WEIGHTED NORM INEQUALITIES FOR PARAPRODUCTS AND BILINEAR PSEUDODIFFERENTIAL OPERATORS WITH MILD REGULARITY

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Abstract. We establish boundedness properties on products of weighted Lebesgue, Hardy, and amalgam spaces of certain paraproducts and bilinear pseudodifferential operators with mild regularity. We do so by showing that these operators can be realized as generalized bilinear Calderón-Zygmund operators.

1. Bilinear pseudodifferential operators with mild regularity

Let us motivate our main result on bilinear pseudodifferential operators (ΨDOs) by revisiting some facts from the linear theory. A sufficiently regular function $\sigma(x,\xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ has an associated ΨDO T_σ defined by

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \qquad x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).$$

Here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class and $\hat{f}$ denotes the Fourier transform of f,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^n.$$

For $m \in \mathbb{R}$, $0 \le \delta, \rho \le 1$, the *symbol* $\sigma(x,\xi)$ belongs to Hörmander's class $S^m_{\rho,\delta}$ if

$$(1.1) \qquad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}, \quad x,\xi \in \mathbb{R}^n,$$

where α, $\beta \in \mathbb{Z}^n$ and $|\alpha|$, $|\beta|$ depend on the context.

The exploration of classes of smooth symbols, in particular the classes $S^m_{\rho,\delta}$, appears to be predominant in the ΨDO literature. However, as diverse problems in Analysis and PDEs demand, the case in which the symbol has mild or no regularity in x has received considerable attention, see for instance [33], [34], [35], [42], [44], [45], and references therein. For $\omega, \Omega : [0,\infty) \to [0,\infty)$, $m \in \mathbb{R}$ and $\rho \in (0,1)$, we write $\sigma \in S^m_{\rho,\omega,\Omega}$ (this notation is not standard, we introduce it for the sake of presentation) if

$$(1.2) \qquad |\partial_\xi^\beta \sigma(x,\xi)| \le C_\beta(1+|\xi|)^{m-\rho|\beta|}, \qquad x,\xi \in \mathbb{R}^n,$$

and

$$(1.3) \qquad |\partial_\xi^\beta \sigma(x+h,\xi) - \partial_\xi^\beta \sigma(x,\xi)| \le C_\beta \omega(|h|)\Omega(|\xi|)(1+|\xi|)^{m-\rho|\beta|}, \qquad x,\xi \in \mathbb{R}^n.$$

Again, the number of derivatives $|\beta|$ depends on the context.

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For $a > 0$, we write $\omega \in Dini(a)$ if $\omega : [0, \infty) \to [0, \infty)$, ω is non-decreasing, concave, and

$$|\omega|_{Dini(a)} := \int_0^1 \omega^a(t) \frac{dt}{t} < \infty.$$

The classes $S^m_{\rho,\omega,\Omega}$ were originally motivated by a question posed by L. Nirenberg about whether symbols verifying (1.2) with $m = 0$, $\rho = 1$, and all multi-indices β (no a priori regularity in x involved) produce L^2-bounded ΨDOs. In [10], C.-H. Ching resolved this question in the negative. Afterwards, a number of authors showed that this lack of L^2-boundedness could be circumvented if a mild regularity assumption on the x-variable were assumed. Indeed, R. Coifman and Y. Meyer proved (see [14, Theorem 9, p.38]) that if $\sigma \in S^0_{1,\omega,\Omega}$ with $\Omega \equiv 1$ then T_σ is bounded in $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ if and only if $\omega \in Dini(2)$. M. Nagase proved (Theorem B in [38]) that T_σ is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ when $\omega(t) = t^\tau$ and $\Omega(t) = t^\gamma$ for some $0 \le \gamma < \tau \le 1$, and $|\beta| \le n + 2$. In [7], G. Bourdaud proved that if $\sigma \in S^0_{1,\omega,\Omega}$, then T_σ is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if

$$\sum_{j\in\mathbb{N}} \omega^2(2^{-j})\Omega^2(2^j) < \infty.$$

On the other hand, it is known that the Hörmander class $S^0_{1,1}$ is maximal with respect to the property of producing ΨDOs with Calderón-Zygmund kernels, however these ΨDOs need not be bounded in $L^2(\mathbb{R}^n)$ and "they must remain forbidden fruit" ([43, Chapter VII]). Notice that the class of forbidden symbols $S^0_{1,1}$ satisfies $S^0_{1,1} \subset S^0_{1,\omega_0,\Omega_0}$, where $\omega_0(t) = t$ and $\Omega_0(t) = 1 + t$. In [47] and [48], K. Yabuta developed the notion of Calderón-Zygmund operator of type $\omega(t)$ (which includes the classical Calderón-Zygmund operators), and determined conditions on a symbol $\sigma \in S^0_{1,\omega,\Omega}$, and on the functions ω, Ω, so that T_σ can be realized as a Calderón-Zygmund operator of type $\omega(t)$. As a consequence he also obtained L^∞-BMO and weighted L^p-estimates for T_σ. Similar estimates were obtained, using different methods, by S. Nishigaki in [39] and S. Sato in [41].

Let us now describe the relevant objects of the bilinear theory of ΨDOs. A sufficiently regular function $\sigma(x, \xi, \eta)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ has an associated bilinear pseudodifferential operator T_σ defined by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix\cdot(\xi+\eta)} \sigma(x, \xi, \eta) \hat{f}(\xi)\hat{g}(\eta)\, d\xi d\eta, \qquad x \in \mathbb{R}^n, f, g \in \mathcal{S}(\mathbb{R}^n).$$

We say that the bilinear symbol $\sigma(x, \xi, \eta)$ belongs to the bilinear Hörmander class $BS^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \le C_{\alpha,\beta}(1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}, \quad x, \xi, \eta \in \mathbb{R}^n. \tag{1.4}$$

We also write $\sigma \in BS^m_{\rho,\omega,\Omega}$ if

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \sigma(x, \xi, \eta)\right| \le C_{\alpha,\beta}(1 + |\xi| + |\eta|)^{m-\rho(|\alpha|+|\beta|)}, \tag{1.5}$$

$$\left|\partial_\xi^\alpha \partial_\eta^\beta (\sigma(x + h, \xi, \eta) - \sigma(x, \xi, \eta))\right| \le C_{\alpha,\beta}\, \omega(|h|)\Omega(|\xi| + |\eta|)(1 + |\xi| + |\eta|)^{m-\rho(|\alpha|+|\beta|)}, \tag{1.6}$$

for all x, $\xi, \eta \in \mathbb{R}^n$. As usual, the sizes of the multiindices α, $\beta \in \mathbb{Z}^n$ will depend on the context.

The study of bilinear ΨDOs grew from the seminal work of R. Coifman and Y. Meyer [13], [14] who used them as models to represent Calderón-Zygmund commutators. Further applications now include the study of compensated compactness, see [11], [12], and [50], and, as bilinear ΨDOs also model expressions of the type $\sum_{\alpha,\beta} c_{\alpha,\beta}\partial_x^\alpha f \partial_x^\beta g$, they are useful in generalizing Leibnitz's rule in the spirit of the Kato-Ponce inequality, see [4] and [37].

The bilinear setting is translucent to various well-known linear ΨDO estimates that project their natural bilinear analogues (see, for instance, [1], [2], [4], [5], and [14]), but, at the same time, it is opaque to some other. For example, a celebrated theorem of A. Calderón and R. Vaillancourt establishes the L^2-boundedness of ΨDOs with smooth symbols in the class $S^0_{0,0}$. In contrast, as Á. Bényi and R. Torres showed in [5], the class $BS^0_{0,0}$ does not mimic that mapping behavior in the corresponding function space scene of $L^2 \times L^2 \to L^1$, even for x-independent, tensor-like symbols. Another example is the linear Marcinkiewicz multiplier theorem, whose natural bilinear version also fails, as shown by L. Grafakos and N. Kalton in [24]. This semitransparency phenomenon adds to the interest in bilinear ΨDO estimates.

Clearly, we have $BS^0_{1,1} \subset BS^0_{1,\omega_0,\Omega_0}$. The class $BS^0_{1,1}$ produces bilinear ΨDOs with bilinear Calderón-Zygmund kernels in the sense of L. Grafakos and R. Torres [26], and, as proved by Á. Bényi and R. Torres [4], it also remains forbidden. Here we implement a bilinear interpretation of Yabuta's scheme [47], [48]. In Section 3, we introduce the notion of bilinear Calderón-Zygmund operator of type $\omega(t)$. In Section 4 we show that under suitable assumptions on ω and Ω the ΨDOs with symbols in the class $BS^0_{1,\omega,\Omega}$ can be realized as bilinear Calderón-Zygmund operators of type $\omega(t)$. As a consequence we obtain our first main theorem. Namely,

Theorem 1.1. *Let $a \in (0,1)$, $\omega \in Dini(a/2)$, and $\Omega : [0,\infty) \to [0,\infty)$ non-decreasing such that*

$$\sup_{0<t<1} \omega^{1-a}(t)\Omega(1/t) < \infty.$$

Consider $1 \le p,\ q \le \infty$ and $\frac{1}{2} \le r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, if $\sigma \in BS^0_{1,\omega,\Omega}$, with $|\alpha|+|\beta| \le 4n+4$, the bilinear pseudo-differential operator T_σ has the following boundedness properties:

(i) if $1 < p,\ q$, then

$$\|T_\sigma(f,g)\|_{L^r(\mathbb{R}^n)} \le C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}\,,$$

where $L^p(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$ should be replaced by $L^\infty_c(\mathbb{R}^n)$ (bounded functions with compact support) if $p=\infty$ or $q=\infty$, respectively;

(ii) if $p=1$ or $q=1$, then

$$\|T_\sigma(f,g)\|_{L^{r,\infty}(\mathbb{R}^n)} \le C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}\,,$$

where $L^p(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$ should be replaced by $L^\infty_c(\mathbb{R}^n)$ if $p=\infty$ or $q=\infty$, respectively;

(iii)

$$\|T_\sigma(f,g)\|_{BMO(\mathbb{R}^n)} \le C\, \|f\|_{L^\infty(\mathbb{R}^n)}\, \|g\|_{L^\infty(\mathbb{R}^n)}\,.$$

(iv) If $1 < p,q < \infty$, and $w \in A_{\min(p,q)}$, then

$$\|T_\sigma(f,g)\|_{L^r_w(\mathbb{R}^n)} \le C\, \|f\|_{L^p_w(\mathbb{R}^n)}\, \|g\|_{L^q_w(\mathbb{R}^n)}\,,$$

where A_r, $1 \le r \le \infty$, denotes the Muckenhoupt weight class (see Section 6.2 for the definition).

(v) *If $w \in A_1$, the following endpoint estimates hold*

$$\|T_\sigma(f,g)\|_{L^{1/2,\infty}_w(\mathbb{R}^n)} \leq C\,\|f\|_{L^1_w(\mathbb{R}^n)}\,\|g\|_{L^1_w(\mathbb{R}^n)}$$

and

$$\|T_\sigma(f,g)\|_{L^{1/2}_w(\mathbb{R}^n)} \leq C\,\|f\|_{H^1_w(\mathbb{R}^n)}\,\|g\|_{H^1_w(\mathbb{R}^n)}\,.$$

(vi) *Finally, if $1 < p,q < \infty$, $1 < s_1, s_2 < \infty$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_{\min(s_1,s_2)}(\mathbb{Z}^n)$, then T_σ verifies the following inequality on weighted amalgam spaces*

$$\|T_\sigma(f,g)\|_{(L^r,l^{s_3}_w)} \leq C\,\|f\|_{(L^p,l^{s_1}_w)}\,\|g\|_{(L^q,l^{s_2}_w)}\,.$$

Remark 1. To the best of our knowledge, the only result on bilinear ΨDOs with mild regularity previous to Theorem 1.1 is Theorem 12 in Coifman-Meyer [14, p.55] where the symbol $\sigma(x,\xi,\eta)$ belongs to $BS^0_{1,\omega,\Omega}$ with $\omega \in Dini(2)$ and $\Omega \equiv 1$.[1] Theorem 12 in Coifman-Meyer [14, p.55] deals with unweighted Lebesgue spaces and asserts that the associated bilinear ΨDO maps $L^p \times L^q$ into L^r for $1/r = 1/p + 1/q$, $1 < p,q,r < \infty$. In the case of unweighted Lebesgue spaces, Theorem 1.1 allows for more general choices of Ω and brings the exponent r down to $1/2$ (with weak type when $r = 1/2$), although it requires the stronger condition $\omega \in Dini(a/2)$. For the particular choices $\omega(t) = t^\tau$ and $\Omega(t) = t^\gamma$ $(0 < \gamma < \tau \leq 1)$, Theorem 1.1 lifts Theorem B in Nagase [38] to the bilinear context. Finally, we point out that, in the mentioned literature, it has been customary to ask for some kind of doubling condition on Ω and here we dispose of such a hypothesis.

2. Paraproducts with mild regularity

We now discuss the boundedness properties of paraproducts built from Dini-continuous molecules. Some notation is in order. For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $P_{\nu k}$ be the dyadic cube

$$P_{\nu k} := \{(x_1,\dots,x_n) \in \mathbb{R}^n : k_i \leq 2^\nu x_i < k_i + 1, i = 1,\dots,n\}. \tag{2.1}$$

The *lower left-corner* of $P = P_{\nu k}$ is $x_P = x_{\nu k} := 2^{-\nu}k$, the Lebesgue measure of P is $|P| = 2^{-\nu n}$, and its characteristic function is denoted by $\chi_{P_{\nu k}}$. We set

$$\mathcal{D} := \{P_{\nu k} : \nu \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

as the collection of all dyadic cubes.

Definition 2.1. Let $\omega : [0,\infty) \to [0,\infty)$ be a non-decreasing function. An *ω-molecule* associated to a dyadic cube $P = P_{\nu k}$ is a function $\phi_P = \phi_{\nu k} : \mathbb{R}^n \to \mathbb{C}$ such that, for some $A > 0$ and $N > n$, it satisfies the *decay* (or *concentration*) condition

$$|\phi_P(x)| \leq \frac{A2^{\nu n/2}}{(1 + 2^\nu|x - x_P|)^N}, \;\; x \in \mathbb{R}^n, \tag{2.2}$$

and the *mild regularity condition*

$$|\phi_P(x) - \phi_P(y)| \leq A2^{\nu n/2}\omega(2^\nu|x-y|)\left[\frac{1}{(1+2^\nu|x-x_P|)^N} + \frac{1}{(1+2^\nu|y-x_P|)^N}\right] \tag{2.3}$$

for all $x, y \in \mathbb{R}^n$.

[1]Notice that there is an omission of the factor $(1 + |\xi| + |\eta|)^{-(|\alpha|+|\beta|)}$ on the right hand side of conditions (51) on page 55 of [14] (compare with Theorem 9 on page 38 and Theorem 1 in [13]). With that factor, Theorem 12 follows from the techniques developed in the linear case that prove Theorem 9. Without that factor the result is not true, since there exist x-independent symbols with bounded derivatives that do not produce ΨDOs mapping any $L^p \times L^q$ into L^r for $1/r = 1/p + 1/q$, $1 \leq p,q,r < \infty$, see Proposition 1 in [5].

Definition 2.2. Given three families of ω-molecules $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j=1,2,3$, the *paraproduct* $\Pi(f,g)$ associated to these families is defined by

$$\Pi(f,g)=\sum_{Q\in\mathcal{D}}|Q|^{-1/2}\langle f,\phi_Q^1\rangle\langle g,\phi_Q^2\rangle\phi_Q^3,\quad f,g\in\mathcal{S}(\mathbb{R}^n). \tag{2.4}$$

The term paraproduct was coined by J.M. Bony in [6] and ever since it has been used to denote superpositions of various time-frequency components of two functions. Paraproducts have found plenty of inspired applications: from Bony's paradifferential calculus (see [6]) and David-Journé's remarkable $T(1)$-theorem (see [16]), to their alliance with wavelet analysis in the study of PDEs (see, for instance, [8], [9], [24], [44], and [45]), and their role as toy models or building blocks of classical operators in Fourier Analysis (see, for instance, [21], [22], [31], [32] [37], [36], [40], and [46]), just to mention a few. The paraproducts we treat (in Section 5) are built from mildly regular molecules which come to cover the gap between the smooth molecules and paraproducts in [3], [18], [19], and the (discontinuous) Haar molecules and paraproducts studied, for instance, in [46].

In [3], sufficient conditions on *smooth* molecules are given so that smooth paraproducts of the form (2.4) can be realized as bilinear Calderón-Zygmund operators. In Section 5 we analyze ω-molecules and prove that the paraproducts they build can be realized as bilinear Calderón-Zygmund operators of type $\omega(t)$, provided that they have enough decay, suitable cancelation, and $\omega\in Dini(1/2)$. This allows us to prove our second main result. Namely,

Theorem 2.3. *Consider $\omega\in Dini(1/2)$ and let $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j=1,2,3$ be three families of ω-molecules with decay $N>10n$ and such that at least two of them, say $j=1,2$, enjoy the cancelation property*

$$\int_{\mathbb{R}^n}\phi_Q^j(x)\,dx=0,\quad Q\in\mathcal{D}, j=1,2.$$

Then, the paraproduct $\Pi(f,g)$ defined in (2.4) *verifies the inequalities* (i)-(vi) *in Theorem* 1.1.

3. Bilinear Calderón-Zygmund operators of type $\omega(t)$

Definition 3.1. Let $\omega:[0,\infty)\to[0,\infty)$ be a non-decreasing function. We say that $K(x,y,z)$ defined on $\mathbb{R}^{3n}\setminus\{(x,y,z)\in\mathbb{R}^{3n}:x=y=z\}$ is a *bilinear Calderón-Zygmund kernel of type $\omega(t)$* if for some constants $0<\tau<1$ (the specific value of $\tau\in(0,1)$ is immaterial in the development of the theory), $C_K>0$, and every $(x,y,z)\in\mathbb{R}^{3n}\setminus\{(x,y,z)\in\mathbb{R}^{3n}:x=y=z\}$ it holds

$$|K(x,y,z)|\le\frac{C_K}{(|x-y|+|x-z|)^{2n}}, \tag{3.1}$$

and

(3.2)

$$|K(x+h,y,z)-K(x,y,z)|+|K(x,y+h,z)-K(x,y,z)|+|K(x,y,z+h)-K(x,y,z)|$$
$$\le\frac{C_K}{(|x-y|+|x-z|)^{2n}}\,\omega\left(\frac{|h|}{|x-y|+|x-z|}\right),$$

whenever $|h|\le\tau\max(|x-y|,|x-z|)$. A bilinear operator $T:\mathcal{S}\times\mathcal{S}\to\mathcal{S}'$ is said to be associated to a bilinear Calderón-Zygmund kernel of type $\omega(t)$, $K(x,y,z)$, if

$$T(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K(x,y,z)f(y)g(z)\,dydz$$

whenever $x \notin \operatorname{supp}(f) \cap \operatorname{supp}(g)$ and $f, g \in C_0^\infty$. If, besides, T maps

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^{r,\infty}(\mathbb{R}^n),$$

for some $1 < p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1/r$; or

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^1(\mathbb{R}^n),$$

for some $1 < p, q < \infty$ with $1/p+1/q = 1$, T is called a *bilinear Calderón-Zygmund operator of type* $\omega(t)$.

The multilinear Calderón-Zygmund theory was introduced by R. Coifman and Y.Meyer in [13], [14], and [15]. This theory was then further investigated by L. Grafakos and R. Torres [26], [27], who considered the case in which $\omega(t) = t^\epsilon$ for some $\epsilon \in (0, 1]$, and C. Kenig and E. Stein [29].

The plan of the proofs of Theorems 1.1 and 2.3 is as follows: Sections 4 and 5, respectively, are devoted to showing that the bilinear ΨDO operator T_σ in Theorem 1.1 and the paraproduct Π in Theorem 2.3 are bilinear Calderón-Zygmund operators of type $\omega(t)$ for suitable ω. In Section 6 we prove that bilinear Calderón-Zygmund operators of type $\omega(t)$ satisfy the boundedness properties (i)-(vi) in Theorem 1.1, which completes the plan.

4. Proof of Theorem 1.1

In this section we consider the bilinear pseudo-differential operator

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi, \eta) e^{ix(\xi+\eta)} \hat{f}(\xi) \hat{g}(\eta) \, d\xi \, d\eta,$$

x, ξ, $\eta \in \mathbb{R}^n$, f, $g \in \mathcal{S}(\mathbb{R}^n)$, whose symbol $\sigma(x, \xi, \eta)$ satisfies the following conditions:

$$\left| \partial_\xi^\alpha \partial_\eta^\beta \sigma(x, \xi, \eta) \right| \le \frac{C_{\alpha,\beta}}{(1 + |\xi| + |\eta|)^{|\alpha|+|\beta|}}, \tag{4.1}$$

$$\left| \partial_\xi^\alpha \partial_\eta^\beta (\sigma(x + h, \xi, \eta) - \sigma(x, \xi, \eta)) \right| \le C_{\alpha,\beta} \; \omega(|h|) \; \frac{\Omega(|\xi| + |\eta|)}{(1 + |\xi| + |\eta|)^{|\alpha|+|\beta|}}, \tag{4.2}$$

for all x, $\xi, \eta \in \mathbb{R}^n$, and for a certain number of multi-indices α, $\beta \in \mathbb{Z}^n$. The following theorem establishes sufficient conditions on ω and Ω so that the class $BS^0_{1,\omega,\Omega}$ produces ΨDOs with bilinear Calderón-Zygmund kernels of type $\omega^a(t)$, for some $a \in (0, 1)$.

Theorem 4.1. *Let $\omega, \Omega : [0, \infty) \to [0, \infty)$ be non-decreasing functions with ω concave. Suppose that there exists $a \in (0, 1)$ such that ω and Ω verify*

$$B := \sup_{0<t<1} \omega^{1-a}(t) \Omega(1/t) < \infty. \tag{4.3}$$

If $\sigma(x, \xi, \eta)$ verifies (4.1) *and* (4.2) *with $|\alpha|+|\beta| \le 2n+2$, then T_σ has a bilinear Calderón-Zygmund kernel of type $\omega^a(t)$.*

Proof of Theorem 4.1. It is enough to assume that σ has compact support in the variables ξ and η, uniformly in x, and to show that the constants involved do not depend on the support of $\sigma(x, \cdot, \cdot)$ (see [43, Chapter VII]). We have the following kernel representation for T_σ,

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z) \, f(y) \, g(z) \, dy \, dz, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

where

$$K(x,y,z) = \hat{\sigma}(x, y-x, z-x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta) e^{-i\xi(y-x)} e^{-i\eta(z-x)}\, d\xi\, d\eta.$$

We will show that $K(x,y,z)$ satisfies conditions (3.1) and (3.2) with ω^a and $\tau = 1/3$. In terms of the symbol σ, conditions (3.1) and (3.2) follow from

$$|\hat{\sigma}(x,y,z)| \le \frac{C}{(|y|+|z|)^{2n}}, \quad x,y,z \in \mathbb{R}^n, \tag{4.4}$$

$$|\hat{\sigma}(x+h,y,z) - \hat{\sigma}(x,y,z)| \le \frac{C}{(|y|+|z|)^{2n}}\, \omega^a\left(\frac{|h|}{|y|+|z|}\right), \quad |h| \le \frac{1}{2}\max\{|y|,|z|\}, \tag{4.5}$$

$$|\hat{\sigma}(x,y+h,z) - \hat{\sigma}(x,y,z)| \le \frac{C}{(|y|+|z|)^{2n}}\, \omega^a\left(\frac{|h|}{|y|+|z|}\right), \quad |h| \le \frac{1}{2}\max\{|y|,|z|\}, \tag{4.6}$$

$$|\hat{\sigma}(x,y,z+h) - \hat{\sigma}(x,y,z)| \le \frac{C}{(|y|+|z|)^{2n}}\, \omega^a\left(\frac{|h|}{|y|+|z|}\right), \quad |h| \le \frac{1}{2}\max\{|y|,|z|\}. \tag{4.7}$$

We will now show condition (4.5). For $j \in \mathbb{N}_0$ consider $\psi_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ infinitely differentiable with

$$\begin{aligned} &\operatorname{supp}(\psi_j) \subset \{(\xi,\eta) : 2^{j-1} \le |(\xi,\eta)| \le 2^{j+1}\} \text{ if } j \ge 1,\\ &\operatorname{supp}(\psi_0) \subset \{(\xi,\eta) : |(\xi,\eta)| \le 2\},\\ &\sum_{j\ge 0} \psi_j(\xi,\eta) = 1, \quad \xi,\eta \in \mathbb{R}^n. \end{aligned}$$

Fix h, y, and z in $\mathbb{R}^n$ such that $|h| \le \frac{1}{2}\max\{|y|,|z|\}$ and define

$$\sigma_j(x,\xi,\eta) := \psi_j(\xi,\eta)\,\sigma(x,\xi,\eta)$$

$$L_j^h(x,y,z) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\sigma_j(x+h,\xi,\eta) - \sigma_j(x,\xi,\eta)) e^{-i\xi y} e^{-i\eta z}\, d\xi\, d\eta. \tag{4.8}$$

Note that properties (4.1) and (4.2) are satisfied by σ_j uniformly on j, and that $\sum_{j\ge 0} L_j^h(x,y,z) = \hat{\sigma}(x+h,y,z) - \hat{\sigma}(x,y,z)$. Also, the concavity of ω and (4.3) imply

$$\omega(t)/t \text{ is monotone non-increasing, } t > 0, \tag{4.9}$$

$$\omega(2t) \le 2\omega(t), \quad t > 0, \text{ (i.e., } \omega \text{ is doubling with constant 2)}, \tag{4.10}$$

$$\Omega(s) \le B\omega(1)^{a-1} s^{1-a}, \quad s \ge 1. \tag{4.11}$$

From now on, given two quantities F and G we will write $F \lesssim G$ if $F \le CG$ where C is a structural constant that, according to the context, will depend on n, C_K, B, $\omega(1)$, etc.

Integrating by parts $2n+1$ times in (4.8) and applying (4.2), we have

$$\left|L_j^h(x,y,z)\right| \lesssim (|y|+|z|)^{-(2n+1)}\, \omega(|h|) \int\int_{(\xi,\eta)\in\operatorname{supp}(\psi_j)} \frac{\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^{2n+1}}\, d\xi\, d\eta.$$

Since Ω is non-decreasing, we obtain

$$\left|L_j^h(x,y,z)\right| \lesssim (|y|+|z|)^{-(2n+1)}\, \omega(|h|)\, \frac{\Omega(2^{j+2})}{2^j}. \tag{4.12}$$

We now consider two cases according to $|y|+|z|$ being greater or smaller than 1.

First Case: $|y|+|z|\geq 1$. By (4.9),

$$\omega(|h|)=\omega^{1-a}(|h|)\omega^{a}(|h|)\leq \omega^{1-a}(|h|)(|y|+|z|)^{a}\omega^{a}\left(\frac{|h|}{|y|+|z|}\right).$$

Also, since ω is non-decreasing, $|h|\leq \frac{1}{2}\max\{|y|,|z|\}$, (4.9), and $|y|+|z|\geq \frac{1}{2^j}$, we obtain

$$\omega^{1-a}(|h|)\leq \omega^{1-a}(|y|+|z|)\leq \omega^{1-a}\left(\frac{1}{2^j}\right)2^{j(1-a)}(|y|+|z|)^{1-a}.$$

Putting all together,

$$\omega(|h|)\leq (|y|+|z|)\,\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\omega^{1-a}\left(\frac{1}{2^j}\right)2^{j(1-a)}.$$

Plugging this into (4.12), and using (4.10) and (4.3),

$$\begin{aligned}\left|L_j^h(x,y,z)\right| &\lesssim (|y|+|z|)^{-2n}\,\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\frac{\Omega(2^{j+2})}{2^{ja}}\,\omega^{1-a}\left(\frac{1}{2^j}\right)\\ &\lesssim 4^{1-a}(|y|+|z|)^{-2n}\,\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\frac{\Omega(2^{j+2})}{2^{ja}}\,\omega^{1-a}\left(\frac{1}{2^{j+2}}\right)\\ &\lesssim B4^{1-a}(|y|+|z|)^{-2n}\,\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\frac{1}{2^{ja}}.\end{aligned}$$

Then,

$$|\hat{\sigma}(x+h,y,z)-\hat{\sigma}(x,y,z)|=\left|\sum_{j\geq 0}L_j^h(x,y,z)\right|\lesssim (|y|+|z|)^{-2n}\,\omega^{a}\left(\frac{|h|}{|y|+|z|}\right),$$

for $|h|\leq \frac{1}{2}\max(|y|,|z|)$ and $|y|+|z|\geq 1$.

Second Case: $|y|+|z|<1$. Assume without any loss of generality that $|z|\leq|y|$. We split the sum in j as follows

$$\begin{aligned}|\hat{\sigma}(x+h,y,z)-\hat{\sigma}(x,y,z)| &= \left|\sum_{j\geq 0}L_j^h(x,y,z)\right|\\ &\leq \sum_{1\leq 2^j|z|}\left|L_j^h\right|+\sum_{1>2^j|y|}\left|L_j^h\right|+\sum_{\substack{1\leq 2^j|y|\\ 1>2^j|z|}}\left|L_j^h\right|\\ &=: I+II+III\end{aligned}$$

Noting that $\omega(|h|)\leq\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\omega^{1-a}(|y|+|z|)$, and recalling (4.12) and (4.9),

$$\begin{aligned}I &\lesssim (|y|+|z|)^{-2n}\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\sum_{1\leq 2^j|z|}\frac{\Omega(2^{j+2})\omega^{1-a}(|y|+|z|)}{2^j(|y|+|z|)}\\ &\lesssim (|y|+|z|)^{-2n}\omega^{a}\left(\frac{|h|}{|y|+|z|}\right)\sum_{1\leq 2^j|z|}\frac{\Omega(2^{j+2})}{2^{ja}}\,\omega^{1-a}\left(\frac{2}{2^j}\right)(|y|+|z|)^{-a}.\end{aligned}$$

By (4.3) and (4.10),

$$\Omega(2^{j+2})\,\omega^{1-a}\left(\frac{2}{2^j}\right) \leq 8^{1-a}\Omega(2^{j+2})\omega^{1-a}\left(\frac{1}{2^{j+2}}\right) \leq 8^{1-a}B.$$

We then obtain,

$$I \lesssim (|y|+|z|)^{-2n}\omega^a\left(\frac{|h|}{|y|+|z|}\right).$$

We now estimate II. Integrating by parts p times in (4.8), with $n+1 \leq p < 2n$, using (4.2), and recalling that ω is non-decreasing, we have

$$\begin{aligned}
II &\lesssim (|y|+|z|)^{-p}\,\omega(|h|)\int_{|\xi|\leq 2|y|^{-1}}\int_{|\eta|\leq 2|z|^{-1}}\frac{\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^p}\,d\eta\,d\xi\\
&\lesssim (|y|+|z|)^{-p}\,\omega^a\left(\frac{|h|}{|y|+|z|}\right)\,\omega^{1-a}(|h|)\int_{|\xi|\leq 2|y|^{-1}}\int_{|\eta|\leq 2|z|^{-1}}\frac{\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^p}\,d\eta\,d\xi\\
&= (|y|+|z|)^{-p}\,\omega^a\left(\frac{|h|}{|y|+|z|}\right)\\
&\times\left(\omega^{1-a}(|h|)\int_{|\xi|\leq 2|y|^{-1}}\int_{|\eta|\leq|\xi|}\cdots\,d\eta\,d\xi + \omega^{1-a}(|h|)\int_{|\xi|\leq 2|y|^{-1}}\int_{|\eta|>|\xi|}\cdots\,d\eta\,d\xi\right)\\
&=: (|y|+|z|)^{-p}\,\omega^a\left(\frac{|h|}{|y|+|z|}\right)(II_1+II_2).
\end{aligned}$$

We will show that II_1 and II_2 are bounded by $C\,(|y|+|z|)^{p-2n}$. Using that ω and Ω are non-decreasing, $|z|\leq|y|$, (4.10), (4.3), and that $p<2n$, we get

$$II_1 \lesssim \Omega\left(\frac{8}{|y|+|z|}\right)\,\omega^{1-a}(|y|+|z|)\int_{|\xi|\leq 2|y|^{-1}}\frac{|\xi|^n}{(1+|\xi|)^p}\,d\xi \lesssim B(|y|+|z|)^{p-2n}.$$

To bound II_2 we change to polar coordinates and integrate by parts, to get

$$\begin{aligned}
II_2 &\lesssim \omega^{1-a}(|y|+|z|)\int_{|\xi|\leq 2|y|^{-1}}\int_{|\xi|<|\eta|}\frac{\Omega(2\,|\eta|)}{(1+|\eta|)^p}\,d\eta\,d\xi\\
&\lesssim \omega^{1-a}(|y|+|z|)\int_0^{4(|y|+|z|)^{-1}} t^{n-1}\int_t^\infty\frac{\Omega(2\rho)\rho^{n-1}}{(1+\rho)^p}\,d\rho\,dt\\
&\lesssim \omega^{1-a}(|y|+|z|)(|y|+|z|)^{-n}\int_{4(|y|+|z|)^{-1}}^\infty\frac{\Omega(2\rho)\rho^{n-1}}{(1+\rho)^p}\,d\rho\\
&+\omega^{1-a}(|y|+|z|)\int_0^{4(|y|+|z|)^{-1}}\frac{t^{2n-1}\Omega(2t)}{(1+t)^p}\,dt =: II_{2,1}+II_{2,2}.
\end{aligned}$$

Notice that (4.11) implies $\int_0^\infty \frac{\Omega(2\rho)\rho^{n-1}}{(1+\rho)^p}\,d\rho<\infty$, for $p\ge n+1$, which eliminated one of the boundary terms in the integration by parts. For $II_{2,1}$ we use (4.10) and (4.3),

$$\begin{aligned}
II_{2,1} &\le (|y|+|z|)^{-n}\sum_{j=1}^{\infty}\int_{2^j(|y|+|z|)^{-1}}^{2^{j+1}(|y|+|z|)^{-1}}\frac{\omega^{1-a}(|y|+|z|)\Omega(2\rho)\rho^{n-1}}{\rho^p}\,d\rho\\
&\lesssim (|y|+|z|)^{p-2n}\sum_{j=1}^{\infty}\omega^{1-a}(|y|+|z|)\Omega(2^{j+2}(|y|+|z|)^{-1})2^{j(n-p)}\\
&\lesssim (|y|+|z|)^{p-2n}\sum_{j=1}^{\infty}\omega^{1-a}\left(\frac{|y|+|z|}{2^{j+2}}\right)2^{j(1-a)}\Omega(2^{j+2}(|y|+|z|)^{-1})2^{j(n-p)}\\
&\lesssim B(|y|+|z|)^{p-2n}.
\end{aligned}$$

since $p\ge n+1$. Next, since Ω is non-decreasing, $p<2n$, and by (4.10) and (4.3),

$$II_{2,2}\le \omega^{1-a}(|y|+|z|)\Omega(8(|y|+|z|)^{-1})(|y|+|z|)^{p-2n}\lesssim B(|y|+|z|)^{p-2n}.$$

We now estimate the term III. Integrating by parts $p>2n+1$ times in (4.8) we have,

$$\begin{aligned}
III &\lesssim (|y|+|z|)^{-p}\omega^a\left(\frac{|h|}{|y|+|z|}\right)\omega^{1-a}(|h|)\iint_{(2|y|)^{-1}\le|(\xi,\eta)|\le 2|z|^{-1}}\frac{\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^p}\,d\eta\,d\xi\\
&\lesssim (|y|+|z|)^{-p}\omega^a\left(\frac{|h|}{|y|+|z|}\right)\omega^{1-a}(|h|)\iint_{\substack{(2\sqrt{2}|y|)^{-1}\le|\xi|\\ |\eta|\le 2|z|^{-1}\\ |\eta|\le|\xi|}}\frac{\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^p}\,d\eta\,d\xi.
\end{aligned}$$

Next we prove

$$\int_{|\xi|\ge(2\sqrt{2}|y|)^{-1}}\int_{\substack{|\eta|\le 2|z|^{-1}\\ |\eta|\le|\xi|}}\frac{\omega^{1-a}(|y|+|z|)\Omega(|\xi|+|\eta|)}{(1+|\xi|+|\eta|)^p}\,d\eta\,d\xi\lesssim(|y|+|z|)^{p-2n}, \tag{4.13}$$

from which the bound for III follows. The left hand side of (4.13) is bounded by

$$\begin{aligned}
&\int_{|\xi|\ge(2\sqrt{2}(|y|+|z|))^{-1}}\int_{|\eta|\le|\xi|}\frac{\omega^{1-a}(|y|+|z|)\Omega(2\,|\xi|)}{(1+|\xi|)^p}d\eta\,d\xi\\
&\sim\int_{(2\sqrt{2}(|y|+|z|))^{-1}}^{\infty}\frac{\omega^{1-a}(|y|+|z|)\Omega(2\rho)\,\rho^{2n-1}}{(1+\rho)^p}\,d\rho\lesssim(|y|+|z|)^{p-2n},
\end{aligned}$$

where the last inequality is proved as in the case dealing with $II_{2,1}$, but here $p>2n+1$.

We now turn to the proof of (4.4) and (4.6), the proof of (4.7) being identical to the proof of (4.6). To prove (4.6) it is enough to show that

$$|\hat{\sigma}(x,y+h,z)-\hat{\sigma}(x,y,z)|\le C\frac{|h|}{(|y|+|z|)^{2n+1}},\quad |h|\le\frac{1}{2}\max\{|y|,|z|\},$$

since by the concavity of ω, we have $t\lesssim\omega^a(t)$ (assuming $\omega(1)>0$, of course). This last inequality will be a consequence of

$$|\nabla_y\hat{\sigma}(x,y,z)|\lesssim\frac{1}{(|y|+|z|)^{2n+1}}. \tag{4.14}$$

We notice now that conditions (4.4) and (4.14) follow, respectively, from

$$\left|\widehat{\partial^\alpha_{\xi,\eta}\sigma}(x,y,z)\right| \lesssim 1, \quad |\alpha| = 2n, \tag{4.15}$$

$$\left|\widehat{\partial^\beta_{\xi,\eta}\xi_j\sigma}(x,y,z)\right| \lesssim 1, \quad |\beta| = 2n+1, \tag{4.16}$$

where the hat is always Fourier transform with respect to (ξ,η). Actually, it is enough to prove (4.15) for $\alpha = 2n\,\vec{e_j}$, and (4.16) for $\beta = (2n+1)\vec{e_j}$, $j = 1,\ldots,2n$, where $\vec{e_j} \in \mathbb{R}^{2n}$ is the unit vector with 1 in the component j and zero otherwise. In order to prove (4.15) and (4.16) we use the following lemma (see Journè [28], p. 65)

Lemma 4.2. *If $h \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ satisfies $|h(x)| \le \frac{C(h)}{|x|^d}$ and $|\nabla h(x)| \le \frac{C(h)}{|x|^{d+1}}$ for all $x \in \mathbb{R}^d$, and*

$$\sup_{0<r<R}\left|\int_{r<|x|<R} h(x)\,dx\right| \le C(h),$$

then $\left\|\hat{h}\right\|_\infty \le C(h)$.

By using (4.1), it follows that the hypotheses of the lemma are satisfied, with $d = 2n$, for the functions $h_1(\xi,\eta) = \partial^\alpha_{\xi,\eta}\sigma(x,\xi,\eta)$ and $h_2(\xi,\eta) = \partial^\beta_{\xi,\eta}\xi_j\sigma(x,\xi,\eta)$, $|\alpha| = 2n$, $|\beta| = 2n+1$, uniformly on x.

Finally, the estimates (3.1) and (3.2) with ω^a and $\tau = 1/3$ now follow from (4.4), (4.5), (4.6), (4.7). We mention that the choice $\tau = 1/3$ is made because $|x-x'| \le \frac{1}{3}\max(|x-y|,|x-z|)$ yields $|x-x'| \le \frac{1}{2}\max(|x'-y|,|x'-z|)$ and $|x-x'| \le \frac{1}{2}\max(|x-y|,|x'-z|)$. Then we can use (4.5), (4.6), (4.7) to obtain

$$\begin{aligned}
|K(x',y,z) - K(x,y,z)| &\le |\hat\sigma(x',y-x',z-x') - \hat\sigma(x,y-x',z-x')| \\
&\quad + |\hat\sigma(x,y-x',z-x') - \hat\sigma(x,y-x,z-x')| \\
&\quad + |\hat\sigma(x,y-x,z-x') - \hat\sigma(x,y-x,z-x)| \\
&\le C\omega^a\left(\frac{|x-x'|}{|x-y|+|x-z|}\right)\frac{1}{(|x-y|+|x-z|)^{2n}}.
\end{aligned}$$

For the regularity of $K(x,y,z)$ in the y and z variables, $\tau = 1/2$ is sufficient. □

Theorem 4.3. *Let $\Omega : [0,\infty) \to [0,\infty)$ be a non-decreasing function, $a \in (0,1)$, and $\omega \in Dini(a/2)$ such that (4.3) holds. If $\sigma(x,\xi,\eta)$ verifies (4.1) and (4.2) with $|\alpha|+|\beta| \le 4n+4$, then T_σ is a bilinear Calderón-Zygmund operator of type $\omega^a(t)$.*

Proof of Theorem 4.3. By Theorem 4.1 the operator T_σ has a bilinear Calderón-Zygmund kernel of type $\omega^a(t)$. It is enough to show that T_σ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for some $1 < p,q < \infty$, $1 \le r < \infty$, satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Following the same proofs as in Coifman-Meyer [14] one obtains boundedness for any $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \in (0,1)$. For the reader's convenience we summarize the argument pointing out the appropriate changes.

First one shows that boundedness holds for reduced symbols σ.

Lemma 4.4. *Let ω and Ω be as in Theorem 4.3. Consider a symbol σ of the form*

$$\sigma(x,\xi,\eta) = \sum_{j=0}^{\infty} m_j(x)\phi(2^{-j}\xi, 2^{-j}\eta), \quad x,\xi,\eta \in \mathbb{R}^n, \tag{4.17}$$

$$m_j \in \mathcal{C}(\mathbb{R}^n), \quad \sup_{j\in\mathbb{N}_0} \|m_j\|_{L^\infty(\mathbb{R}^n)} \le C, \tag{4.18}$$

$$\|m_j(\cdot + h) - m_j(\cdot)\|_{L^\infty(\mathbb{R}^n)} \le C\,\omega(|h|)\,\Omega(2^j), \quad h \in \mathbb{R}^n,\, j \in \mathbb{N}_0, \tag{4.19}$$

$$\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{2n}), \quad supp(\phi) \subset \{\tfrac{1}{3} \le |(\xi,\eta)| \le 3\}, \tag{4.20}$$

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \phi(\xi,\eta)\right| \le C, \quad 0 \le |\alpha| \le n+1,\, 0 \le |\beta| \le n+1, \tag{4.21}$$

where C is a positive finite constant. Then T_σ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for any $1 < p < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \in (0,1)$. The boundedness constants depend only on ω, Ω and the constants appearing on the right hand sides of (4.18)-(4.21).

The proof of Lemma 4.4 is analogous to the one in Coifman-Meyer [14, Theorem 12, p.55] as long as one has the following version of the almost orthogonality lemma.

Lemma 4.5. *Consider functions ω and Ω satisfying the hypothesis of Theorem 4.3. Let C_1 be a positive constant and $m_j : \mathbb{R}^n \to \mathbb{C}$, $j \in \mathbb{N}$, be a sequence of continuous functions such that*

$$\sup_{j\in\mathbb{N}} \|m_j\|_{L^\infty(\mathbb{R}^n)} \le C_1, \tag{4.22}$$

$$\|m_j(\cdot + h) - m_j(\cdot)\|_{L^\infty(\mathbb{R}^n)} \le C_1\,\omega(|h|)\,\Omega(2^j), \quad h \in \mathbb{R}^n,\, j \in \mathbb{N}. \tag{4.23}$$

Then, for $1 < p < \infty$, there exists a constant C_2 depending only on ω, Ω, C_1, p, and n such that for any sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n)$ with

$$supp(\widehat{f_j}) \subset \{\tfrac{2^j}{3} \le |\xi| \le 3\,2^j\},$$

we have

$$\left\|\sum_{j=1}^{\infty} m_j(x)\, f_j(x)\right\|_{L^p(\mathbb{R}^n)} \le C_2 \left\|\left(\sum_{j=1}^{\infty} |f_j(x)|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}.$$

To prove this lemma one can proceed as in Coifman-Meyer [14, Proposition 4, p.42] provided that $\sum_{j=1}^{\infty} \omega^2(2^{-j})\Omega^2(2^j) < \infty$. This inequality follows from the hypotheses on ω and Ω. We have $\sum_{j=1}^{\infty} \omega(2^{-j})\Omega(2^j) \lesssim \sum_{j=1}^{\infty} \omega^a(2^{-j}) \sim \int_0^1 \frac{\omega^a(t)}{t}\,dt < \infty$.

Finally, one shows that every symbol σ satisfying the conditions in Theorem 4.3 can be expressed in terms of reduced symbols. More precisely,

Lemma 4.6. *Let ω, Ω and σ satisfy the hypothesis of Theorem 4.3. Then*

$$\sigma(x,\xi,\eta) = \tau(x,\xi,\eta) + \sum_{k,l\in\mathbb{Z}^n} \frac{\sigma_{k,l}(x,\xi,\eta)}{(1+|k|)^{n+1}(1+|l|)^{n+1}},$$

where $\tau(x,\xi,\eta) = 0$ for $|(\xi,\eta)| > 1$, and $\sigma_{k,l}$ are reduced symbols with the constants on the right hand sides of (4.18)-(4.21) *uniform on k and l .*

For the proof of this lemma see Coifman-Meyer [14, p.46] and Bényi-Torres [4]. Note that $T_\tau(f,g)(x) = \int\int L(x, x-y, x-z)\, f(y)g(z)\, dydz$, where

$$|L(x, x-y, x-z)| \lesssim \frac{1}{(1+|x-y|)^{n+1}(1+|x-z|)^{n+1}}.$$

Therefore T_τ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, $1\le p,q\le\infty$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. The boundedness for T_σ from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, $1<p<\infty$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\in(0,1)$, follows from the uniform boundedness for $T_{\sigma_{k,l}}$ and the boundedness for T_τ in the same spaces. □

5. Proof of Theorem 2.3

The first step towards the proof of Theorem 2.3 is the following quadratic estimate for ω-molecules with cancelation. Notice that (2.2) and (2.3) imply

$$\begin{aligned}(5.1)\qquad &|\phi_P(x)-\phi_P(y)|\\ &\le A2^{\nu n/2}\min(1,\omega(2^\nu|x-y|))\left[\frac{1}{(1+2^\nu|x-x_P|)^N}+\frac{1}{(1+2^\nu|y-x_P|)^N}\right].\end{aligned}$$

Lemma 5.1. *Assume that $\omega\in Dini(1)$ and that $\{\phi_Q\}_{Q\in\mathcal{D}}$ is a family of ω-molecules with the cancelation property*

$$\int_{\mathbb{R}^n}\phi_Q(x)\,dx = 0,\quad \textit{for every cube } Q\in\mathcal{D}. \tag{5.2}$$

Then, there exists a constant $C=C(A,|\omega|_{Dini(1)},N,n)$ such that for every $f\in L^2(\mathbb{R}^n)$, we have

$$\sum_{Q\in\mathcal{D}}|\langle\phi_Q,f\rangle|^2\le C\,\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{5.3}$$

Proof of Lemma 5.1. It is enough to show that there is a constant C such that for every $Q\in\mathcal{D}$

$$\sum_{R\in\mathcal{D}}|\langle\phi_Q,\phi_R\rangle|\le C. \tag{5.4}$$

Indeed, given $f\in L^2(\mathbb{R}^n)$ and assuming $\|f\|_{L^2(\mathbb{R}^n)}=1$ we have

$$\begin{aligned}\left(\sum_{Q\in\mathcal{D}}|\langle\phi_Q,f\rangle|^2\right)^2 &= \left(\sum_{Q\in\mathcal{D}}\langle\phi_Q,f\rangle\langle f,\phi_Q\rangle\right)^2 = \left(\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{D}}\langle f,\phi_Q\rangle\phi_Q(x)\bar f(x)\,dx\right)^2\\ &\le \left\|\sum_{Q\in\mathcal{D}}\langle f,\phi_Q\rangle\phi_Q\right\|^2_{L^2(\mathbb{R}^n)} = \sum_{Q,R\in\mathcal{D}}\langle f,\phi_Q\rangle\langle\phi_Q,\phi_R\rangle\langle\phi_R,f\rangle\\ &\le \left(\sum_{Q,R\in\mathcal{D}}|\langle f,\phi_Q\rangle|^2|\langle\phi_Q,\phi_R\rangle|\right)^{1/2}\left(\sum_{Q,R\in\mathcal{D}}|\langle\phi_R,f\rangle|^2|\langle\phi_Q,\phi_R\rangle|\right)^{1/2}\\ &\le C\sum_{Q\in\mathcal{D}}|\langle\phi_Q,f\rangle|^2,\end{aligned}$$

which yields (5.3). In order to prove (5.4) fix $Q = Q(\nu, k)$ and split the sum in $R = R(\mu, m)$ as follows

$$\sum_{R\in\mathcal{D}} |\langle\phi_Q, \phi_R\rangle| = \sum_{\substack{\mu\in\mathbb{Z}\\ m\in\mathbb{Z}^n}} |\langle\phi_Q, \phi_R\rangle| = \sum_{\substack{\mu\leq\nu\\ m\in\mathbb{Z}^n}} |\langle\phi_Q, \phi_R\rangle| + \sum_{\substack{\mu>\nu\\ m\in\mathbb{Z}^n}} |\langle\phi_Q, \phi_R\rangle| =: S_1 + S_2.$$

We first estimate S_1. The cancelation property (5.2) and inequality (5.1) allow to write

$$\begin{aligned}
S_1 &= \sum_{\substack{\mu\leq\nu\\ m\in\mathbb{Z}^n}} |\langle\phi_Q, \phi_R\rangle| = \sum_{\substack{\mu\leq\nu\\ m\in\mathbb{Z}^n}} |\int \phi_Q(x)(\overline{\phi_R(x) - \phi_R(x_Q)})|\, dx\\
&\leq \sum_{\mu\leq\nu} \int \frac{A^2 2^{\nu n/2} 2^{\mu n/2}}{(1 + 2^\nu|x - x_Q|)^N} \min(1, \omega(2^\mu|x - x_Q|))\\
&\times \sum_{m\in\mathbb{Z}^n} \left[\frac{1}{(1 + 2^\mu|x - 2^{-\mu}m|)^N} + \frac{1}{(1 + 2^\mu|x_Q - 2^{-\mu}m|)^N}\right] dx\\
&= A^2 C_N \sum_{\mu\leq\nu} \int \frac{2^{\nu n/2} 2^{\mu n/2}}{(1 + 2^\nu|x - x_Q|)^N} \min(1, \omega(2^\mu|x - x_Q|))\, dx\\
&= A^2 C_N \sum_{\mu\leq\nu} \int_{1\leq\omega(2^\mu|x - x_Q|)} \frac{2^{\nu n/2} 2^{\mu n/2}}{(1 + 2^\nu|x - x_Q|)^N}\, dx\\
&+ A^2 C_N \sum_{\mu\leq\nu} \int_{1>\omega(2^\mu|x - x_Q|)} \frac{2^{\nu n/2} 2^{\mu n/2}}{(1 + 2^\nu|x - x_Q|)^N} \omega(2^\mu|x - x_Q|)\, dx\\
&=: S_{1,1} + S_{1,2}.
\end{aligned}$$

By multiplying by a constant (if needed) we can assume that $\omega(1) = 1$. Hence,

$$S_{1,1} \leq A^2 C_N \sum_{\mu\leq\nu} \int_{|x - x_Q|\geq 2^{-\mu}} \frac{2^{\nu n/2} 2^{\mu n/2}\, dx}{2^{\nu N}|x - x_Q|^N} = A^2 C_N \sum_{\mu\leq\nu} 2^{(\mu-\nu)(N - n/2)} = A^2 \tilde{C}_N.$$

On the other hand,

$$\begin{aligned}
S_{1,2} &= A^2 C_N \sum_{\mu\le\nu} \int\limits_{\substack{|x-x_Q|<2^{-\mu}\\ 2^\nu|x-x_Q|\ge 1}} \frac{2^{\nu n/2}2^{\mu n/2}\omega(2^\mu|x-x_Q|)}{(1+2^\nu|x-x_Q|)^N}\,dx \\
&+ A^2 C_N \sum_{\mu\le\nu} \int\limits_{\substack{|x-x_Q|<2^{-\mu}\\ 2^\nu|x-x_Q|<1}} \frac{2^{\nu n/2}2^{\mu n/2}\omega(2^\mu|x-x_Q|)}{(1+2^\nu|x-x_Q|)^N}\,dx \\
&\le A^2 C_N \sum_{\mu\le\nu} \int\limits_{2^{\mu-\nu}\le 2^\mu|x-x_Q|<1} \frac{2^{\nu n/2}2^{\mu n/2}\omega(2^\mu|x-x_Q|)}{(1+2^\nu|x-x_Q|)^N}\,dx \\
&+ A^2 C_N \sum_{\mu\le\nu} \int\limits_{2^\nu|x-x_Q|<1} 2^{\nu n/2}2^{\mu n/2}\omega(2^\mu|x-x_Q|)\,dx \\
&= A^2 C_N \sum_{\mu\le\nu}\sum_{\lambda=0}^{\nu-\mu-1} \int\limits_{2^{-\lambda-1}\le 2^\mu|x-x_Q|<2^{-\lambda}} \frac{2^{\nu n/2}2^{\mu n/2}\omega(2^\mu|x-x_Q|)}{(1+2^\nu|x-x_Q|)^N}\,dx \\
&+ A^2 C_N \sum_{\mu\le\nu} \int\limits_{2^\nu|x-x_Q|<1} 2^{\nu n/2}2^{\mu n/2}\omega(2^{\mu-\nu}2^\nu|x-x_Q|)\,dx \\
&\le A^2 C_N \sum_{\mu\le\nu}\sum_{\lambda=0}^{\nu-\mu-1} 2^{\nu n/2}2^{\mu n/2}\omega(2^{-\lambda}) \int\limits_{2^{-\lambda-1}\le 2^\mu|x-x_Q|<2^{-\lambda}} \frac{dx}{2^{\nu N}|x-x_Q|^N} \\
&+ A^2 C_N \sum_{\mu\le\nu} \int\limits_{2^\nu|x-x_Q|<1} 2^{\nu n/2}2^{\mu n/2}\omega(2^{\mu-\nu})\,dx \\
&\le A^2 C_N \sum_{\mu\le\nu}\sum_{\lambda=0}^{\nu-\mu-1} 2^{\nu n}2^{-\nu N}2^{(-\mu-\lambda)(n-N)}\omega(2^{-\lambda}) + A^2 C_N \sum_{\mu\le\nu}\omega(2^{\mu-\nu}) \\
&\le |\omega|_{Dini(1)} A^2 \tilde{\tilde{C}}_N.
\end{aligned}$$

The estimate for S_2 follows analogously by interchanging the roles of ϕ_R and ϕ_Q. □

The following lemma is a particular case of a discrete bilinear almost orthogonality result whose proof can be found in [3].

Lemma 5.2. *For every $N > n+1$ there is a constant C_N, depending only on N and n, such that for any $\nu \in \mathbb{Z}$ and any $x, y, z \in \mathbb{R}^n$ the following inequality holds*

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \frac{1}{[(1+2^\nu|x-2^{-\nu}k|)(1+2^\nu|y-2^{-\nu}k|)(1+2^\nu|z-2^{-\nu}k|)]^{5N}} \\
&\le \frac{C_N}{[(1+2^\nu|x-y|)(1+2^\nu|y-z|)(1+2^\nu|x-z|)]^N}.
\end{aligned}$$

The following is the main theorem of this section.

Theorem 5.3. *Assume $\omega \in Dini(1/2)$ and let $\{\phi_Q^j\}_{Q\in\mathcal{D}}$, $j = 1, 2, 3$ be three families of ω-molecules with decay $N > 10n$ and such that at least two of them have cancelation. Then, the paraproduct Π defined in (2.4) has a bilinear Calderón-Zygmund kernel of type $\theta(t)$ with*

$$\theta(t) = A^3 A_N \omega(C_N t), \quad t > 0,$$

for some constants A_N and C_N (hence, $\theta \in Dini(1/2)$). Moreover, Π has the mapping property

$$\Pi : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n).$$

In particular, Π is a bilinear Calderón-Zygmund operator of type $\theta(t)$.

Proof of Theorem 5.3. The kernel of Π is given by

$$K(x, y, z) = \sum_{Q\in\mathcal{D}} |Q|^{-1/2}\overline{\phi_Q^1(y)\phi_Q^2(z)}\phi_Q^3(x).$$

In order to prove the size estimate for $K(x, y, z)$ we index the dyadic cubes by $Q = Q(\nu, k)$ and use (2.2) and Lemma 5.2, to obtain

$$\begin{aligned}
&|K(x,y,z)| \le \sum_{Q\in\mathcal{D}} \frac{A^3|Q|^{-1/2}2^{\nu 3n/2}}{(1+2^\nu|y-2^{-\nu}k|)^N(1+2^\nu|z-2^{-\nu}k|)^N(1+2^\nu|x-2^{-\nu}k|)^N}\\
&= A^3\sum_{\nu\in\mathbb{Z}} 2^{2\nu n}\sum_{k\in\mathbb{Z}^n} \frac{1}{(1+2^\nu|y-2^{-\nu}k|)^N(1+2^\nu|z-2^{-\nu}k|)^N(1+2^\nu|x-2^{-\nu}k|)^N}\\
&\le A^3\sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}}{(1+2^\nu|y-x|)^{N/5}(1+2^\nu|z-y|)^{N/5}(1+2^\nu|x-z|)^{N/5}}\\
&\le A^3\sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}}{[1+2^\nu(|y-x|+|z-y|+|x-z|)]^{N/5}} \le \frac{A^3C_N}{(|x-y|+|y-z|+|x-z|)^{2n}}
\end{aligned}$$

The ω-regularity of the kernel involves the concavity of ω. Take $x, y, z, h \in \mathbb{R}^n$ such that $|h| \leq 1/2 \max(|x-y|, |x-z|)$ and do

$$
\begin{aligned}
|K(x,y,z) - K(x+h,y,z)| &\leq \sum_{Q\in\mathcal{D}} |Q|^{-1/2}|\phi_Q^1(y)||\phi_Q^2(z)||\phi_Q^3(x) - \phi_Q^3(x+h)| \\
&\leq A^3 \sum_{\substack{\nu\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{(1+2^\nu|y-2^{-\nu}k|)^N(1+2^\nu|z-2^{-\nu}k|)^N(1+2^\nu|x-2^{-\nu}k|)^N} \\
&+ A^3 \sum_{\substack{\nu\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{(1+2^\nu|y-2^{-\nu}k|)^N(1+2^\nu|z-2^{-\nu}k|)^N(1+2^\nu|x+h-2^{-\nu}k|)^N} \\
&\leq A^3 \sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{(1+2^\nu|y-x|)^{N/5}(1+2^\nu|z-y|)^{N/5}(1+2^\nu|x-z|)^{N/5}} \\
&+ A^3 \sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{(1+2^\nu|y-x+h|)^{N/5}(1+2^\nu|z-y|)^{N/5}(1+2^\nu|x+h-z|)^{N/5}} \\
&\leq A^3 \sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{[1+2^\nu(|y-x|+|z-y|+|x-z|)]^{N/5}} \\
&+ A^3 \sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{[1+2^\nu(|y-x+h|+|z-y|+|x+h-z|)]^{N/5}}.
\end{aligned}
$$

Since the condition $|h| \leq \frac{1}{2}\max(|x-y|, |x-z|)$ implies

$$
\frac{1}{4}(|x-y|+|x-z|) \leq |y-x+h| + |x-z+h| \leq \frac{3}{2}(|x-y|+|x-z|),
$$

we only need to bound one of the above sums. Let $\alpha \in \mathbb{Z}$ such that

$$
2^\alpha \leq |x-y| + |y-z| + |z-x| \leq 2^{\alpha+1}.
$$

Then

$$
\sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^\nu|h|)}{[1+2^\nu(|y-x|+|z-y|+|x-z|)]^{N/5}} \leq 2^{-2\alpha n} \sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^{\nu-\alpha}|h|)}{(1+2^\nu)^{N/5}},
$$

where we used the change of variables $\nu + \alpha \mapsto \nu$. Set $A_N := \sum_{\nu\in\mathbb{Z}} 2^{2\nu n}(1+2^\nu)^{-N/5}$. By concavity of ω we have

$$
\begin{aligned}
\sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}\omega(2^{\nu-\alpha}|h|)}{(1+2^\nu)^{N/5}} &\leq A_N\omega\left(\frac{1}{A_N}\sum_{\nu\in\mathbb{Z}} \frac{2^{2\nu n}2^{\nu-\alpha}|h|}{(1+2^\nu)^{N/5}}\right) \\
&\leq A_N\omega\left(C_N 2^{-\alpha}|h|\right) \leq A_N\omega\left(\frac{2C_N|h|}{|x-y|+|y-z|+|x-z|}\right).
\end{aligned}
$$

Finally, set $\theta(t) := 2A^3 A_N \omega(2C_N t)$. The regularity in the y and z coordinates follows similarly. The $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness is a consequence of Lemma 5.1. We can assume (by taking transposes of Π, if necessary) that $\{\phi_Q^1\}$ and $\{\phi_Q^2\}$ are the families of molecules with cancelation. Given $f, g \in L^2(\mathbb{R}^n)$ and $h \in L^\infty(\mathbb{R}^n)$ we use duality and

(2.2) to obtain

$$
\begin{aligned}
|\langle \Pi(f,g),h\rangle| &\leq \sum_{Q\in\mathcal{D}} |Q|^{-1/2}|\langle f,\phi^1_Q\rangle||\langle g,\phi^2_Q\rangle||\langle \phi^3_Q,h\rangle| \\
&\leq \left(\sum_{Q\in\mathcal{D}} |\langle f,\phi^1_Q\rangle|^2\right)^{1/2}\left(\sum_{Q\in\mathcal{D}} |\langle g,\phi^2_Q\rangle|^2\right)^{1/2} \sup_{\substack{\nu\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \int_{\mathbb{R}^n} \frac{2^{n\nu}h(x)\,dx}{(1+2^\nu|x-2^{-\nu}k|)^N} \\
&\leq AC_N \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} \|h\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

□

Remark 2. If instead of the regularity condition (2.3) we require from a molecule to verify the weaker inequality

$$
|\phi_P(x)-\phi_P(y)| \leq A2^{\nu n/2}\omega(2^\nu|x-y|), \quad x,y\in\mathbb{R}^n, \tag{5.5}
$$

then, (2.2) and (5.5) imply

$$
\begin{aligned}
&|\phi_P(x)-\phi_P(y)| \\
&\leq A2^{\nu n/2}\min(1,\omega^{1/2}(2^\nu|x-y|))\left[\frac{1}{(1+2^\nu|x-x_P|)^{N/2}}+\frac{1}{(1+2^\nu|y-x_P|)^{N/2}}\right].
\end{aligned} \tag{5.6}
$$

The proof of Theorem 2.3 also applies with condition (5.5) instead of (2.3), since we can replace the use of (5.1) by utilizing (5.6). In this case, we require the weaker assumption that $\omega^{1/2}$ be concave instead of ω be concave. However, we also need the stronger assumptions $\int_0^1 \omega^{1/4}(t)dt/t < \infty$ (instead of $\int_0^1 \omega^{1/2}(t)dt/t < \infty$) and $N > 10n+10$ (instead of $N > 5n+5$).

Remark 3. One could be tempted to think that paraproducts associated to ω-molecules with enough decay can be realized as pseudo-differential operators in the class $BS^0_{1,\omega,\Omega}$ for some choice of Ω. If that were the case, the results in this section would just follow from the ones in Section 4. However, such a realization is not true in general, as the following example shows. Consider three functions $\psi^j \in \mathcal{S}(\mathbb{R}^n)$, $j=1,2,3$, and, given a dyadic cube $Q=Q_{\nu k}$, set

$$
\psi^j_Q(x) = 2^{\nu n/2}\psi^j(2^\nu x-k), \quad x\in\mathbb{R}^n, j=1,2,3.
$$

Also assume that $\widehat{\psi^j}$ is supported in $\{\xi\in\mathbb{R}^n : 1/2\leq|\xi|\leq 2\}$ and equals one in $\{\xi\in\mathbb{R}^n : 1\leq|\xi|\leq 3/2\}$, for $j=1,2$; and $\operatorname{supp}(\psi^3)\subset[0,1]^n$. The paraproduct Π built from these ψ^j's can be written as a ΨDO with symbol

$$
\sigma(x,\xi,\eta) = \sum_{\nu\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} e^{-i(x-2^{-\nu}k)\cdot(\xi+\eta)}\widehat{\psi^1}(2^{-\nu}\xi)\widehat{\psi^2}(2^{-\nu}\eta)\psi^3(2^\nu x-k).
$$

The support hypotheses on the smooth molecules ψ^j allow to easily estimate

$$
|\partial^\alpha_\xi\partial^\beta_\eta\sigma(x,\xi,\eta)| \simeq (|\xi|+|\eta|)^{-(|\alpha|+|\beta|)}, \quad \xi,\eta\in\mathbb{R}^n\setminus\{0\}.
$$

Hence, condition (4.1) does not hold. A closer look at the example also shows that condition (4.2) cannot hold either for any choice of ω, Ω due to the blow-up of the ξ and η derivatives of $\partial^\gamma_x\sigma(x,\xi,\eta)$, $|\gamma|=1$, at $(\xi,\eta)=(0,0)$.

Remark 4. We point out that the realization of paraproducts as bilinear Calderón-Zygmund operators of type $\omega(t)$ described in Theorem 2.3 complements the approaches in [21], [22], [31], [32] [37], [36], [40], and [46], where, in turn, classical Fourier Analysis operators are reduced or decomposed into paraproducts.

6. Boundedness of bilinear Calderón-Zygmund operators of type $\omega(t)$

In this section we elaborate on a bilinear theory for Calderón-Zygmund operators of type $\omega(t)$ with Dini continuous ω. In subsections 6.1 and 6.2 we have deemed it appropriate to provide the reader with either complete proofs or detailed outlines of proofs of the boundedness properties of bilinear Calderón-Zygmund operators of type $\omega(t)$ on Lebesgue spaces, although they sometimes follow the known proofs in the case of $\omega(t) = t^\epsilon$ by L. Grafakos and R. Torres in [26] and [27]. On the other hand, we point out that the results in subsections 6.3 and 6.4 concerning weighted Hardy spaces and weighted amalgam spaces respectively, are new even in the case $\omega(t) = t^\epsilon$. For simplicity, we have also fixed the value of τ in Definition 3.1 to be the usual 1/2.

6.1. Boundedness on unweighted Lebesgue spaces, H^1, and BMO.

Theorem 6.1. *Consider $\omega \in Dini(1/2)$ and let T be a bilinear operator associated to a bilinear Calderón-Zygmund kernel of type $\omega(t)$, $K(x,y,z)$. Assume that for some $1 \le p, q \le \infty$ and $0 < r < \infty$ satisfying*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

T maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$. Then, T can be extended to a bounded operator from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into $L^{\frac{1}{2},\infty}(\mathbb{R}^n)$.

Proof. We write $l(Q)$ to denote its side length and Q^* to indicate the cube with the same center as Q and $l(Q^*) = (2n+1)\, l(Q)$. The arguments in this proof are similar to those in Grafakos-Torres [26], but they are also slightly different as we make no use of the boundedness properties of the Marcinkiewicz operator. Fix $\lambda > 0$ and $f_1, f_2 \in L^1(\mathbb{R}^n)$. Assuming, without loss of generality, that $\|f_j\|_1 = 1$, $j = 1, 2$, we have to prove that

$$|\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > \lambda\}| \le C\, \lambda^{-\frac{1}{2}},$$

for some constant C independent of f_1, f_2 and λ. Consider the Calderón-Zygmund decomposition of each function f_j at height $\lambda^{1/2}$. Then, for $j = 1, 2$, we have

$$f_j = g_j + b_j, \tag{6.1}$$

$$\|g_j\|_p \le (2^n \lambda^{\frac{1}{2}})^{1-\frac{1}{p}},\ 1 \le p \le \infty, \tag{6.2}$$

$$b_j = \sum_k b_{j,k}, \text{ where each } b_{j,k} \text{ is supported in a dyadic cube } Q_{j,k}, \tag{6.3}$$

$$\text{For } k \ne k',\ \text{the interiors of } Q_{j,k} \text{ and } Q_{j,k'} \text{ are disjoint,} \tag{6.4}$$

$$\int_{Q_{j,k}} b_{j,k}(x)\, dx = 0, \tag{6.5}$$

$$\|b_{j,k}\|_1 \le 2^{n+1} \lambda^{\frac{1}{2}} |Q_{j,k}|, \tag{6.6}$$

$$\sum_k |Q_{j,k}| \le \lambda^{-\frac{1}{2}}. \tag{6.7}$$

The set $\{x \in \mathbb{R}^n : |T(f_1, f_2)(x)| > \lambda\}$ is contained in the union of the sets

$$\{x \in \mathbb{R}^n : |T(h_1, h_2)| > 4^{-1}\lambda\}$$

where $h_j \in \{g_j, b_j\}, j = 1, 2$. Therefore, we have to show that

$$\left|\{x \in \mathbb{R}^n : |T(h_1, h_2)(x)| > 4^{-1}\lambda\}\right| \le C\, \lambda^{-\frac{1}{2}},$$

where $h_j \in \{g_j, b_j\}$ and C is independent of λ and f_j, $j = 1, 2$. Let us first consider the easy case where $h_j = g_j$, $j = 1, 2$. Using the boundedness of T from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ (with norm A) and (6.2), we do

$$\left|\{x \in \mathbb{R}^n : |T(g_1, g_2)(x)| > 4^{-1}\lambda\}\right| \leq \left(\frac{4\,A}{\lambda}\,\|g_1\|_p\,\|g_2\|_q\right)^r \leq \frac{4^r\,A^r}{\lambda^r} 2^{n(2r-1)}\lambda^{r-\frac{1}{2}} \leq C_{n,r}A^r\lambda^{-\frac{1}{2}}.$$

We address now the rest of the cases, when there is at least one function $h_j = b_j$. Let $\mathcal{B} \subset \{1, 2\}$, $\#(\mathcal{B}) = l \geq 1$. Assume that $h_j = b_j$ if $j \in \mathcal{B}$, and $h_j = g_j$ if $j \notin \mathcal{B}$. We have

$$\left|\{x \in \mathbb{R}^n : |T(h_1, h_2)(x)| > 4^{-1}\lambda\}\right| \leq \left|\{x \in \cup_{j\in\mathcal{B}} \cup_k Q^*_{j,k}\}\right| + \left|\{x \notin \cup_{j\in\mathcal{B}} \cup_k Q^*_{j,k} : |T(h_1, h_2)(x)| > 4^{-1}\lambda\}\right|.$$

In view of (6.7), we only need to work on the measure of the set $E_{\mathcal{B}} := \{x \notin \cup_{j\in\mathcal{B}} \cup_k Q^*_{j,k} : |T(h_1, h_2)| > 4^{-1}\lambda\}$. Denoting by $c_{j,k}$ the center of $Q_{j,k}$, we will show that

$$|T(h_1, h_2)(x)| \leq D\,\lambda \prod_{j\in\mathcal{B}} \mathcal{M}^{\omega}_{j,l}(x), \tag{6.8}$$

where $x \notin \cup_{j\in\mathcal{B}} \cup_k Q^*_{j,k}$, D is a constant independent of λ and f_j, $j = 1, 2$, and

$$\mathcal{M}^{\omega}_{j,l}(x) := \sum_k \omega\left(\frac{l(Q_{j,k})}{|x - c_{j,k}|}\right)^{\frac{1}{l}} \frac{l(Q_{j,k})^n}{|x - c_{j,k}|^n}.$$

Assuming that (6.8) holds, Chebychev's and Hölder inequality yield

$$|E_{\mathcal{B}}| \leq \left|\left\{x \notin \cup_{m\in\mathcal{B}} \cup_k Q^*_{m,k} : \prod_{j\in\mathcal{B}} \mathcal{M}^{\omega}_{j,l}(x) > (4D)^{-1}\right\}\right|$$

$$\leq (4D)^{\frac{1}{l}} \int_{x\notin\cup_{m\in\mathcal{B}}\cup_k Q^*_{m,k}} \prod_{j\in\mathcal{B}} \left(\mathcal{M}^{\omega}_{j,l}(x)\right)^{\frac{1}{l}}\,dx \leq (4D)^{\frac{1}{l}} \prod_{j\in\mathcal{B}} \left(\int_{x\notin\cup_{m\in\mathcal{B}}\cup_k Q^*_{m,k}} \mathcal{M}^{\omega}_{j,l}(x)\,dx\right)^{\frac{1}{l}}.$$

We now estimate each of the above integrals by using polar coordinates.

$$\int_{x\notin\cup_{m\in\mathcal{B}}\cup_k Q^*_{m,k}} \mathcal{M}^{\omega}_{j,l}(x)\,dx \leq \sum_k \int_{|x-c_{j,k}|>l(Q_{j,k})} \omega\left(\frac{l(Q_{j,k})}{|x - c_{j,k}|}\right)^{\frac{1}{l}} \frac{l(Q_{j,k})^n}{|x - c_{j,k}|^n}\,dx$$

$$= C_n \sum_k \int_{l(Q_{j,k})}^{\infty} \omega\left(\frac{l(Q_{j,k})}{\rho}\right)^{\frac{1}{l}} \frac{l(Q_{j,k})^n}{\rho^n}\rho^{n-1}d\rho = C_n \sum_k l(Q_{j,k})^n \int_0^1 \frac{\omega(t)^{\frac{1}{l}}}{t}\,dt$$

$$\leq C_{n,\omega,l}\,\lambda^{-\frac{1}{2}}.$$

We thus obtain $|E_{\mathcal{B}}| \leq (4D)^{\frac{1}{l}} C_{n,\omega,l}\,\lambda^{-\frac{1}{2}}$, and the theorem is then proved. We will now proceed to prove (6.8). In what follows $x \notin \cup_{j\in\mathcal{B}} \cup_k Q^*_{j,k}$.

First case: $l = 2$. Since x is away from the support of b_1 and b_2, by (6.3) we can write

$$|T(b_1, b_2)(x)| \leq \sum_{k_1} \sum_{k_2} \left|\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z) b_{1,k_1}(y) b_{2,k_2}(z)\,dydz\right|.$$

Fix for a moment k_1, k_2 and assume, without loss of generality, that $l(Q_{1,k_1}) \leq l(Q_{2,k_2})$. Using the cancelation (6.5) of b_{1,k_1} and the regularity (3.2) of the kernel K,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} K(x,y,z) b_{1,k_1}(y)\, dy\right| = \left|\int_{\mathbb{R}^n} (K(x,y,z) - K(x,c_{1,k_1},z)) b_{1,k_1}(y)\, dy\right| \\
&\leq \int_{\mathbb{R}^n} \frac{C_K}{(|x-y|+|x-z|)^{2n}}\, \omega\left(\frac{|y-c_{1,k_1}|}{|x-y|+|x-z|}\right) |b_{1,k_1}(y)|\ dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{C_K}{(|x-y|+|x-z|)^{2n}}\, \omega\left(\frac{l(Q_{1,k_1})}{|x-y|+|x-z|}\right) |b_{1,k_1}(y)|\ dy
\end{aligned}$$

Note that the condition $|y-c_{1,k_1}| \leq \frac{1}{2}\max(|x-y|,|x-z|)$ is satisfied since $y \in Q_{1,k_1}$ and $x \notin Q^*_{1,k_1}$. Actually $|y-c_{1,k_1}| \leq \frac{\sqrt{n}}{2} l(Q_{1,k_1}) \leq \frac{\sqrt{n}}{2n}\left|x-c_{1,k_1}\right|$. We then have,

$$\begin{aligned}
&|T(b_1,b_2)(x)| \leq \sum_{k_1}\sum_{k_2} \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} K(x,y,z) b_{1,k_1}(y)\right| |b_{2,k_2}(z)|\, dydz \\
&\lesssim \sum_{k_1}\sum_{k_2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{C_K}{(|x-y|+|x-z|)^{2n}}\, \omega\left(\frac{l(Q_{1,k_1})}{|x-y|+|x-z|}\right) |b_{1,k_1}(y)||b_{2,k_2}(z)| dydz.
\end{aligned}$$

Note that for $y \in Q_{1,k_1}$ and $x \notin Q^*_{1,k_1}$ we have $|x-y| \geq \frac{1}{2}\left|x-c_{1,k_1}\right|$. Similarly for $z \in Q_{1,k_2}$. Then, using that ω is nondecreasing and doubling,

$$\frac{\omega\left(\frac{l(Q_{1,k_1})}{|x-y|+|x-z|}\right)}{(|x-y|+|x-z|)^{2n}} \lesssim \frac{\omega\left(\frac{l(Q_{1,k_1})}{|x-c_{1,k_1}|+|x-c_{2,k_2}|}\right)}{(|x-c_{1,k_1}|+|x-c_{2,k_2}|)^{2n}} \lesssim \prod_{i=1}^{2} \frac{\omega\left(\frac{l(Q_{i,k_i})}{\left|x-c_{i,k_i}\right|}\right)^{\frac{1}{2}}}{\left|x-c_{i,k_i}\right|^n}$$

This and (6.6) give

$$\begin{aligned}
&|T(b_1,b_2)(x)| \\
&\lesssim C_K \sum_{k_1}\sum_{k_2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \prod_{i=1}^{2} \frac{1}{\left|x-c_{i,k_i}\right|^n}\, \omega\left(\frac{l(Q_{i,k_i})}{\left|x-c_{i,k_i}\right|}\right)^{\frac{1}{2}} |b_{1,k_1}(y)||b_{2,k_2}(z)| dydz \\
&\lesssim C_K\,\lambda \sum_{k_1}\sum_{k_2} \prod_{i=1}^{2} \frac{l(Q_{i,k_i})^n}{\left|x-c_{i,k_i}\right|^n}\, \omega\left(\frac{l(Q_{i,k_i})}{\left|x-c_{i,k_i}\right|}\right)^{\frac{1}{2}} \sim C_K\,\lambda \prod_{j=1}^{2} \mathcal{M}^{\omega}_{j,2}(x),
\end{aligned}$$

which is (6.8).

Second case: $l = 1$. Suppose $h_1 = b_1$ and $h_2 = g_2$. Then, since x is away from the support of b_1, we use (6.3), (6.5), (6.2), (6.6), and the properties of K and ω, to write

$$
\begin{aligned}
|T(b_1, g_2)(x)| &\leq \sum_{k_1} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} K(x,y,z) b_{1,k_1}(y)\, dy \right| |g_2(z)| dz \\
&= \int_{\mathbb{R}^n} \sum_{k_1} \left| \int_{\mathbb{R}^n} (K(x,y,z) - K(x, c_{1,k_1}, z)) b_{1,k_1}(y)\, dy \right| |g_2(z)| dz \\
&\leq \|g_2\|_{L^\infty} \int_{\mathbb{R}^n} \sum_{k_1} \int_{\mathbb{R}^n} \frac{C_K}{(|x-y| + |x-z|)^{2n}} \omega\left(\frac{|y - c_{1,k_1}|}{|x-y| + |x-z|} \right) |b_{1,k_1}(y)|\ dydz \\
&\lesssim C_K \lambda^{1/2} \int_{\mathbb{R}^n} \sum_{k_1} \int_{\mathbb{R}^n} \frac{1}{(|x-y| + |x-z|)^{2n}} \omega\left(\frac{l(Q_{1,k_1})}{|x-y|} \right) |b_{1,k_1}(y)|\ dydz \\
&\lesssim C_K \lambda^{1/2} \sum_{k_1} \int_{\mathbb{R}^n} \frac{1}{|x-y|^n} \omega\left(\frac{l(Q_{1,k_1})}{|x-y|} \right) |b_{1,k_1}(y)|\ dy \\
&\lesssim C_K \lambda \sum_{k_1} \frac{l(Q_{1,k_1})^n}{|x - c_{1,k_1}|^n} \omega\left(\frac{l(Q_{1,k_1})}{|x - c_{1,k_1}|} \right) \sim C_K \lambda\, \mathcal{M}^{\omega}_{j,1}(x),
\end{aligned}
$$

which is (6.8). The case $h_1 = g_1$ and $h_2 = b_2$ follows similarly. □

Theorem 6.2. *Consider $\omega \in Dini(1/2)$ and T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$ with kernel K. Let $1 \leq p,\ q \leq \infty$, $\frac{1}{2} \leq r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then we have*

(i) *If $p,\ q > 1$, then T can be extended to a bounded operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, where $L^p(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$ should be replaced by $L^\infty_c(\mathbb{R}^n)$ if $p = \infty$ or $q = \infty$, respectively;*
(ii) *If $p = 1$ or $q = 1$, then T can be extended to a bounded operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$, where $L^p(\mathbb{R}^n)$ or $L^q(\mathbb{R}^n)$ should be replaced by $L^\infty_c(\mathbb{R}^n)$ if $p = \infty$ or $q = \infty$, respectively;*
(iii) *T can be extended to a bounded operator from $L^\infty_c(\mathbb{R}^n) \times L^\infty_c(\mathbb{R}^n)$ into BMO.*

The proof of Theorem 6.2 can be carried out using duality and multilinear interpolation techniques as in Grafakos-Torres [26, Theorem 3] (case $\omega(t) = t^\epsilon$), if the following holds:

(i) T is bounded from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into $L^{\frac{1}{2},\infty}(\mathbb{R}^n)$ (this is our Theorem 6.1),
(ii) for each $h \in L^\infty_c(\mathbb{R}^n)$, $T^1_h(f) = T(f,h)$ and $T^2_h(f) = T(h,g)$ are bounded operators from $L^s(\mathbb{R}^n)$ into $L^s(\mathbb{R}^n)$ for $1 < s < \infty$ and from L^∞ to BMO with both norms bounded by a constant multiple of $\|h\|_{L^\infty(\mathbb{R}^n)}$. This follows from the fact that T^1_h and T^2_h are linear Calderón-Zygmund operators of type $\omega(t)$ as described in Yabuta [47], where these boundedness properties are proved.

Corollary 6.3. *Under the hypothesis of Theorem 6.2, T can be extended to a bounded operator from $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into BMO, from $L^\infty(\mathbb{R}^n) \times H^1$ into $L^1(\mathbb{R}^n)$, and from $H^1 \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

The extension of T to $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ can be done in the usual way once T is defined on $L^\infty_c(\mathbb{R}^n) \times L^\infty_c(\mathbb{R}^n)$ (see Grafakos-Torres [26]).

6.2. **Boundedness on weighted Lebesgue spaces.** Let $\mathbf{Q}$ denote the collection of all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. The Hardy-Littlewood maximal function $\mathcal{M}$ is defined for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$\mathcal{M}f(x) = \sup_{Q\in\mathbf{Q}:x\in Q} \frac{1}{|Q|}\int_Q |f(x)|\, dx.$$

A nonnegative weight $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the A_p Muckenhoupt class, for $1 < p < \infty$ if

$$|w|_{A_p} := \sup_{Q\in\mathbf{Q}} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} < \infty.$$

We write $w \in A_1$ if there exists a constant C such that $\mathcal{M}w(x) \le Cw(x)$ for a.e. $x \in \mathbb{R}^n$ and set $A_\infty = \cup_{p\ge1} A_p$. Recall also that a weight w is in the class A_∞ if and only if there exist positive constants c and θ such that for every cube $Q \in \mathbf{Q}$ and every measurable set $E \subset Q$,

$$\frac{w(E)}{w(Q)} \le c\left(\frac{|E|}{|Q|}\right)^\theta, \tag{6.9}$$

where $w(S) = \int_S w(x)\, dx$ for any measurable set $S \subset \mathbb{R}^n$.

We denote by $L^p_w(\mathbb{R}^n)$ the weighted Lebesgue space of all functions f on $\mathbb{R}^n$ such that $\|f\|_{L^p_w(\mathbb{R}^n)} := (\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx)^{1/p} < \infty$. In this subsection we study weighted norm inequalities for a bilinear Calderón-Zygmund operator of type $\omega(t)$ and its corresponding maximal truncated operator.

Let T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ associated to a kernel $K(x, y, z)$. The maximal truncated operator is defined as

$$T_*(f, g)(x) = \sup_{\delta>0} |T_\delta(f, g)(x)|\,.$$

where

$$T_\delta(f, g)(x) = \int_{|x-y|^2+|x-z|^2>\delta^2} K(x, y, z)\, f(y)g(z)\, dydz.$$

Note that condition (3.1) guarantees that T_δ is well defined for $(f, g) \in L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$, $1 \le p,\, q \le \infty$, since the integral is absolutely convergent in this case. In what follows, W denotes de norm of T as a bounded operators from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into $L^{1/2,\infty}(\mathbb{R}^n)$ (see Theorem 6.1).

6.2.1. *Cotlar's inequality.*

Theorem 6.4. *Let $\omega \in Dini(1/2)$ and T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$ with kernel K. Then for all $\eta > 0$, there exists a constant $C_{\eta,\omega,n}$ such that*

$$T_*(f, g)(x) \le C_{\eta,\omega,n}\left((\mathcal{M}(|T(f, g)|^\eta)(x))^{1/\eta} + (C_K + W)\,\mathcal{M}f(x)\,\mathcal{M}g(x)\right), \quad x \in \mathbb{R}^n, \tag{6.10}$$

for all (f, g) in any product space $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ with $1 \le p,\, q < \infty$.

Proof. Define

$$\widetilde{T}_*(f, g)(x) = \sup_{\delta>0} \left|\widetilde{T}_\delta(f, g)(x)\right|$$

where

$$\widetilde{T}_\delta(f, g)(x) = \int_{A_\delta(x)} K(x, y, z)\, f(y)g(z)\, dydz, \quad A_\delta(x) = \{(y, z) : \max\{|x - y|\,, |x - z|\} > \delta\}$$

It is enough to prove (6.10) with T_* replaced by $\widetilde{T}_*$, since by (3.1),

$$\sup_{\delta>0}\left|\int_{\substack{\max(|x-y|,|x-z|)\leq\delta\\ |x-y|^2+|x-z|^2>\delta^2}} K(x,y,z)f(y)g(z)\right| \lesssim \mathcal{M}f(x)\,\mathcal{M}g(x).$$

We will show that

$$\left|\widetilde{T}_\delta(f,g)(x)\right| \lesssim C_K\,\mathcal{M}f(x)\,\mathcal{M}g(x) + \left|T(f,g)(x') - T(f_0,g_0)(x')\right|, \quad \left|x-x'\right| < \frac{\delta}{2}, \tag{6.11}$$

where $f_0 = f\,\chi_{B(x,\delta)}$ and $g_0 = g\,\chi_{B(x,\delta)}$. Once (6.11) is proved, we have for each fixed $\eta > 0$,

$$\begin{aligned}\left|\widetilde{T}_\delta(f,g)(x)\right|^\eta &\lesssim (C_K\,\mathcal{M}f(x)\,\mathcal{M}g(x))^\eta + \mathcal{M}\left(|T(f,g)|^\eta\right)(x)\\ &\quad + \frac{1}{\left|B(x,\frac{\delta}{2})\right|}\int_{B(x,\frac{\delta}{2})}\left|T(f_0,g_0)(x')\right|^\eta\,dx'.\end{aligned}$$

The last term in the above inequality can be shown to be bounded by $C_\eta W^\eta\left(\mathcal{M}f(x)\,\mathcal{M}g(x)\right)^\eta$, $0<\eta<\frac{1}{2}$, using only the boundedness of T from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ into $L^{1/2,\infty}(\mathbb{R}^n)$ (Theorem 6.1) and it follows as in Grafakos-Torres [27].

To prove (6.11), note that $T(f,g)(x') - T(f_0,g_0)(x') = \int_{A_\delta(x)} K(x',y,z)\,f(y)g(z)\,dydz$ for $|x-x'| < \frac{\delta}{2}$. It is then enough to show that

$$\left|\widetilde{T}_\delta(f,g)(x) - \int_{A_\delta(x)} K(x',y,z)\,f(y)g(z)\,dydz\right| \lesssim C_K\,\mathcal{M}f(x)\,\mathcal{M}g(x), \quad \left|x-x'\right| \leq \frac{\delta}{2}.$$

Noting that $|x-x'| \leq \frac{1}{2}\max\{|x-y|\,,|x-z|\}$, for $(y,z)\in A_\delta(x)$, we can use the regularity of the kernel (3.2) to obtain,

$$\begin{aligned}&|\widetilde{T}_\delta(f,g)(x) - \int_{A_\delta(x)} K(x',y,z)\,f(y)g(z)\,dydz|\\ &\leq \int_{A_\delta(x)} \frac{C_K}{(|x-y|+|x-z|)^{2n}}\,\omega\left(\frac{|x-x'|}{|x-y|+|x-z|}\right) f(y)g(z)\,dydz\\ &= \int_{|x-y|>\delta,|x-z|>\delta}\cdots + \int_{|x-y|>\delta,|x-z|\leq\delta}\cdots + \int_{|x-y|\leq\delta,|x-z|>\delta}\cdots.\end{aligned}$$

For the first term we have, using that ω is non-decreasing and that $|x-x'| < \frac{\delta}{2}$,

$$\begin{aligned}\int_{|x-y|>\delta,|x-z|>\delta}\cdots &\leq C_K\int_{|x-y|>\delta}\frac{\omega^{\frac{1}{2}}\left(\frac{\delta}{|x-y|}\right)}{|x-y|^n}\,f(y)\,dy\int_{|x-z|>\delta}\frac{\omega^{\frac{1}{2}}\left(\frac{\delta}{|x-z|}\right)}{|x-z|^n}\,g(z)\,dz\\ &\lesssim C_K\left(\omega^{\frac{1}{2}}(1) + \int_0^1\frac{\omega^{\frac{1}{2}}(t)}{t}\,dt\right)^2 \mathcal{M}f(x)\,\mathcal{M}g(x).\end{aligned}$$

For the second term (and similarly for the third term) we have,

$$\begin{aligned}\int_{|x-y|>\delta,|x-z|\leq\delta}\cdots &\leq C_K\int_{|x-y|>\delta}\frac{\omega\left(\frac{\delta}{|x-y|}\right)}{|x-y|^{2n}}\,f(y)\,dy\int_{|x-z|\leq\delta} g(z)\,dz\\ &\lesssim C_K\left(\omega(1) + \int_0^1\omega(t)\,t^{n-1}\,dt\right)\mathcal{M}f(x)\,\mathcal{M}g(x).\end{aligned}$$

□

Corollary 6.5. *Let $\omega \in Dini(1/2)$ and T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$. Then T_* is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ for $1 < p, q \le \infty$, $1/2 < r < \infty$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^{r,\infty}(\mathbb{R}^n)$ for $p = 1$ or $q = 1$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Moreover, in any case, $\|T_*\| \lesssim (C_K + W)$.*

6.2.2. *Weighted norm inequalities for T_*.*

Theorem 6.6. *Let $1 \le p, q < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $w \in A_\infty$. Consider $\omega \in Dini(1/2)$ and let T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$. Then,*

(i) if $\|T_*(f,g)\|_{L^r_w(\mathbb{R}^n)} < \infty$,

$$\|T_*(f,g)\|_{L^r_w(\mathbb{R}^n)} \le C_{p,n}(C_K + W) \|\mathcal{M}f\|_{L^p_w(\mathbb{R}^n)} \|\mathcal{M}g\|_{L^q_w(\mathbb{R}^n)}. \tag{6.12}$$

(ii) if $\min(p,q) > 1$ and $w \in A_{\min(p,q)}$, then we have $\|T_*(f,g)\|_{L^r_w(\mathbb{R}^n)} < \infty$ and

$$\|T_*(f,g)\|_{L^r_w(\mathbb{R}^n)} \le C_{p,n}(C_K + W) \|f\|_{L^p_w(\mathbb{R}^n)} \|g\|_{L^q_w(\mathbb{R}^n)}. \tag{6.13}$$

Theorem 6.6 will be a consequence of the following good-lambda inequality and the boundedness properties of T_* in the unweighted case (Corollary 6.5).

Theorem 6.7. *Consider $\omega \in Dini(1/2)$ and let T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$ with kernel K. Let $w \in A_\infty$ and θ be as in (6.9). Then there exists a positive constant C, such that for all $\alpha > 0$, all $\gamma > 0$ sufficiently small, and all $(f,g) \in L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$, $1 \le p, q < \infty$, the following inequality holds,*

$$w\left(\{\widetilde{T}_*(f,g) > 2^3\alpha\} \cap \{\mathcal{M}f\mathcal{M}g \le \gamma\alpha\}\right) \le C(C_K + W)^{\theta/2}\gamma^{\theta/2} w\left(\{\widetilde{T}_*(f,g) > \alpha\}\right). \tag{6.14}$$

Proof. We set

$$P := \{x : \widetilde{T}_*(f,g)(x) > \alpha\} = \cup_j Q_j$$

where Q_j are Whitney cubes. Since $w \in A_\infty$ it is enough to prove that

$$\left|Q_j \cap \{\widetilde{T}_*(f,g) > 2^3\alpha\} \cap \{\mathcal{M}f\mathcal{M}g \le \gamma\alpha\}\right| \le C(C_K + W)^{1/2}\gamma^{1/2} |Q_j|. \tag{6.15}$$

Let Q_j^* be an appropriate large multiple of Q_j and $x_j \in Q_j^* \cap P^c$ such that

$$\max_{u \in Q_j} |x_j - u| \le \tfrac{1}{4} \operatorname{dist}(x_j, (Q_j^*)^c). \tag{6.16}$$

Also, consider $\xi_j \in Q_j$ such that $\mathcal{M}f(\xi_j)\mathcal{M}g(\xi_j) \le \gamma\alpha$. For $h \in \{f,g\}$, define $h^0 = h\chi_{Q_j^*}$ and $h^\infty = h - h^0$. Then

$$\begin{aligned} &\left|Q_j \cap \{\widetilde{T}_*(f,g) > 2^3\alpha\} \cap \{\mathcal{M}f\mathcal{M}g \le \gamma\alpha\}\right| \\ &\le \sum_{i,k \in \{0,\infty\}} \left|Q_j \cap \{\widetilde{T}_*(f^i,g^k) > 2\alpha\} \cap \{\mathcal{M}f\mathcal{M}g \le \gamma\alpha\}\right|. \end{aligned}$$

The term corresponding to $i = k = 0$ is shown to be bounded by $C(C_K + W)^{1/2}\gamma^{1/2} |Q_j|$ (see Grafakos-Torres [27, Theorem 3.1]). This only uses the fact that $\widetilde{T}_*$ is bounded from $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n)$ into $L^{1/2,\infty}(\mathbb{R}^n)$ with norm bounded by $C(C_K + W)$ (see Corollary 6.5). The terms corresponding to $i = 0$, $j = \infty$ and $i = \infty$, $j = 0$, can be made zero for γ small enough (see Grafakos-Torres [27, Theorem 3.1]). This only uses the hypothesis (3.1) on the size of the kernel. When $i = k = \infty$ we can make the corresponding term equal to zero by using the regularity (3.2) of the kernel in the following way. We show first that

$$\left|\widetilde{T}_\delta(f^\infty, g^\infty)(x) - \widetilde{T}_\delta(f^\infty, g^\infty)(x_j)\right| \lesssim C_K\, \mathcal{M}f(\xi_j)\mathcal{M}g(\xi_j), \quad x \in Q_j. \tag{6.17}$$

We have

$$\left|\widetilde{T}_\delta(f^\infty, g^\infty)(x) - \widetilde{T}_\delta(f^\infty, g^\infty)(x_j)\right| \le \int_{A_\delta(x)} |K(x,y,z) - K(x_j,y,z)|\, |f^\infty(y)g^\infty(z)|\, dydz$$
$$+ \int_{A_\delta(x)\setminus A_\delta(x_j)} |K(x_j,y,z)\, f^\infty(y)g^\infty(z)|\, dydz + \int_{A_\delta(x_j)\setminus A_\delta(x)} |K(x_j,y,z)\, f^\infty(y)g^\infty(z)|\, dydz.$$

Note that (6.16) implies that $|x - x_j| \le \frac{1}{2}\max(|x-y|, |x-z|)$ for $y, z \in (Q_j^*)^c$, $x \in Q_j$. Then we can apply (3.2) to obtain

$$\begin{aligned}
&\int_{A_\delta(x)} |K(x,y,z) - K(x_j,y,z)|\, |f^\infty(y)g^\infty(z)|\, dydz \\
&\qquad \le C_K \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\omega\left(\frac{|x-x_j|}{|x-y|+|x-z|}\right)}{(|x-y|+|x-z|)^{2n}}\, |f^\infty(y)\, g^\infty(z)|\, dydz \\
&\qquad \lesssim C_K \int_{(Q_j^*)^c} \frac{\omega^{1/2}\left(\frac{l(Q_j^*)}{|x-y|}\right)}{|x-y|^n}\, |f(y)|\, dy \int_{(Q_j^*)^c} \frac{\omega^{1/2}\left(\frac{l(Q_j^*)}{|x-z|}\right)}{|x-z|^n}\, |g(z)|\, dz.
\end{aligned}$$

Noting that $|x-y| \sim |\xi_j - y|$ for $y \in (Q_j^*)^c$, $x \in Q_j$, we have

$$\begin{aligned}
\int_{(Q_j^*)^c} \frac{\omega^{1/2}\left(\frac{l(Q_j^*)}{|x-y|}\right)}{|x-y|^n}\, |f(y)|\, dy &\lesssim \int_{|\xi_j-y|>l(Q_j^*)} \frac{\omega^{1/2}\left(\frac{l(Q_j^*)}{|\xi_j-y|}\right)}{|\xi_j-y|^n}\, |f(y)|\, dy \\
&\lesssim \left(\omega^{\frac{1}{2}}(1) + \int_0^1 \frac{\omega^{1/2}(t)}{t}\, dt\right) \mathcal{M}f(\xi_j).
\end{aligned}$$

We have a similar estimate for the factor corresponding to g.

Since $|x-u| \sim |x_j - u| \sim |\xi_j - u|$ for $u \in (Q_j^*)^c$ and $x \in Q_j$, using the size (3.1) of the kernel K, we get,

$$\begin{aligned}
\int_{A_\delta(x)\setminus A_\delta(x_j)} |K(x_j,y,z)\, f^\infty(y)g^\infty(z)|\, dydz &\lesssim \int_{|\xi_j-y|\sim\delta} \frac{|f(y)|}{|\xi_j-y|^n}\, dy \int_{|\xi_j-z|\sim\delta} \frac{|g(z)|}{|\xi_j-z|^n}\, dz \\
&\quad + \int_{|\xi_j-y|\sim\delta} \frac{|f(y)|}{|\xi_j-y|^{2n}}\, dy \int_{|\xi_j-z|\lesssim\delta} |g(z)|\, dz \\
&\quad + \int_{|\xi_j-z|\sim\delta} \frac{|g(z)|}{|\xi_j-z|^{2n}}\, dz \int_{|\xi_j-y|\lesssim\delta} |f(y)|\, dy \\
&\lesssim \mathcal{M}f(\xi_j)\, \mathcal{M}g(\xi_j),
\end{aligned}$$

with a similar estimate for $\int_{A_\delta(x_j)\setminus A_\delta(x)} |K(x_j,y,z)\, f^\infty(y)g^\infty(z)|\, dydz$. Therefore (6.17) follows.

One also has that

$$\left|\widetilde{T}_\delta(f^\infty, g^\infty)(x_j)\right| \le \widetilde{T}_*(f,g)(x_j) + C\, C_K\, \mathcal{M}f(\xi_j)\mathcal{M}g(\xi_j), \quad \delta > 0. \tag{6.18}$$

The proof of this uses the condition (3.1) on the size of the kernel, and follows the steps in Grafakos-Torres [27, Theorem 3.1]. Using (6.17), (6.18), $\mathcal{M}f(\xi_j)\mathcal{M}g(\xi_j) \leq \gamma\alpha$, and $\widetilde{T}_*(f,g)(x_j) < \alpha$, we get $\widetilde{T}_*(f^\infty, g^\infty)(x) \leq 2\alpha$ for $x \in Q_j$ by choosing γ small enough. □

6.2.3. *Weighted norm inequalities for T.* The weighted norm inequalities above can be extended for a bilinear Calderón-Zygmund operator T of type $\omega(t)$ by controlling T by T^* and a bounded bilinear pointwise multiplier operator. More precisely, for f and g bounded and compactly supported,

$$|T(f,g)(x)| \lesssim T_*(f,g)(x) + \|b\|_{L^\infty(\mathbb{R}^n)} |f(x)g(x)|, \tag{6.19}$$

where b is a function satisfying $\|b\|_{L^\infty(\mathbb{R}^n)} \lesssim (C_K + W)$. This is proved using the arguments from the linear case (see [27]).

Theorem 6.8. *Let $1 < p,\ q < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $w \in A_\infty$. Consider $\omega \in Dini(1/2)$ and let T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ in $\mathbb{R}^n$ with kernel K. Then for f and g bounded and compactly supported,*

$$\|T(f,g)\|_{L^r_w(\mathbb{R}^n)} \leq C_{p,n}(C_K + W) \|\mathcal{M}f\|_{L^p_w(\mathbb{R}^n)} \|\mathcal{M}g\|_{L^q_w(\mathbb{R}^n)}. \tag{6.20}$$

In particular, if $w \in A_{\min(p,q)}$ we have

$$\|T(f,g)\|_{L^r_w(\mathbb{R}^n)} \leq C_{p,n}(C_K + W) \|f\|_{L^p_w(\mathbb{R}^n)} \|g\|_{L^q_w(\mathbb{R}^n)}, \tag{6.21}$$

and therefore, T extends as a bounded operator from $L^p_w(\mathbb{R}^n) \times L^q_w(\mathbb{R}^n)$ into $L^r_w(\mathbb{R}^n)$.

Remark 5. Adapting Remark 3.6 of Grafakos-Torres [27], for $w \in A_1$ and f and g bounded and compactly supported, one can prove that $\left\|\widetilde{T}_*(f,g)\right\|_{L^{1/2,\infty}_w(\mathbb{R}^n)} < \infty$. Using the good-lambda inequality (6.14) and that $\left|T_*(f,g)(x) - \widetilde{T}_*(f,g)(x)\right| \lesssim \mathcal{M}f(x)\mathcal{M}g(x)$, we then obtain that

$$\|T_*(f,g)\|_{L^{1/2,\infty}_w(\mathbb{R}^n)} \lesssim \|\mathcal{M}f\|_{L^{1,\infty}_w(\mathbb{R}^n)} \|\mathcal{M}g\|_{L^{1,\infty}_w(\mathbb{R}^n)}.$$

As a consequence of this and (6.19), T extends as a bounded operator from $L^1_w(\mathbb{R}^n)\times L^1_w(\mathbb{R}^n)$ into $L^{1/2,\infty}_w(\mathbb{R}^n)$ if $w \in A_1$.

6.3. **Boundedness on weighted Hardy spaces.** In this subsection we present a weighted version of the Hardy space estimates for bilinear Calderón-Zygmund operators established by L. Grafakos and N. Kalton in [23].

Theorem 6.9. *Let $\frac{n}{n+1} < p_1, p_2 \leq 1$, $0 < p \leq 1$, with $1/p = 1/p_1 + 1/p_2$, and $\omega : [0,\infty) \to [0,\infty)$ non-decreasing, concave such that*

$$\int_0^1 t^{np_j - n} \omega^{p_j/2}(t)\, \frac{dt}{t} < \infty, \quad j = 1, 2. \tag{6.22}$$

If $w \in A_1$ and T is a bilinear Calderón-Zygmund operator of type $\omega(t)$, then

$$T : H^{p_1}_w(\mathbb{R}^n) \times H^{p_2}_w(\mathbb{R}^n) \to L^p_w(\mathbb{R}^n).$$

Remark 6. The *critical index* q_w of a weight $w \in A_\infty$ is defined as the $\inf\{q > 1 : w \in A_q\}$. A well-known result in the linear theory of Calderón-Zygmund singular integrals asserts that if the kernel of a Calderón-Zygmund operator T has smoothness $\omega(t) = t^\gamma$, for some $0 < \gamma \leq 1$, and $0 < r \leq 1$, then T maps $H^r_w(\mathbb{R}^n) \to L^r_w(\mathbb{R}^n)$ provided that $q_w/r < 1 + \gamma/n$ (see Theorem 2.8 in [20]). In our situation, the bilinear kernels possess moduli of continuity of Dini type instead of Hölder type. This essentially compels (by letting γ go to 0) the choice of the class A_1 (i.e., $q_w = 1$) and $r = 1$. Also, notice that if $p_1 = p_2 = 1$, then (6.22) reduces to $\omega \in Dini(1/2)$.

Proof of Theorem 6.9. Let $w \in A_1$ and $n/n+1 < p_1, p_2 \leq 1$. By the atomic decomposition of the weighted Hardy spaces $H^{p_j}_w$, $j = 1, 2$, (see Proposition 1.5 in [20]) we can consider the dense class of *finite* sums of the form $f_j = \sum_k \lambda_{j,k} a_{j,k}$, where the functions $a_{j,k}$ (called p_j-atoms) and the coefficients λ_{j,k_j} satisfy

$$\operatorname{supp}(a_{j,k}) \subset Q_{j,k}, \tag{6.23}$$

$$\|a_{j,k}\|_{L^\infty} \leq w(Q_{j,k})^{-1/p_j}, \tag{6.24}$$

$$\int_{Q_{j,k}} a_{j,k}(x)dx = 0, \tag{6.25}$$

$$\left(\sum_k |\lambda_{j,k}|^{p_j}\right)^{1/p_j} \leq 2\, \|f_j\|_{H^{p_j}_w}. \tag{6.26}$$

As in [23], in order to estimate the L^p_w-norm of

$$T(f_1, f_2)(x) = \sum_{k_1} \sum_{k_2} \lambda_{1,k_1} \lambda_{2,k_2} T(a_{1,k_1}, a_{2,k_2})(x),$$

we fix k_1, k_2, $x \in \mathbb{R}^n$. For $j = 1, 2$, let P_{j,k_j} be the cube concentric with Q_{j,k_j} such that $l(P_{j,k_j}) = \frac{1}{2} l(Q_{j,k_j})$, and pick $c_{j,k_j} \in P_{j,k_j}$. Suppose first that $x \notin Q^*_{1,k_1} \cup Q^*_{2,k_2}$ and assume (by switching the roles of Q_{1,k_1} and Q_{2,k_2}, if necessary) that $l(Q_{1,k_1}) \leq l(Q_{2,k_2})$. By (6.24) and (6.25),

$$\begin{aligned}
&|T(a_{1,k_1}, a_{2,k_2})(x)| \\
&= \left| \int\!\!\int a_{2,k_2}(z) a_{1,k_1}(y)(K(x,y,z) - K(x, c_{1,k_1}, z)\, dy\, dz \right| \\
&\leq \int |a_{2,k_2}(z)| \int \frac{|a_{1,k_1}(y)|}{(|x-y| + |x-z|)^{2n}} \omega\left(\frac{|y - c_{1,k_1}|}{|x-y| + |x-z|}\right) dy\, dz \\
&\lesssim \int\!\!\int \frac{|a_{2,k_2}(z)||a_{1,k_1}(y)|}{(|x - c_{1,k_1}| + |x - c_{2,k_2}|)^{2n}} \omega\left(\frac{l(Q_{1,k_1})}{|x - c_{1,k_1}| + |x - c_{2,k_2}|}\right) dy\, dz \\
&\lesssim \frac{w(Q_{1,k_1})^{-1/p_1} w(Q_{2,k_2})^{-1/p_2} |Q_{1,k_1}||Q_{2,k_2}|}{|x - c_{1,k_1}|^n |x - c_{2,k_2}|^n} \omega\left(\frac{l(Q_{1,k_1})}{|x - c_{1,k_1}| + |x - c_{2,k_2}|}\right) dy\, dz \\
&\lesssim \prod_{j=1}^{2} \frac{|Q_{j,k_j}| w(Q_{j,k_j})^{-1/p_j}}{(|x - c_{j,k_j}| + l(Q_{j,k_j}))^n} \omega^{1/2}\left(\frac{l(Q_{j,k_j})}{|x - c_{j,k_j}| + l(Q_{j,k_j})}\right).
\end{aligned}$$

Now suppose $x \in Q^*_{2,k_2} \setminus Q^*_{1,k_1}$. Again, by (6.24) and (6.25),

$$\begin{aligned}|T(a_{1,k_1}, a_{2,k_2})(x)| &\leq \int |a_{2,k_2}(z)| \int \frac{|a_{1,k_1}(y)|}{(|x-y|+|x-z|)^{2n}} \omega\left(\frac{|y-c_{1,k_1}|}{|x-y|+|x-z|}\right) dy\, dz \\ &\lesssim w(Q_{2,k_2})^{-1/p_2} \int\int \frac{|a_{1,k_1}(y)|}{(|x-c_{1,k_1}|+|x-z|)^{2n}} \omega\left(\frac{l(Q_{1,k_1})}{|x-c_{1,k_1}|}\right) dy\, dz \\ &\lesssim \frac{w(Q_{2,k_2})^{-1/p_2} w(Q_{1,k_1})^{-1/p_1} |Q_{1,k_1}|}{|x-c_{1,k_1}|^n} \omega\left(\frac{l(Q_{1,k_1})}{|x-c_{1,k_1}|}\right) \\ &\simeq \frac{w(Q_{2,k_2})^{-1/p_2} w(Q_{1,k_1})^{-1/p_1} |Q_{1,k_1}|}{(|x-c_{1,k_1}|+l(Q_{1,k_1}))^n} \omega\left(\frac{l(Q_{1,k_1})}{|x-c_{1,k_1}|+l(Q_{1,k_1})}\right).\end{aligned}$$

Since $x \in Q^*_{2,k_2}$ and concavity of ω we have

$$1 \lesssim \frac{l(Q_{2,k_2})}{|x-c_{2,k_2}|+l(Q_{2,k_2})} \lesssim \omega\left(\frac{l(Q_{2,k_2})}{|x-c_{2,k_2}|+l(Q_{2,k_2})}\right).$$

Therefore, if $x \in Q^*_{2,k_2} \setminus Q^*_{1,k_1}$, and by symmetry, whenever $x \in Q^*_{2,k_2} \Delta Q^*_{1,k_1}$,

$$|T(a_{1,k_1}, a_{2,k_2})(x)| \lesssim \prod_{j=1}^{2} \frac{|Q_{j,k_j}| w(Q_{j,k_j})^{-1/p_j}}{(|x-c_{j,k_j}|+l(Q_{j,k_j}))^n} \omega\left(\frac{l(Q_{j,k_j})}{|x-c_{j,k_j}|+l(Q_{j,k_j})}\right). \tag{6.27}$$

Combining the bounds above, and since $\omega(t) \leq C\omega^{1/2}(t)$, for $0 < t < 1$, we get that if $x \notin Q^*_{1,k_1} \cap Q^*_{2,k_2}$,

$$|T(a_{1,k_1}, a_{2,k_2})(x)| \lesssim \prod_{j=1}^{2} \frac{|Q_{j,k_j}| w(Q_{j,k_j})^{-1/p_j}}{(|x-c_{j,k_j}|+l(Q_{j,k_j}))^n} \omega^{1/2}\left(\frac{l(Q_{j,k_j})}{|x-c_{j,k_j}|+l(Q_{j,k_j})}\right). \tag{6.28}$$

Consequently,

$$\begin{aligned}|T(f_1, f_2)(x)| &\lesssim \sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}||T(a_{1,k_1}, a_{2,k_2})(x)| \chi_{Q^*_{1,k_1} \cap Q^*_{2,k_2}}(x) \\ &\quad + \prod_{j=1}^{2}\left(\sum_{k_j} \frac{|\lambda_{j,k_j}||Q_{j,k_j}| w(Q_{j,k_j})^{-1/p_j}}{(|x-c_{j,k_j}|+l(Q_{j,k_j}))^n} \omega^{1/2}\left(\frac{l(Q_{j,k_j})}{|x-c_{j,k_j}|+l(Q_{j,k_j})}\right)\right) \\ &=: G_1(x) + G_2(x).\end{aligned}$$

In order to bound the L^p_w-norm of the first summand we use the following real analysis lemma (see Lemma 2.1 in [23])

Lemma 6.10. *Fix $p \in (0,1]$ and let w be a doubling weight in $\mathbb{R}^n$. Then, there is a constant C, depending only on p and the doubling constant of w, such that for all finite collections $\{Q_k\}_{k=1}^K \subset \mathbf{Q}$ and all nonnegative integrable functions g_k supported on Q_k we have*

$$\left\|\sum_{k=1}^{K} g_k\right\|_{L^p_w} \leq C \left\|\sum_{k=1}^{K}\left(\frac{1}{w(Q_k)}\int_{Q_k} g_k(x) w(x)\, dx\right) \chi_{Q^*_k}\right\|_{L^p_w}.$$

Fix atoms a_{1,k_1} and a_{2,k_2} and suppose $Q^*_{1,k_1} \cap Q^*_{2,k_2} \neq \emptyset$ (the case of empty intersection being trivial). Assume, without loss of generality, that $l(Q_{1,k_1}) \leq l(Q_{2,k_2})$ and pick $R_{k_1,k_2} \in \mathbf{Q}$ such that

$$Q^*_{1,k_1} \cap Q^*_{2,k_2} \subset R_{k_1,k_2} \subset R^*_{k_1,k_2} \subset Q^{**}_{1,k_1} \cap Q^{**}_{2,k_2},$$

and $w(R_{k_1,k_2}) \geq cw(Q_{1,k_1})$. By Theorem 6.2 and the linear case treated in Yabuta [47], T maps $L^\infty_c \times L^2_w \to L^2_w$ and $L^2_w \times L^\infty_c \to L^2_w$, hence

$$\begin{aligned}\int_{R_{k_1,k_2}} |T(a_{1,k_1}, a_{2,k_2})(x)|w(x)\,dx &\leq \left(\int |T(a_{1,k_1}, a_{2,k_2})(x)|^2 w(x)\,dx\right)^{1/2} w(R_{k_1,k_2})^{1/2}\\ &\lesssim \|a_{1,k_1}\|_{L^2_w} \|a_{2,k_2}\|_{L^\infty}\, w(R_{k_1,k_2})^{1/2}\\ &\leq w(Q_{1,k_1})^{1/2-1/p_1} w(Q_{2,k_2})^{-1/p_2} w(R_{k_1,k_2})^{1/2}\\ &\leq w(Q_{1,k_1})^{-1/p_1} w(Q_{2,k_2})^{-1/p_2} w(R_{k_1,k_2}).\end{aligned}$$

By Lemma 6.10, and recalling that $0 < p_j \leq 1$ for $j = 1, 2$, we obtain

$$\begin{aligned}\|G_1\|_{L^p_w} &\leq \left\|\sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}||T(a_{1,k_1}, a_{2,k_2})|\chi_{R_{k_1,k_2}}\right\|_{L^p_w}\\ &\lesssim \left\|\sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}| \left(\frac{1}{w(R_{k_1,k_2})}\int |T(a_{1,k_1}, a_{2,k_2})|(x)w(x)\,dx\right)\chi_{R^*_{k_1,k_2}}\right\|_{L^p_w}\\ &\lesssim \left\|\sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}| w(Q_{1,k_1})^{-1/p_1} w(Q_{2,k_2})^{-1/p_2}\chi_{Q^{**}_{1,k_1}}\chi_{Q^{**}_{2,k_2}}\right\|_{L^p_w}\\ &\lesssim \prod_{j=1}^{2}\left\|\sum_{k_j} |\lambda_{j,k_j}| w(Q_{j,k_j})^{-1/p_j}\chi_{Q^{**}_{j,k_j}}\right\|_{L^{p_j}_w} \lesssim \prod_{j=1}^{2}\left(\sum_{k_j}|\lambda_{j,k_j}|^{p_j}\right)^{1/p_j} \lesssim \prod_{j=1}^{2}\|f_j\|_{H^{p_j}_w}.\end{aligned}$$

To estimate G_2 we begin with the simple

Lemma 6.11. *Let $l > 0$ and $0 < q \leq 1$. Suppose that $\omega : [0,\infty) \to [0,\infty)$ is non-decreasing and it verifies*

$$C(q,\omega) := \int_0^1 u^{qn-n}\omega^{q/2}(u)\,\frac{du}{u} < \infty. \tag{6.29}$$

Then, the function

$$h_l(x) := \frac{l^{qn-n}}{(|x|+l)^{qn}}\omega^{q/2}\left(\frac{l}{|x|+l}\right), \quad x \in \mathbb{R}^n,$$

belongs to $L^1(\mathbb{R}^n)$ and $\|h_l\|_{L^1} \lesssim C(q,\omega)$, (uniform in l.)

Proof of Lemma 6.11. Changing to polar coordinates we obtain

$$\begin{aligned}\|h_l\|_{L^1} &= \int_0^\infty \frac{l^{qn-n}\rho^{n-1}}{(\rho+l)^{qn}}\omega^{q/2}\left(\frac{l}{\rho+l}\right)d\rho\\ &\simeq \int_0^l \frac{l^{qn-n}\rho^{n-1}\omega^{q/2}(1)}{l^{qn}}\,d\rho + \int_l^\infty \frac{l^{qn-n}\rho^{n-1}}{\rho^{qn}}\omega^{q/2}(l/\rho)\,d\rho\\ &\lesssim \int_0^l l^{-n}\rho^{n-1}\,d\rho + \int_l^\infty \frac{l^{qn-n}\rho^{n-1}}{\rho^{qn}}\omega^{q/2}\left(\frac{l}{\rho}\right)d\rho\\ &\leq 1 + \int_0^1 u^{qn-n}\omega^{q/2}(u)\,\frac{du}{u} \lesssim C(q,\omega). \quad \square\end{aligned}$$

Using Hölder's inequality first, and recalling that $0 < p_j \leq 1$ for $j = 1, 2$, we obtain

$$\|G_2\|_{L^p_w} \leq \prod_{j=1}^{2} \left\| \sum_{k_j} \frac{|\lambda_{j,k_j}||Q_{j,k_j}|w(Q_{j,k_j})^{-1/p_j}}{(|x - c_{j,k_j}| + l(Q_{j,k_j}))^n} \omega^{1/2} \left(\frac{l(Q_{j,k_j})}{|x - c_{j,k_j}| + l(Q_{j,k_j})} \right) \right\|_{L^{p_j}_w}$$

$$\leq \prod_{j=1}^{2} \left(\int \sum_{k_j} \frac{|\lambda_{j,k_j}|^{p_j}|Q_{j,k_j}|^{p_j} w(Q_{j,k_j})^{-1}}{(|x - c_{j,k_j}| + l(Q_{j,k_j}))^{np_j}} \omega^{p_j/2} \left(\frac{l(Q_{j,k_j})}{|x - c_{j,k_j}| + l(Q_{j,k_j})} \right) w(x)\, dx \right)^{1/p_j}$$

$$= \prod_{j=1}^{2} \left(\sum_{k_j} |\lambda_{j,k_j}|^{p_j} \frac{|Q_{j,k_j}|}{w(Q_{j,k_j})} (h_{j,k_j} * w)(c_{j,k_j}) \right)^{1/p_j},$$

where

$$h_{j,k_j}(x) := \frac{|Q_{j,k_j}|^{p_j - 1}}{(|x| + l(Q_{j,k_j}))^{np_j}} \omega^{p_j/2} \left(\frac{l(Q_{j,k_j})}{|x| + l(Q_{j,k_j})} \right).$$

By Lemma 6.11, $h_{j,k_j} \in L^1(\mathbb{R}^n)$ with norm uniform in (j, k_j), $j = 1, 2$. Then,

$$\|G_2\|^p_{L^p_w} \lesssim \prod_{j=1}^{2} \left(\sum_{k_j} |\lambda_{j,k_j}|^{p_j} \frac{|Q_{j,k_j}|}{w(Q_{j,k_j})} \mathcal{M}w(c_{j,k_j}) \right)^{p/p_j}$$

(6.30)
$$\lesssim \prod_{j=1}^{2} \left(\sum_{k_j} |\lambda_{j,k_j}|^{p_j} \frac{|Q_{j,k_j}|}{w(Q_{j,k_j})} w(c_{j,k_j}) \right)^{p/p_j}.$$

Considering $\frac{|Q_{j,k_j}|}{w(Q_{j,k_j})} w(c_{j,k_j})$ as a function of c_{j,k_j} and taking its average (with respect to Lebesgue measure) over the cube P_{j,k_j} we get

$$\frac{|Q_{j,k_j}| w(P_{j,k_j})}{|P_{j,k_j}| w(Q_{j,k_j})},$$

which is bounded by a constant depending only on n. Averaging (6.30) (in the Lebesgue measure) over P_{j,k_j} with respect to each c_{j,k_j} (there are finitely many of them) and using Hölder's inequality with exponents p_1/p and p_2/p, we finally obtain

$$\|G_2\|_{L^p_w} \lesssim \prod_{j=1}^{2} \left(\sum_{k_j} |\lambda_{j,k_j}|^{p_j} \right)^{1/p_j} \lesssim \prod_{j=1}^{2} \|f_j\|_{H^{p_j}_w}. \quad \square$$

Remark 7. One can prove Lemma 6.10 proceeding as in the proof of Lemma 2.1 in [23]. The doubling contion of w is used to insure that $w(Q_k) \simeq w(Q_{kj})$ in the notation of [23].

6.4. **Boundedness on weighted amalgam spaces.** Amalgam spaces have been intensively considered in several areas of Analysis (see [17] for an excellent survey), as they allow for a better understanding of the global and local features of functions. In this subsection we prove boundedness properties on products of weighted amalgam spaces for bilinear Calderón-Zygmund operators of type $\omega(t)$. The results are obtained as a consequence of the properties (3.1) and (3.2) of the kernel, the boundedness of T on unweighted Lebesgue spaces (Theorem 6.2), and the boundedness of the corresponding truncated operators on weighted Lebesgue spaces (Theorem 6.6). The behavior of linear Calderón-Zygmund operators of type $\omega(t)$ on amalgam spaces was studied by Kikuchi et al [30].

For $1 < p < \infty$, the discrete variant of Muckenhoupt's A_p class is denoted by $A_p(\mathbb{Z}^n)$ and consists of the positive sequences $\{w_z\}_{z\in\mathbb{Z}^n}$ such that

$$|w|_{A_p(\mathbb{Z}^n)} := \sup_{Q\in\mathbf{Q}} \left(\frac{1}{\#(Q\cap\mathbb{Z}^n)} \sum_{z\in Q\cap\mathbb{Z}^n} w_z \right) \left(\frac{1}{\#(Q\cap\mathbb{Z}^n)} \sum_{z\in Q\cap\mathbb{Z}^n} w_z^{1-p'} \right)^{p-1} < \infty.$$

For $z \in \mathbb{Z}^n$ set $Q_z := \{x \in \mathbb{R}^n : |x_i - z_i| \le 1/2, i = 1, \dots, n\}$. Consider $1 \le p, q \le \infty$ and a positive sequence $\{w_z\}_{z\in\mathbb{Z}^n}$. We denote by l^q_w the space of all sequences $\{a_z\}_{z\in\mathbb{Z}^n}$ such that $\|a\|_{l^q_w} := (\sum_{z\in\mathbb{Z}^n} |a_z|^q\, w_z)^{1/q} < \infty$. In particular we write l^q instead of l^q_w when $w \equiv 1$. The *weighted amalgam space* (L^p, l^q_w) consists of the locally integrable functions f on $\mathbb{R}^n$ such that $\left\{\|f\|_{L^p(Q_z)}\right\}_{z\in\mathbb{Z}^n} \in l^q_w$, with norm

$$\|f\|_{(L^p,l^q_w)} := \left(\sum_{z\in\mathbb{Z}^n} \|f\|^q_{L^p(Q_z)}\, w_z \right)^{1/q}.$$

The usual interpretation applies when $q = \infty$. The main result in this subsection is the following

Theorem 6.12. *Consider $\omega \in Dini(1/2)$ and let T be a bilinear Calderón-Zygmund operator of type $\omega(t)$ with kernel K. If $1 < p, q < \infty$, $1 < s_1, s_2 < \infty$, $1/r = 1/p + 1/q$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_s(\mathbb{Z}^n)$, $s = \min\{s_1, s_2\}$, then*

$$\|T(f,g)\|_{(L^r,l^{s_3}_w)} \le C\, \|f\|_{(L^p,l^{s_1}_w)}\, \|g\|_{(L^q,l^{s_2}_w)}\,. \tag{6.31}$$

Remark 8. Note that $w = \{w_z\}_{z\in\mathbb{Z}^n} \in A_s(\mathbb{Z}^n)$ if and only if $W = \sum_z w_z \chi_{Q_z} \in A_s$, and that $(L^t, l^t_w) = L^t_W(\mathbb{R}^n)$ with $\|f\|_{(L^t,l^t_w)} = \|f\|_{L^t_W(\mathbb{R}^n)}$. Therefore the result of Theorem 6.12 for the case $p = s_1$, $q = s_2$ and $r = s_3$ is a particular case of Theorem 6.8.

We state here some definitions and known results that will be used in the proof of Theorem 6.12.

Lemma 6.13. *(see Kikuchi et al* [30]*) Let $w \in A_t(\mathbb{Z}^n)$, $1 < t < \infty$. Then, for $\mu \in \mathbb{Z}^n$ and all cubes Q containing μ,*

$$\sum_{\zeta\in\mathbb{Z}^n\cap Q} w_\zeta \le w_\mu\, |w|_{A_t(\mathbb{Z}^n)} \left(\#(\mathbb{Z}^n \cap Q)\right)^t.$$

For a sequence $a = \{a_\mu\}_{\mu\in\mathbb{Z}^n}$ we consider the discrete maximal function

$$(\mathcal{M}_d a)_\mu = \sup_{\mu\in Q} \frac{1}{\#(\mathbb{Z}^n\cap Q)} \sum_{\nu\in\mathbb{Z}^n\cap Q\in\mathbf{Q}} |a_\nu|\,, \qquad \mu \in \mathbb{Z}^n.$$

The following properties for $\mathcal{M}_d$ are well-known:

Lemma 6.14. *If $w \in A_s(\mathbb{Z}^n)$ and $1 < s < \infty$, then $\mathcal{M}_d$ is bounded in l^s_w.*

For sequences $h = \{h_\mu\}_{\mu\in\mathbb{Z}^n}$ and $a = \{a_\nu\}_{\nu\in\mathbb{Z}^n}$ define the convolution

$$(h * a)_\mu := \sum_{\nu\in\mathbb{Z}^n} h_{\mu-\nu} a_\nu, \qquad \mu \in \mathbb{Z}^n.$$

Lemma 6.15. *Let $h = \{h_\mu\}_{\mu\in\mathbb{Z}^n}$ be a sequence in $l^1(\mathbb{Z}^n)$ which is nonnegative, radial and non-increasing (i.e., $h_\mu = h_{\mu'}$ if $|\mu| = |\mu'|$, and $h_\mu \le h_{\mu'}$ if $|\mu| \ge |\mu'|$). Then for any sequence $a = \{a_\mu\}_{\mu\in\mathbb{Z}^n}$*

$$|(h * a)_\mu| \lesssim \|h\|_{l^1}\, (\mathcal{M}_d a)_\mu, \qquad \mu \in \mathbb{Z}^n.$$

Proof of Theorem 6.12. Let $f, g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. For Q_ζ, $\zeta \in \mathbb{Z}^n$, and $b \in \{f, g\}$, we consider

$$b = b_1 + b_2, \qquad b_i \in \mathcal{C}_0^\infty(\mathbb{R}^n), \qquad |b_i(x)| \le |b(x)|, \quad i = 1, 2,$$
$$\operatorname{supp}(b_1) \subset 2\,Q_\zeta^*, \qquad \operatorname{supp}(b_2) \cap Q_\zeta^* = \emptyset,$$

where Q_ζ^* is the closed cube centered at ζ and such that $l(Q_\zeta^*) = (2n+1)\, l(Q_\zeta)$. We then have $T(f, g) = T(f_1, g_1) + T(f_2, g_1) + T(f_1, g_2) + T(f_2, g_2)$.

By the boundedness of T from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ (Theorem 6.2), we have

$$\|T(f_1, g_1)\|_{L^r(Q_\zeta)} \lesssim \|f_1\|_{L^p(\mathbb{R}^n)}\ \|g_1\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{\mu \in \mathbb{Z}^n \cap 2\,Q_\zeta^*} \|f\|_{L^p(Q_\mu)} \sum_{\nu \in \mathbb{Z}^n \cap 2\,Q_\zeta^*} \|g\|_{L^q(Q_\nu)}\,. \tag{6.32}$$

For the term $T(f_2, g_1)$ (and similarly for $T(f_1, g_2)$), we have for $x \in Q_\zeta$,

$$\begin{aligned}
T(f_2, g_1)(x) &= \int_{z \in 2\,Q_\zeta^*} \int_{y \in (Q_\zeta^*)^c} K(x, y, z) f_2(y) g_1(z)\, dy\, dz \\
&= \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \int_{z \in 2\,Q_\zeta^*} \int_{y \in Q_\mu} K(x, y, z) f_2(y) g_1(z)\, dy\, dz \\
&= \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \int_{z \in 2\,Q_\zeta^*} \int_{y \in Q_\mu} \left(K(x, y, z) - K(\zeta, \mu, z)\right) f_2(y) g_1(z)\, dy\, dz \\
&\quad + \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \int_{Q_\mu} f_2(y)\, dy \int_{2\,Q_\zeta^*} K(\zeta, \mu, z)\, g_1(z)\, dz.
\end{aligned}$$

Using (3.2) and (3.1), we get

$$\begin{aligned}
\|T(f_2, g_1)\|_{L^r(Q_\zeta)} &\lesssim \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \frac{1}{|\zeta - \mu|^{2n}} \omega\left(\frac{c_n}{|\zeta - \mu|}\right) \|f\|_{L^p(Q_\mu)}\ \|g\|_{L^q(2\,Q_\zeta^*)} \\
&\quad + \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \frac{1}{|\zeta - \mu|^{2n}} \|f\|_{L^p(Q_\mu)}\ \|g\|_{L^q(2\,Q_\zeta^*)}\,.
\end{aligned} \tag{6.33}$$

For $x \in Q_\zeta$, we have

$$\begin{aligned}
T(f_2, g_2)(x) &= \int_{(Q_\zeta^*)^c} \int_{(Q_\zeta^*)^c} K(x, y, z) f_2(y) g_2(z)\, dy\, dz \\
&= \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \int_{z \in Q_\nu} \int_{y \in Q_\mu} K(x, y, z) f_2(y) g_2(z)\, dy\, dz \\
&= \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \int_{z \in Q_\nu} \int_{y \in Q_\mu} \left(K(x, y, z) - K(\zeta, \mu, \nu)\right) f_2(y) g_2(z)\, dy\, dz \\
&\quad + \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} K(\zeta, \mu, \nu) \int_{Q_\mu} f_2(y)\, dy \int_{Q_\nu} g_2(z)\, dz.
\end{aligned}$$

Using (3.2), we obtain

(6.34)

$$\|T(f_2,g_2)\|_{L^r(Q_\zeta)}$$
$$\lesssim \sum_{\nu\in\mathbb{Z}^n-Q_\zeta^*}\sum_{\mu\in\mathbb{Z}^n-Q_\zeta^*}\frac{1}{(|\zeta-\mu|+|\zeta-\nu|)^{2n}}\,\omega\left(\frac{c_n}{|\zeta-\mu|+|\zeta-\nu|}\right)\|f\|_{L^p(Q_\mu)}\,\|g\|_{L^q(Q_\nu)}$$
$$+\left|\sum_{\nu\in\mathbb{Z}^n-Q_\zeta^*}\sum_{\mu\in\mathbb{Z}^n-Q_\zeta^*}K(\zeta,\mu,\nu)\int_{Q_\mu}f_2(y)\,dy\int_{Q_\nu}g_2(z)\,dz\right|.$$

We now proceed to estimate the $l_w^{s_3}$-norm of the terms on the right hand side of (6.32), (6.34), and (6.33). For the right hand side of (6.32), we apply Hölder's inequality and then observe that, using Jensen's inequality and Lemma 6.13, we have

$$\left(\sum_{\zeta\in\mathbb{Z}^n}\left(\sum_{\mu\in\mathbb{Z}^n\cap 2\,Q_\zeta^*}\|f\|_{L^p(Q_\mu)}\right)^{s_1}w_\zeta\right)^{\frac{1}{s_1}}$$
$$\le (4n+3)^{2(1-\frac{1}{s_1})}\left(\sum_{\mu\in\mathbb{Z}^n}\|f\|_{L^p(Q_\mu)}^{s_1}\sum_{\zeta\in\mathbb{Z}^n}w_\zeta\,\chi_{\mathbb{Z}^n\cap 2\,Q_\zeta^*}(\mu)\right)^{\frac{1}{s_1}}$$
$$\le (4n+3)^{2(1-\frac{1}{s_1})}\left(\sum_{\mu\in\mathbb{Z}^n}\|f\|_{L^p(Q_\mu)}^{s_1}\,w_\mu\,|w|_{A_s(\mathbb{Z}^n)}\,\#(\mathbb{Z}^n\cap Q)^s\right)^{\frac{1}{s_1}},$$

where for each fixed μ, Q is a cube containing μ and all those $\zeta\in\mathbb{Z}^n$ such that $\mu\in\mathbb{Z}^n\cap 2\,Q_\zeta^*$. It is clear that $\#(\mathbb{Z}^n\cap Q)$ is independent of μ, therefore we get

$$\left(\sum_{\zeta\in\mathbb{Z}^n}\left(\sum_{\mu\in\mathbb{Z}^n\cap 2\,Q_\zeta^*}\|f\|_{L^p(Q_\mu)}\right)^{s_1}w_\zeta\right)^{\frac{1}{s_1}}\lesssim |w|_{A_s(\mathbb{Z}^n)}^{\frac{1}{s_1}}\,\|f\|_{(L^p,l_w^{s_1})}.$$

We have a similar bound for the factor corresponding to g in (6.32).

For the first term on the right hand side of (6.33) consider a nonnegative, radial, decreasing sequence $h=\{h_\mu\}_{\mu\in\mathbb{Z}^n}$ defined by

$$h_\mu=\frac{1}{|\mu|^{2n}}\omega\left(\frac{c_n}{|\mu|}\right),\qquad \mu\in\mathbb{Z}^n-\{0\}.$$

Note that $\|h\|_{l^1}<\infty$. Then, by Lemma 6.15 with $a=\{\|f\|_{L^p(Q_\mu)}\}_{\mu\in\mathbb{Z}^n}$,

$$\sum_{\mu\in\mathbb{Z}^n-Q_\zeta^*}\frac{1}{|\zeta-\mu|^{2n}}\omega\left(\frac{c_n}{|\zeta-\mu|}\right)\|f\|_{L^p(Q_\mu)}\,\|g\|_{L^q(2\,Q_\zeta^*)}\lesssim (h*a)_\zeta\,\|g\|_{L^q(2\,Q_\zeta^*)}$$
$$\lesssim \|h\|_{l^1}\,(\mathcal{M}_d a)_\zeta\,\|g\|_{L^q(2\,Q_\zeta^*)}.$$

Using Hölder's inequality, recalling that $w\in A_{s_1}(\mathbb{Z}^n)$ and Lemma 6.14, we have

$$\left\|(\mathcal{M}_d a)_\zeta\,\|g\|_{L^q(2\,Q_\zeta^*)}\right\|_{l_w^{s_3}}\lesssim \|(\mathcal{M}_d a)_\zeta\|_{l_w^{s_1}}\,\left\|\|g\|_{L^q(2\,Q_\zeta^*)}\right\|_{l_w^{s_2}}\lesssim \|f\|_{(L^p,l_w^{s_1})}\,\|g\|_{(L^q,l_w^{s_2})},$$

where the inequality $\left\| \|g\|_{L^q(2\,Q_\zeta^*)} \right\|_{l_w^{s_2}} \lesssim \|g\|_{(L^q, l_w^{s_2})}$ follows as in the treatment of (6.32). The second term in (6.33) is treated in the same way using $h_\mu = \frac{1}{|\mu|^{2n}}$, $\mu \in \mathbb{Z}^n - \{0\}$.

The first term in (6.34) is bounded by

$$C \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \frac{1}{|\zeta - \mu|^n}\, \omega^{\frac{1}{2}} \left(\frac{c_n}{|\zeta - \mu|} \right) \|f\|_{L^p(Q_\mu)} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} \frac{1}{|\zeta - \nu|^n}\, \omega^{\frac{1}{2}} \left(\frac{c_n}{|\zeta - \nu|} \right) \|g\|_{L^q(Q_\nu)},$$

We can proceed in a similar way as in (6.33) with $h_\mu = \frac{1}{|\mu|^n}\, \omega^{\frac{1}{2}} \left(\frac{c_n}{|\mu|} \right)$, $\mu \in \mathbb{Z}^n - \{0\}$. Observe that $h \in l^1$ since $\int_0^1 \frac{\omega^{1/2}(t)}{t}\, dt < \infty$, then Lemma 6.15 can be applied. Also the fact that $w \in A_s(\mathbb{Z}^n)$ allows us to use Lemma 6.14.

Finally, we will show that the second term on the right hand side of (6.34) satisfies the desired estimates. Consider the truncated operator

$$T_{\sqrt{n}}(u,v)(x) = \int_{|x-y|^2 + |x-z|^2 > n} K(x,y,z)\, u(y)\, v(z)\, dy\, dz$$

Note that $B(x, \sqrt{n}) \subset Q_\zeta^*$ for every $x \in Q_\zeta$. We will see that there is a non-increasing function $h : [0, \infty) \to [0, \infty)$, such that $h(|x|)$ is integrable in $\mathbb{R}^n$, and

$$\left| T_{\sqrt{n}}(u,v)(x) - \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} K(\zeta, \mu, \nu) \int_{Q_\mu} u(y)\, dy \int_{Q_\nu} v(z)\, dz \right| \lesssim (h * |u|)(x)\, (h * |v|)(x), \quad x \in Q_\zeta. \tag{6.35}$$

Assume (6.35) for the moment. Applying (6.35) to $u = \sum_{\mu \in \mathbb{Z}^n} \int_{Q_\mu} f_2(x) dx\, \chi_{Q_\mu}$, and $v = \sum_{\mu \in \mathbb{Z}^n} \int_{Q_\mu} g_2(x) dx\, \chi_{Q_\mu}$, we get,

$$\left| \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} K(\zeta, \mu, \nu) \int_{Q_\mu} f_2(y)\, dy \int_{Q_\nu} g_2(z)\, dz \right|^{s_3}$$
$$\lesssim \int_{Q_\zeta} \left| T_{\sqrt{n}}(u,v)(x) \right|^{s_3} dx + \|h\|_{L^1(\mathbb{R}^n)}^{2 s_3} \int_{Q_\zeta} (\mathcal{M}u(x)\, \mathcal{M}v(x))^{s_3}\, dx.$$

Recalling that $W = \sum_{\mu \in \mathbb{Z}^n} w_\mu \chi_{Q_\mu} \in A_s(\mathbb{R}^n)$, and using the boundedness properties of $T_{\sqrt{n}}$ and $\mathcal{M}$ in the weighted Lebesgue spaces, we get

$$\left\| \sum_{\nu \in \mathbb{Z}^n - Q_\zeta^*} \sum_{\mu \in \mathbb{Z}^n - Q_\zeta^*} K(\zeta, \mu, \nu) \int_{Q_\mu} f_2(y)\, dy \int_{Q_\nu} g_2(z)\, dz \right\|_{l_w^{s_3}}$$
$$\lesssim \left\| T_{\sqrt{n}}(u,v) \right\|_{L_W^{s_3}(\mathbb{R}^n)} + \|h\|_{L^1(\mathbb{R}^n)}^2 \, \|\mathcal{M}u\|_{L_W^{s_1}(\mathbb{R}^n)} \, \|\mathcal{M}v\|_{L_W^{s_2}(\mathbb{R}^n)}$$
$$\lesssim \|u\|_{L_W^{s_1}(\mathbb{R}^n)} \, \|v\|_{L_W^{s_2}(\mathbb{R}^n)} \lesssim \|f\|_{(L^p, l_w^{s_1})} \, \|g\|_{(L^q, l_w^{s_2})}.$$

We now prove (6.35). Define $S_x = \{(y,z) : |x-y|^2 + |x-z|^2 > n\}$. Fix $x \in Q_\zeta$, the left hand side of (6.35) is equal to

$$\left| \int_{y \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} F_{\zeta, x}(y,z)\, u(y) v(z)\, dy dz \right|,$$

where

$$F_{\zeta,x}(y,z) = \left(K(x,y,z)\,\chi_{S_x}(y,z) - \sum_{\nu\in\mathbb{Z}^n - Q^*_\zeta}\ \sum_{\mu\in\mathbb{Z}^n - Q^*_\zeta} K(\zeta,\mu,\nu)\,\chi_{Q_\mu}(y)\chi_{Q_\nu}(z) \right).$$

If $y,z \in \mathbb{R}^n \setminus Q^*_\zeta$, then $(y,z)\in S_x$, and using the regularity (3.2) of the kernel K, and that $y\in Q_\mu$ and $z\in Q_\nu$ for unique $Q_\mu \subset \mathbb{R}^n \setminus Q^*_\zeta$ and $Q_\nu \subset \mathbb{R}^n \setminus Q^*_\zeta$,

$$\begin{aligned} |F_{\zeta,x}(y,z)| &= |K(x,y,z) - K(\zeta,\mu,\nu)| \\ &\lesssim \frac{1}{(|x-y|+|x-z|)^{2n}}\,\omega\left(\frac{c_n}{|x-y|+|x-z|}\right) \\ &\lesssim \frac{1}{|x-y|^n}\,\omega^{\frac12}\left(\frac{c_n}{|x-y|}\right)\frac{1}{|x-z|^n}\,\omega^{\frac12}\left(\frac{c_n}{|x-z|}\right) \end{aligned}$$

If $y\in Q^*_\zeta$ and $z\in\mathbb{R}^n\setminus Q^*_\zeta$, then using the size assumption (3.1) on the kernel K, and that $t\lesssim\omega(t)$ (since ω is concave),

$$|F_{\zeta,x}(y,z)| = |K(x,y,z)| \lesssim \frac{1}{(|x-y|+|x-z|)^{2n}} \lesssim \frac{1}{|x-z|^n}\,\omega^{\frac12}\left(\frac{c_n}{|x-z|}\right),$$

Similarly for $z\in Q^*_\zeta$ and $y\in\mathbb{R}^n\setminus Q^*_\zeta$. Finally, if $y\in Q^*_\zeta$ and $z\in Q^*_\zeta$ we have, using again (3.1),

$$|F_{\zeta,x}(y,z)| \le |K(x,y,z)\,\chi_{S_x}(y,z)| \lesssim C_n.$$

Then we can define for $T\in\mathbb{R}^n$,

$$h(T) = \begin{cases} \tilde{C}_n & |T|\le\sqrt{n}, \\ \frac{1}{|T|^n}\,\omega^{\frac12}\left(\frac{c_n}{|T|}\right), & |T|>\sqrt{n}. \end{cases}$$

□

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