A New Regularization of Coulomb Friction

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We present a new regularization of Coulomb's law of friction that permits a straightforward incorporation of frictional forces within existing numerical simulations. Similar to existing regularizations, the proposed modification to Coulomb friction leads to a continuous representation of friction and does not require the identification of transitions between slip and stick. However, unlike more common regularizations, the current reformulation maintains a structure at zero contact velocity that is identical to the classical, discontinuous form of Coulomb friction. The implementation of this regularization is presented through two examples in which slip-stick motion induced by sliding friction is of primary importance. The first is a simple one degree-of-freedom system and illustrates the existence of nontrivial equilibrium states. The second example is a multi-degree-of-freedom system in which the present model provides a computationally efficient scheme for simulating the dissipation arising from sliding friction. For systems in which slip-stick transitions are important the proposed regularization provides a computationally efficient scheme to obtain time-accurate simulations. [DOI: 10.1115/1.1760564]

1 Introduction

Frictional contact problems exist in a wide variety of mechanical systems and give rise to energy dissipation, slip-stick instabilities, and even chaotic motion (see for example [1]). For single point contact between two bodies the force of friction, representing the interaction between the bodies in contact, is commonly modeled through Coulomb's law of friction, which can be represented as:

$$\begin{aligned} \mathbf{f} &= -\mu N \,\mathrm{sgn}(\dot{x})\hat{\mathbf{t}}, \quad \dot{x} \neq 0, \\ |\mathbf{f}| &\leq \mu N, \quad \dot{x} = 0, \end{aligned} \tag{1}$$

where $\mathbf{f}$ is the frictional force and $\dot{x}\hat{\mathbf{t}}$ is the relative velocity of the points of contact between the two bodies in question. However, for zero contact velocity the inequality can be removed by considering the frictional force to be a function of $\mathbf{f}_{\mathrm{eq}}$, the value of the friction force necessary to balance the gross force arising from all other sources:

$$\left.\begin{aligned} \mathbf{f} &= \mathbf{f}_{\mathrm{eq}}, && |\mathbf{f}_{\mathrm{eq}}| \leq \mu N, \\ &= \mu N \frac{\mathbf{f}_{\mathrm{eq}}}{|\mathbf{f}_{\mathrm{eq}}|}, && |\mathbf{f}_{\mathrm{eq}}| > \mu N, \end{aligned}\right\} \quad \dot{x} = 0. \tag{2}$$

Therefore, when Eqs. (1) and (2) are considered together, the frictional force is written as a unique function of two quantities, namely $\dot{x}$ and $\mathbf{f}_{\mathrm{eq}}$, although this latter quantity only affects the frictional force for $\dot{x}=0$. This viewpoint is consistent with the force-balance friction model originally proposed by Karnopp [2]. The classical definition of Coulomb friction is discontinuous at $\dot{x}=0$, giving rise to difficulties in the numerical simulation of mechanical systems with friction when slip-stick behavior plays a central role [3,4]. However for $|\mathbf{f}_{\mathrm{eq}}\cdot\hat{\mathbf{t}}| \equiv |f_{\mathrm{eq}}| > \mu N$ we note that the discontinuity is only one sided. For $f_{\mathrm{eq}} < -\mu N$ the friction force is discontinuous at $\dot{x}=0^-$, that is:

$$f(\dot{x})|_{\dot{x}=0} = \lim_{\dot{x}\to 0^+} f(\dot{x}) \neq \lim_{\dot{x}\to 0^-} f(\dot{x}),$$

while for $f_{\mathrm{eq}} > \mu N$ the friction force is discontinuous at $\dot{x}=0^+$. As a unique function of these two variables, the form of Coulomb friction can be represented as shown in Fig. 1 (with $f=|\mathbf{f}|$).

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The predicted response of mechanical systems subject to Coulomb friction is frequently very sensitive to the tuning of the numerical algorithm used within the simulation. Indeed, standard techniques are notoriously bad at simulating the response of systems with Coulomb friction in the presence of repeated transitions between stick and slip. Quite often, the application of standard methods (such as Runge-Kutta algorithms or stiff solvers) leads to numerical chatter in the direction of both the velocity and frictional force. Instead, numerous specialized numerical algorithms and approaches tailored for nonsmooth dynamical systems have been developed [5–7]. However, their widespread use is limited because of their complexity and the computational requirements often needed to resolve each transition between sticking and sliding [8,9]. Unfortunately, without such specialized routines numerical simulations often fail not only to provide time-accurate solutions, but to simply resolve the equilibrium state of the system. The present work formulates an alternative representation of friction that removes the discontinuity noted above. Although this model is developed for planar problems, the results can be directly applied to more general three dimensional contacts.

2 Alternative Friction Models

As an alternative to the development and use of specialized numerical routines, another common approach to simulating mechanical systems with friction is to develop a model of the frictional force that reduces the numerical problems associated with the discontinuities in Coulomb friction at zero velocity. Such regularizations provide a modified representation for the force of friction so that the numerical algorithm is no longer required to separately identify the transitions between sticking and slipping. These regularizations come at the expense of time-accuracy when compared to the implementation of specialized numerical algorithms for the discontinuous model of Coulomb friction. Because the representation of friction has been modified, solutions to the regularized problem are not identical to those of the original discontinuous system. However, for a well-posed regularization, as some fixed smoothing parameter ϵ is decreased, the two solutions converge.

Karnopp [2] introduced a threshold contact velocity magnitude, below which the kinematic degree-of-freedom of the system associated with the frictional contact is eliminated—above this critical relative speed Coulomb friction applies. In terms of the quantities $\dot{x}$ and f_{eq}, this model can be represented graphically as seen in Fig. 2. This method was extended by several researchers, includ-

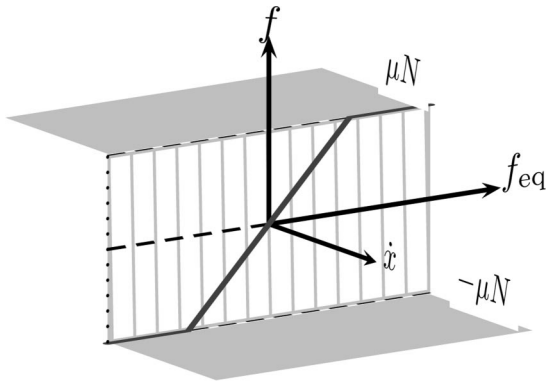

Fig. 1 Representation of Coulomb's law of friction. The heavy line represents the friction force at zero velocity.

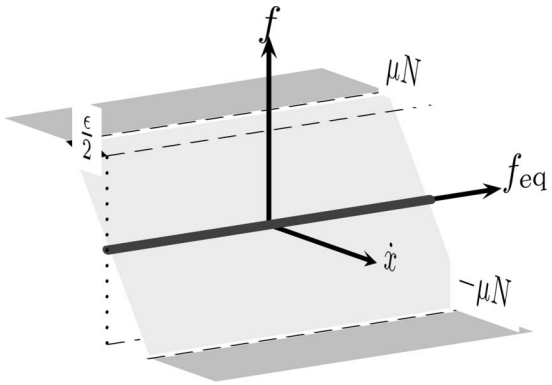

Fig. 3 Velocity-based regularization of the classical law of friction (see Martins and Oden [12])

ing Tan and Rogers [10], Tariku and Rogers [4], and Leine et al. [11] to enhance the accuracy and numerical stability of Karnopp's original model. However, in each of these reformulations, the frictional force continues to possess discontinuities, particularly in the transition from slip to stick.

Finally, researchers have attempted to remove the discontinuities identified above. Such elimination allows for the force of friction to be incorporated into standard numerical and analytical techniques without the necessity for specialized treatment. One such regularization proposed by Martins and Oden [12] takes the form:

$$\begin{aligned} f &= -(\mu N/\epsilon)\dot{x}, \quad |\dot{x}| \leq \epsilon, \\ &= -\mu N \operatorname{sgn}(\dot{x}), \quad |\dot{x}| > \epsilon, \end{aligned} \tag{3}$$

where ϵ is a fixed regularization parameter and the function $\operatorname{sgn}(\cdot)$ gives the sign of its argument. In this model the force of friction is equivalent to a viscous damper near zero contact velocity ($|\dot{x}| < \epsilon$), saturating at $|\dot{x}| = \epsilon$. As a function of $(\dot{x}, f_{\text{eq}})$ this proposed regularization is shown in Fig. 3. However, the force of friction proposed in Eq. (3) must vanish at zero contact velocity so that the velocity-limited regularization does not coincide with the classical definition of Coulomb friction for $\dot{x} = 0$ (cf. Figs. 1 and 3). In fact, as $\epsilon \to 0$, Eq. (3) does not converge pointwise to a description of Coulomb friction.

Smooth regularizations, similar in form Eq. (3), have also been proposed [13,14], although they suffer from the same discrepancy between their value at zero contact velocity and the predictions of Coulomb friction. Thus, while such regularizations allow one to easily incorporate friction into computational models, the resulting behavior can be qualitatively different from that predicted when using the classical definition of Coulomb friction. Finally, alternative friction models also can be based on physical considerations and may introduce additional internal variables to represent experimentally observed frictional phenomena, such as shear resistance or relaxation [15–17].

As an alternative regularization of Coulomb friction, the following model is proposed:

$$\begin{aligned} f &= -\frac{\mu N}{\epsilon}(\dot{x} - \zeta), \quad |\dot{x} - \zeta| \leq \epsilon, \\ &= -\mu N \operatorname{sgn}\left(\frac{\dot{x} - \zeta}{\epsilon}\right), \quad |\dot{x} - \zeta| > \epsilon. \end{aligned} \tag{4a}$$

where:

$$\begin{aligned} \zeta &= \epsilon(f_{\text{eq}}/(\mu N)), \quad |f_{\text{eq}}/(\mu N)| \leq 1, \\ &= \epsilon \operatorname{sgn}(f_{\text{eq}}/(\mu N)) \quad |f_{\text{eq}}/(\mu N)| > 1. \end{aligned} \tag{4b}$$

Although this is similar in form to Eq. (3), for $\dot{x} = 0$ this reduces to a description of friction identical to standard Coulomb friction, as in Eq. (2). Therefore, this modification eliminates the discontinuities in the classical definition of friction while maintaining the structure of the friction model at zero contact velocity. In Fig. 4 this proposed model of friction is shown as a function of the relative contact velocity $\dot{x}$ and f_{eq}, the force required to balance the gross (non-frictional) force. In contrast to the velocity-limited regularization, the frictional model given in Eqs. (4) converges pointwise to the classical definition of Coulomb friction as $\epsilon \to 0$ and in particular the structure of the regularization at zero velocity is identical to that of Coulomb friction.

Because the force of friction has been given an analytic representation near zero contact velocity, the solution to dynamical systems subject to Eqs. (4) can no longer reach a state of sticking

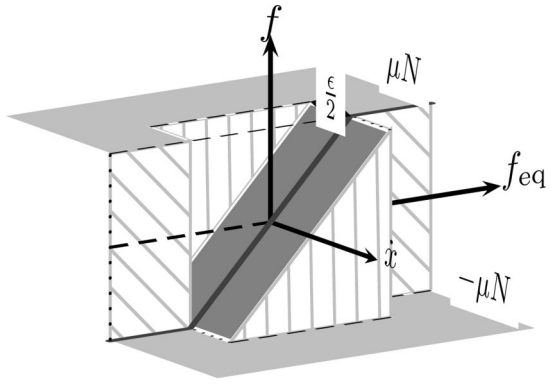

Fig. 2 Threshold contact velocity model as proposed by Karnopp [2], represented in terms of $\dot{x}$ and f_{eq}

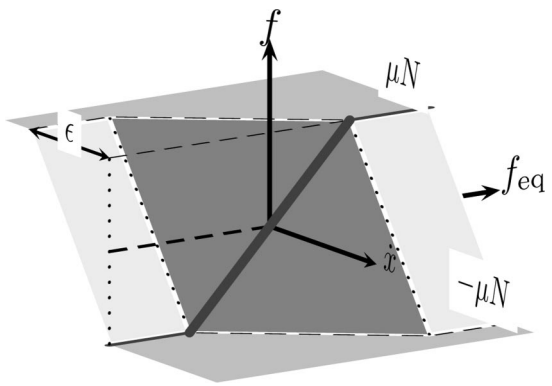

Fig. 4 Modified friction law proposed in Eqs. (4)

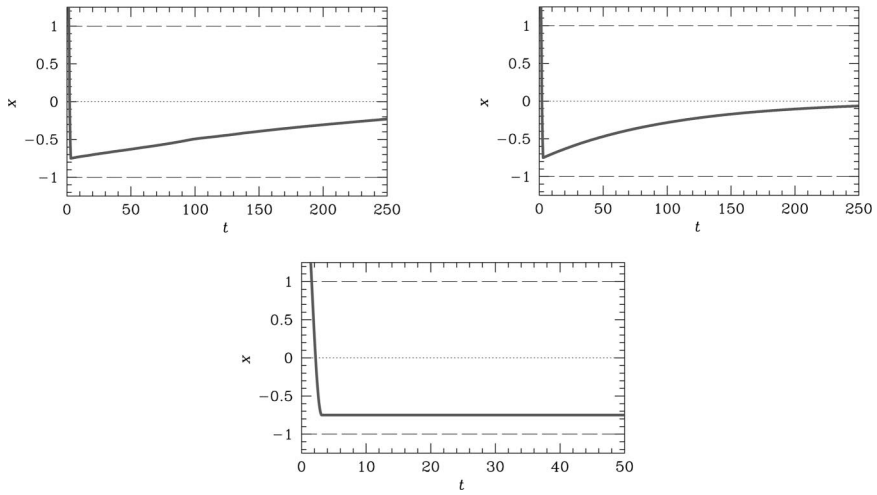

Fig. 5 Numerical simulations of Eq. (5) (4th-order Runge-Kutta method, $\Delta t=0.01$). Each integration begins with the initial conditions $(x(0),\dot{x}(0))=(2.75,0.00)$. Note the different scale used in Fig. 5(c).

in finite times. Instead, such systems exhibit exponential decay near the zero contact velocity state. However, the numerical implementation of Eqs. (4) leads to a stiff system of differential equations near the sticking state that rapidly approaches the asymptotic state, with the possible zero velocity states to which a response can approach being identical to those that exist with the use of the discontinuous model of Coulomb friction.

3 Numerical Simulations

The significant advantage in the use of Eqs. (4) comes with the numerical simulation of frictional systems by simple, commonly used numerical algorithms. Because the force of friction described by Eqs. (4) possesses a continuous representation, simulations based on fixed step sizes are significantly less sensitive to numerical error. Moreover, the present regularization does not require the simulator to resolve the transitions between intervals of stick and slip. The model of Eqs. (4) shares these features with that given in Eq. (3). However, unlike common regularizations of friction, Eqs. (4) maintains a structure at zero contact velocity that is identical to that described by Coulomb friction, therefore allowing a straightforward and computationally efficient simulation of slip-stick behavior. In the numerical studies below, a 4th-order Runge-Kutta method is applied with fixed step size. However, simulations with adaptive step size as well as stiff integrators exhibit the same behavior (e.g., force and velocity chatter for the classical discontinuous implementation) as shown below. In Appendix A the pseudo-code is given for the numerical evaluation of Eqs. (4).

3.1 1-dof Oscillator. To illustrate these properties, we first consider the numerical simulation of a simple forced harmonic oscillator, subject to the representations of friction given in Eqs. (1), (2), (3), and (4), that is:

$$\ddot{x}+x=f_{\text{friction}}+\alpha\sin(\omega t). \tag{5}$$

For this system, the force required to balance the nonfrictional forces is simply:

$$f_{\text{eq}}=x-\alpha\sin(\omega t).$$

In each computation we use a 4th-order Runge-Kutta method with fixed step size of Δt, and with $\mu N=1.00$. Unless noted, $\Delta t=0.01$. This algorithm is one of the most commonly used methods for numerically solving systems of equations.

Finally, in Eqs. (3) and (4), we choose $\epsilon=2\Delta t$. All variables are of double precision.

3.1.1 Unforced Response. We first consider the unforced response ($\alpha=0$). The system is subject to the initial conditions $(x(0),\dot{x}(0))=(2.75,0.00)$ and the response is shown in Fig. 5. For the initial interval of sliding, each simulation is nearly identical. However, once the trajectory ''sticks'' (that is, reaches a state for which $\dot{x}(t)$ is approximately 0), the trajectories diverge. Because of the discontinuity in the standard definition of Coulomb friction, the numerical simulation suffers from discretization errors and begins to drift toward the origin (see Fig. 5(a)). This can be explained by noting that due to the discrete nature of the numerical simulation, the system never exactly satisfies the condition $\dot{x}=0$. Instead, as the system approaches the equilibrium the frictional force jumps erratically between $+1$ and -1 (force chatter) as the velocity never takes a numerical value of *exactly* zero. Likewise, using Karnopp's original force-balance model (not shown) the response fails to reach a nonzero equilibrium point.

Simulating with the regularization presented in Eq. (3) the model no longer possesses nontrivial equilibrium points. Therefore, the system can never equilibrate at a nonzero displacement as illustrated in Fig. 5(b). It suffers the same long-term fate as the direct implementation of Coulomb friction, although because Eq. (3) is continuous the frictional force does not chatter, and the drift toward the origin is smooth.

In contrast, using the regularization presented in Eqs. (4), the system approaches a nontrivial equilibrium at $x_{\text{eq,num}}=-0.750090$. This value can be compared with the exact value of $x_{\text{eq,exact}}=-0.750000$. The numerical integration no longer suffers from the long-term drift that characterizes the results shown in Figs. 5(a) and 5(b) and moreover, since the friction model is continuous the simulation does not suffer from force chatter, as illustrated in Fig. 6.

Regularization Error. As illustrated in the above example, the response of the system with the friction regularization given in Eqs. (4) is qualitatively identical to the closed-form solution of Eq. (5). However, the regularization does alter the form of Coulomb friction so that the regularized response, defined as $x_\epsilon(t)$, will differ from the exact solution $x_{\text{ex}}(t)$. *s* we show below, this error due to the introduction of the regularized friction model is minimal.

Provided the sliding velocity is sufficiently large, the regularized and exact friction force will be identical. Therefore, we compare the resulting solutions beginning at the point where the friction models first differ. We consider the response subject to the initial conditions:

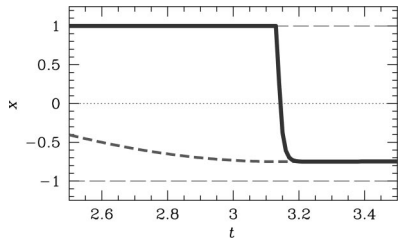

Fig. 6 Force evolution for the proposed friction model (Eqs. (4)) near the point of sticking (cf. Fig. 5(c)). Note that $2.50 \leq t \leq 3.50$. The dashed line represents $x(t)$ while the solid line reflects $f(t)$, the value of the frictional force as a function of time.

$$x(0)=x_0, \quad \dot{x}(0)=\frac{\epsilon}{2\mu N}\left(\frac{x_0}{\mu N}-1\right)<0,$$

with $|x_0|<(\mu N-o(\epsilon))$. The exact solution can easily be found to be:

$$x_{\text{ex}}(t)=A\cos(t-\phi)+\mu N,$$

with:

$$A=\sqrt{(x_0-\mu N)^2+\frac{\epsilon^2}{4}\left(\frac{x_0}{\mu N}-1\right)^2}$$

$$\tan\phi=\frac{\epsilon}{2\mu N}.$$

Therefore, the system comes to rest at a value $x_{\text{eq},f}$, with:

$$x_{\text{ex},f}=\sqrt{(x_0-\mu N)^2+\frac{\epsilon^2}{4}\left(\frac{x_0}{\mu N}-1\right)^2}+\mu N,$$

$$=x_0+\epsilon^2\frac{x_0-\mu N}{8(\mu N)^2}+\mathcal{O}(\epsilon^4).$$

Using the regularized friction model, in this interval the friction force becomes:

$$f=-\frac{2\mu N}{\epsilon}\dot{x}+f_{\text{eq}}.$$

With $f_{\text{eq}}=x$, in this interval of motion Eq. (5) reduces to:

$$\ddot{x}_\epsilon=-\frac{2\mu N}{\epsilon}\dot{x}_\epsilon.$$

For the above initial conditions the solution becomes:

$$x_\epsilon(t)=\left(x_0+\epsilon^2\frac{x_0-\mu N}{4(\mu N)^2}\right)-\epsilon^2\left(\frac{x_0-\mu N}{4(\mu N)^2}\right)e^{-(2\mu N/\epsilon)t},$$

and the system comes to rest at a value $x_{\epsilon,f}$, with:

$$x_{\epsilon,f}=x_0+\epsilon^2\frac{x_0-\mu N}{4(\mu N)^2}.$$

The error due to the regularization $e_r=x_{\text{exact},f}-x_{\epsilon,f}$ is simply:

$$e_r=-\epsilon^2\frac{x_0-\mu N}{4(\mu N)^2}+\mathcal{O}(\epsilon^4). \tag{6}$$

Therefore the position error induced by this regularized model is $\mathcal{O}(\epsilon^2)$ for fixed μN. Also, as noted above, the regularized model does not come to rest in finite time, as is predicted with the use of Coulomb friction. However, the system does decay exponentially fast to the equilibrium point, with a decay rate $\sim 1/\epsilon$.

3.1.2 Forced Response. Like the unforced response, for nonzero forcing the behavior of Eq. (5) subject to the classical definition of Coulomb friction can be solved in closed form over any time interval of constant sliding direction. The transition between intervals is determined by solving for the time at which the sliding velocity vanishes—the solution to a transcendental equation that must be performed numerically. The result is a semi-analytical solution for the dynamical behavior of Eq. (5). In what follows this semi-analytical solution is convergent to the 12th significant digit. We refer to this as the ''exact'' solution, in contrast to purely numerical solutions generated by solving Eq. (5) using for example, as in this work, Runge-Kutta methods. With sufficiently small Δt, each of the friction models considered above can accurately resolve the motion of the system, provided the system does not undergo permanent sticking (as was illustrated in Section 3.1.1). However, the rate at which the predicted response converges to the exact solution varies for each of the models.

The frictional dissipation per unit forcing cycle can be written as:

$$D=-\int_0^{2\pi/\omega}\dot{x}(t)f_{\text{friction}}(t)dt.$$

The value of D is sensitive to the representation of friction and, in particular, the behavior of the system at the transitions between stick and slip. In Fig. 7 we calculate the percent error in the frictional dissipation per unit forcing cycle, defined as:

$$\%E=\frac{D-D_{\text{exact}}}{D_{\text{exact}}}$$

for each of the frictional models as Δt is decreased. The frictional oscillator was integrated for 49 forcing cycles to remove any transient dynamics. The frictional dissipation was then calculated on the following forcing cycle. The exact value, D_{exact}, is calculated from the semi-analytical solution described above.

In Fig. 7(a), the forcing frequency and amplitude were chosen so that the system undergoes a sustained period of sticking during the response, and the response exhibits slip-stick transitions. The regularization tolerance was chosen to be $\epsilon=2\Delta t$. Although this regularization leads to a stiff system of equations near the equilibria, the sliding velocities are $\mathcal{O}(\epsilon)$ and for this value of ϵ (as a function of Δt) numerical instability is not encountered.

As expected, for the proposed model the percent error in D scales as $(\Delta t)^2$. In contrast, both the classical implementation of Coulomb friction and the velocity limited regularization scale as Δt. We note that regularized models vary smoothly, while the classical implementation varies irregularly due to the discontinuity in its definition. The classical implementation is also sensitive to, for example, the initial conditions of the system.

In Fig. 7(b), the forcing frequency and amplitude were chosen so that the system does not undergo an interval of sticking during the response. In this simulation, the regularized models give comparable results while the straightforward implementation of the classical definition performs best. However, again the results from the regularized models vary smoothly, while the result from the classical implementation varies irregularly. Notice that the dissipation error in each model scales as $(\Delta t)^2$.

3.2 *n-dof Oscillator.* We consider an n-degree-of-freedom model composed of an elastic chain of elements on a frictional surface, as illustrated in Fig. 8. This system is a version of the Burridge-Knopff model used to study earthquake faults [18,19] and more recently the dissipation induced by mechanical joints in structural systems [20,21]. Berger et al. [9] have formulated a similar problem using a mixed differential-algebraic formulation in which the transitions between stick and slip are explicitly determined.

The discrete equations of motion are written as:

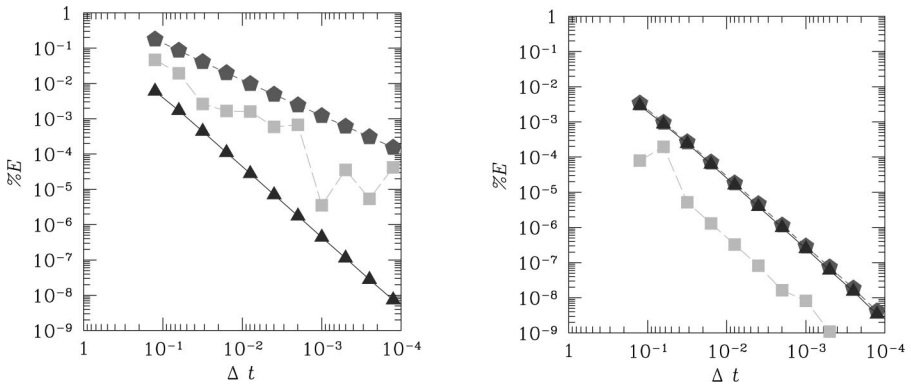

Fig. 7 Percent dissipation error per forcing cycle (μ=1). D_{exact} is the value calculated from the semi-analytical solution. The proposed friction law is marked with triangles and connected by solid lines, the classical definition of Coulomb friction is marked with squares and connected by long dashes, while the velocity-limited friction law is marked with pentagons and connected by short dashes. In panel b, the proposed and velocity-limited regularizations are almost coincident.

$$\frac{1}{n}\ddot{u}_1+n(u_1-u_2)=\frac{f_1}{n}-G(t),$$

$$\frac{1}{n}\ddot{u}_2+n(-u_1+2u_2-u_3)=\frac{f_2}{n},$$

$$\vdots$$

$$\frac{1}{n}\ddot{u}_j+n(-u_{j-1}+2u_j-u_{j+1})=\frac{f_j}{n},$$

$$\vdots$$

$$\frac{1}{n}\ddot{u}_n+n\left(-\frac{4}{3}u_{n-1}+4u_n\right)=\frac{f_n}{n},$$

We restrict the external loading to be harmonic, of the form $G(t)=\alpha\sin(\omega t)$. The description of the friction force f_i follows from the appropriate model of friction (Eqs. (1), (2), (3), or (4)). Likewise, $f_{\mathrm{eq},i}$ represents the force required to impose zero acceleration on the ith element. For example, $f_{\mathrm{eq},1}$ and $f_{\mathrm{eq},2}$ can be written as:

$$f_{\mathrm{eq},1}=n^2(u_1-u_2)+nG(t),$$

$$f_{\mathrm{eq},2}=n^2(-u_1+2u_2-u_3),$$

As the number of blocks increases, the computational requirements necessary to solve the exact problem, that is, to explicitly identify the transitions of each block between stick and slip, increase significantly and one could recast this system into the form of a Linear Complementarity Problem [22]. Instead, we apply the regularization presented above, allowing us to solve this system using a straightforward, fixed step size algorithm, without altering the formulation of the problem.

For this system, the lowest natural frequency of the chain approaches unity as $n\to\infty$. Also, the chain of oscillators is expected to undergo partial slip for loading amplitudes $|\alpha|<1$. The dissipation induced by the frictional interface per cycle of the forcing is:

$$D=-\frac{1}{n}\sum_{i=1}^{n}\left(\int_0^{2\pi/\omega}\dot{u}_i(t)f_i(t)dt\right).$$

The dissipation is very sensitive to the representation of the friction force. As in the one degree-of-freedom example, this system is integrated using a 4th-order Runge-Kutta method with fixed step size of Δt. In the regularized models (Eqs. (3) and (4)), we choose $\epsilon=10\omega\cdot\Delta t$. All variables are of double precision. With this value of ϵ, as the step size approaches zero, each friction regularization converges to the classical definition of Coulomb friction.

In Fig. 9 we show the dissipation per forcing cycle as the numerical step size is varied. In the calculation of D we begin the simulation with zero initial conditions and integrate for three forcing cycles to remove any transient dynamics. (Numerically we find that for the low forcing frequencies considered, the dissipation per cycle settles down after the second forcing cycle.) The value of D shown is then determined from the fourth forcing cycle. For each model of friction, the dissipation per cycle converges to $D_0=0.0105$ as Δt is decreased. However, as seen in the figure, as Δt is increased, the implementation of both the classical definition of friction as well as the velocity-limited friction model exhibit significant errors. In contrast, the dissipation calculated using the proposed model given in Eqs. (4) remains much more accurate over several orders of magnitude (up to $\Delta t=0.01$ is this simulation). Note that unlike the single degree-of-freedom example described above, the application of a semi-analytical solution for this problem is, although possible, computationally intensive.

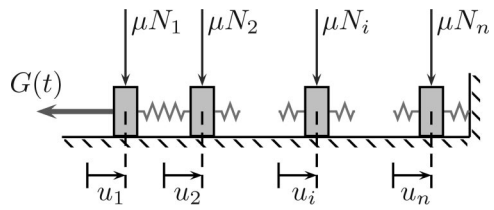

Fig. 8 *n*-dof discrete model. We consider uniform normal loads, with μN_i=1.

3.3 Multiple Frictional Contacts. In the examples of Secs. 3.1 and 3.2, the gross force f_{eq} across each interface is uniquely determined by the configuration of the system. However, for problems in which multiple frictional contacts occur on the same rigid body the force f_{eq} at each contact necessary to balance the remaining forces on that body depends on the state of the remaining contacts (slip vs. stick). Karnopp overcame this by separately formulating different sets of equations depending on the state of the contacts [2]. Similarly, with the current regularization described by Eqs. (4), f_{eq} must be determined in a way consistent with the contact states [22,23].

Determining $\mathbf{f}_{\mathrm{eq}}$ is particularly straightforward if the equations of motion can be partitioned into those variables associated with states of potential sticking, including near-sticking, and those de-

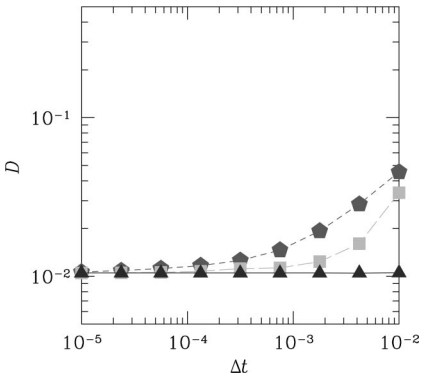

Fig. 9 Friction-induced dissipation for the *n*-dof chain ($n=32$, $\alpha=0.25$, $\omega=0.50$) as the numerical step size Δt is varied. The proposed friction law is marked with triangles and connected by solid lines, the classical definition of Coulomb friction is marked with squares and connected by long dashes, while the velocity-limited friction law is marked with pentagons and connected by short dashes.

scribing sliding motions (including all nonfrictional contacts). For a system with n frictional contacts, the equations for the relative motion across the interfaces can be written as:

$$\begin{bmatrix} \ddot{z}_{\text{slip}} \\ \ddot{z}_{\text{stick}} \end{bmatrix} + \begin{bmatrix} \mathbf{f}_{\text{ex,slip}} \\ \mathbf{f}_{\text{ex,stick}} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{f}_{\text{r}} \end{bmatrix}$$

where $\ddot{z}_{\text{slip}} \in \mathbb{R}^{\ell}$ represents the acceleration variables that are currently sliding with $[\dot{z}_{\text{slip}}]_i > \epsilon$, and $\ddot{z}_{\text{stick}} \in \mathbb{R}^{m}$ denotes those that are near sticking so that $[\dot{z}_{\text{stick}}]_j \leqslant \epsilon$. Here ϵ again represents the regularization parameter and $\ell + m = n$. Also, $\boldsymbol{\mu} \in \mathbb{R}^{\ell}$ is the sliding friction force at each slipping contact while $\mathbf{f}_{\text{r}} \in \mathbb{R}^{m}$ is the regularized frictional force for those contacts with incipient slip. The terms $\mathbf{f}_{\text{ex}}$ contain all non-frictional forces partitioned according to the slip and stick variables. With this formulation, $\mathbf{f}_{\text{eq}} \in \mathbb{R}^{m}$ is determined by setting $\dot{z}_{\text{stick}} \equiv \mathbf{0}$, so that:

$$\mathbf{f}_{\text{eq}} = \mathbf{R}_{22}^{-1}(\mathbf{f}_{\text{ex,stick}} - \mathbf{R}_{21} \cdot \boldsymbol{\mu}).$$

Once $\mathbf{f}_{\text{eq}}$ is determined the proposed friction regularization can be used and the dynamic equations of motion remain continuous.

As an example of multiple contacts, we consider the system shown in Fig. 10, consisting of two blocks resting one on the other. The lower block then rests on a moving belt and each mass is connected to the ground by a linear spring. The nondimensional equations of motion for this system can be written as:

$$\ddot{x}_1 + x_1 = f_1 - f_2, \tag{7a}$$

$$\ddot{x}_2 + \kappa x_2 = f_2, \tag{7b}$$

where f_1 represents the friction force between the belt and the lower surface, while f_2 is the friction force acting between the two blocks. The coefficient of friction between the two blocks is μ_2 while the coefficient of friction between the lower block and the belt is μ_1. In terms of the relative velocities across the friction interfaces:

$$\dot{z}_1 = \dot{x}_1 - \dot{u}(t), \quad \dot{z}_2 = \dot{x}_2 - \dot{x}_1,$$

the equations for the relative accelerations across the interfaces are:

$$\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} \ddot{u} + x_1 \\ \kappa x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

If the lower surface is at or near a sticking state, so that $|\dot{z}_1| \leqslant \epsilon$, then the gross force can easily be determined as:

$$f_{1,\text{eq}} = (\ddot{u} + x_1) + f_2,$$

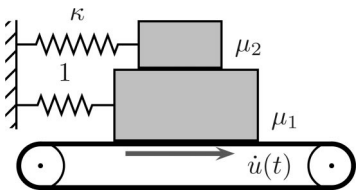

Fig. 10 Multiple frictional contacts. The lower block rests on a surface moving with velocity $\dot{u}(t)$.

while if the upper interface is near a state of sticking ($|\dot{z}_2| \leqslant \epsilon$):

$$f_{2,\text{eq}} = \frac{1}{2}(f_1 + \kappa x_2 - x_1).$$

We emphasize that for the lower interface, the force $f_{1,\text{eq}}$ depends on f_2, which is only equal to $f_{2,\text{eq}}$ if the second interface is also sticking. If instead the second interface is undergoing slip with $\dot{z}_2 > \epsilon$, then f_2 takes on its sliding friction value. A similar statement holds for the force at the upper interface. With the values of f_{eq} specified, the actual frictional forces used in Eqs. (7) are determined by the proposed regularization. A numerical simulation of the stick-slip motions that this system can undergo is shown in Fig. 11 for $u(t) = \sin(0.50t)$. In this simulation each block is initially at rest and the system parameters are listed in the figure caption. Figure 11(b) shows the relative velocities across each interface and one can clearly see substantial intervals of sticking across both interfaces. Finally, these results are indistinguishable from simulations performed using the classical definition of Coulomb friction.

4 Discussion

The proposed regularization of Coulomb's law of friction allows for a straightforward incorporation of friction within existing, commonly used numerical algorithms. The frictional model is a continuous function of the system state (the contact velocity and the force required to instantaneously impose zero acceleration) and therefore does not exhibit numerical chatter near transitions between stick and slip for sufficiently small step sizes. In addition, this reformulation allows for nontrivial equilibrium states without the need to separately identify these transitions between states of slip and stick within the numerical algorithm [4].

The computational requirements associated with Eqs. (4) are almost identical to both the classical definition of Coulomb friction as well as velocity-limited representations, as in Eq. (3). The calculation of f from Eqs. (4) requires a single additional calculation of the quantity ζ (c.f., Eq. (4)). In the examples considered, the use of constant step size algorithms, without the need to explicitly identify the transitions between slip and stick, more than compensates for this minimal cost. Finally, as evidenced in Fig. 9, one can often use step sizes that are orders of magnitude larger than in simulations with alternative friction laws.

The proposed regularization does not seek to develop a more fundamental, physics-based model of friction (like those using internal variables) nor does it attempt to *exactly* represent the discontinuous Coulomb model (as do many complementarity-based formulations). Instead, it provides a computationally efficient model that converges to Coulomb friction as the regularization parameter ϵ is decreased, which does not suffer from many of the numerical problems seen in other, more commonly used regularizations (primarily near the equilibrium values). For problems in which slip-stick transitions play a central role, the proposed model may be used with minimal additional effort by the simulator and to obtain reasonable quantitative accuracy (depending on ϵ).

As for any regularization technique, the parameter ϵ should be chosen with care. Examining Eq. (6), ϵ should be chosen so that the quantity $\epsilon/(\mu N)$ is small to minimize the regularization error.

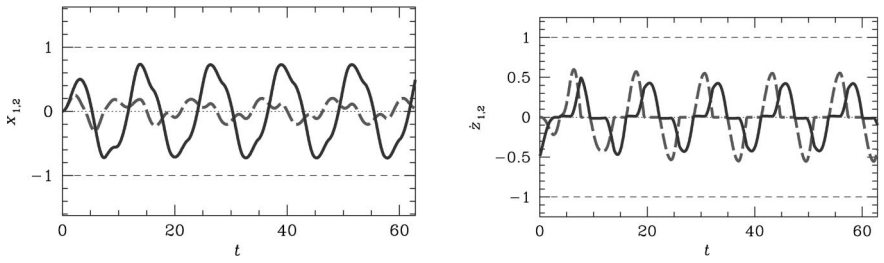

Fig. 11 Stick-slip dynamics generated with multiple frictional contacts with $\kappa=2$, $\mu_1=0.50$, $\mu_2=0.25$, $u(t)=\sin(0.50t)$. In each panel the lower block $(x_1,\dot{z}_1)$ is indicated by the solid line while the response of the upper block $(x_2,\dot{z}_2)$ is dashed.

However, the regularization constant should be chosen large enough so that numerical instability associated with the discretization of the stiff equations near the equilibria does not occur. In this work, $\epsilon=2\Delta t$. More generally, the optimal choice depends on the problem to be solved.

The proposed form of the frictional force is composed of affine functions (see Fig. 4). This model could easily be modified to meet any necessary smoothness requirements, similar in spirit to the quadratic velocity-limited friction model of Oden and Martins [13]. If the regularization is non-Lipschitz near the equilibrium state (for example $f-f_{\text{eq}}\sim(\dot{x})^{1/3}$), the analytic response will stick in finite time, qualitatively similar to the response of a system subject to the classical definition of Coulomb friction. The proposed regularization of friction leads to analytic solutions that undergo exponential decay to the equilibrium. With finite precision computations this exponential decay is indistinguishable from finite time sticking (see Fig. 6). Finally, the frictional force away from $\dot{x}=0$ could be modified to include static and kinetic coefficients of friction, as well as dependence on contact velocity, internal state variables [15,24], and normal loads (see [13] and the references therein).

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A Pseudo-Code for Eq. (4)

The modified frictional law is easily implemented into existing numerical models. The pseudo-code for Eq. (4) can be given as:

```
xdot: contact velocity, ẋ;
feq: force required to balance the gross force, f_eq;
muN: frictional intensity, μN;
eps: regularization parameter, ϵ;
f: resulting frictional force, f;
if (abs(feq)>muN)
      zeta=sgn(feq/muN);
else
      zeta=feq/muN;
endif
if (abs(xdot−eps*zeta)>eps)
      f=−muN*sgn(xdot−eps*zeta);
else
      f=−muN*(xdot/eps−zeta);
endif
```

The variable f is subsequently used in the numerical routine as the frictional force.

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