\right] u(y)\,\mathrm{d}s(y), \quad x \in D_- \tag{C6}$$

if the boundary conditions (iii)(a) are satisfied;

$$u(x) = \int_\Gamma G_i(x,y)\,\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}s(y), \quad x \in D_- \tag{C7}$$

if the boundary conditions (iii)(b) are satisfied;

$$u(x) = -\int_\Gamma \frac{\partial G_i(x,y)}{\partial \nu(y)} u(y)\,\mathrm{d}s(y), \quad x \in D_- \tag{C8}$$

if the boundary conditions (iii)(c) are satisfied; and

$$u(x) = -\int_\Gamma \left[\frac{\partial G_i(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G_i(x,y)\right] u(y)\,\mathrm{d}s(y), \quad x \in D_- \tag{C9}$$

if the boundary conditions (iii)(d) are satisfied.

Lemma C.4
Let $k>0$. If u is a solution of the mixed problem (MP) with the boundary condition (iii)(b), then $\partial u/\partial \nu \in L^2(\Gamma)$.

Proof
Note that, by Remark C.1, $u \in C^2(D_-) \cap C^1(\bar{D}_-)$. Multiply (C1) by $2\partial_2 \bar{u}$, integrate over $D(A)$, and take the real part of the equation thus obtained. Noting that $2\Re[\partial_2\bar{u}(\Delta u + k^2 u)] = 2\Re[\nabla \cdot (\partial_2 \bar{u} \nabla u)] - \partial_2(|\nabla u|^2) + k^2 \partial_2(|u|^2)$, we find, on applying the divergence theorem in $D_-(A)$, that

$$\int_{\Gamma(A)} [\nu_2 |\nabla u|^2 - 2\Re(\partial_2 \bar{u} \partial_\nu u)]\,\mathrm{d}s + \int_{\Gamma_0(A)} (|\partial_1 u|^2 - 2k^2|u|^2)\,\mathrm{d}s = -2\Re \int_{T_A} \partial_2 \bar{u} \partial_n u\,\mathrm{d}s \tag{C10}$$

where $\partial_n := \partial/\partial n$ is the partial derivative in the direction of the normal directed out of $D_-(A)$, $\partial_\nu := \partial/\partial \nu = -\partial/\partial n$ on Γ, and $\nu = (\nu_1, \nu_2)$. As $u=0$ on Γ, we have that $\partial_2 u = \nu_2 \partial_\nu u$ and $|\nabla u| = |\partial_\nu u|$ on Γ. It follows from (C.10), since $\nu_2 = -(1+|f'|^2)^{-1/2} \leqslant -[1 + \|f\|^2_{C^{1,1}(\mathbb{R})}]^{-1/2} =: -C_f$ on Γ, that

$$\begin{aligned} C_f \int_{\Gamma(A)} |\partial_\nu u|^2\,\mathrm{d}s &\leqslant -\int_{\Gamma(A)} \nu_2 |\partial_\nu u|^2\,\mathrm{d}s \\ &\leqslant 2k^2 \int_{\Gamma_0(A)} |u|^2\,\mathrm{d}s - 2\Re \int_{T_A} \partial_2 \bar{u} \partial_n u\,\mathrm{d}s \end{aligned} \tag{C11}$$

Now, by Remark C.1, u satisfies (C2) with $\theta > 1/2$, so that $|\int_{T_A} \partial_2 \bar{u} \partial_n u\,\mathrm{d}s|$ exists for all $A>0$ and is bounded uniformly in A. Thus, and since, by Lemma C.2, $u \in L^2(\Gamma_0)$, it follows from (C11) that $\partial_\nu u \in L^2(\Gamma)$. The lemma is proved. □

Proof of Theorem C.1
Note first that the case when condition (iii)(c) is satisfied can be proved in exactly the same way as in the proof of Chandler-Wilde *et al.* [5, Theorem 5.1] by utilizing (C8). So we only consider the other three cases. Also, in view of Lemma C.2 we only need to consider

the $k>0$ case. Our proof will make use of the integral representation results (Lemma C.3) combined with the fact that $u\in L^2(\Gamma)$ for the cases when conditions (iii)(a) or (iii)(d) hold (see Lemma C.4) and $\partial u/\partial \nu\in L^2(\Gamma)$ for the case with condition (iii)(b) (Lemma C.4). We prove the theorem for the case with condition (iii)(a). Other cases can be shown similarly, for example making use of (C5), (C7), and (C11) in the case (iii)(b).

For $x\in D_-$ with $|x_1|\geqslant 1$, we have, by (C6) and (8) along with the Cauchy–Schwarz inequality,

$$\begin{aligned}|u(x)|^2 &\leqslant 2\left\{\int_{\Gamma\setminus\Gamma(|x_1|/2)} |u(y)|\left|\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right| \mathrm{d}s(y)\right\}^2 \\ &\quad + 2\left\{\int_{\Gamma(|x_1|/2)} |u(y)|\left|\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right| \mathrm{d}s(y)\right\}^2 \\ &\leqslant C_1\int_{\Gamma\setminus\Gamma(|x_1|/2)} |u|^2\,\mathrm{d}s + C_2\left\{\frac{|x_1|}{2}\right\}^{-3}\int_\Gamma |u|^2\,\mathrm{d}s\end{aligned}$$

where

$$C_1 = 2\sup_{x\in D_-}\int_\Gamma \left|\frac{\partial G(x,y)}{\partial \nu(y)} + \mathrm{i}\eta G(x,y)\right|^2 \mathrm{d}s(y) < \infty$$

by (8). Thus, $u(x)\to 0$ as $x_1\to\infty$ with $x\in D_-$, uniformly in x_2. From this and (C4) it follows that $u=0$ on Γ and hence, from (C6), that $u\equiv 0$ in D_-. Theorem C.1 is thus complete.

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