Control Synthesis of T–S Fuzzy Systems Based on a New Control Scheme

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***Abstract*—This paper studies the control synthesis problem of Takagi–Sugeno (T–S) fuzzy systems. By splitting the premise variable spaces and using the properties of fuzzy sets, a new control scheme is proposed based on a new class of fuzzy Lyapunov functions, and a convex condition for designing fuzzy controllers is given, where the new fuzzy Lyapunov functions and fuzzy controllers are constructed based on the split subspaces. In particular, some existing fuzzy Lyapunov functions and control schemes are special cases of the new Lyapunov function and control scheme, respectively. Numerical examples are given to illustrate the effectiveness of the proposed method.**

***Index Terms*—Fuzzy control, fuzzy Lyapunov function, linear matrix inequalities (LMIs), nonlinear systems, Takagi–Sugeno (T–S) fuzzy models.**

I. Introduction

IN THE NONLINEAR control area, there is no systematic mathematical technique to obtain necessary and sufficient conditions to guarantee the stability and performance of nonlinear systems. In general, control of nonlinear systems is often very difficult, and various control methods have been exploited for the nonlinear control systems [1]. In particular, an important approach to nonlinear control system design is to model the considered nonlinear systems as Takagi and Sugeno (T–S) fuzzy systems, which are locally linear time-invariant systems connected by IF–THEN rules. As a result, the conventional linear system theory can be applied for analysis and synthesis of the nonlinear control systems. In recent years, control synthesis problems of T–S fuzzy systems have been well studied, where quadratic Lyapunov function approaches [2]–[8] are widely employed. Since a common Lyapunov matrix is used for all local models of fuzzy systems, the quadratic Lyapunov function approach often leads to conservative results. Then, parameter-dependent Lyapunov functions (or called fuzzy Lyapunov functions) [9]–[12], piecewise Lyapunov functions [13], [14], and k-sample variation Lyapunov functions [15] are, respectively, proposed to reduce the conservatism introduced by using quadratic Lyapunov functions. In most of the fuzzy control designs based on T–S fuzzy models, the parallel distributed compensation (PDC) control scheme in [16], i.e., the controller shares the same fuzzy rules with the considered fuzzy model, plays an important role. In addition, a number of alternative control schemes, such as the non-PDC control scheme in [11], the switching constant controller gain scheme in [17], and the switching PDC (SPDC) control scheme in [18] and [19], are also developed for designing fuzzy controllers.

Although many important progresses have been achieved in the fuzzy control area, the properties about the structure or shape of membership functions are often neglected in some of the literature. Therefore, a great deal of effort has recently been devoted to exploiting the properties about the structure or shape of membership functions for less-conservative results [7], [20]–[30]. In [24], by using the knowledge of the membership functions' shape to introduce slack variables, relaxed stability conditions are presented. A systematic design approach of T–S fuzzy control systems is presented by searching a common positive-definite symmetric matrix in each maximal overlapped-rules group of fuzzy rules in [28]. By separating the original plant rules into several fuzzy regions, the T–S fuzzy region control approach and the regional-membership-function-shape-dependent approach are, respectively, proposed in [25] and [30]. By exploiting the dependence of the stability upon membership functions, a fuzzy control design approach is given based on Kharitonov's theorem in [23]. In [26], a line-integral function is introduced as a fuzzy Lyapunov function without the association with the time derivatives of membership functions, and then, relaxed stability conditions are achieved. Moreover, by exploiting the properties of multidimensional fuzzy summation, relaxed stability analysis and synthesis conditions are proposed in [6], [7], and [31], and an asymptotically necessary and sufficient condition is achieved in [6]. In particular, Cao *et al.* [20]–[22] first partition the premise variable space into some subspaces by fuzzy membership functions, and then, some relaxed stability analysis and synthesis conditions are presented based on piecewise Lyapunov functions.

Motivated by these works, where the properties about the structure or shape of membership functions are exploited for less-conservative results, this paper will further study the fuzzy control design technique by exploiting some new properties about the structure and shape of membership functions.

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According to the role of fuzzy sets, the premise variable space is split into a set of subspaces, where there is one and only one fuzzy set playing a dominant role on each premise variable v_j-axis. By switching the parameters of a class of matrix functions with some special constraints between the split subspaces, the continuity of the class of matrix functions can be guaranteed. Further, by using the class of matrix functions, a new fuzzy Lyapunov function (which are known as dominant fuzzy Lyapunov functions) and a new control scheme can be obtained. Then, the proposed technique can continuously switch Lyapunov functions and control gains based on the role of the different fuzzy sets, which has the potential to give less-conservative results. The fact will be illustrated by numerical examples. In particular, the fuzzy Lyapunov functions in [9] and [32] and the non-PDC control scheme in [11] can be viewed as special cases of the dominant fuzzy Lyapunov functions and the new proposed control scheme, respectively. The comparison will be given to illustrate the effectiveness of the new technique by numerical examples.

The rest of this paper is organized as follows. T–S fuzzy models are given by multiple-dimensional summations in Section II. Based on the split subspaces of the premise variable space, a new class of continuous functions, which is used to obtain new Lyapunov functions and control schemes, are constructed in Section III. In Section IV, a convex control design condition is presented. Section V gives three numerical examples to illustrate the effectiveness of the proposed method. Section VI concludes the paper.

Notation: For a square matrix E, $\mathtt{He}(E)$ is defined as $\mathtt{He}(E) = E + E^T$, and E^{-T} denotes $(E^{-1})^T$ if E is nonsingular.

II. System Descriptions

The nonlinear systems under consideration is described by the following fuzzy system model:

Plant Rule $(i_1 i_2 \ldots i_p)$:

IF $v_1(t)$ is M_{1i_1} and $v_2(t)$ is $M_{2i_2}, \ldots, v_p(t)$ is M_{pi_p}

$$\text{THEN } \dot{x}(t) = A_{i_1 i_2 \ldots i_p} x(t) + B_{i_1 i_2 \ldots i_p} u(t) \tag{1}$$

$x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $v(t) = [v_1(t)\ v_2(t)\ \ldots\ v_p(t)]^T \in \mathbb{R}^p$ are the premise variables and assumed to be measurable, and M_{ij} denotes an $v_j(t)$-based fuzzy set. Let r_j be the number of $v_j(t)$-based fuzzy sets. Then, $r = \Pi_{i=1}^{p} r_i$ is the number of IF–THEN rules.

Given a pair of $(x(t), u(t))$, by using the fuzzy inference method with a singleton fuzzifier, product inference, and center average defuzzifiers, the final output of the T–S fuzzy model is obtained as in (2), shown at the bottom of the page.

In order to make full use of the properties of fuzzy membership functions, the T–S fuzzy model is rewritten as follows:

$$\dot{x}(t) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} (\Pi_{j=1}^{p} \mu_{ji_j}(v_j(t))) \times (A_{i_1 i_2 \ldots i_p} x(t) + B_{i_1 i_2 \cdots i_p} u(t)) \tag{3}$$

where

$$\mu_{ji_j}(v_j(t)) = \frac{M_{ji_j}(v_j(t))}{\sum_{l_j=1}^{r_j} M_{jl_j}(v_j(t))}. \tag{4}$$

Remark 1: In (3), the fuzzy system is represented by multidimensional fuzzy summations. The description is helpful for making full use of the properties of fuzzy membership functions, and the description in (3) has been used in [7], [26], and [31] etc., where many special properties of fuzzy membership functions are exploited to obtain relaxed analysis and synthesis conditions.

In this paper, we consider a particular type of fuzzy system models, which is often used to represent nonlinear systems. In the fuzzy model, it is assumed that the v_j-based fuzzy sets M_{ji_j}, $1 \le i_j \le r_j$ are normal, consistent, and complete in $W_j \subset R$ with pseudotrapezoid membership functions, where W_j is the universe of discourse on the v_j-axis, which contains all the possible elements of concern in each particular context or application on the premise variable $v_j(t)$. In particular, some familiar membership functions (triangular, trapezoidal, or Gaussian membership functions, etc.) are all pseudotrapezoid membership functions [33]. The concepts and properties about fuzzy sets are shown in the Appendix, or see [33].

Then, these types of fuzzy systems have the following nice properties.

1) By virtue of Lemma 3 in the Appendix, we can always assume that $M_{j1} < M_{j2} < \cdots < M_{jr_j}$, where the signification "<" refers to Definition 1(v).
2) According to Lemma 3 (ii) in the Appendix, there are at most two fuzzy sets that are fired on the v_j-axis at some moment.
3) From (4), we have that $0 \le \mu_{ji_j} \le 1$, $\sum_{i_j=1}^{r_j} \mu_{ji_j} = 1$.

In the following, we further illustrate these properties with a concrete example. For simplicity of explanation, a scale nonlinear system $\dot{x} = f(x)x$ is considered, where $f(x)$ is a nonlinear function on the compact set $W = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$. To approximate $f(x)$, define two sets of normal, consistent, complete fuzzy sets M_{1i}, $1 \le i \le 3$ in $[\alpha_1, \alpha_2]$ and M_{2i}, $1 \le i \le 3$ in $[\beta_1, \beta_2]$ with pesudotrapezoid membership functions (see Fig. 1). Then, a fuzzy system with nine rules can be constructed to approximate $f(x)$ as follows:

Rule $(i_1 i_2)$:

IF $v_1(t)$ is M_{1i_1} and $v_2(t)$ is M_{2i_2}

$$\text{THEN } \dot{x}(t) = A_{i_1 i_2} x(t). \tag{5}$$

$$\dot{x}(t) = \frac{\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} (\Pi_{j=1}^{p} M_{ji_j})(A_{i_1 i_2 \ldots i_p} x(t) + B_{i_1 i_2 \ldots i_p} u(t))}{\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \Pi_{j=1}^{p} M_{ji_j}}. \tag{2}$$

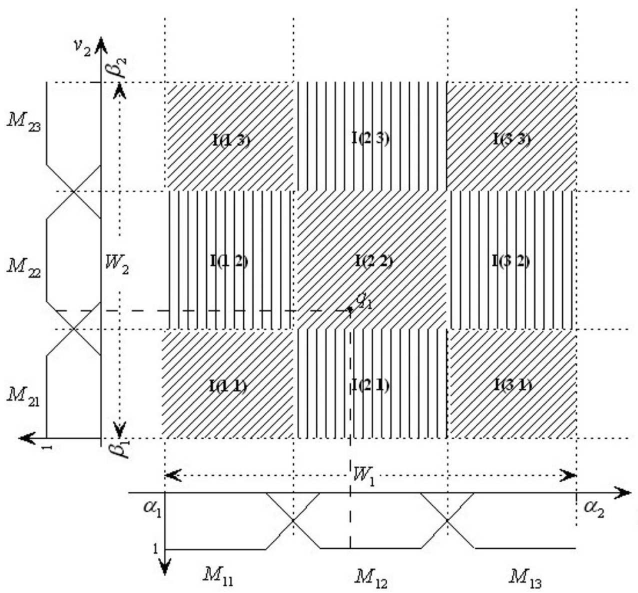


Fig. 1. Normal, consistent, and complete fuzzy sets in v_1- and v_2-axes.

Using a singleton fuzzifier, product inference, and center average defuzzifiers

$$\dot{x}(t) = \sum_{i_1=1}^{3}\sum_{i_2=1}^{3} \mu_{1i_1}(v_1(t))\mu_{2i_2}(v_2(t))A_{i_1 i_2}x(t) \tag{6}$$

where $\mu_{li_l}(v_l(t))$ are the grade of membership of $v_l(t)$ in M_{li_l}, $l = 1, 2$. Among v_1-based fuzzy sets, the overlapped fuzzy sets can be M_{11} and M_{12} or M_{12} and M_{13}, which implies that $\mu_{11}(v_1(t)) + \mu_{12}(v_1(t)) = 1$ or $\mu_{12}(v_1(t)) + \mu_{13}(v_1(t)) = 1$.

Given a point in the universe of discourse $W = W_1 \times W_2$, there are at most two fuzzy sets firing on the v_1-axis (v_2-axis). Consider that the area $I(21)$, (M_{11}, M_{12}) or M_{12} or (M_{12}, M_{13}) are fired on v_1-axis at the same moment, which implies that $\mu_{11} + \mu_{12} = 1$ or $\mu_{12} = 1$, or $\mu_{12} + \mu_{13} = 1$ at some moment when the premise variable $v(t) \in I(21)$.

In the next section, these properties will be used to construct a class of continuous functions.

III. Construct Continuous Functions to Obtain New Lyapunov Functions and Control Schemes

A. Class of Continuous Functions on the Split Subspaces

Assume a T–S fuzzy system is constructed by normal, consistent, and complete fuzzy sets with pseudotrapezoid membership functions, the premise variable is p dimensions, and the premise variable space is denoted as $W = W_1 \times W_2 \cdots \times W_p$. Then, for a point belonging to W, there exists a fuzzy set M_{ji_j} on the v_j-axis such that M_{ji_j} plays a dominant role. For example, consider the point q_1 in Fig. 1; it can be seen that M_{12} plays a dominant role on the v_1-axis, and M_{22} plays a dominant role on the v_2-axis. Then, according to the dominant role of fuzzy sets, we can split the space $W = W_1 \times W_2$ in Fig. 1 into nine subspaces, i.e., $I(ij)$, $1 \le i, j \le 3$. For the all points in $I(ij)$, the fuzzy sets M_{1i} on the v_1-axis and M_{2j} on the v_2-axis play the dominant role. Note that there are at most two fuzzy sets simultaneously firing in one axis, and the sum of the two corresponding membership functions is equal to 1. Then, the space is spilt by whether or not the value of the fuzzy membership function on v_1- and v_2-axes is equal to 0.5 (see Fig. 1).

For the general case that p premise variables are considered, we can split the premise variable space W as follows:

$$I(i_1 i_2 \cdots i_p) = \left\{ v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_p(t) \end{bmatrix} : \mu_{ji_j}(v_j(t)) \ge 0.5, 1 \le j \le p \right\}.$$

Obviously

$$W = \bigcup_{\substack{1 \le i_1 \le r_1 \\ \vdots \\ 1 \le i_p \le r_p}} I(i_1 i_2 \cdots i_p). \tag{7}$$

Moreover, before constructing continuous functions, some new significations are also given for clear descriptions.

Denote

$$i_1 = i_{1\to 1}, i_1 i_2 \cdots i_{k-1} i_k = i_{1\to k}, 2 \le k \le p \tag{8}$$

$$\begin{aligned}
&U_D(i_{1\to k}) = \{l_{1\to k} m_{1\to k} || l_\delta - i_\delta| \le 1, |m_\delta - l_\delta| \le 1 \\
&|m_\delta - i_\delta| \le 1, l_\delta, m_\delta \in \{1, 2, \cdots, r_\delta\}, 1 \le \delta \le k\} \\
&2 \le k \le p
\end{aligned} \tag{9}$$

$$\begin{aligned}
&U_D^0(i_{1\to k}) = \{l_{1\to k} m_{1\to k} || l_\delta - i_\delta| \le 1, |m_\delta - l_\delta| \le 1 \\
&|m_\delta - i_\delta| \le 1, l_\delta, m_\delta \in \{1, 2, \cdots, r_\delta\}, 1 \le \delta \le k \\
&l_k - m_k = -1\}, 2 \le k \le p
\end{aligned} \tag{10}$$

$$\begin{aligned}
&U_D^k(i_k) = \{l_k m_k || l_k - i_k| \le 1, |m_k - i_k| \le 1 \\
&|l_k - m_k| \le 1, l_k, m_k \in \{1, 2, \cdots, r_k\}\}, 1 \le k \le p \\
&U_D^{k0}(i_k) = \{l_k m_k || l_k - i_k| \le 1, |m_k - i_k| \le 1 \\
&l_k - m_k = -1, l_k, m_k \in \{1, 2, \cdots, r_k\}\}, 1 \le k \le p \\
&U_S(i_{1\to k}) = \{l_{1\to k} || l_\delta - i_\delta| \le 1, l_\delta \in \{1, 2, \cdots, r_\delta\} \\
&1 \le \delta \le k\}, 2 \le k \le p \\
&U_S^k(i_k) = \{l_k || l_k - i_k| \le 1, l_k \in \{1, 2, \cdots, r_k\}\} \\
&1 \le k \le p.
\end{aligned} \tag{11}$$

Remark 2: Note that we define some notations in the earlier discussion, for example, $U_D(i_{1\to k})$, $U_D^0(i_{1\to k})$, etc. These special notations are given to obtain compact descriptions and clear proofs of the subsequent lemmas and theorems.

In what follows, these notations are further explained by the two examples in Figs. 1 and 10. In Fig. 10, the space is split as three subspaces $I(1)$, $I(2)$, and $I(3)$. $U_S^1(i_1)|_{i_1=1} = \{1,2\}$, $U_S^1(i_1)|_{i_1=2} = \{1,2,3\}$, $U_S^1(i_1)|_{i_1=3} = \{2,3\}$, and $U_D^1(i_1)|_{i_1=1} = \{11,12,21,22\}$, $U_D^1(i_1)|_{i_1=2} = \{11,12,21,22,23,32,33\}$, $U_D^1(i_1)|_{i_1=3} = \{22,23,32,33\}$.

In Fig. 1, the given universe of discourse $W = W_1 \times W_2$, where $W_1 = [\alpha_1, \alpha_2]$, and $W_2 = [\beta_1, \beta_2]$. The space W is split into nine subspaces, i.e., $I(11)$, $I(12)$, $I(13)$, $I(21)$, $I(22)$, $I(23)$, $I(31)$, $I(32)$, $I(33)$. $U_D(i_1 i_2)|_{i_1=1, i_2=1} = \{1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222\}$.

Note that the $v_j(t)$-based fuzzy sets M_{ji_j}, $1 \le i_j \le r_j$ are normal, consistent, and complete in $W_j \subset R$ with pseudo-trapezoid membership functions; therefore, at some moment, if the premise variables $v(t) = [\, v_1(t) \quad v_2(t) \quad \cdots \quad v_p(t)\,] \in I(i_1 i_2 \cdots i_p)$, then only the M_{jl_j} with $|l_j - i_j| \le 1$ on the $v_j(t)$-axis can be fired. The fact implies that $\mu_{jl_j}(v_j(t))$ with $|l_j - i_j| \le 1$, i.e., the grade of membership of $v_j(t)$ in M_{jl_j}, can be fired and the $\mu_{jl_j}(v_j(t)) = 0$ with $|l_j - i_j| > 2$. Thus, for $v(t) \in I(i_1 i_2 \cdots i_p)$

$$\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} (\Pi_{j=1}^{p} \mu_{ji_j}) A_{i_1 i_2 \cdots i_p} = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1} \mu_{2l_2} \cdots \mu_{pl_p} A_{l_{1\to p}}.$$

Then, the system (3) can be described by the following compact form:

$$\dot{x}(t) = A(\mu)x(t) + B(\mu)u(t)) \tag{12}$$

where

$$A(\mu) = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1} \mu_{2l_2} \cdots \mu_{pl_p} A_{l_{1\to p}}$$
$$B(\mu) = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1} \mu_{2l_2} \cdots \mu_{pl_p} B_{l_{1\to p}}$$
$$\text{for } v(t) \in I(i_1 i_2 \cdots i_p). \tag{13}$$

Now, based on the split subspaces of W, a class of continuous functions are constructed as follows:

$$F(\mu(v(t))) = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1} \mu_{2l_2} \cdots \mu_{pl_p} F_{l_{1\to p}}^{(i_{1\to p})}$$
$$\text{for } v(t) \in I(i_1 i_2 \cdots i_p) \tag{15}$$

whose parameters satisfy the following equalities:

$$\sum_{l_\tau \in \{i_\tau, i_\tau+1\}} F_{l_{1\to p}}^{(i_{1\to p})} = \sum_{l_\tau \in \{i_\tau, i_\tau+1\}} F_{l_{1\to p}}^{(i_1 \cdots i_{\tau-1}(i_\tau+1) i_{\tau+1} \cdots i_p)}$$
$$\text{for } 1 \le \tau \le p, 1 \le i_\tau \le r_\tau - 1 \tag{15}$$

where $F_{l_{1\to p}}^{(i_{1\to p})}$, $l_{1\to p} \in U_S(i_{1\to p})$ are constant matrices.

The continuity of the function (14) can be proved by the following theorem.

Theorem 1: The function (14) with (15) is continuous.

Proof: From the definition of $F(\mu(v(t)))$ in (14), it follows that the function $F(\mu(v(t)))$ is continuous if the continuity of the function can be guaranteed on the bound of $I(i_1 i_2 \cdots i_p)$. In what follows, we will show that the function is continuously across the bound.

Assume the function $F(\mu(v(t)))$ goes into $I(j_1 j_2 \cdots j_p)$ from $I(i_1 i_2 \cdots i_p)$; then, their intersection is a no-empty set. In order to give a simple proof, we also assume that

$$|i_\tau - j_\tau| = \begin{cases} 1, & 1 \le \tau \le k \\ 0, & k+1 \le \tau \le p. \end{cases}$$

For the other cases, the same technique can be used.

When $v(t) \in I(i_1 i_2 \cdots i_p)$ and reaches the bound, there are only two fuzzy membership functions, i.e., $\mu_{\tau i_\tau}(v_\tau(t)) = \mu_{\tau j_\tau}(v_\tau(t)) = 0.5$, that are fired on each v_τ-axis $1 \le \tau \le k$. Then, the function $F(\mu(v(t)))$ can be written as follows:

$$\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(i_1 i_2 i_3 \cdots i_p)}. \tag{16}$$

When $v(t)$ leaves the bound and enters into the subspace $I(j_1 j_2 \cdots j_p)$, the function $F(\mu(v(t)))$ is the following form:

$$\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 j_2 j_3 \cdots j_p)}. \tag{17}$$

Now, we will show that (16) is equal to (17), which implies that the function $F(\mu(v(t)))$ is continuously across the bound.

Consider (16), we have

$$\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(i_1 i_2 i_3 \cdots i_p)}$$
$$= \sum_{\substack{l_2 \in \{i_2, j_2\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} \sum_{l_1 \in \{i_1, j_1\}} F_{l_1 l_2 l_3 \cdots l_p}^{(i_1 i_2 i_3 \cdots i_p)}.$$

Combining it, (15), and $|i_1 - j_1| = 1$ yields

$$\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(i_1 i_2 i_3 \cdots i_p)}$$

$$
\begin{aligned}
&= \sum_{\substack{l_2 \in \{i_2, j_2\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} \sum_{l_1 \in \{i_1, j_1\}} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 i_2 i_3 \cdots i_p)} \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 i_2 i_3 \cdots i_p)}.
\end{aligned}
$$

Further, on applying the same technique to the earlier equality, then it follows that

$$
\begin{aligned}
&\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 i_2 i_3 \cdots i_p)} \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ l_3 \in \{i_3, j_3\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} \sum_{l_2 \in \{i_2, j_2\}} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 i_2 i_3 \cdots i_p)} \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ l_3 \in \{i_3, j_3\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} \sum_{l_2 \in \{i_2, j_2\}} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 j_2 i_3 \cdots i_p)} \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 j_2 i_3 \cdots i_p)} \\
&\ \vdots \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 \cdots j_k i_{k+1} \cdots i_p)}.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&\sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(i_1 \cdots i_k i_{k+1} \cdots i_p)} \\
&= \sum_{\substack{l_1 \in \{i_1, j_1\} \\ \vdots \\ l_k \in \{i_k, j_k\} \\ l_{k+1} \in U_S^{k+1}(i_{k+1}) \\ \vdots \\ l_p \in U_S^p(i_p)}} 0.5^k \mu_{(k+1)l_{k+1}} \cdots \mu_{pl_p} F_{l_1 l_2 l_3 \cdots l_p}^{(j_1 \cdots j_k i_{k+1} \cdots i_p)}.
\end{aligned}
$$

Note that $|i_\tau - j_\tau| = 0,\ k+1 \le \tau \le p$; then, it follows that (16) is equal to (17) from the aforementioned equality, which implies that the function $F(\mu(v(t)))$ is continuously across the bound. Thus, the proof is complete. ■

Remark 3: Note that the function (14) is a summation of some continuous functions $\mu_{1l_1}\mu_{2l_2}\cdots\mu_{pl_p}F_{l_{1\to p}}^{(i_{1\to p})}$ in one subregion of the premise variable space, which implies that the function (14) is continuous in each subregion of the premise variable space. Moreover, from the Proof of Theorem 1, the condition (15) guarantees that the function (14) is continuously across the bound. Therefore, the function (14) is continuous in the global variable space.

A simple example is introduced further to illustrate the conclusion of Theorem 1. Consider Fig. 10, and denote $\mu_{1i}(v_1(t))$ as the grade of membership of $v_1(t)$ in A^i, $i = 1, 2, 3$. Then, (14) and (15) are of the following forms:

$$
F(\mu(v(t))) = \begin{cases} \mu_{11}(v_1(t))F_1^{(1)} + \mu_{12}(v_1(t))F_2^{(1)}, & v_1(t) \in I(1) \\ \mu_{11}(v_1(t))F_1^{(2)} + \mu_{12}(v_1(t))F_2^{(2)} \\ \quad + \mu_{13}(v_1(t))F_3^{(2)}, & v_1(t) \in I(2) \\ \mu_{12}(v_1(t))F_2^{(3)} + \mu_{13}(v_1(t))F_3^{(3)}, & v_1(t) \in I(3) \end{cases}
$$

with

$$
\begin{aligned}
F_1^{(1)} + F_2^{(1)} &= F_1^{(2)} + F_2^{(2)} \\
F_2^{(2)} + F_3^{(2)} &= F_2^{(3)} + F_3^{(3)}.
\end{aligned}
$$

When $v_1(t) \in I(1) \cap I(2)$, $\mu_{11}(v_1(t)) = \mu_{12}(v_1(t)) = 0.5$, and $\mu_{13}(v_1(t)) = 0$, then

$$\begin{aligned} &\mu_{11}(v_1(t))F_1^{(1)} + \mu_{12}(v_1(t))F_2^{(1)} \\ &\quad = 0.5(F_1^{(1)} + F_2^{(1)}) \\ &\quad = 0.5(F_1^{(2)} + F_2^{(2)}) \\ &\quad = \mu_{11}(v_1(t))F_1^{(2)} + \mu_{12}(v_1(t))F_2^{(2)} + \mu_{13}(v_1(t))F_3^{(2)}. \end{aligned}$$

When $v_1(t) \in I(2) \cap I(3)$, $\mu_{12}(v_1(t)) = \mu_{13}(v_1(t)) = 0.5$, and $\mu_{11}(v_1(t)) = 0$, then

$$\begin{aligned} &\mu_{11}(v_1(t))F_1^{(2)} + \mu_{12}(v_1(t))F_2^{(2)} + \mu_{13}(v_1(t))F_3^{(2)} \\ &\quad = 0.5(F_2^{(2)} + F_3^{(2)}) \\ &\quad = 0.5(F_2^{(3)} + F_3^{(3)}) \\ &\quad = \mu_{12}(v_1(t))F_2^{(3)} + \mu_{13}(v_1(t))F_3^{(3)}. \end{aligned}$$

Thus, the function $F(\mu(v(t)))$ is continuous.

B. New Control Scheme and Dominant Fuzzy Lyapunov Functions

In this section, based on the split subspaces of the premise variable space, a new class of continuously switching fuzzy Lyapunov functions are given. Because the premise variable space is split according to whether or not the fuzzy sets on each v_j-axis, $1 \le j \le p$, play the dominant roles, we call the new class of fuzzy Lyapunov functions dominant fuzzy Lyapunov functions in this paper. Moreover, a new continuously switching fuzzy control scheme is also presented.

Assume symmetric matrices $0 < Q_{l_1 l_2 \ldots l_p}^{(i_{1\to p})} \in R^{n\times n}$, $l_{1\to p} \in U_S(i_{1\to p})$; then from Theorem 1, it follows that

$$Q(\mu) = Q^{(i_{1\to p})}(\mu) = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1}\mu_{2l_2}\cdots\mu_{pl_p} Q_{l_{1\to p}}^{(i_{1\to p})}$$
$$\text{for } v(t) \in I(i_1 i_2 \cdots i_p)$$

with

$$\sum_{l_\tau \in \{i_\tau, i_\tau+1\}} Q_{l_{1\to p}}^{(i_{1\to p})} = \sum_{l_\tau \in \{i_\tau, i_\tau+1\}} Q_{l_{1\to p}}^{(i_1\cdots i_{\tau-1}(i_\tau+1)i_{\tau+1}\cdots i_p)}$$
$$\text{for } 1 \le \tau \le p, 1 \le i_\tau \le r_\tau - 1 \tag{18}$$

is continuous and $Q(\mu) > 0$ on

$$W = \bigcup_{\substack{1 \le i_1 \le r_1 \\ \vdots \\ 1 \le i_p \le r_p}} I(i_1 i_2 \cdots i_p).$$

Let

$$P(\mu) = Q^{-1}(\mu). \tag{19}$$

Then, $P(\mu) > 0$ is continuous on W. Therefore, we may choose $P(\mu)$ as a Lyapunov matrix.

Moreover, we also give a new continuously switching control scheme as follows.

Control scheme:

$$u(t) = K(\mu)x(t) \tag{20}$$

with $K(\mu) = F(\mu)Q^{-1}(\mu)$, and

$$F(\mu) = F^{(i_{1\to p})}(\mu(v(t))) = \sum_{l_{1\to p} \in U_S(i_{1\to p})} \mu_{1l_1}\mu_{2l_2}\cdots\mu_{pl_p} \times F_{l_{1\to p}}^{(i_{1\to p})}, \text{ for } v(t) \in I(i_1 i_2 \cdots i_p) \tag{21}$$

whose parameters satisfy the following equalities:

$$\sum_{l_\tau \in \{i_\tau, i_\tau+1\}} F_{l_{1\to p}}^{(i_{1\to p})} = \sum_{l_\tau \in \{i_\tau, i_\tau+1\}} F_{l_{1\to p}}^{(i_1\cdots i_{\tau-1}(i_\tau+1)i_{\tau+1}\cdots i_p)}$$
$$\text{for } 1 \le \tau \le p, 1 \le i_\tau \le r_\tau - 1.$$

Remark 4: (i) Based on Theorem 1, a new class of fuzzy Lyapunov matrices are proposed in this paper. For the split subspace $I(i_1 i_2 \cdots i_p)$, the membership functions $\mu_{1i_1}(v_1(t)) \ge 0.5$, $\mu_{2i_2}(v_2(t)) \ge 0.5, \ldots, \mu_{pi_p}(v_p(t)) \ge 0.5$, which implies that the fuzzy set $M_{\tau i_\tau}$ on v_τ-axis $1 \le \tau \le p$ plays a dominant role; therefore, we call it a dominant fuzzy Lyapunov matrix in this paper. If the matrices $Q_{l_{1\to p}}^{(i_{1\to p})} = Q_{l_{1\to p}}^{(11\cdots 1)}$ in (19), i.e., the $Q_{l_{1\to p}}^{(i_{1\to p})}$ will remain a constant in all the split subspaces, then the dominant fuzzy Lyapunov matrix is reduced to the fuzzy Lyapunov matrix. The fact implies that the condition based on the dominant fuzzy Lyapunov function has the potential to give less-conservative results.

(ii) Note that a new control scheme (20), which continuously switches gains on the bound of the split subspaces $I(i_1 i_2 \cdots i_p)$, is presented. If $F_{l_{1\to p}}^{(i_{1\to p})} = F_{l_{1\to p}}^{(j_{1\to p})}$, then the new control scheme can be reduced to the non-PDC control scheme in [11].

IV. Control Synthesis Condition

In this section, based on the proposed dominant fuzzy Lyapunov matrix (19) and the new fuzzy control scheme (20), a convex controller design condition will be developed for the T–S fuzzy system (3). To give the controller design condition, we need the following lemmas.

Lemma 1: Assuming $0 \le \eta_1 \le 0.5, 0.5 \le \eta_2 \le 1, \eta_1 + \eta_2 = 1$, if symmetric matrices R_1, R_2, and R_3 satisfy the following inequalities:

$$\begin{aligned} &R_1 \le 0 \\ &R_1 + R_2 + R_3 \ge 0 \\ &R_3 \ge 0 \end{aligned}$$

then

$$\eta_1^2 R_1 + \eta_1\eta_2 R_2 + \eta_2^2 R_3 \ge 0.$$

Proof: Consider the following function:

$$f(\lambda) = \lambda^2 x^T R_1 x + \lambda x^T R_2 x + x^T R_3 x.$$

Then, $d^2(f)/d\lambda^2 = x^T R_1 x \le 0$, and $f(0) = x^T R_3 x \ge 0$, $f(1) = x^T R_1 x + x^T R_2 x + x^T R_3 x = x^T(R_1+R_2+R_3)x \ge 0$, for $x \ne 0$. Then $f(\lambda) \ge 0$, for $\lambda \in [0, 1]$.

For $0 \leq \eta_1 \leq 0.5, 0.5 \leq \eta_2 \leq 1$, it follows that $0 \leq \eta_1/\eta_2 \leq 1$. Further, $f(\eta_1/\eta_2) \geq 0$, i.e.,

$$\left(\frac{\eta_1}{\eta_2}\right)^2 x^T R_1 x + \frac{\eta_1}{\eta_2} x^T R_2 x + x^T R_3 x \geq 0$$

Combining it and $0.5 \leq \eta_2 \leq 1$ then yields

$$\eta_1^2 x^T R_1 x + \eta_1 \eta_2 x^T R_2 x + \eta_2^2 x^T R_3 x \geq 0$$

which implies that

$$\eta_1^2 R_1 + \eta_1 \eta_2 R_2 + \eta_2^2 R_3 \geq 0.$$

Thus, the proof is complete. ■

Lemma 2: If there exist symmetric matrices $R^{(i_{1\to p})}_{l_{1\to k-1} l_k m_{1\to k-1} l_k}$ and matrices $R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}$, $l_{1\to k-1} m_{1\to k-1} \in U_D(i_{1\to k-1})$, $l_k m_k \in U_D^{k0}(i_k)$, $1 \leq k \leq p$, satisfying the following inequalities:

$$\begin{aligned}
&R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k a} \geq 0\\
&R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1}(i_k-1)} \leq 0\\
&R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k a} + \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1} i_k})\\
&\quad + R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1}(i_k-1)} \geq 0\\
&R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k b} \geq 0\\
&R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1}(i_k+1)} \leq 0\\
&R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k b} + \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1}(i_k+1)})\\
&\quad + R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1}(i_k+1)} \geq 0
\end{aligned} \tag{23}$$

then

$$\sum_{l_k m_k \in U_D^k(i_k)} \mu_{kl_k} \mu_{km_k} R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} \geq 0,$$
$$\text{for } l_{1\to k-1} m_{1\to k-1} \in U_D(i_{1\to k-1}) \tag{24}$$

where

$$R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k} = R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k a} + R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k b}$$

μ_{kl_k}, $1 \leq k \leq p$, $1 \leq l_k \leq r_k$ are the grade of membership of $v_k(t)$ in M_{ki_k} and the v_k-based fuzzy sets M_{kl_k}, $1 \leq l_k \leq r_k$ are normal, consistent, and complete in $W_k \subset R$ with pseudotrapezoid membership functions.

Proof: Because the v_k-based fuzzy sets M_{kl_k}, $1 \leq l_k \leq r_k$ are normal, consistent, and complete in $W_k \subset R$ with pseudotrapezoid membership functions $\mu_{kl_k}(v_k(t))$, from Lemma 3 (ii), there are at most two fuzzy sets that are fired on the v_j-axis, which implies that $\mu_{ki_k}(v_k(t)) + \mu_{k(i_k-1)}(v_k(t)) = 1$, or $\mu_{ki_k}(v_k(t)) + \mu_{k(i_k+1)}(v_k(t)) = 1$ for $v(t) \in I(i_1 \cdots i_k \cdots i_p)$. Then, the left side of (24) can be rewritten as follows:

$$\begin{aligned}
&\sum_{l_k m_k \in U_D^k(i_k)} \mu_{kl_k} \mu_{km_k} R^{(i_{1\to p})}_{l_1 \cdots l_k m_1 \cdots m_k}\\
&= \mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k}\\
&\quad + \mu^2_{k(i_k-1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1}(i_k-1)}\\
&\quad + \mu_{k(i_k-1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1} i_k})\\
&\quad \text{or}\\
&= \mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k}\\
&\quad + \mu^2_{k(i_k+1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1}(i_k+1)}\\
&\quad + \mu_{k(i_k+1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1} i_k}).
\end{aligned} \tag{25}$$

From $v(t) \in I(i_1 \cdots i_k \cdots i_p)$, we have $0.5 \leq \mu_{ki_k}(v_k(t)) \leq 1$, $0 \leq \mu_{k(i_k-1)}(v_k(t)) \leq 0.5$, and $0 \leq \mu_{k(i_k+1)}(v_k(t)) \leq 0.5$. Further from Lemma 1 and (23), we have

$$\begin{aligned}
&\mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k a}\\
&\quad + \mu^2_{k(i_k-1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1}(i_k-1)}\\
&\quad + \mu_{k(i_k-1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1} i_k})\\
&\quad \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k b}\\
&\quad + \mu^2_{k(i_k+1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1}(i_k+1)}\\
&\quad + \mu_{k(i_k+1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1} i_k})\\
&\quad \geq 0.
\end{aligned}$$

From $R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k a} \geq 0$, $R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k b} \geq 0$, and the aforementioned two inequalities, then we can obtain

$$\begin{aligned}
&\mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k}\\
&\quad + \mu^2_{k(i_k-1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1}(i_k-1)}\\
&\quad + \mu_{k(i_k-1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k-1) m_{1\to k-1} i_k})\\
&\quad \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\mu^2_{ki_k}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1} i_k m_{1\to k-1} i_k}\\
&\quad + \mu^2_{k(i_k+1)}(v_k(t)) R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1}(i_k+1)}\\
&\quad + \mu_{k(i_k+1)}(v_k(t)) \mu_{ki_k}(v_k(t)) \mathtt{He}(R^{(i_{1\to p})}_{l_{1\to k-1}(i_k+1) m_{1\to k-1} i_k})\\
&\quad \geq 0.
\end{aligned}$$

Combining them and (25), then (24) holds. Thus, the proof is complete. ■

By using Lemma 2, a convex condition for designing fuzzy controllers is obtain as follows.

Theorem 2: Assume $\dot{\mu}_{\tau l_\tau}(v_\tau(t)) \geq \phi_{\tau l_\tau}$, for $1 \leq \tau \leq p$, $1 \leq l_\tau \leq r_\tau$. If there exists symmetric matrices $Q^{(i_{1\to p})}_{l_{1\to p}} > 0$, $T^{(i_{1\to p})}_{l_{1\to p}}$, $l_{1\to p} \in U_S(i_{1\to p})$, $R^{(i_{1\to p})}_{l_{1\to k-1} l_k m_{1\to k-1} l_k}$, $X^{(i_{1\to p})}_{l_{1\to k-1} l_k m_{1\to k-1} l_k}$, $l_{1\to k-1}$, $m_{1\to k-1} \in U_S(i_{1\to k-1})$, $l_k \in U^k_S(i_k)$, and matrices $F^{(i_{1\to p})}_{l_{1\to p}}$, $l_{1\to p} \in U_S(i_{1\to p})$, $R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} = (R^{(i_{1\to p})}_{l_{1\to k-1} m_k m_{1\to k-1} l_k})^T$, $X^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} = (X^{(i_{1\to p})}_{l_{1\to k-1} m_k m_{1\to k-1} l_k})^T$, $l_{1\to k} m_{1\to k} \in U^0_D(i_{1\to k})$, $1 \leq i_k \leq r_k$, $1 \leq k \leq p$, satisfying (15), (18), (23), and the following linear matrix inequalities (LMIs):

$$\begin{aligned}
&Y^{(i_{1\to p})}_{l_{1\to(k-1)} l_k m_{1\to(k-1)} l_k} + R^{(i_{1\to p})}_{l_{1\to(k-1)} l_k m_{1\to(k-1)} l_k}\\
&\leq X^{(i_{1\to p})}_{l_{1\to(k-1)} l_k m_{1\to(k-1)} l_k}, l_{1\to k-1} m_{1\to k-1} \in U_D(i_{1\to k-1})\\
&l_k \in U^k_S(i_k), 1 \leq k \leq p \qquad (26)\\
&Y^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} + Y^{(i_{1\to p})}_{l_{1\to k-1} m_k m_{1\to k-1} l_k} + \mathrm{He}(R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}})\\
&\leq \mathrm{He}(X^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}) l_{1\to k-1 m_{1\to k-1}} \in U_D(i_{1\to k-1})\\
&l_k m_k \in U^{k0}_D(i_k), 1 \leq k \leq p \qquad (27)\\
&Y^{(i_{1\to p})} \leq 0 \qquad (28)\\
&Q^{(i_{1\to p})}_{\eta_{1\to p}} - T^{(i_{1\to p})}_{\eta_1 \cdots \eta_{\tau-1} i_\tau \eta_{\tau+1} \cdots \eta_p} \geq 0, \eta_{1\to p} \in U_S(i_{1\to p})\\
&1 \leq \tau \leq p \qquad (29)
\end{aligned}$$

where $Y^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}$ is as given in (30a), shown at the bottom of the page, and

$$Y^{(i_{1\to p})} = \begin{bmatrix} X^{(i_{1\to p})}_{\tau_1\tau_1} & X^{(i_{1\to p})}_{\tau_1\tau_2} & \cdots & X^{(i_{1\to p})}_{\tau_1\tau_w} \\ X^{(i_{1\to p})}_{\tau_2\tau_1} & X^{(i_{1\to p})}_{\tau_2\tau_2} & \cdots & X^{(i_{1\to p})}_{\tau_2\tau_w} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(i_{1\to p})}_{\tau_w\tau_1} & X^{(i_{1\to p})}_{\tau_w\tau_2} & \cdots & X^{(i_{1\to p})}_{\tau_w\tau_w} \end{bmatrix}$$

$$\tau_1 < \tau_2 < \cdots < \tau_w \in U^1_S(i_1) \text{ and } w = |U^1_S(i_1)| \qquad (30b)$$

then the fuzzy system (3) is asymptotically stable via the controller (20).

Proof: Multiplying (26) and (27), respectively, by $\mu^2_{kl_k}$ and $\mu_{kl_k}\mu_{km_k}$, then summing them, yields

$$\begin{aligned}&\sum_{l_k m_k \in U^k_D(i_k)} \mu_{kl_k}\mu_{km_k}(Y^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} + R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}})\\
&\leq \sum_{l_k m_k \in U^k_D(i_k)} \mu_{kl_k}\mu_{km_k} X^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}. \qquad (31)\end{aligned}$$

From Lemma 2 and (23), we have

$$\sum_{l_k m_k \in U^k_D(i_k)} \mu_{kl_k}\mu_{km_k} R^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} \geq 0.$$

Combining it and (31), then it follows that

$$\begin{aligned}&\sum_{l_k m_k \in U^k_D(i_k)} \mu_{kl_k}\mu_{km_k} Y^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}\\
&\leq \sum_{l_k m_k \in U^k_D(i_k)} \mu_{kl_k}\mu_{km_k} X^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}. \qquad (32)\end{aligned}$$

Define

$$\vec{\mu}_{i_k} = \begin{bmatrix} \mu_{k\tau_1} E^{i_{1\to p}}(i_k) \\ \mu_{k\tau_2} E^{i_{1\to p}}(i_k) \\ \vdots \\ \mu_{k\tau_w} E^{i_{1\to p}}(i_k) \end{bmatrix}, \tau_1 < \tau_2 < \cdots < \tau_w \in U^k_S(i_k)$$

$$w = |U^k_S(i_k)|, 1 \leq k \leq p-1.$$

$E^{i_{1\to p}}(i_k)$ is an identity matrix with the same dimensions of $X^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}}$.

Pre- and postmultiplying (28) by $\vec{\mu}^T_{i_1}$ and $\vec{\mu}_{i_1}$ then yields

$$\sum_{l_1 m_1 \in U^1_D(i_1)} \mu_{1l_1}\mu_{1m_1} X^{(i_{1\to p})}_{l_1 m_1} \leq 0.$$

Combining it and (32) yields

$$\sum_{l_1 m_1 \in U^1_D(i_1)} \mu_{1l_1}\mu_{1m_1} Y^{(i_{1\to p})}_{l_1 m_1} \leq 0. \qquad (33)$$

$$Y^{(i_{1\to p})}_{l_{1\to k} m_{1\to k}} = \begin{cases} \mathrm{He}(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}) \\ \quad -\sum_{\tau=1}^{p} \sum_{\eta_\tau \in U_S(i_\tau)} \phi_{\tau\eta_\tau}(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}), \quad \text{for } k = p \\ \begin{bmatrix} X^{(i_{1\to p})}_{l_{1\to k}\tau_1 m_{1\to k}\tau_1} & X^{(i_{1\to p})}_{l_{1\to k}\tau_1 m_{1\to k}\tau_2} & \cdots & X^{(i_{1\to p})}_{l_{1\to k}\tau_1 m_{1\to k}\tau_w} \\ X^{(i_{1\to p})}_{l_{1\to k}\tau_2 m_{1\to k}\tau_1} & X^{(i_{1\to p})}_{l_{1\to k}\tau_2 m_{1\to k}\tau_2} & \cdots & X^{(i_{1\to p})}_{l_{1\to k}\tau_2 m_{1\to k}\tau_w} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(i_{1\to p})}_{l_{1\to k}\tau_w m_{1\to k}\tau_1} & X^{(i_{1\to p})}_{l_{1\to k}\tau_w m_{1\to k}\tau_2} & \cdots & X^{(i_{1\to p})}_{l_{1\to k}\tau_w m_{1\to k}\tau_w} \end{bmatrix} \\ \tau_1 < \tau_2 < \cdots < \tau_w \in U^{k+1}_S(i_{k+1}) \text{ and } w = |U^{k+1}_S(i_{k+1})|, \quad \text{for } 1 \leq k < p \end{cases} \qquad (30a)$$

From the aforementioned inequality and (30), then it follows that

$$\sum_{l_1 m_1 \in U_D^1(i_1)} \mu_{1l_1}\mu_{1m_1} \times \begin{bmatrix} X^{(i_{1\to p})}_{l_1\tau_1 m_1\tau_1} & X^{(i_{1\to p})}_{l_1\tau_1 m_1\tau_2} & \cdots & X^{(i_{1\to p})}_{l_1\tau_1 m_1\tau_w} \\ X^{(i_{1\to p})}_{l_1\tau_2 m_1\tau_1} & X^{(i_{1\to p})}_{l_1\tau_2 m_1\tau_2} & \cdots & X^{(i_{1\to p})}_{l_1\tau_2 m_1\tau_w} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(i_{1\to p})}_{l_1\tau_w m_1\tau_1} & X^{(i_{1\to p})}_{l_1\tau_w m_1\tau_2} & \cdots & X^{(i_{1\to p})}_{l_1\tau_w m_1\tau_w} \end{bmatrix} \le 0$$

$$\tau_1 < \tau_2 < \cdots < \tau_w \in U_S^2(i_2), w = |U_S^2(i_2)|$$

Pre- and postmultiplying the aforementioned inequality by $\vec{\mu}_{i_2}^T$ and $\vec{\mu}_{i_2}$, then we can obtain

$$\sum_{l_1 m_1 \in U_D^1(i_1)} \mu_{1l_1}\mu_{1m_1} \sum_{l_2 m_2 \in U_D^2(i_2)} \mu_{2l_2}\mu_{2m_2} X^{(i_{1\to p})}_{l_1 l_2 m_1 m_2} \le 0.$$

From (32) with $k = 2$ and considering the aforementioned inequality, then we have

$$\sum_{l_1 m_1 \in U_D^1(i_1)} \mu_{1l_1}\mu_{1m_1} \sum_{l_2 m_2 \in U_D^2(i_2)} \mu_{2l_2}\mu_{2m_2} Y^{(i_{1\to p})}_{l_1 l_2 m_1 m_2} \le 0. \tag{34}$$

Further, the technique from (33) to (34) is recursively applied for reaching $k = p$, then we can obtain (35), shown at the bottom of the page. From (29), it follows that

$$\prod_{h=1,h\neq\tau}^{p} \mu_{hl_h}(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}) \ge 0.$$

Combining it and $\dot{\mu}_{\tau\eta_\tau}(v_\tau(t)) \ge \phi_{\tau\eta_\tau}$ yields

$$\dot{\mu}_{\tau\eta_\tau} \prod_{h=1,h\neq\tau}^{p} \mu_{hl_h}(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p})$$
$$\ge \phi_{\tau\eta_\tau} \prod_{h=1,h\neq\tau}^{p} \mu_{hl_h}(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}).$$

Then, we have that, as in (36), shown at the bottom of the next page, holds. The left side of (36) can be rewritten as (37), shown at the bottom the next page. Since $\sum_{\eta_\tau \in U_S^\tau(i_\tau)} \mu_{\tau\eta_\tau} = 1$, $\sum_{\eta_\tau \in U_S^\tau(i_\tau)} \dot{\mu}_{\tau\eta_\tau} = 0$; further, the left side of (36) is equal

$$\begin{aligned}
&\sum_{l_1 m_1 \in U_D^1(i_1)} \mu_{1l_1}\mu_{1m_1} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{pl_p}\mu_{pm_p} Y^{(i_{1\to p})}_{l_{1\to p} m_{1\to p}} \\
&= \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} Y^{(i_{1\to p})}_{l_{1\to p} m_{1\to p}} \\
&= \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \\
&\quad \times \left(\mathtt{He}\left(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\right) - \sum_{\tau=1}^{p}\sum_{\eta_\tau \in U_S^\tau(i_\tau)} \phi_{\tau\eta_\tau}\left(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}\right)\right) \\
&= \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \mathtt{He}\left(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\right) \\
&\quad - \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \sum_{\tau=1}^{p}\sum_{\eta_\tau \in U_S^\tau(i_\tau)} \phi_{\tau\eta_\tau}\left(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}\right)\Big) \\
&= \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \mathtt{He}\left(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\right) \\
&\quad - \sum_{l_1 \in U_S^1(i_1)} \cdots \sum_{l_p \in U_S^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p} \sum_{\tau=1}^{p}\sum_{\eta_\tau \in U_S^\tau(i_\tau)} \phi_{\tau\eta_\tau}\left(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}\right) \\
&= \sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \mathtt{He}\left(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\right) \\
&\quad - \sum_{\tau=1}^{p} \sum_{l_1 \in U_S^1(i_1)} \cdots \sum_{l_{\tau-1} \in U_S^{\tau-1}(i_{\tau-1})} \sum_{\eta_\tau \in U_S^\tau(i_\tau)} \sum_{l_{\tau+1} \in U_S^{\tau+1}(i_{\tau+1})} \cdots \sum_{l_p \in U_S^p(i_p)} \phi_{\tau\eta_\tau} \prod_{h=1,h\neq\tau}^{p} \mu_{hl_h} \\
&\quad \times \left(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p}\right) \le 0.
\end{aligned} \tag{35}$$

to

$$\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\,\Pi\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}.$$

Combining it and (36), we have

$$\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\,\Pi\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}$$

$$\geq\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\phi_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\left(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}-T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p}\right)$$

i.e.,

$$\frac{d\left(\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_p\in U_S^p(i_p)}\prod_{h=1}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}l_\tau l_{\tau+1}\cdots l_p}\right)}{dt}$$

$$\geq\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}$$

$$\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}-T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p})$$
$$\geq\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\phi_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}$$
$$\times(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}-T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p})\tag{36}$$

$$\begin{aligned}
&\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}\\
&-\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p}\\
=&\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}\\
&-\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\left(\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p}\right)\\
=&\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}\\
&-\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\left(\left(\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p}\right)\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\dot{\mu}_{\tau\eta_\tau}\right)\\
=&\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\dot{\mu}_{\tau\eta_\tau}\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p}\\
&-\sum_{\tau=1}^{p}\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_{\tau-1}\in U_S^{\tau-1}(i_{\tau-1})}\sum_{l_{\tau+1}\in U_S^{\tau+1}(i_{\tau+1})}\cdots\sum_{l_p\in U_S^{p}(i_p)}\left(\sum_{\eta_\tau\in U_S^{\tau}(i_\tau)}\dot{\mu}_{\tau\eta_\tau}\right)\left(\prod_{h=1,h\neq\tau}^{p}\mu_{hl_h}\,T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}i_\tau l_{\tau+1}\cdots l_p}\right)
\end{aligned}\tag{37}$$

$$\cdots \sum_{l_p \in U_S^p(i_p)} \phi_{\tau\eta_\tau} \prod_{h=1,h\neq\tau}^{p} \mu_{hl_h} \Big(Q^{(i_{1\to p})}_{l_1\cdots l_{\tau-1}\eta_\tau l_{\tau+1}\cdots l_p} - T^{(i_{1\to p})}_{l_1\cdots l_{\tau-1} i_\tau l_{\tau+1}\cdots l_p} \Big).$$

Combining it and (35), then we have

$$\sum_{l_1 m_1 \in U_D^1(i_1)} \cdots \sum_{l_p m_p \in U_D^p(i_p)} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \times \mathtt{He}\big(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\big) - \frac{d\left(\sum_{l_1\in U_S^1(i_1)}\cdots\sum_{l_p\in U_S^p(i_p)} \prod_{h=1}^{p}\mu_{hl_h} Q^{(i_{1\to p})}_{l_{1\to p}}\right)}{dt} \le 0.$$

From the definitions of $U_D^k(i_k)$, $U_S^k(i_k)$, $U_D(i_{1\to k})$, and $U_S(i_{1\to k})$ in (8)–(11), the aforementioned inequality can be rewritten as follows:

$$\sum_{l_{1\to p} m_{1\to p} \in U_D(i_{1\to p})} \mu_{1l_1}\cdots\mu_{pl_p}\mu_{1m_1}\cdots\mu_{pm_p} \times \mathtt{He}\big(A_{l_{1\to p}} Q^{(i_{1\to p})}_{m_{1\to p}} + B_{l_{1\to p}} F^{(i_{1\to p})}_{m_{1\to p}}\big) - \frac{d\left(\sum_{l_{1\to p}\in U_S(i_{1\to p})} \prod_{h=1}^{p}\mu_{hl_h} Q^{(i_{1\to p})}_{l_{1\to p}}\right)}{dt} \le 0$$

i.e.,

$$\mathtt{He}(A(\mu)Q^{(i_{1\to p})}(\mu) + B(\mu)F^{(i_{1\to p})}(\mu)) - \dot{Q}^{(i_{1\to p})}(\mu) \le 0$$

where $A(\mu)$, $B(\mu)$, $Q^{(i_{1\to p})}(\mu)$, and $F^{(i_{1\to p})}(\mu)$ are, respectively, the same as in (13), (19), and (21).

Letting $P^{(i_{1\to p})}(\mu) = (Q^{(i_{1\to p})}(\mu))^{-1}$ and then multiplying both sides of the aforementioned inequality by $P^{(i_{1\to p})}(\mu)$, it follows that

$$\mathtt{He}(P^{(i_{1\to p})}(\mu)A(\mu) + P^{(i_{1\to p})}(\mu)B(\mu)F^{(i_{1\to p})}(\mu) \times P^{(i_{1\to p})}(\mu)) - P^{(i_{1\to p})}(\mu)\dot{Q}^{(i_{1\to p})}(\mu)P^{(i_{1\to p})}(\mu) \le 0$$

which can be rewritten as follows:

$$\mathtt{He}(P^{(i_{1\to p})}(\mu)A(\mu) + P^{(i_{1\to p})}(\mu)B(\mu)K^{(i_{1\to p})}(\mu)) + \dot{P}^{(i_{1\to p})}(\mu) \le 0 \tag{38}$$

where $K^{(i_{1\to p})}(\mu)$ is the same as in (20).

Choosing Lyapunov function

$$V(t) = x^T(t)P^{(i_{1\to p})}(\mu)x(t), \text{ for } v(t)\in I(i_{1\to p})$$

then for $v(t)\in I(i_{1\to p})$

$$\begin{aligned}\dot{V}(t) &= 2x^T(t)P^{(i_{1\to p})}(\mu)(A(\mu)+B(\mu)K^{(i_{1\to p})}(\mu))x(t) \\ &\quad + x^T(t)\dot{P}^{(i_{1\to p})}(\mu)x(t) \\ &= x^T(t)(\mathtt{He}(P^{(i_{1\to p})}(\mu)(A(\mu)+B(\mu)K^{(i_{1\to p})}(\mu))) \\ &\quad + \dot{P}^{(i_{1\to p})}(\mu))x(t).\end{aligned}$$

Combining it and (38) then yields

$$\dot{V}(t) < 0$$

which implies that the system is asymptotically stable. Thus, the proof is complete. ■

Remark 5: (i) Based on the dominant fuzzy Lyapunov function and the control scheme (20), a convex condition is presented in Theorem 2. In contrast with the existing approach, if the Lyapunov matrices do not switch between the spilt space of the premise variables, i.e., $Q^{(i_{1\to p})}_{l_{1\to p}} = Q^{(11\cdots1)}_{l_{1\to p}}$, then the dominant fuzzy Lyapunov function is reduced to the fuzzy Lyapunov function in [32]. On the other hand, if the parameters $F^{(i_{1\to p})}_{l_{1\to p}} = F^{(11\cdots1)}_{l_{1\to p}}$ in the fuzzy controller (20), then the controller is reduced to the types of non-PDC controllers in [11]. It has since been proved that the fuzzy Lyapunov function approach can give less-conservative results than the quadratic Lyapunov function approach in [32]. Moreover, it has also been shown that the non-PDC control scheme can give less-conservative design than the conventional PDC control scheme in [11]. Therefore, the condition in Theorem 2 has the potential to give less-conservative results than the fuzzy Lyapunov function and quadratic Lyapunov function approaches. Examples in Section V are given to illustrate the fact.

(ii) In Theorem 2, by splitting the premise variable space into a set of subspaces, the new properties about the structure or shape of membership functions are exploited; then, a relaxed condition is obtained. On the other hand, we would like to point out that more variables and inequalities in Theorem 2 lead to a big burden on the programming and computation. The technique, which is with a light computational burden and small conservativeness, will be exploited in the future.

V. Example

Example 1: Consider the following nonlinear system, which is the same as in [32]:

$$\begin{aligned}\dot{x}_1 &= x_2(t) \\ \dot{x}_2 &= -2x_1(t) - x_2(t) - f(t)x_1(t)\end{aligned}$$

where $f(t)$ is a C^1 function. Thus, the nonlinear system can be described by the following fuzzy model:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(v(t))(A_i x(t) + B_i u(t))$$

with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the membership functions are $h_1(v(t)) = k - f(t)/k$ and $h_2(v(t)) = f(t)/k$.

The common Lyapunov approach guarantees the stability of the aforementioned system for $k \le 3.82$. In [34], a two-term piecewise quadratic Lyapunov functions approach is presented for linear time-varying systems, which can be used for T–S fuzzy systems. The condition in [34] guarantees the stability for $k \le 4.7$. Moreover, by considering the different values of ϕ_1, the fuzzy Lyapunov function method in [32] (i.e., Corollary 3 in [32]) and Theorem 2 with $T_1^1 = Q_2^1$ and $T_2^2 = Q_2^2$ can be used (if $T_1^1 = Q_2^1$ and $T_2^2 = Q_2^2$, then $\phi_2(Q_2^1 - T_1^1) = 0$ and

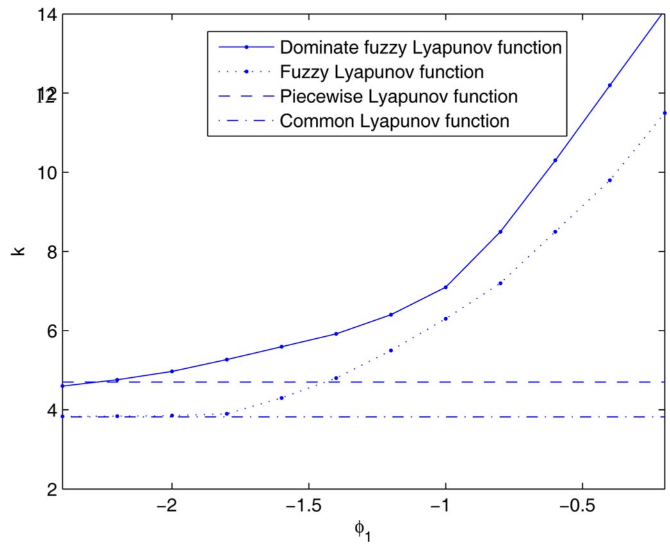


Fig. 2. Maximum values of k guaranteeing feasibility.

$\phi_2(Q_2^2 - T_2^2) = 0$, which implies that Theorem 2 can be used without the knowledge of the bound ϕ_2 of $\dot{\mu}_{12}$. The constraint is imposed in order to give a fair comparison with the method in [32]). The obtained maximum values of k guaranteeing stability are shown in Fig. 2.

From Fig. 2, it can been seen that the common Lyapunov function and the piecewise Lyapunov function approaches are feasible, respectively, for the largest $k_{\max} = 3.82$ and $k_{\max} = 4.7$. The fuzzy Lyapunov approach in [32] and the dominant fuzzy Lyapunov function method guarantee larger feasible (stable) areas for larger ϕ_1. When $\phi_1 \geq -2.3$, the dominant fuzzy Lyapunov function method in this paper guarantees the largest feasible area for the parameter k. When $\phi_1 < -2.3$, the piecewise Lyapunov function approach in [34] can guarantee the largest feasible area. In particular, it can be seen from Fig. 2 that the dominant fuzzy Lyapunov function approach always can give less-conservative results than the fuzzy Lyapunov function and the common Lyapunov function approaches. The fact shows that the new technique can give less-conservative results than some existing approaches.

Example 2: Consider the following problem of balancing an inverted pendulum on a cart [14]. The equations of motion are as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin(x_1) - (amlx_2^2\sin(2x_1)/2) - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}$$

where x_1 denotes the angle of the pendulum from the vertical, x_2 is the angular velocity, $g = 9.8$ m/s^2 is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, and u is the force applied to the cart $a = 1/(m + M)$. We choose $m = 2.0$ kg, $M = 8.0$ kg, and $2l = 1.0$ m in the simulation and approximate the system by the following two-rule fuzzy model.

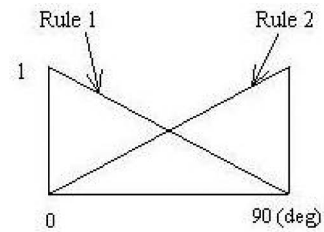


Fig. 3. Membership functions for Example 2.

Plant Rule 1:

IF $|x_1(t)|$ is about 0, THEN
$$\dot{x}(t) = A_1x(t) + B_{21}u(t)$$

Plant Rule 2:

IF $|x_1(t)|$ is about $\pi/2$, THEN
$$\dot{x}(t) = A_2x(t) + B_{22}u(t)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{4l/3 - aml} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix}$$
$$B_{21} = \begin{bmatrix} 0 \\ -\dfrac{a}{4l/3 - aml} \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ -\dfrac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix}$$

$\beta = \cos(88^o)$ and membership functions for Rules 1 and 2 are shown in Fig. 3.

Assuming $\dot{\mu}_{1l_1} \geq -50, l_1 = 1, 2$, i.e., $\phi_{11} = \phi_{12} = -50$ and applying the condition of Theorem 2 to the example, then we can obtain the following results:

$$F_1^1 = [\,260.2790 \quad -67.9682\,], \quad F_2^1 = [\,534.2046 \quad -184.1360\,]$$
$$Q_1^1 = \begin{bmatrix} 1.9948 & -4.9157 \\ -4.9157 & 17.0504 \end{bmatrix}, \quad Q_2^1 = \begin{bmatrix} 1.9951 & -4.9168 \\ -4.9168 & 17.0493 \end{bmatrix}$$
$$F_1^2 = [\,219.7715 \quad -433.2055\,], \quad F_2^2 = [\,574.7122 \quad 181.1013\,]$$
$$Q_1^2 = \begin{bmatrix} 1.8418 & -5.3816 \\ -5.3816 & 16.5956 \end{bmatrix}, \quad Q_2^2 = \begin{bmatrix} 2.1481 & -4.4508 \\ -4.4508 & 17.5040 \end{bmatrix}.$$

By using the controller (20), the state and input trajectories are shown in Figs. 4 and 5. From Figs. 4 and 5, it can be seen that the system with the controller designed by Theorem 2 is asymptotically stable, which further shows the effectiveness of the new technique.

Example 3: Consider the following six-rule fuzzy system:

Rule $(i_1 i_2)$*:*

IF $x_1(t)$ is M_{1i_1} and $x_2(t)$ is M_{2i_2}

THEN $\dot{x}(t) = A_{i_1i_2}x(t) + B_{i_1i_2}u(t)$

where $x(t) = [x_1(t) \;\; x_2(t)]^T \in R^2$ is the state, and $u(t) \in R^1$ is the control input. M_{1i_1} and M_{2i_2} are fuzzy sets, and $i_1 = 1$,

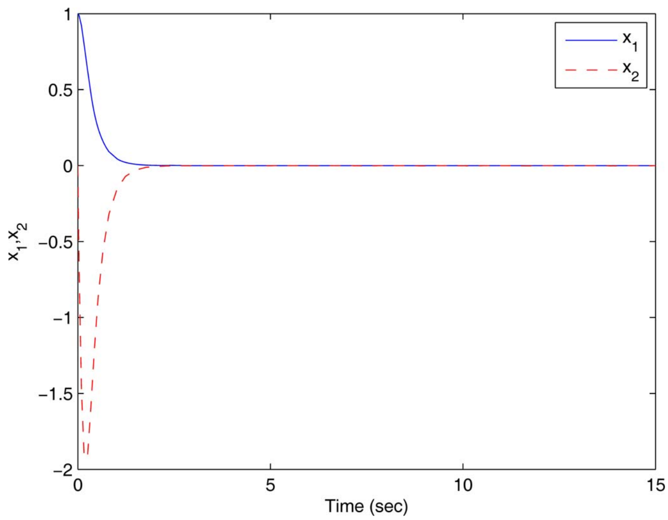


Fig. 4. Trajectories of the states $x(t)$.

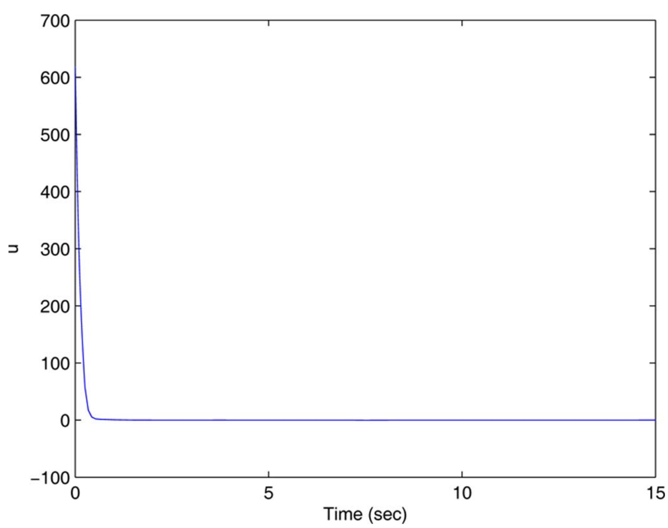


Fig. 5. Trajectories of the input $u(t)$.

$2, i_2 = 1, 2, 3$, with

$$M_{11} = \begin{cases} 1, & x_1 \in (-\infty, -1) \\ -0.5x_1 + 0.5, & x_1 \in [-1, 1] \\ 0, & x_1 \in (1, +\infty) \end{cases}$$

$$M_{12} = \begin{cases} 0, & x_1 \in (-\infty, -1) \\ 0.5x_1 + 0.5, & x_1 \in [-1, 1] \\ 1, & x_1 \in (1, +\infty) \end{cases}$$

$$M_{21} = \begin{cases} 1, & x_2 \in (-\infty, -2) \\ -x_2 - 1, & x_2 \in [-2, -1] \\ 0, & x_2 \in (-1, +\infty) \end{cases}$$

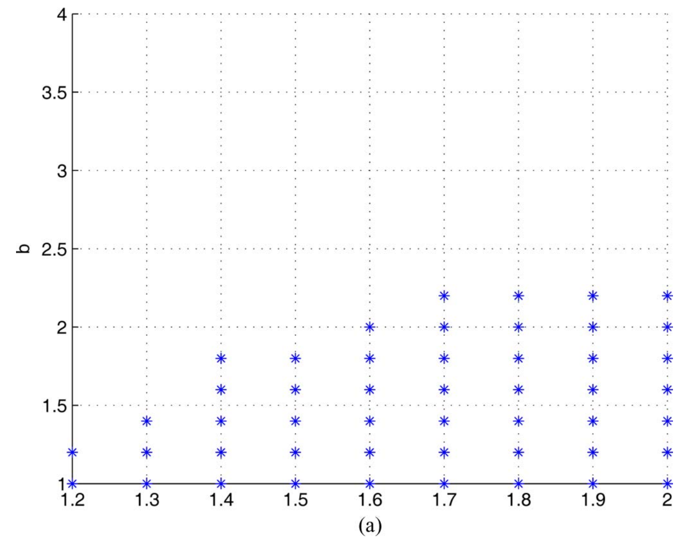


Fig. 6. Feasible area for the common Lyapunov approach in [3].

$$M_{22} = \begin{cases} 0, & x_2 \in (-\infty, -2) \\ x_2 + 2, & x_2 \in [-2, -1] \\ 1, & x_2 \in [-1, 1] \\ -x_2 + 2, & x_2 \in (1, 2] \\ 0, & x_2 \in (2, +\infty) \end{cases}$$

$$M_{23} = \begin{cases} 0, & x_2 \in (-\infty, 1) \\ x_2 - 1, & x_2 \in [1, 2] \\ 1, & x_2 \in (2, +\infty) \end{cases}$$

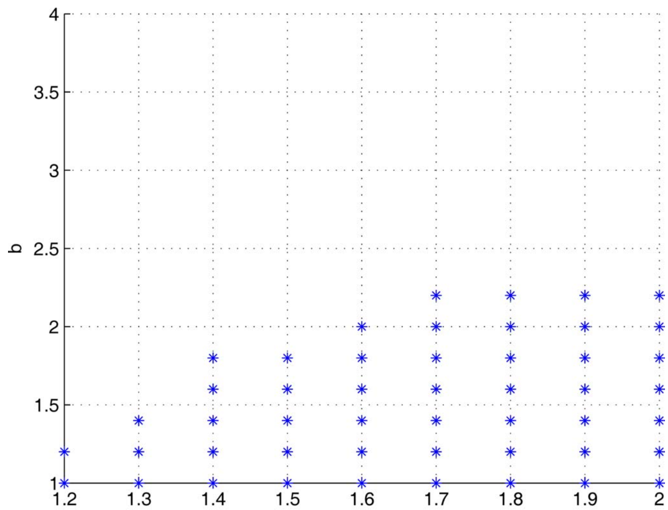
$$A_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 1.5 & 4 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 2 \\ 1.3 & 4 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 2 \\ 1.8 & 4 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 1 & 2 \\ 2.3 & 4 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0.5 \\ a \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ a \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 0.5 + b \\ a \end{bmatrix}$$

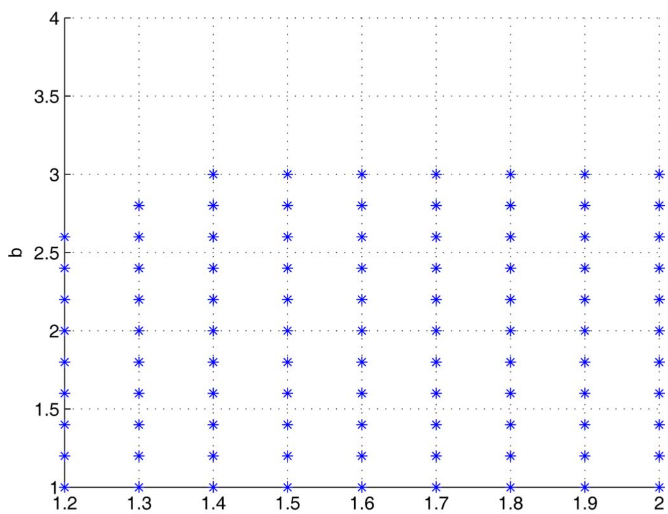
$$B_{21} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad B_{23} = \begin{bmatrix} -1 + b \\ 1 \end{bmatrix}.$$

Assume $\dot{\mu}_{il_i} \geq -1 \times 10^8$, $i = 1, 2$, $l_1 = 1, 2$, and $l_2 = 1, 2, 3$, i.e., $\phi_{11} = \phi_{12} = \phi_{21} = \phi_{22} = \phi_{23} = -1 \times 10^8$. We compare the dominant-dependent fuzzy Lyapunov approach with the common Lyapunov approach in [3], the fuzzy Lyapunov approach in [32], and the non-PDC control approach (which can be obtained from Theorem 2 with $Q_{l_{1\to p}}^{(i_{1\to p})} = Q_{l_{1\to p}}^{(11\cdots 1)}$ and $F_{l_{1\to p}}^{(i_{1\to p})} = F_{l_{1\to p}}^{(11\cdots 1)}$). We design stabilizing controllers for several combinations of a and b using the aforementioned approaches. Figs. 6–9 show the feasible areas. From Figs. 6–9, it can be seen that the new proposed dominant-dependent fuzzy Lyapunov approach provides more relaxed conditions than the common Lyapunov approaches in [3], fuzzy Lyapunov approaches in [32], and non-PDC control approaches.



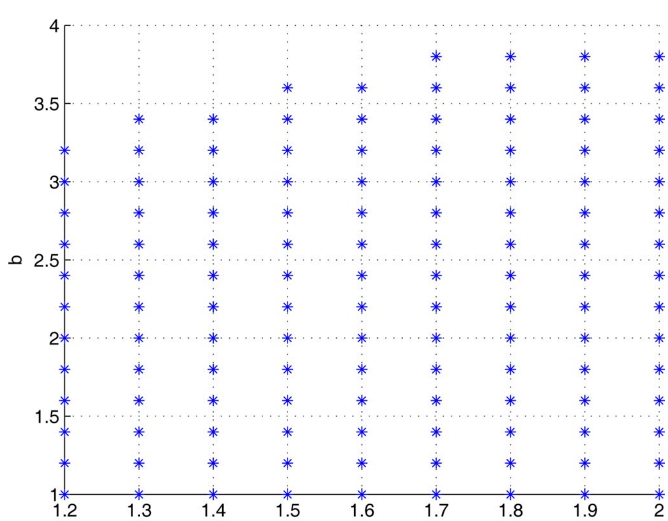
(a)

Fig. 7. Feasible area for the fuzzy Lyapunov function approach in [32].



(a)

Fig. 8. Feasible area for using non-PDC controllers.



(a)

Fig. 9. Feasible area for Theorem 2 in this paper.

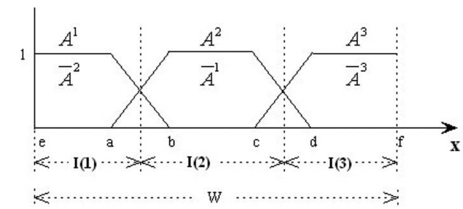


Fig. 10. Examples of pseudotrapezoid membership functions.

VI. Conclusion

In this paper, we have addressed the problem of state feedback control for T–S fuzzy systems via fuzzy Lyapunov functions. By splitting the premise variable spaces into some subspaces and using the properties of fuzzy sets, a new control scheme is proposed based on the dominant fuzzy Lyapunov function, and an LMI-based condition for designing fuzzy controllers has been given. Some existing fuzzy Lyapunov functions and non-PDC controllers are special cases of the dominant fuzzy Lyapunov functions and fuzzy controllers, respectively. Numerical examples have been given to illustrate the effectiveness of the proposed method.

Appendix

Some Existing Preliminary Concepts on Fuzzy Sets

Definition 1: [33] (i) *Normal pseudotrapezoid membership function:* Let $[a, d] \subset R$. The pseudotrapezoid membership function of fuzzy set A is a continuous function in R given by

$$\mu_A(x, a, b, c, d) = \begin{cases} H(x), & x \in [a, b) \\ 1, & x \in [b, c] \\ D(x), & x \in (c, d] \\ 0, & x \in R - (a, d) \end{cases}$$

where $a \le b \le c \le d$, $0 \le H(x) \le 1$ is a nondecreasing function in $[a, b)$, and $0 \le D(x) \le 1$ is a nonincreasing function in $(c, d]$.

(ii) *Completeness of fuzzy sets:* Fuzzy sets $A^1, A^2, \ldots, A^N$ in $W \subset R$ are said to be complete on W if for any $x \in W$, there exists A^j such that $\mu_{A^j}(x) > 0$, where W is the universe of discourse.

(iii) *Consistency of fuzzy sets:* $A^1, A^2, \ldots, A^N$ in $W \subset R$ are said to be consistent on W if $\mu_{A^j}(x) = 1$ for some $x \in W$ implies that $\mu_{A^i}(x) = 0$ for all $i \ne j$.

(iv) *High set of fuzzy set:* The high set of a fuzzy set A in $W \subset R$ is a subset in W defined by

$$\mathrm{hgh}(A) = \{x \in W | \mu_A(x) = \sup_{x' \in W} \mu_A(x')\}.$$

If A is a normal fuzzy set with pseudotrapezoid membership function $\mu_A(x, a, b, c, d)$, then $\mathrm{hgh}(A) = [b, c]$.

(v) *Order between fuzzy sets:* For two fuzzy sets A and B in $W \subset R$, we say $A > B$ if $\mathrm{hgh}(A) > \mathrm{hgh}(B)$ (that is, if $x \in \mathrm{hgh}(A)$ and $x' \in \mathrm{hgh}(B)$, then $x > x'$).

Fig. 10 shows some examples of pseudotrapezoid membership functions. If the universe of discourse is bounded, then

a, b, c, and d are finite numbers. Pseudotrapezoid membership functions include many commonly used membership functions as specia1 cases. For example, if we choose

$$H(x) = \frac{x-a}{b-a}, \quad D(x) = \frac{x-d}{c-d} \tag{39}$$

then the pseudotrapezoid membership functions become the trapezoid membership functions. If $b = c$, and $H(x)$ and $D(x)$ are the same as in (39), we obtain the triangular membership functions. If we choose $a = \infty$, $b = c = \bar{x}$, $d = \infty$, and

$$H(x) = D(x) = \exp\left(-\left(\frac{x-\bar{x}}{\sigma}\right)^2\right)$$

then the pseudotrapezoid membership functions become the Gaussian membership functions. Therefore, the pseudotrapezoid membership functions constitute a very rich family of membership functions. See [33] for more details.

Note that there are three fuzzy sets in Fig. 10, i.e., A^1, A^2, and A^3 in the universe of discourse $W \subset R$, and it follows from Definition 1 that A^1, A^2, and A^3 are consistent and complete in W.

Based on Definition 1, some properties of fuzzy sets with pseudotrapezoid membership functions are shown in the following lemma, which is useful for the development of this paper.

Lemma 3: [33] (i) If $A^1, A^2, \ldots, A^N$ are consistent and normal fuzzy sets in $W \subset R$ with pseudotrapezoid membership functions $\mu_{A^i}(x_i, a_i, b_i, c_i, d_i)$, $(i = 1, 2, \ldots, N)$, then there exists a rearrangement $\{i_1, i_2, \ldots, i_N\}$ of $\{1, 2, \ldots, N\}$ such that $A^1 < A^2 < \cdots < A^N$.

(ii) Let the fuzzy sets A^1, A^2, ..., A^N in $W \subset R$ be normal, consistent, and complete with pseudotrapezoid membership functions $\mu_{A^i}(x_i, a_i, b_i, c_i, d_i)$. If $A^1 < A^2 < \cdots < A^N$, then

$$c_i \le a_{i+1} < d_i \le b_{i+1}.$$

Based on the definition of the order between fuzzy sets, Lemma 3 shows that the consistent and normal fuzzy sets in $W \subset R$ with pseudotrapezoid membership functions can be rearranged from "small" to "big," which will be helpful for obtaining the controller design conditions and simplifying the description of the obtained conditions in the later theorems. For example, there are three fuzzy sets $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$, and they can be rearranged as A^2, A^1, A^3 such that $A^1 < A^2 < A^3$.

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