On the Optimal Convergence Speed of Wireless Scheduling for Fair Resource Allocation

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***Abstract*—In this paper, we study the design of joint flow rate control and scheduling policies in multi-hop wireless networks for achieving maximum network utility *with provably optimal convergence speed*. Fast convergence is especially important in wireless networks which are dominated by the dynamics of incoming and outgoing flows as well as the time sensitive applications. Yet, the design of fast converging policies in wireless networks is complicated by: (i) the interference-constrained communication capabilities, and (ii) the *finite* set of transmission rates to select from due to operational and physical-layer constraints.**

We tackle these challenges by explicitly incorporating such *discrete* constraints to understand their impact on the convergence speed at which the running average of the received service rates and the network utility converges to their limits. In particular, we establish a fundamental fact that the convergence speed of any feasible policy cannot be faster than $\Omega\left(\frac{1}{T}\right)$ under both the rate and utility metrics. Then, we develop an algorithm that achieves this optimal convergence speed in both metrics. We also show that the well-known dual algorithm can achieve the optimal convergence speed in terms of its utility value.

These results reveal the interesting fact that the convergence speed of rates and utilities in wireless networks is dominated by the discrete choices of scheduling and transmission rates, which also implies that the use of higher-order flow rate controllers with fast convergence guarantees cannot overcome the aforementioned fundamental limitation.

I. Introduction

Wireless networks are expected to serve users in an efficient and fair way, which requires careful flow control and interference management among simultaneous transmissions. These design goals can be achieved by solving the Network Utility Maximization (NUM) problem, where the utility of *long-term average* flow rates is maximized under stability constraints. Moreover, it is desired that these optimal solutions are reached rapidly due to the dynamic nature of flows and the time-sensitive nature of many wireless applications.

However, the design of controllers with *fast* convergence speed in most wireless networks is complicated by two natural constraints: (i) interference constraints leading to *discrete* link scheduling choices; and (ii) a *finite* set of choices for the transmission rate selection over the scheduled links. The latter constraint is caused by both digital communication (e.g., modulation, coding, etc.) and hardware design principles. For example, in IEEE 802.11b standard, there are four transmission rates: 1Mpbs, 2Mpbs, 5.5Mpbs and 11Mbps.

Previous works mainly focus on the design and analysis of policies with optimal *limiting* behavior. A large body of works (e.g. [2], [3], [4], [5], [6], [7]) has utilized *dual* and *primal-dual* methods to develop cross-layer policies with long-term optimality guarantees. Such solutions are amenable to distributed implementation due to their natural decomposition into loosely coupled components. However, being *first-order* methods, they suffer from the slow convergence speed shared by all such methods (e.g. [8], [9], [10]).

This speed deficiency of dual methods has recently spurred an exciting thread of research activity in the design of distributed Interior Point (e.g. [11]) and Newton's (e.g. [12], [13], [14]) methods for network utility maximization. However, these works do not incorporate two aforementioned features of wireless networks, namely the discreteness in the scheduling and transmission rate selections. We explicitly incorporate these intrinsic characteristics of wireless networks in our analysis and algorithm design. To the best of our knowledge, this is the first work that systematically analyzes and designs algorithms in terms of their converge speed in wireless networks with such discrete constraints. Next, we list our main contributions, along with references on where they appear in the text.

- We show that the convergence speed[1] at which the running average of the received service rates (see Section IV-A) and their utility (see Section IV-A) over T time slots cannot be faster than $\Omega\left(\frac{1}{T}\right)$. This fundamental limitation on the convergence speed is caused by the discrete nature of the allowable transmission rates under the operation of any stabilizing and asymptotically optimal flow control and scheduling policy.
- We develop a generic algorithm that can work with a range of flow rate controllers, and achieves the optimal convergence speed in both rate (see Section IV-B) and utility (see Section V-B) metrics.
- Somewhat surprisingly, we also show that even a first-order method such as the well-known dual algorithm can achieve the aforementioned optimal convergence speed in terms of its utility value (see Section V-C).
- These results collectively reveal that, under wireless networks subject to discrete scheduling and rate constraints, the convergence speed of cross-layer algorithms is dominated by the convergence speed of the *scheduling* component, and not

An earlier version of this paper has appeared in the Proceedings of the 32th IEEE International Conference on Computer Communications (INFOCOM) Conference, Turin, Italy, April 2013 [1].

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[1]The following standard notations are used to describe the rates of growth of two real-valued sequences $\{a_n\}$ and $\{b_n\}$:$a_n = O(b_n)$ if $\exists c > 0$ such that $|a_n| \leq c|b_n|$; $a_n = \Omega(b_n)$ if $b_n = O(a_n)$.

the flow rate controller. As such, the speed improvements in the flow rate convergence, unfortunately, cannot extend to the received service rates or utilities in wireless networks. On the bright side, however, with careful design we can achieve the optimal convergence speed under such constraints.

This work extends our earlier work [1] in several key aspects: (1) we provide a key example that illustrates the fundamental speed limitation caused by the discreteness of transmission rates; (2) we generalize the assumption of consecutive integer-valued transmission rates to any discrete integer-valued set in the analysis of convergence speed, and extend the convergence speed results under utility benefit metric to the case with heterogeneous transmission rates; (3) we include extensive comparison between the proposed algorithms and the traditional dual algorithm in terms of the convergence speed and the average delay through simulations.

II. System Model

We consider a multi-hop fading wireless network with a set $\mathcal{L} = \{1, 2, ..., L\}$ of links that operates in a time-slotted fashion, where all links transmit at the beginning of each time slot subject to interference constraints. Due to modulation, coding, as well as other practical constraints, each link has to transmit at one of a **finite** set of rates[2]. We use $\mathbf{S}[t] = (S_l[t])_{l=1}^L$ to denote the service rate vector offered to the links in slot t, which must be selected from a **feasible** set of transmission rates at the time. The feasible set, in turn, depends on the *network fading state* and the interference constraints amongst the links. Using $\mathcal{J}$ to denote the set of global channel states (with finite cardinality), we let $\mathcal{S}^j$ denote the set of *feasible service rate vectors* when the channel is in state $j \in \mathcal{J}$. We assume that the fading process is stationary and ergodic with π_j denoting the stationary probability of the channel state being in state j. Then, the *capacity region* can be defined as

$$\mathcal{R} \triangleq \sum_{j \in \mathcal{J}} \pi_j \cdot \mathrm{CH}\{\mathcal{S}^j\}, \tag{1}$$

where $\mathrm{CH}\{\mathcal{A}\}$ denotes the convex hull of the set $\mathcal{A}$. We note that $\mathcal{R}$ is a polyhedron due to the discreteness of the transmission rate choices, and hence can be written as $\mathcal{R} = \{\mathbf{y} \geq 0 : \mathbf{H}\mathbf{y} \leq \mathbf{b}\}$, where $\mathbf{y} \in \mathbb{R}^L$ and $\mathbf{H}$ is some positive matrix. Note that $\mathbf{H}$ has L columns and the number of rows in $\mathbf{H}$ is equal to the dimension of $\mathbf{b}$ associated with the number of interference constraints. As a special case, when $|\mathcal{J}| = 1$ we obtain the *non-fading* scenario.

To capture the heterogeneous and potentially inter-dependent preferences of users, we define a utility function $U : \mathbb{R}_+^L \to \mathbb{R}_+$ that measures the total network utility when link l receives an average service rate of r_l, where $\mathbf{r} = (r_l)_{l=1}^L$. We assume that $U(\mathbf{r})$ to be a strictly concave function that is non-decreasing in each coordinate. The objective of Network Utility Maximization (NUM), then, is to design a congestion control and scheduling algorithm such that the average service rate vector $\mathbf{r}$ solves the following optimization problem:

Definition 1: (**Network Utility Maximization (NUM)**)

$$\max_{\mathbf{r}=(r_l)_{l=1}^L} \quad U(\mathbf{r}) \tag{2}$$

$$\text{Subject to} \quad \mathbf{r} \in \mathcal{R}, \tag{3}$$

where $\mathcal{R}$ is defined in (1). $\diamond$

The strict concavity of $U(\cdot)$ together with the convexity of $\mathcal{R}$ guarantees the uniqueness of the solution of NUM, which is denoted as $\mathbf{r}^* = (r_l^*)_{l=1}^L$. Also, due to the non-decreasing nature of $U(\cdot)$, $\mathbf{r}^*$ must lie on the boundary of $\mathcal{R}$.

It is important to note that $\mathbf{r}^*$ is the optimal average **offered service rate** to the links. The purpose of the flow rate controller, however, is to determine the optimal **injection rate** of traffic into the network while maintaining network stability. To define network stability more rigorously, let $Q_l[t]$ denote the queue-length at link $l \in \mathcal{L}$ at the beginning of slot t, let $X_l[t]$ denote the amount of injected data into Queue-l in slot t under a given *flow rate controller*, and recall that $S_l[t]$ denotes the service rate offered to link l in slot t under a given *scheduler*. Then, the evolution of Q_l can be expressed as

$$Q_l[t+1] \;=\; (Q_l[t] + X_l[t] - S_l[t])^+, \qquad t \geq 1,$$

where $(y)^+ \triangleq \max\{0, y\}$, and Queue-$l$ is said to be *stable* if

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[Q_l[t]] < \infty, \tag{4}$$

and the *network is stable* if all queues are stable.

In this work, we are interested in the convergence speed of a broad class of joint flow rate control and scheduling policies $\mathcal{P}$ that are both **stabilizing** and **asymptotically rate optimal**. To define this class of policies abstractly, we introduce the parameter $\epsilon > 0$ as a generic term to characterize the performance of the joint policy under specific design choices. Accordingly, the average injection rate of a given policy under parameter ϵ lies in a set $\overline{\mathcal{X}}^{(\epsilon)}$. Similarly, we will use the superscript $\cdot^{(\epsilon)}$ over $(Q_l[t])_l, (X_l[t])_l, (S_l[t])_l$, etc. to express the queue-lengths, injections, offered service rates, etc. under the policy with parameter ϵ. The *stability condition* requires that $\overline{\mathcal{X}}^{(\epsilon)} \subset \mathcal{R}$ for all $\epsilon > 0$, and the *asymptotic rate optimality condition* requires that $\lim_{\epsilon\downarrow 0} \overline{\mathcal{X}}^{(\epsilon)} = \{\mathbf{r}^*\}$, i.e., the asymptotically optimal policy achieves the optimal service rate vector in the limit. Thus, the parameter ϵ captures the closeness of the injection rate to the optimal service rate $\mathbf{r}^*$ under the class of joint policies parametrized by ϵ. We note that this abstraction includes a wide range of joint control and scheduling policies in the literature. For example, in the well-known subgradient-based designs (e.g., [2], [3], [4], [5], [6]) the generic term ϵ maps to the particular design parameter that corresponds to the step-size on the subgradient iteration.

The stability condition of the joint flow rate control and

[2]For example, IEEE 802.11a standard uses OFDM transmission technique and can support rates in Mega bits per second selected from the finite set {6, 9, 12, 18, 24, 36, 48, 54}; In CDMA2000 1xEV-DO specification, the forward link transmission rate in kilo bits per second is chosen from the finite set $\{38.4, 76.8, 153.6, 307.2, 614.4, 921.6, 1228.8, 1843.2, 2457.6\}$.

scheduling policies in $\mathcal{P}$ implies that the running average of **departures** over time must also converge to the set[3] $\overline{\mathcal{X}}^{(\epsilon)}$. Since the running average of departures[4] up to time T is the real measure of **received** service until that time, we are interested in its convergence speed to $\overline{\mathcal{X}}^{(\epsilon)}$. To be more precise, for the policy with parameter ϵ we use $D_l^{(\epsilon)}[t] \triangleq \min(S_l^{(\epsilon)}[t], Q_l^{(\epsilon)}[t])$ to denote the departures in slot t for link $l \in \mathcal{L}$, and define its *running average* until $T \geq 1$ as

$$\overline{d}_l^{(\epsilon)}[T] \triangleq \frac{1}{T}\sum_{t=1}^{T} D_l^{(\epsilon)}[t], \qquad \forall l \in \mathcal{L}, \tag{5}$$

and use $\overline{\mathbf{d}}^{(\epsilon)}[T] \triangleq (\overline{d}_l^{(\epsilon)}[T])_l$. Next, we introduce the metrics of interest in our study of convergence speed, both in the running average departure rate and its corresponding utility value.

Definition 2: (**Metrics of Interest**) For any policy in $\mathcal{P}$ with parameter ϵ, we define the *rate deviation* $\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathcal{X}}^{(\epsilon)})$ between $\overline{\mathbf{d}}^{(\epsilon)}[T]$ and the set $\overline{\mathcal{X}}^{(\epsilon)}$ at time T as

$$\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathcal{X}}^{(\epsilon)}) \triangleq \inf_{\overline{\mathbf{x}}^{(\epsilon)} \in \overline{\mathcal{X}}^{(\epsilon)}} \left\| \overline{\mathbf{d}}^{(\epsilon)}[T] - \overline{\mathbf{x}}^{(\epsilon)} \right\|, \tag{6}$$

and the *utility benefit received until time* T as $U(\overline{\mathbf{d}}^{(\epsilon)}[T])$, where $\|\mathbf{y}\|$ is the l_2 norm of the vector $\mathbf{y}$. $\diamond$

In the rest of paper, we will: (i) provide an example showing the fundamental speed limitation exerted by the discrete choice of transmission rates (Section III); (ii) establish fundamental limits on the speed at which $\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathcal{X}}^{(\epsilon)})]$ converges to zero as T increases (Section IV-A); (iii) develop joint flow control and scheduling policy with provably optimal convergence speed in terms of rate deviation (Section IV-B); (iv) derive fundamental limits on the speed at which the utility benefit converges to the optimal utility value of NUM when sources of randomness are eliminated (Section V-A); (v) show that our proposed algorithm, as well as the well-known dual algorithm, achieves the optimal convergence speed in terms of utility benefit (Section V-B,V-C); and finally (vi) provide the detailed comparison between the proposed algorithms and the traditional dual algorithm in terms of the convergence speed and the average delay through simulations (Section VI).

III. A Motivating example

In this section, we study a simple example to see how the convergence speed of a sequence is limited by the discreteness of its elements. In particular, we consider the convergence speed of any zero-one sequence converging to 0.5. For any zero-one sequence $\{D[t] : D[t] \in \{0, 1\}\}_{t\geq 0}$, we have

$$\left| \frac{1}{T}\sum_{t=1}^{T} D[t] - 0.5 \right| = \frac{1}{2T}\left| 2\sum_{t=1}^{T} D[t] - T \right|. \tag{7}$$

[3] Note that the convergence of a sequence to a set is the convergence of its minimum distance to the set.

[4] Due to the discreteness of transmission rate choices, it is unlikely that the departure rate in slot T converges to the set $\overline{\mathcal{X}}^{(\epsilon)}$, as T increases.

Noting that both $2\sum_{t=1}^{T} D[t]$ and T are integers, we have

$$\begin{aligned}\phi(\overline{d}[T], 0.5) &= \left| \frac{1}{T}\sum_{t=1}^{T} D[t] - 0.5 \right| \\ &= \begin{cases} \geq \frac{1}{2T}, & \text{if } T \neq 2\sum_{t=1}^{T} D[t]; \\ = 0, & \text{if } T = 2\sum_{t=1}^{T} D[t]. \end{cases}\end{aligned} \tag{8}$$

Hence, the subsequence $\{\phi(\overline{d}[T_k], 0.5) : T_k \text{ is odd}\}$ is always lower-bounded by $\frac{1}{2T_k}$ and thus the convergence speed of any zero-one sequence cannot be faster than $\Omega\left(\frac{1}{T}\right)$. To validate this result, we consider an independently and identically distributed (i.i.d.) Bernoulli random sequence with mean 0.5. Figure 1 shows one realization of this random sequence. From this figure, we can see that $\phi(\overline{d}[T], 0.5)$ hits 0 for some T, and is always non-zero when T is odd. The subsequence $\{\phi(\overline{d}[T_k], 0.5) : T_k \text{ is odd}\}$ is always lower-bounded by $\frac{1}{2T_k}$.

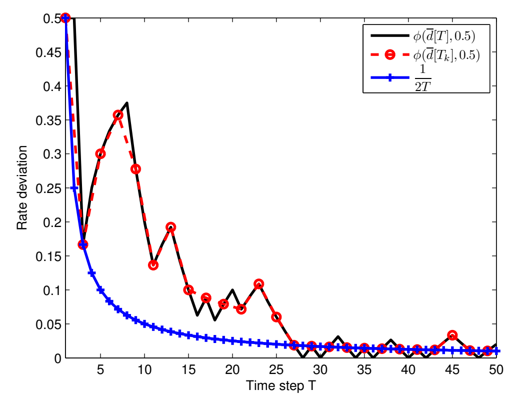


Fig. 1: The convergence speed of an i.i.d Bernoulli sequence.

This simple example suggests that the discreteness in the choice of elements in the sequence exerts a fundamental limitation on the speed with which its running average over time can approach its limit. In what follows, we will show that this observation indeed holds even in the wider context of a multi-hop fading wireless network with a finite selection of transmission rates.

IV. Convergence Speed in Rate Deviation

In this section, we study the optimal convergence speed in terms of rate deviation over wireless fading channels. To that end, we first give the fundamental lower bound on the expected rate deviation for any algorithm. Then, we provide an algorithm that can achieve this lower bound and establish the optimality of the proposed algorithm.

A. A lower bound on the expectation of rate deviation

In this subsection, we show that for any policy in $\mathcal{P}$, the convergence speed of expected rate deviation is $\Omega\left(\frac{1}{T}\right)$. To that end, we need the following integer assumption on the transmission rate, which measures the number of packets that can be transmitted in one time unit.

Assumption 1: The service rate S_l for each link $l \in \mathcal{L}$ is selected from a finite and nonnegative-integer-valued set $\{B_{l,1}, B_{l,2}, ..., B_{l,K_l}\}$, where $0 \leq B_{l,1} < B_{l,2} < ... < B_{l,K_l}$ and K_l is some positive integer. ⋄

Next, we give the following key lemma, which will also be useful in the later section.

Lemma 1: Let $\mathcal{I} \triangleq \{a_1, a_2, ..., a_K\}$, where $0 \leq a_1 < a_2 < ... < a_K$ and K is some positive integer. If $r \in (a_i, a_{i+1})$ for some $i = 1, ..., K-1$, then for any sequence $\{I[t] : I[t] \in \mathcal{I}\}_{t\geq 1}$, there exists a constant $c_r \in (0, \min\{\frac{r-a_i}{2}, \frac{a_{i+1}-r}{2}\})$ such that if $|\frac{1}{T}\sum_{t=1}^{T} I[t] - r| \leq \frac{c_r}{T}$, then

$$\left|\frac{1}{T+1}\sum_{t=1}^{T+1} I[t] - r\right| \geq \frac{c_r}{T+1}. \tag{9}$$

Remark: Note that K can be as large as ∞.

Proof: See Appendix A for the proof. ■

Proposition 1: Under Assumption 1, for any policy in $\mathcal{P}$ with parameter ϵ, if the closure of set $\overline{\mathcal{X}}^{(\epsilon)}$ does not contain a vector with all integer-valued coordinates, then convergence speed of the expected rate deviation to zero is $\Omega\left(\frac{1}{T}\right)$, i.e., there exists a strictly positive constant c and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that

$$\phi(\overline{\mathbf{d}}^{(\epsilon)}[T_k], \overline{\mathcal{X}}^{(\epsilon)}) \geq \frac{c}{T_k}, \qquad \forall k \geq 1, \tag{10}$$

holds for any sample path of departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$, which also implies that

$$\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T_k], \overline{\mathcal{X}}^{(\epsilon)})] \geq \frac{c}{T_k}, \qquad \forall k \geq 1. \tag{11}$$

Remark: If the optimal rate vector $\mathbf{r}^*$ has at least one non-integer-valued coordinate, then the condition for Proposition 1 holds when ϵ is sufficiently small. Moreover, since the region $\mathcal{R}$ is compact, there are finitely many rate vectors with all coordinates being integer in $\mathcal{R}$. Thus, Proposition 1 holds in almost all cases.

Proof: See Appendix B for the proof. ■

Proposition 1 indicates that the discrete structure of the transmission rates intrinsically limits the convergence speed for any algorithm in class $\mathcal{P}$. Thus, the search for higher-order numerical optimization methods cannot overcome this fundamental limitation in wireless networks. Despite pessimism of this observation, we are still interested in designing an algorithm that can achieve this fundamental bound and establish the optimality of this algorithm in terms of its convergence speed. To that end, we define the rate deviation optimality for an algorithm in class $\mathcal{P}$.

Definition 3: (**Rate Deviation Optimality**) An algorithm in class $\mathcal{P}$ with parameter ϵ is called *rate deviation optimal*, if its departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$ satisfies

$$\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathcal{X}}^{(\epsilon)})] \leq \frac{F_1^{(\epsilon)}}{T}, \qquad \forall T \geq 1, \tag{12}$$

where $F_1^{(\epsilon)}$ is a positive constant and $\overline{\mathbf{d}}^{(\epsilon)}[T]$ is defined in (5).

Next, we propose an algorithm with rate deviation optimality.

B. A Rate Deviation Optimal Policy

In this subsection, we propose a rate deviation optimal algorithm with parameter $\epsilon > 0$ that converges to the injection rate $\overline{\mathbf{x}}^{(\epsilon)}$ solving the following optimization problem.

Definition 4: (ϵ-NUM)

$$\max_{\mathbf{r}=(r_l)_{l=1}^{L}} \quad U(\mathbf{r}) \tag{13}$$

$$\text{Subject to} \quad \mathbf{r} \in \mathcal{R}^{(\epsilon)}, \tag{14}$$

where $\mathcal{R}^{(\epsilon)} \triangleq \{\mathbf{y} \geq 0 : \mathbf{H}\mathbf{y} \leq \mathbf{b} - \epsilon\}$.

Since U is strictly concave and $\mathcal{R}^{(\epsilon)}$ is convex, $\overline{\mathbf{x}}^{(\epsilon)}$ is unique. In addition, as $\epsilon \to 0$, $\overline{\mathbf{x}}^{(\epsilon)}$ converges to the optimal rate vector $\mathbf{r}^*$. Without loss of generality, we assume $\|\overline{\mathbf{x}}^{(\epsilon)} - \mathbf{r}^*\| \leq \rho^{(\epsilon)}$, where $\lim_{\epsilon\downarrow 0} \rho^{(\epsilon)} = 0$. The relationship between $\mathcal{R}^{(\epsilon)}$ and $\mathcal{R}$ in two-dimensional case is shown in Figure 2.

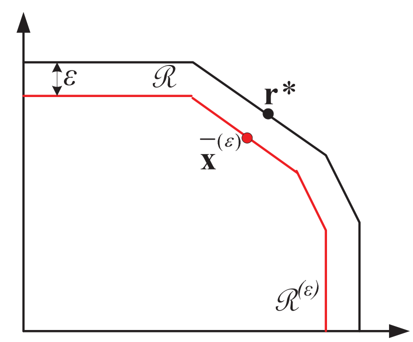


Fig. 2: The relationship between $\mathcal{R}^{(\epsilon)}$ and $\mathcal{R}$.

Each link maintains a data queue and a virtual queue. Let $Y_l^{(\epsilon)}[t]$ denote the virtual queue length at link l in each slot t.

Algorithm 1: (**Rate Deviation Optimal (RDO) Algorithm with parameter** ϵ): At each time slot t,

Flow control: $\{\mathbf{x}^{(\epsilon)}[t] = (x_l^{(\epsilon)}[t])_l\}_{t\geq 1}$ is a sequence generated by a numerical optimization algorithm solving ϵ-NUM. Note that $\mathbf{x}^{(\epsilon)}[t] \in \mathcal{R}^{(\epsilon)}, \forall t \geq 1$.

Arrival generation: For each link l,

(1) if $t = 1$, then $X_l^{(\epsilon)}[t] = B_{l,K_l}$;

(2) else if $\sum_{i=1}^{t-1} X_l^{(\epsilon)}[i] < \sum_{i=1}^{t-1} x_l^{(\epsilon)}[i]$, then $X_l^{(\epsilon)}[t] = B_{l,K_l}$; $X_l^{(\epsilon)}[t] = 0$, otherwise.

Then, inject $X_l^{(\epsilon)}[t]$ packets into each data queue l and increase virtual queue length $Y_l^{(\epsilon)}[t]$ by $x_l^{(\epsilon)}[t]$.

Scheduling: Perform Maximum Weight Scheduling (MWS) algorithm among virtual queues, that is,

$$\mathbf{S}^{(\epsilon)}[t] \in \operatorname*{argmax}_{\boldsymbol{\eta}=(\eta_l)_{l=1}^{L}\in\mathcal{S}^{J[t]}} \sum_{l=1}^{L} Y_l^{(\epsilon)}[t]\eta_l, \tag{15}$$

where $J[t] \in \mathcal{J}$ denotes the channel state at time t. Use $\mathbf{S}^{(\epsilon)}[t]$ to serve data queues.

Queue evolution: Let $Q_l^{(\epsilon)}[1] = Y_l^{(\epsilon)}[1] = 0, \forall l$, and for $t \geq 2$, update the data queue length and virtual queue length as follows:

$$Q_l^{(\epsilon)}[t+1] = \left(Q_l^{(\epsilon)}[t] + X_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t]\right)^+, \forall l, \quad (16)$$

$$Y_l^{(\epsilon)}[t+1] = \left(Y_l^{(\epsilon)}[t] + x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t]\right)^+, \forall l. \quad (17)$$

Remarks: (1) Recent advances in the design of distributed Newton's method (e.g, [13], [14]) show the promise in generating sequence $\{\mathbf{x}^{(\epsilon)}[t]\}_{t\geq 1}$ in quick and distributed way. In addition, we can also use Gradient Projection method to solve ϵ-NUM.

(2) The purpose of maintaining the virtual queue is to help show the stability of data queues. In fact, directly performing MWS among data queues does not hurt the convergence speed, as we will see in the simulations. However, the Lyapunov drift argument to show the stability of the proposed algorithm does not work in such a case, since the deterministic arrivals to each link l are alternating between 0 and B_{l,K_l}, which leads to the potential positiveness of the one-step Lyapunov drift given the current queue length state.

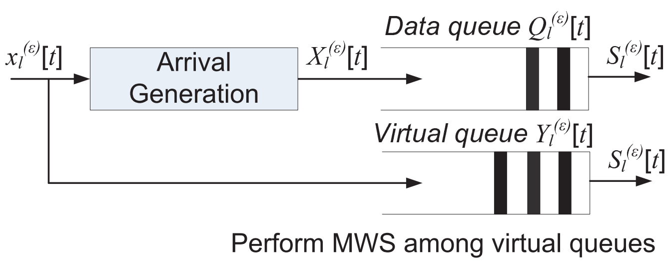


Fig. 3: The operation of the RDO Algorithm at link l.

Figure 3 shows the operation of the RDO Algorithm at link l. Next, we show that the RDO Algorithm can achieve rate deviation optimality if the generated sequence $\{\mathbf{x}^{(\epsilon)}[t]\}_{t\geq 1}$ converges fast enough. To that end, we need the following lemma exhibiting that the generated arrivals closely track the generated sequence $\{\mathbf{x}^{(\epsilon)}[t]\}_{t\geq 1}$.

Lemma 2: For each link l, we have

$$\left|\sum_{t=1}^{T}(X_l^{(\epsilon)}[t] - x_l^{(\epsilon)}[t])\right| \leq B_{l,K_l}, \qquad \forall T \geq 1. \quad (18)$$

Proof: See Appendix C for the proof. ■

Based on Lemma 2, we can show that for each link, the data queue length is upper-bounded by the sum of some constant and the virtual queue length for all sample paths, which is useful in establishing the rate deviation optimality of the RDO Algorithm.

Lemma 3: For each link l, the data queue length is upper-bounded by the sum of the virtual queue length and $2B_{l,K_l}$ for all sample paths, i.e.,

$$Q_l^{(\epsilon)}[T] \leq Y_l^{(\epsilon)}[T] + 2B_{l,K_l}, \qquad \forall T \geq 1, \quad (19)$$

holds for all sample paths.

Proof: See Appendix D for the proof. ■

We are now ready to establish the rate deviation optimality of the RDO Algorithm.

Proposition 2: For the RDO Algorithm with parameter $\epsilon > 0$, as long as the flow controller satisfies $\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{x}^{(\epsilon)}[t] - \overline{\mathbf{x}}^{(\epsilon)}\| \leq \frac{R_1}{T}$, for all $T \geq 1$, the generated link departure sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$ satisfies

$$\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathbf{x}}^{(\epsilon)})] \leq \frac{R_1^{(\epsilon)}}{T}, \qquad \forall T \geq 1, \quad (20)$$

where R_1 and $R_1^{(\epsilon)}$ are some positive constants.

Remark: $\{\mathbf{x}^{(\epsilon)}[t]\}_{t\geq 1}$ generated by the distributed Newton's method (e.g., [12], [13], [14]) satisfies the condition for Proposition 2.

Proof: See Appendix E for the proof. ■

Proposition 3: For the RDO Algorithm with parameter $\epsilon > 0$, if at least one coordinate of $\overline{\mathbf{x}}^{(\epsilon)}$ is a non-integer and the same condition in Proposition 2 holds, then the RDO Algorithm is rate deviation optimal (cf. Definition 3).

Proof: The result directly follows from Propositions 1, 2 and the definition of rate deviation optimality. ■

So far, we have observed that the discrete choice of transmission rates significantly limits the convergence speed to $\Omega(\frac{1}{T})$ and provided an algorithm that can achieve the optimal convergence speed in terms of rate deviation. In [15], the authors showed that for dual algorithm, the convergence speed of the running average of primal variables over T slots can be as fast as $\Omega\left(\frac{1}{T}\right)$ in terms of utility benefit. To the best of our knowledge, there does not exist a convergence speed analysis of dual methods in terms of rate deviation metric due to the non-smoothness of the dual function (see [16]). This motivates us to investigate the optimality of dual algorithm in terms of its convergence speed of the utility benefit metric under additional assumptions of non-randomness. These assumptions are necessary to establish the fundamental upper bound on the utility benefit under random environment, since the aggregation over links and the randomness (such as random arrivals, randomized scheduling or channel fading) distort the discrete structure. Thus, we focus on the deterministic system in next section, where there is no randomness in the system. It is still quite difficult and non-trivial to establish the convergence speed optimality in terms of utility benefit in such a system.

V. Convergence speed in Utility benefit

In this section, we first establish the fundamental upper bound on the utility benefit for any algorithm in the deterministic system. Then, we show that both deterministic version of the RDO Algorithm and the well-known dual algorithm can achieve this upper bound and establish their optimality under utility benefit metric.

A. An upper bound on the utility benefit

In this subsection, we establish an upper bound on the utility benefit $U(\overline{\mathbf{d}}^{(\epsilon)}[T])$ for any algorithm in class $\mathcal{P}$ with parameter ϵ. We do not require the integer Assumption 1 for the deterministic system. Without loss of generality, we assume that each link has a finite set of transmission rates $\mathcal{F}_l \triangleq \{b_{l,1}, b_{l,2}, ..., b_{l,K_l}\}$, where $0 \le b_{l,1} < b_{l,2} < ... < b_{l,K_l}$. To establish the fundamental upper bound on the utility benefit, we need the following assumption on the scheduling:

Assumption 2: Each link l with queue length less than b_{l,K_l} is not to be scheduled. $\diamond$

Remark: This scheduling assumption helps establish the fundamental bound on the utility benefit. Removing this assumption does not speedup the convergence, which is validated through simulations

We also need the following assumptions on the utility function:

Assumption 3: (1) The utility function $U(\mathbf{r})$ is additive, that is, $U(\mathbf{r}) = \sum_{l=1}^{L} U_l(r_l)$, where $U_l(y)$ is a concave and non-decreasing function of y;
(2) $h_{\min} \le U_l'(y) \le h_{\max}$, $\forall y$, where $0 < h_{\min} < h_{\max} < \infty$;
(3) $-\beta_{\max} \le U_l''(y) \le -\beta_{\min}$, $\forall y$, where $0 < \beta_{\min} < \beta_{\max}$. $\diamond$

Examples of such utility functions include $U_l(y) = \log(y+\gamma)$ and $U_l(y) = \frac{(y+\gamma)^{1-m}}{1-m}$, where m and γ are positive constants. Now, we are ready to establish the fundamental upper bound on the utility benefit for any algorithm in class $\mathcal{P}$.

Proposition 4: (1) Under Assumption 2 and 3, for any $\delta \in (0, \max_{\mathbf{r}\in\mathcal{R}} \|\mathbf{r} - \mathbf{r}^*\|)$ and any policy in $\mathcal{P}$ with parameter ϵ, there exists a constant $c^{(\delta)} > 0$ and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that

$$U(\overline{\mathbf{d}}^{(\epsilon)}[T_k]) \le U(\mathbf{r}^*) - \frac{1}{2}\beta_{\min}\sqrt{L}\delta^2 - \frac{c^{(\delta)}}{T_k}, \forall T_k \le \frac{c^{(\delta)}}{H\delta}, \quad (21)$$

where $H \triangleq \sqrt{L}(2\beta_{\max}\max_{l\in\mathcal{L}} b_{l,K_l} + h_{\max})$.

(2) If we further have $\sum_{l=1}^{L} U_l'(r_l^*) r_l^* \notin \mathcal{H}$, then there exists a sequence $\{c^{(\delta)}\}_{\delta>0}$ such that $c^{(0)} \triangleq \lim_{\delta\downarrow 0} c^{(\delta)} > 0$ and $\lim_{\delta\downarrow 0} \frac{c^{(\delta)}}{H\delta} = \infty$, and (21) becomes

$$U(\overline{\mathbf{d}}^{(\epsilon)}[T_k]) \le U(\mathbf{r}^*) - \frac{c^{(0)}}{T_k}, \qquad \forall k \ge 1, \quad (22)$$

where $\mathcal{H} \triangleq \left\{\sum_{l=1}^{L} U_l'(r_l^*) I_l : \mathbf{I} = (I_l)_{l=1}^{L} \in \mathcal{R} \text{ and } I_l \in \mathcal{F}_l, \forall l\right\}$.

Remark: The finiteness of the set $\mathcal{F}_l$ implies that the set $\mathcal{H}$ also has a finite number of elements. Thus, it is unlikely that $\sum_{l=1}^{L} U_l'(r_l^*) r_l^*$ is in the set $\mathcal{H}$ in practice.

Proof: See Appendix F for the proof. ∎

The first part of Proposition 4 establishes a fundamental bound on how close the utility benefit can be to the optimal utility level within a finite range of time. The second part, then, shows that, under an additional mild assumption on $\mathbf{r}^*$, the range over which the bound holds can be made to extend to infinity by letting the error go to zero. To illustrate the nature of this result, Figure 4 shows the utility benefit of an algorithm in class $\mathcal{P}$ over time. It shows that the utility benefit repeatedly falls below the fundamental bound until time $\frac{c^{(\delta)}}{H\delta}$, which, from the second part of the proposition, goes to infinity as δ vanishes.

Note that it is impossible for any policy in class $\mathcal{P}$ with parameter $\epsilon > 0$ that (21) holds for all $T \ge 1$, since the "good" policy (e.g., where ϵ is sufficiently small) can achieve

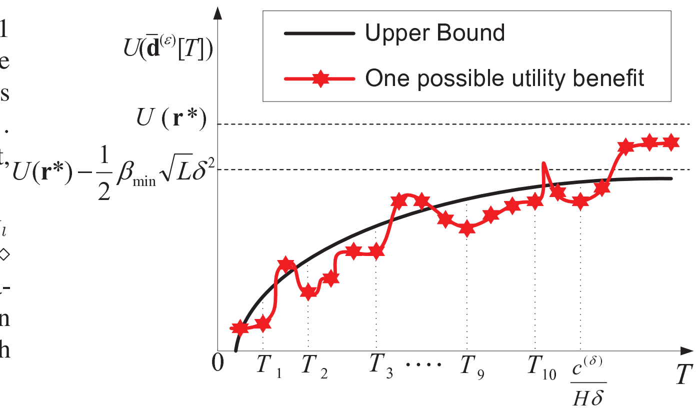


Fig. 4: The utility benefit of an algorithm in class $\mathcal{P}$.

the optimal value at arbitrary accuracy and thus $U(\overline{\mathbf{d}}^{(\epsilon)}[T])$ will exceed $U(\mathbf{r}^*) - \frac{1}{2}\beta_{\min}\sqrt{L}\delta^2$ eventually. In addition, inequality (22) implies that the utility benefit of any algorithm cannot be beyond the optimal value. Thus, these fundamental upper bounds on the utility benefit motivate the definition of utility benefit optimality of an algorithm given next.

Definition 5: (**Utility benefit optimality**) For any $\delta > 0$, an algorithm in class $\mathcal{P}$ with parameter $\epsilon > 0$ is called *utility benefit optimal*, if its generated departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\ge1}$ satisfies

$$U(\overline{\mathbf{d}}^{(\epsilon)}[T]) \ge U(\mathbf{r}^*) - \frac{1}{2}\beta_{\min}\sqrt{L}\delta^2 - \frac{F_3^{(\epsilon)}}{T}, \ \forall T \le F_4^{(\delta)},$$

where $F_3^{(\epsilon)} > 0$ and $F_4^{(\delta)} > 0$ with $\lim_{\delta\to0} F_4^{(\delta)} = \infty$. $\diamond$

Next, we first investigate the utility benefit optimality of the deterministic version of the RDO Algorithm.

B. The utility benefit optimality of the RDO Algorithm

In this subsection, we show that the deterministic version of the RDO Algorithm is utility benefit optimal under Assumptions 2 and 3.

The deterministic version of the RDO Algorithm works as follows:

Algorithm 2: (**Deterministic RDO (DRDO) Algorithm with parameter** $\epsilon > 0$): At each time slot t,

Flow control: $\{\mathbf{x}^{(\epsilon)}[t] = (x_l^{(\epsilon)}[t])_l\}_{t\ge1}$ is a sequence generated by a numerical optimization algorithm solving ϵ-NUM.

Arrival: Inject $x_l^{(\epsilon)}[t]$ amount of data into each queue l;

Scheduling: Perform the MWS algorithm among links whose queue length are no less than b_{l,K_l}, that is,

$$\mathbf{S}^{(\epsilon)}[t] \in \underset{\boldsymbol{\eta}=(\eta_l)_{l=1}^{L}\in\mathcal{R}}{\operatorname{argmax}} \sum_{l=1}^{L} Q_l^{(\epsilon)}[t] \mathbb{1}_{\{Q_l^{(\epsilon)}[t]\ge b_{l,K_l}\}} \eta_l; \quad (23)$$

Queue evolution: Update the queue length as follows:

$$Q_l^{(\epsilon)}[t+1] = (Q_l^{(\epsilon)}[t] + x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t])^+, \qquad \forall l. \quad (24)$$

Next, we give a lower bound of the DRDO Algorithm under utility benefit metric.

Proposition 5: Under Assumption 3 on the utility function U, for the DRDO Algorithm with parameter $\epsilon > 0$, if $\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{x}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)}\| \le \frac{R_2}{T}$, for all $T \ge 1$, then its departure sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\ge 1}$ satisfies

$$U(\overline{\mathbf{d}}^{(\epsilon)}[T]) \ge U(\mathbf{r}^*) - h_{\max}\sqrt{L}\rho^{(\epsilon)} - \frac{\sqrt{L}h_{\max}R_2^{(\epsilon)}}{T}, \forall T \ge 1,$$

where R_2 and $R_2^{(\epsilon)}$ are some positive constants.

Proof: See Appendix G for the proof. ■

Proposition 6: Under Assumptions 2 and 3, the DRDO Algorithm is utility benefit optimal (c.f. Definition 5), i.e., for any $\delta > 0$, by choosing $\epsilon > 0$ such that $\rho^{(\epsilon)} \le \frac{\beta_{\min}\delta^2}{2h_{\max}}$, the DRDO Algorithm can achieve the upper bound in (21).

Proof: The proof immediately follows from Propositions 4, 5 and the definition of utility benefit optimality. ■

Next, we study the utility benefit optimality of the well-known dual algorithm.

C. Utility benefit optimality of dual algorithm

In this subsection, we establish the utility benefit optimality of the well-known dual algorithm (e.g., [2], [3], [5], [6]). The dual algorithm can be obtained by Lagrangian relaxation and naturally decomposes the network function into the two main components: the congestion control and the scheduling. Next, we give the definition of the dual algorithm for completeness.

Definition 6: (**Dual Algorithm with parameter** $\epsilon > 0$):

Flow control: Given $\mathbf{Q}^{(\epsilon)}[t] = (Q_l^{(\epsilon)}[t])_{l=1}^{L}$, solve the following optimization problem:

$$\mathbf{x}^{(\epsilon)}[t] \in \underset{0\le \boldsymbol{w}\le M}{\operatorname{argmax}} \frac{1}{\epsilon}U(\boldsymbol{w}) - \sum_{l=1}^{L} Q_l^{(\epsilon)}[t]w_l; \tag{25}$$

Scheduling: Perform MWS algorithm among links whose queue length are no less than b_{l,K_l}, that is,

$$\mathbf{S}^{(\epsilon)}[t] \in \underset{\boldsymbol{\eta}=(\eta_l)_{l=1}^{L}\in\mathcal{R}}{\operatorname{argmax}} \sum_{l=1}^{L} Q_l^{(\epsilon)}[t]\mathbb{1}_{\{Q_l^{(\epsilon)}[t]\ge b_{l,K_l}\}}\eta_l; \tag{26}$$

Queue evolution:

$$Q_l^{(\epsilon)}[t+1] = (Q_l^{(\epsilon)}[t] + x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t])^+, \forall l, \tag{27}$$

where M is the maximum allowable input rate. ◇

The Dual Algorithm also uses the scheduling assumption as the DRDO Algorithm that does not schedule link l with queue length less than b_{l,K_l}, which helps establish its utility benefit optimality. However, removing this scheduling constraint does not improve the convergence speed, which is validated through simulations.

We are now ready to give the convergence speed of the Dual Algorithm in terms of utility benefit.

Proposition 7: For the Dual Algorithm with parameter $\epsilon > 0$, the generated departure sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\ge 1}$ satisfies

$$U(\overline{\mathbf{d}}^{(\epsilon)}[T]) \ge U(\mathbf{r}^*) - \frac{\epsilon}{2T}\|\mathbf{Q}^{(\epsilon)}[1]\|^2 - \frac{\epsilon L}{2}(M^2 + 3\max_{l\in\mathcal{L}} b_{l,K_l}^2) - \frac{h_{\max}\sqrt{L}}{T}\left(\|\mathbf{Q}^{(\epsilon)}[1]\| + G^{(\epsilon)}\sqrt{L}\right), \forall T \ge 1, \tag{28}$$

where $G^{(\epsilon)} \triangleq \sqrt{W} + \frac{h_{\max}}{\epsilon}$ and $W \triangleq \left(\frac{\beta_{\max}}{\epsilon} + 2\right)LM^2 + \left(\frac{3\beta_{\max}}{\epsilon} + 2\right)L\max_{l\in\mathcal{L}} b_{l,K_l}^2$.

Proof: See Appendix H for the proof. ■

Remark: The difference between our analysis and that in [15] lies in that we add the scheduling component in wireless networks and consider the utility of the running average of departure rate vector sequence rather than that of the running average of primal vector sequence, which makes it more challenge to deal with.

From (28), we can see that the utility benefit $U(\overline{\mathbf{d}}^{(\epsilon)}[T])$ converges to the optimal value $U(\mathbf{r}^*)$ within error level $\frac{\epsilon L}{2}(M^2 + 3\max_{l\in\mathcal{L}} b_{l,K_l}^2)$ with the speed of $\Omega\left(\frac{1}{T}\right)$. When the parameter ϵ decreases, the error level will decrease in the price of the slower convergence speed. Next, we establish the utility benefit optimality of the Dual Algorithm.

Proposition 8: The Dual Algorithm is utility benefit optimal (c.f. Definition 5), i.e., for any $\delta > 0$, by choosing $\epsilon \le \frac{\beta_{\min}\delta^2}{\sqrt{L}(M^2+3\max_{l\in\mathcal{L}} b_{l,K_l}^2)}$, the Dual Algorithm can achieve the upper bound in (21).

Proof: The proof directly follows from propositions 4, 7 and the definition of utility benefit optimality. ■

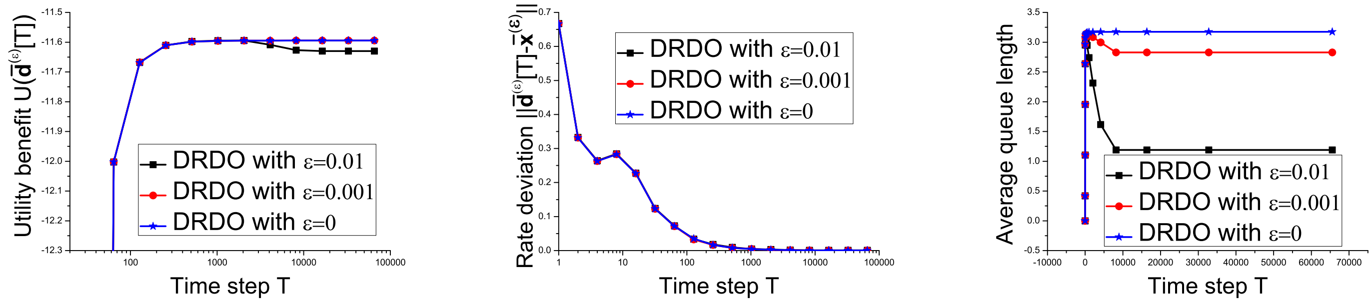
VI. Simulation Results

In this section, we consider a single-hop network topology with $L = 5$ links in both non-fading and fading channels. In each time slot, at most one link can be active. We take the additive utility function with $U_l(y) = \log(y + \gamma), \forall l$, where $\gamma = 10^{-8}$, for both non-fading and fading scenarios. Recall that this function satisfies Assumption 3 on the utility function to establish the utility benefit optimality of both DRDO Algorithm and the Dual Algorithm. For the non-fading scenario, each link has a fixed rate and the link rate vector is $\mathbf{p} = [0.8, 0.4, 0.6, 0.5, 0.3]$. For the fading scenario, each link suffers from ON-OFF channel fading independently and the link ON probability vector is also $\mathbf{p}$. For DRDO and RDO algorithms with parameter ϵ, we use Newton's method (see [9]) to generate sequence $\{\mathbf{x}^{(\epsilon)}[t]\}_{t\ge 1}$ that satisfies the condition for Proposition 2.

A. Non-fading scenario

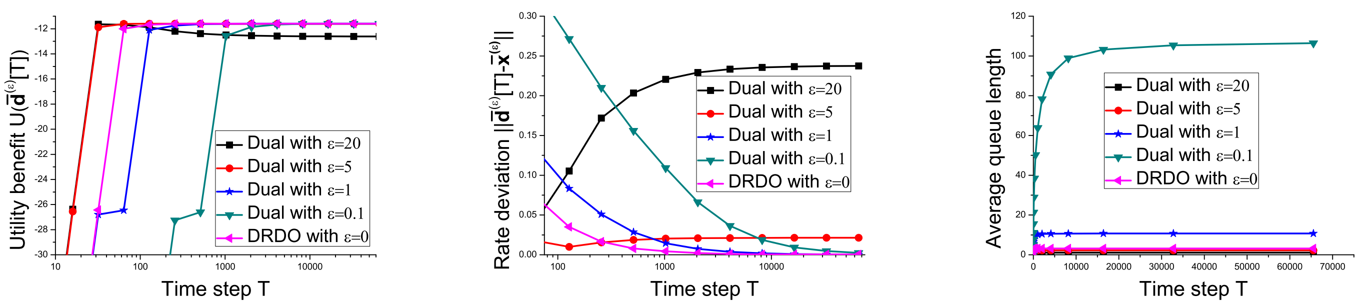
In this subsection, we mainly investigate the impact of parameter ϵ on the performance of DRDO algorithm and Dual Algorithm. Then, we study whether removing scheduling constraint (each link cannot be scheduled if its queue length is less than 1) can speedup the convergence.

Figure 5a, 5b and 5c show the impact of parameter ϵ on the convergence speed and the average queue length per link for DRDO Algorithm. From Figure 5a and 5c, we can observe that as ϵ decreases, the utility benefit converges to a value closer to the optimal value in price of increasing the average queue length per link. Note that the system is still stable even when $\epsilon = 0$, which means the optimal point in the boundary of capacity region can be achieved eventually. Thus, we can



(a) Rate convergence under utility difference metric (b) Rate convergence under rate difference metric (c) Average queue length

Fig. 5: DRDO Algorithm performance with varying ϵ



(a) Rate convergence under utility difference metric (b) Rate convergence under rate difference metric (c) Average queue length

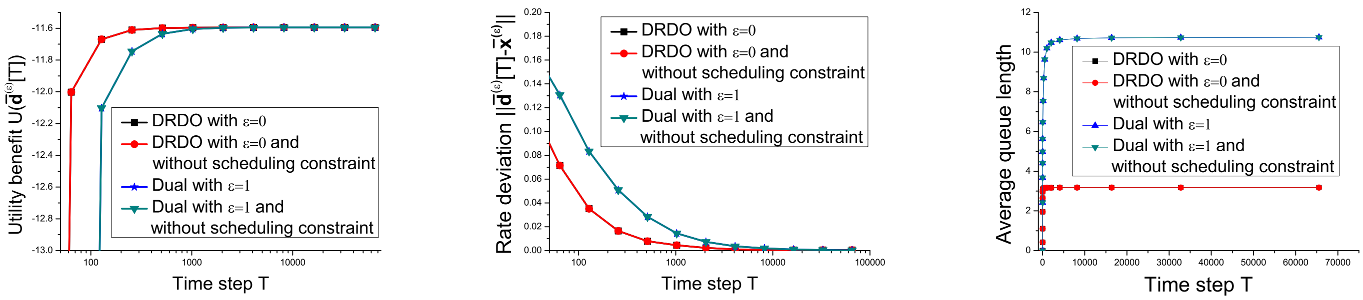
Fig. 6: Dual Algorithm performance with varying ϵ

choose $\epsilon = 0$ for the deterministic system. In addition, we can observe from Figure 5a and 5b that ϵ does not have significant influence on the convergence speed under both utility benefit and rate deviation metrics.

Figure 6a, 6b and 6c show the impact of parameter ϵ on the convergence speed and the average queue length per link for the Dual Algorithm. From Figure 6a, we can observe that when ϵ is too large (e.g., $\epsilon = 20$), the utility benefit cannot converge to the optimal value. From Figure 6b, we can see that the convergence under rate deviation metric requires much smaller ϵ than that under utility benefit metric. Here, it is worth mentioning that the large ϵ (e.g., $\epsilon = 1$) can still lead to the convergence to the optimal value under both metrics of interest. This is a little contradictory with the traditional dual algorithm in wireless networks, where the convergence property requires much smaller ϵ (e.g., $\epsilon = 0.01$). The reason is that we are interested in the time average metric rather than the instantaneous value. If ϵ is relatively large, the instantaneous value oscillates around the optimal value. However, the time average value may still be arbitrary close to the optimal value. In addition, among the set of parameters ϵ guaranteeing the convergence to the optimal value, the smaller ϵ leads to the slower convergence speed under both interest metrics. From Figure 6c, we can observe that the average queue length per link increases as ϵ decreases. This observation matches the theoretical upper bound on the average queue length. Thus, for the Dual Algorithm, we need to choose ϵ as large as possible among the set of parameters ϵ guaranteeing convergence to the optimal value, which not only leads to faster convergence but also enjoys smaller average delay.

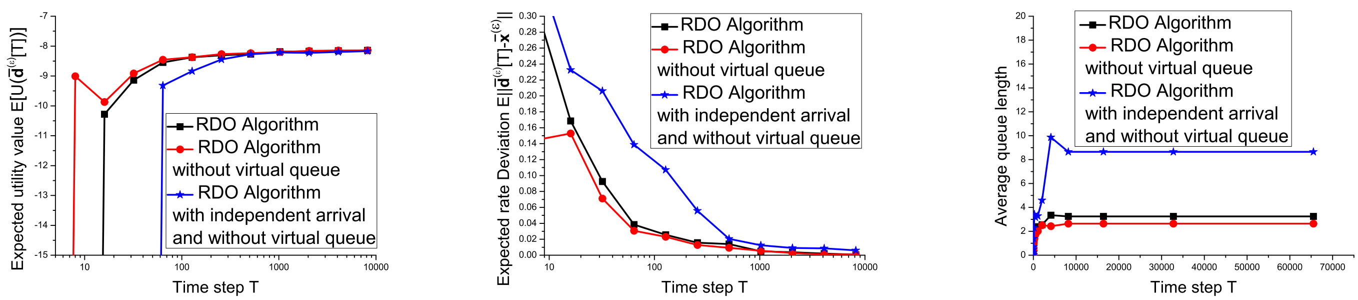
In Figure 6a, 6b and 6c, we also compare the performance between the Dual Algorithm and the DRDO Algorithm with $\epsilon = 0$. We can observe that the Dual Algorithm with proper parameter ϵ (e.g., $\epsilon = 5$) converges slightly faster than the DRDO Algorithm under the utility benefit metric. This does not contradict our result in Section V that both Dual and DRDO algorithms are utility benefit optimal and thus their convergence speed may differ at most a constant factor. However, the DRDO Algorithm converges faster than the Dual Algorithm under rate deviation metric, which matches our theoretical result that the DRDO algorithm is still optimal and the optimality of the Dual Algorithm is unknown under such metric. Finally, we can see from Figure 6c that the DRDO Algorithm has quite small average queue length.

Figure 7a, 7b and 7c study the impact of scheduling constraint, which does not allow link with queue length less than 1 to be scheduled. From there figures, we can see that the scheduling constraint does not have significant impact on the convergence speed and the average queue length for both DRDO and Dual algorithms.



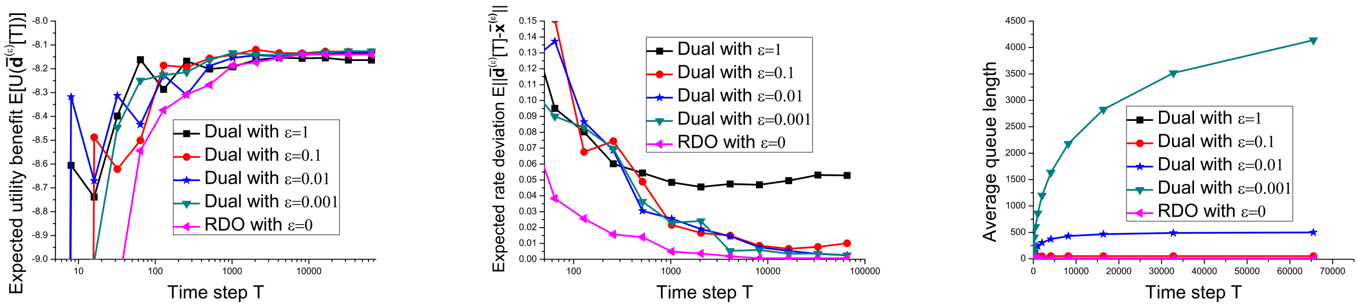
(a) Rate convergence under utility difference metric (b) Rate convergence under rate difference metric (c) Average queue length

Fig. 7: The impact of the scheduling constraint on the performance of the Dual and DRDO Algorithm



(a) Rate convergence under utility difference metric (b) Rate convergence under rate difference metric (c) Average queue length

Fig. 8: Variants of the RDO Algorithm



(a) Rate convergence under utility difference metric (b) Rate convergence under rate difference metric (c) Average queue length

Fig. 9: The performance of Dual Algorithm over wireless fading channels

B. Fading channels

In this subsection, we first consider the performance of variants of the RDO Algorithm over wireless fading channels. Then, we compare the performance between the RDO Algorithm and the Dual Algorithm.

We consider a variant of the RDO Algorithm that does not require maintaining a virtual queue and performs MWS directly among data queues, and another variant of the RDO Algorithm that has independent random arrivals and performs MWS directly among data queues. Recall that the purpose of introducing virtual queues in RDO Algorithm is to show the boundedness of the average data queue length by avoiding the difficulty in using Lyapunov Drift argument. From Figure 8a, 8b and 8c, we can observe that for both rate deviation and utility benefit metrics, the original RDO Algorithm converges faster than and has smaller average queue length than a variant of the RDO Algorithm with independent random arrivals, but converges slower than and has larger average queue length than a variant of the RDO Algorithm without virtual queues. Thus, we suggest to use the arrival generation component in the RDO

Algorithm and perform MWS among data queues directly in practice.

Figure 9a, 9b and 9c show the performance of a version of the Dual Algorithm (see [4]) over wireless fading channels and compare it with the RDO Algorithm with $\epsilon = 0$. To guarantee the convergence of the Dual Algorithm under both metrics of interest under fading, it requires much smaller parameter ϵ than that under non-fading. As in non-fading scenario, the Dual Algorithm with proper parameter can outperform the RDO Algorithm under utility benefit metric and has slower convergence speed than the RDO Algorithm under rate deviation metric, which matches our result in Section IV-B, the RDO Algorithm is rate deviation optimal while the optimality of both RDO and Dual algorithms is unknown under the utility benefit metric in the fading scenario. In addition, the average queue length of the Dual Algorithm increases as the decreasing of ϵ, while the RDO Algorithm always has small average queue length.

VII. Conclusion

In this paper, we considered the convergence speed of joint flow control and scheduling algorithms in Network Utility Maximization (NUM) problem in multi-hop wireless networks. We realized that the discreteness of scheduling constraints and transmission rates are two of the most important features in wireless networks. We incorporated these important characteristics into the analysis and design of algorithms in terms of their convergence speed by defining two metrics of interest: rate deviation and utility benefit.

We showed that the convergence speed of any algorithm cannot be faster than $\Omega\left(\frac{1}{T}\right)$ for both rate deviation and utility benefit metrics due to the discrete choices of transmission rates at each link. This interesting and fundamental finding reveals that designing faster (e.g., Interior-Point or Newton based) algorithms for the flow rate control cannot break the barrier of $\Omega\left(\frac{1}{T}\right)$ in wireless networks caused by the scheduling component. Then, we provided an algorithm that can achieve optimal convergence speed under both rate deviation and utility benefit metrics. Moreover, we showed that the well-known dual algorithm also has optimal convergence speed in terms of utility benefit, which is a somewhat surprising outcome in view of the first-order nature of its iteration.

References

[1] B. Li, A. Eryilmaz, and R. Li. Wireless scheduling for utility maximization with optimal convergence rate. In *Proc. IEEE International Conference on Computer Communications (INFOCOM)*, Turin, Italy, April 2013.

[2] F. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: Shadow prices, proportionalfairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, 1998.

[3] S. Low and D. Lapsley. Optimization flow control, I: Basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, 7(6):861–874, December 1999.

[4] A. Eryilmaz and R. Srikant. Fair resource allocation in wireless networks using queue-length based scheduling and congestion control. In *Proc. IEEE International Conference on Computer Communications (INFOCOM)*, Miami, Florida, March 2005.

[5] A. Eryilmaz and R. Srikant. Joint congestion control, routing and mac for stability and fairness in wireless networks. *IEEE Journal on Selected Areas in Communications, special issue on Nonlinear Optimization of Communication Systems*, 14(8):1514–1524, August 2006.

[6] X. Lin and N. Shroff. The impact of imperfect scheduling on cross-layer rate control in multihop wireless networks. In *Proc. IEEE International Conference on Computer Communications (INFOCOM)*, Miami, Florida, March 2005.

[7] M. Neely, E. Modiano, and C. Li. Fairness and optimal stochastic control for heterogeneous networks. In *Proc. IEEE International Conference on Computer Communications (INFOCOM)*, Miami, Florida, March 2005.

[8] Y. Nesterov. *Introductory lectures on convex optimization: A basic course*. Kluwer, Boston, 2004.

[9] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, 2004.

[10] D. Bertsekas. *Nonlinear Programming*. Athena Scientific, Belmont, MA, 1995.

[11] H. Feng, C. Xia, Z. Liu, and L. Zhang. Linear-speed interior-path algorithms for distributed control of information networks. *Performance Evaluation*, 67(11):1107–1122, 2010.

[12] A. Jadbabaie, A. Ozdaglar, and M. Zargham. A distributed newton method for network optimization. In *Proc. IEEE Conference on Decision and Control (CDC)*, Shanghai, China, December 2009.

[13] E. Wei, A. Ozdaglar, and A. Jadbabaie. Distributed newton method for network utility maximization. In *Proc. IEEE Conference on Decision and Control (CDC)*, Atlanta, Georgia, December 2010.

[14] J. Liu and H. Sherali. A distributed newton's method for joint multi-hop routing and flow control: Theory and algorithm. In *Proc. IEEE International Conference on Computer Communications (INFOCOM)*, Orlando, Florida, March 2012.

[15] A. Nedic and A. Ozdaglar. Subgradient methods in network resource allocation: Rate analysis. In *42nd Annual Conference on Information Sciences and Systems (CISS)*, Princeton, New Jersey, March 2008.

[16] D. Bertsekas, A. Nedic, and A. Ozdaglar. *Convex Analysis and Optimization*. Athena Scientific, Belmont, Mass., 2003.

[17] S. Meyn and R. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, 1993.

Appendix A
Proof of Lemma 1

Note that $\left|\frac{1}{T}\sum_{t=1}^{T} I[t] - r\right| \leq \frac{c_r}{T}$ is equivalent to

$$-c_r \leq \sum_{t=1}^{T} I[t] - Tr \leq c_r. \qquad (29)$$

(i) If $I[t+1] \leq a_i$, then we have

$$\begin{aligned}\sum_{t=1}^{T+1} I[t] - (T+1)r &\leq \sum_{t=1}^{T} I[t] - Tr + a_i - r \\ &\leq c_r + a_i - r \qquad (30)\\ &< -c_r, \qquad (31)\end{aligned}$$

where (30) follows from the inequality (29), and (31) follows from $c_r < \frac{r-a_i}{2}$.

(ii) If $I_l[t+1] \geq a_{i+1}$, then we have

$$\begin{aligned}\sum_{t=1}^{T+1} I[t] - (T+1)r &\geq \sum_{t=1}^{T} I[t] - Tr + a_{i+1} - r \\ &\geq -c_r + a_{i+1} - r \qquad (32)\\ &> c_r, \qquad (33)\end{aligned}$$

where (32) follows from the inequality (29), and (33) follows from $c_r < \frac{a_{i+1}-r}{2}$.

Thus, by combining (31) and (33), we have the desired result.

Appendix B
Proof of Proposition 1

We first show the following claim:

Claim 1: If $\overline{x}_l^{(\epsilon)} \in (B_{l,i}, B_{l,i+1})$, for some $i = 1, 2, ..., K_l - 1$, is not an integer, then, there exists a $c_{\overline{\mathbf{x}}^{(\epsilon)}} \in (0, \min\{\frac{\overline{x}_l^{(\epsilon)} - B_{l,i}}{2}, \frac{B_{l,i+1} - \overline{x}_l^{(\epsilon)}}{2}, \frac{1}{2}\})$ and a positive integer-valued sequence $\{T_k\}_{k\geq 1}$ such that

$$\|\overline{\mathbf{d}}^{(\epsilon)}[T_k] - \overline{\mathbf{x}}^{(\epsilon)}\| \geq \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{T_k}, \qquad \forall k, \tag{34}$$

holds for any sample path of departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$. $\diamond$

Since all elements in the closure of $\overline{\mathcal{X}}^{(\epsilon)}$ have non-integer-valued coordinates, by Claim 1, if $\overline{x}_l^{(\epsilon)} \in (B_{l,i_{\overline{\mathbf{x}}^{(\epsilon)}}}, B_{l,i_{\overline{\mathbf{x}}^{(\epsilon)}}+1})$, for some non-negative integer $i_{\overline{\mathbf{x}}^{(\epsilon)}}$, is not an integer, then we can take $c'_{\overline{\mathbf{x}}^{(\epsilon)}} = \frac{1}{4}\min\{\overline{x}_l^{(\epsilon)} - B_{l,i_{\overline{\mathbf{x}}^{(\epsilon)}}}, B_{l,i_{\overline{\mathbf{x}}^{(\epsilon)}}+1} - \overline{x}_l^{(\epsilon)}, 1\}, \forall \overline{\mathbf{x}}^{(\epsilon)} \in \overline{\mathcal{X}}^{(\epsilon)}$, and there exists a positive integer-valued sequence $\{T_k\}_{k\geq 1}$ such that

$$\|\overline{\mathbf{d}}^{(\epsilon)}[T_k] - \overline{\mathbf{x}}^{(\epsilon)}\| \geq \frac{c'_{\overline{\mathbf{x}}^{(\epsilon)}}}{T_k}, \qquad \forall k, \tag{35}$$

holds for any sample path of departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$.

If $c \triangleq \inf_{\overline{\mathbf{x}}^{(\epsilon)} \in \overline{\mathcal{X}}^{(\epsilon)}} c'_{\overline{\mathbf{x}}^{(\epsilon)}} = 0$, then, one of the limiting points of the set $\overline{\mathcal{X}}^{(\epsilon)}$ has all integer-valued coordinates, which contradicts our assumption that the closure of set $\overline{\mathcal{X}}^{(\epsilon)}$ does not contain a vector with all integer-valued coordinates. Thus, we have $c > 0$ and obtain

$$\phi(\overline{\mathbf{d}}^{(\epsilon)}[T_k], \overline{\mathcal{X}}^{(\epsilon)}) = \inf_{\overline{\mathbf{x}}^{(\epsilon)} \in \overline{\mathcal{X}}^{(\epsilon)}} \|\overline{\mathbf{d}}^{(\epsilon)}[T_k] - \overline{\mathbf{x}}^{(\epsilon)}\| \geq \frac{c}{T_k}, \quad \forall k,$$

holds for any sample path of departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq 1}$, which implies (11).

Next, we prove Claim 1 to complete the proof. Since

$$\|\overline{\mathbf{d}}^{(\epsilon)}[T_k] - \overline{\mathbf{x}}^{(\epsilon)}\| \geq |\overline{d}_l^{(\epsilon)}[T_k] - \overline{x}_l^{(\epsilon)}|, \qquad \forall k, \tag{36}$$

we only need to show

$$|\overline{d}_l^{(\epsilon)}[T_k] - \overline{x}_l^{(\epsilon)}| \geq \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{T_k}, \qquad \forall k. \tag{37}$$

Indeed, since $c_{\overline{\mathbf{x}}^{(\epsilon)}} < \frac{1}{2}$ and $\overline{x}_l^{(\epsilon)} > 0$, we have

$$\frac{n+1}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}} - \left(\frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right) = \frac{1 - 2c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}} > 0, \tag{38}$$

which implies

$$\left(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right) \cap \left(\frac{n+1}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n+1}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right) = \emptyset,$$

for any non-negative integer n. Since $c_{\overline{\mathbf{x}}^{(\epsilon)}} < \frac{\overline{x}_l^{(\epsilon)} - B_{l,i}}{2} \leq \frac{\overline{x}_l^{(\epsilon)}}{2}$, we have

$$\frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}} - \left(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right) = \frac{2c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}} < 1, \tag{39}$$

which implies that each interval $(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}})$ can at most contain one non-negative integer.

If the interval $(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}})$ contains some non-negative integer T for some non-negative integer m, i.e., $|\frac{n}{T} - \overline{x}_l^{(\epsilon)}| \leq \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{T}$, where $\overline{x}_l^{(\epsilon)} \in (B_{l,i}, B_{l,i+1})$ for some $i = 1, 2, ..., K_l - 1$ and $c_{\overline{\mathbf{x}}^{(\epsilon)}} \in (0, \min\{\frac{\overline{x}_l^{(\epsilon)} - B_{l,i}}{2}, \frac{B_{l,i+1} - \overline{x}_l^{(\epsilon)}}{2}, \frac{1}{2}\})$, then, by taking the set $\mathcal{I} = \mathbb{N}^0 \triangleq \{0, 1, 2, ...\}$ in Lemma 1, we have $|\frac{n+l}{T+1} - \overline{x}_l^{(\epsilon)}| \geq \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{T+1}$ for any positive integer l, which implies

$$T + 1 \notin \left(\frac{n+l}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n+l}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right),$$

for any positive integer l. Thus, $\cup_{n=0}^{\infty}(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}})$ does not cover all positive integers and thus there exists a sequence of positive integers $\{T_k\}_{k=1}^{\infty}$ such that

$$T_k \notin \left(\frac{j}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{j}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right) \text{ for any non-negative integer } j.$$

Figure 10 shows an example when $\overline{x}_l^{(\epsilon)} = \frac{1}{2}$, $S_l \in \{0, 1\}$ and $c_{\overline{\mathbf{x}}^{(\epsilon)}} = \frac{1}{4}$. We can see that $\cup_{n=0}^{\infty}(\frac{n}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{n}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}})$ does not cover odd numbers.

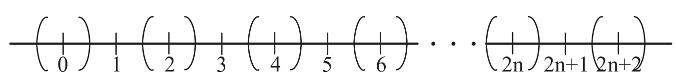


Fig. 10: An example when $\overline{x}_l^{(\epsilon)} = \frac{1}{2}$, $S_l \in \{0, 1\}$ and $c_{\overline{\mathbf{x}}^{(\epsilon)}} = \frac{1}{4}$.

For any sample path of random sequence $\{D_l^{(\epsilon)}[t], t \geq 1\}$, $\sum_{t=1}^{T_k} D_l^{(\epsilon)}[t]$ is an integer and thus we have

$$T_k \notin \left(\frac{\sum_{t=1}^{T_k} D_l^{(\epsilon)}[t]}{\overline{x}_l^{(\epsilon)}} - \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}, \frac{\sum_{t=1}^{T_k} D_l^{(\epsilon)}[t]}{\overline{x}_l^{(\epsilon)}} + \frac{c_{\overline{\mathbf{x}}^{(\epsilon)}}}{\overline{x}_l^{(\epsilon)}}\right),$$

which is equivalent to (37).

Appendix C
Proof of Lemma 2

Since $\mathbf{x}^{(\epsilon)}[t] = (x_l^{(\epsilon)}[t])_{l=1}^{L} \in \mathcal{R}$ for all t and the maximum transmission rate for each link l is B_{l,K_l}, we have $0 \leq x_l^{(\epsilon)}[t] \leq B_{l,K_l}$. Next, we show this lemma by using induction.

(1) If $T = 1$, we have

$$|X_l^{(\epsilon)}[1] - x_l^{(\epsilon)}[1]| \leq \max\{B_{l,K_l} - x_l^{(\epsilon)}[1], x_1^{(\epsilon)}[1]\} \leq B_{l,K_l};$$

(2) Assume $T = n$, (18) holds, that is,

$$-B_{l,K_l} \leq \sum_{t=1}^{n}(X_l^{(\epsilon)}[t] - x_l^{(\epsilon)}[t]) \leq B_{l,K_l}. \tag{40}$$

Then, when $T = n+1$, we consider the following two cases:

(2.1) If $\sum_{t=1}^{n} X_l^{(\epsilon)}[t] \geq \sum_{t=1}^{n} x_l^{(\epsilon)}[t]$, then, by the RDO

Algorithm, we have $X_l^{(\epsilon)}[n+1]=0$. Thus, we have

$$\sum_{t=1}^{n+1}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])=\sum_{t=1}^{n}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])-x_l^{(\epsilon)}[n+1] \leq B_{l,K_l}-x_l^{(\epsilon)}[n+1]\leq B_{l,K_l}, \quad (41)$$

$$\sum_{t=1}^{n+1}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])=\sum_{t=1}^{n}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])-x_l^{(\epsilon)}[n+1] \geq -x_l^{(\epsilon)}[n+1]\geq -B_{l,K_l}. \quad (42)$$

(2.2) If $\sum_{t=1}^{n}X_l^{(\epsilon)}[t] < \sum_{t=1}^{n}x_l^{(\epsilon)}[t]$, then, by the RDO Algorithm, we have $X_l^{(\epsilon)}[n+1]=B_{l,K_l}$. Thus, we have

$$\begin{aligned}&\sum_{t=1}^{n+1}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])\\ =&\sum_{t=1}^{n}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])+B_{l,K_l}-x_l^{(\epsilon)}[n+1]\\ \leq& B_{l,K_l}-x_l^{(\epsilon)}[n+1]\leq B_{l,K_l},\end{aligned} \quad (43)$$

$$\begin{aligned}&\sum_{t=1}^{n+1}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])\\ =&\sum_{t=1}^{n}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])+B_{l,K_l}-x_l^{(\epsilon)}[n+1]\\ \geq& -B_{l,K_l}+B_{l,K_l}-x_l^{(\epsilon)}[n+1]\geq -B_{l,K_l}.\end{aligned} \quad (44)$$

In both cases, we have $\left|\sum_{t=1}^{n+1}(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t])\right|\leq B_{l,K_l}$.

Appendix D
Proof of Lemma 3

By using Lemma 2, we have

$$\begin{aligned}&\left|\sum_{t=n}^{T-1}X_l^{(\epsilon)}[t]-\sum_{t=n}^{T-1}x_l^{(\epsilon)}[t]\right|\\ =&\left|\sum_{t=1}^{T-1}\left(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t]\right)-\sum_{t=1}^{n-1}\left(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t]\right)\right|\\ \leq&\left|\sum_{t=1}^{T-1}\left(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t]\right)\right|+\left|\sum_{t=1}^{n-1}\left(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t]\right)\right|\\ \leq& 2B_{l,K_l},\end{aligned} \quad (45)$$

which implies that

$$\sum_{t=n}^{T-1}X_l^{(\epsilon)}[t]\leq\sum_{t=n}^{T-1}x_l^{(\epsilon)}[t]+2B_{l,K_l}, \quad \forall n=1,2,...,T-1.$$

By using the Lindley's equation, we have

$$\begin{aligned}Q_l^{(\epsilon)}[T]=&\max_{1\leq n\leq T-1}\left\{\sum_{t=n}^{T-1}X_l^{(\epsilon)}[t]-\sum_{t=n}^{T-1}S_l^{(\epsilon)}[t],0\right\}\\ \leq&\max_{1\leq n\leq T-1}\left\{2B_{l,K_l}+\sum_{t=n}^{T-1}x_l^{(\epsilon)}[t]-\sum_{t=n}^{T-1}S_l^{(\epsilon)}[t],0\right\}\\ \leq&\max_{1\leq n\leq T-1}\left\{\sum_{t=n}^{T-1}x_l^{(\epsilon)}[t]-\sum_{t=n}^{T-1}S_l^{(\epsilon)}[t],0\right\}+2B_{l,K_l}\\ =&Y_l^{(\epsilon)}[T]+2B_{l,K_l}.\end{aligned} \quad (46)$$

Appendix E
Proof of Proposition 2

$$\begin{aligned}&\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T],\overline{\mathbf{x}}^{(\epsilon)})]=\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{D}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)}\right\|\\ =&\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{D}^{(\epsilon)}[t]-\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}^{(\epsilon)}[t]+\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}^{(\epsilon)}[t]\right.\\ &\left.-\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]+\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)}\right\|\\ \leq&\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{D}^{(\epsilon)}[t]-\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}^{(\epsilon)}[t]\right\|+\left\|\frac{1}{T}\sum_{t=1}^{T}(\mathbf{x}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)})\right\|\\ &+\left\|\frac{1}{T}\sum_{t=1}^{T}\mathbf{X}^{(\epsilon)}[t]-\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right\|\\ \overset{(a)}{=}&\frac{\mathbb{E}\|\mathbf{Q}^{(\epsilon)}[T+1]\|}{T}+\frac{1}{T}\left\|\sum_{t=1}^{T}\left(\mathbf{X}^{(\epsilon)}[t]-\mathbf{x}^{(\epsilon)}[t]\right)\right\|\\ &+\left\|\frac{1}{T}\sum_{t=1}^{T}(\mathbf{x}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)})\right\|\\ \overset{(b)}{\leq}&\frac{\sum_{l=1}^{L}\mathbb{E}[Q_l^{(\epsilon)}[T+1]]}{T}+\frac{1}{T}\sum_{l=1}^{L}\left|\sum_{t=1}^{T}\left(X_l^{(\epsilon)}[t]-x_l^{(\epsilon)}[t]\right)\right|\\ &+\frac{1}{T}\sum_{t=1}^{T}\left\|\mathbf{x}^{(\epsilon)}[t]-\overline{\mathbf{x}}^{(\epsilon)}\right\|\end{aligned} \quad (47)$$

where the step (a) follows from the fact that $Q_l^{(\epsilon)}[T+1]-Q_l^{(\epsilon)}[1]=\sum_{t=1}^{T}X_l^{(\epsilon)}[t]-\sum_{t=1}^{T}D_l^{(\epsilon)}[t]$ and $Q_l^{(\epsilon)}[1]=0$ for all l; (b) follows from the fact that $\|\mathbf{y}\|\leq\sum_{l=1}^{L}|y_l|$ for any L-dimensional vector $\mathbf{y}$.

First, we will show that

$$\mathbb{E}[Y_l^{(\epsilon)}[T]]\leq M_1, \qquad \forall T\geq 1,\forall l\in\mathcal{L}, \quad (48)$$

where M_1 is some positive number. This will imply $\mathbb{E}[Q_l^{(\epsilon)}[T]]\leq M_1+2B_{l,K_l},\forall T\geq 1,\forall l\in\mathcal{L}$, by using Lemma 3. By choosing the Lyapunov function $V_1(\mathbf{Y}^{(\epsilon)}[t])\triangleq$

$\frac{1}{2}\sum_{l=1}^{L}(Y_l^{(\epsilon)}[t])^2$, it is easy to show that

$$\Delta V_1 \triangleq \mathbb{E}[V_1(\mathbf{Y}^{(\epsilon)}[t+1]) - V_1(\mathbf{Y}^{(\epsilon)}[t]) | \mathbf{Y}^{(\epsilon)}[t] = \mathbf{Y}^{(\epsilon)}] \le -\theta^{(\epsilon)} \sum_{l=1}^{L} Y_l^{(\epsilon)} + M_2, \quad (49)$$

where M_2 and $\theta^{(\epsilon)}$ are some finite positive constants. By using Theorem 14.0.1 in [17], there exists $\overline{Y}_l$ such that $\lim_{T\to\infty}\mathbb{E}[Y_l^{(\epsilon)}[T]] = \mathbb{E}[\overline{Y}_l^{(\epsilon)}]$, where $\mathbb{E}[\overline{Y}_l^{(\epsilon)}] < \infty$. Thus, give $\zeta > 0$, $\exists T_0 \ge 0$ such that $T > T_0$ implies that $\mathbb{E}[Y_l^{(\epsilon)}[T]] \le \mathbb{E}[\overline{Y}_l^{(\epsilon)}] + \zeta$.

For $T \le T_0$, we have

$$\mathbb{E}[Y_l^{(\epsilon)}[T]] \le \sum_{t=1}^{T} x_l^{(\epsilon)}[t] \le \sum_{t=1}^{T_0} x_l^{(\epsilon)}[t] \le T_0 B_{l,K_l}. \quad (50)$$

Hence, by taking $M_1 \triangleq \max_{l\in\mathcal{L}}\{T_0 B_{l,K_l}, \mathbb{E}[\overline{Y}_l^{(\epsilon)}] + \zeta\}$, we have the desired result.

By Lemma 2, we have $\left|\sum_{t=1}^{T}\left(X_l^{(\epsilon)}[t] - x_l^{(\epsilon)}[t]\right)\right| \le B_{l,K_l}$. Thus, we have

$$\mathbb{E}[\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathbf{x}}^{(\epsilon)})] \le \frac{LM_1 + 2\sum_{l=1}^{L} B_{l,K_l}}{T} + \frac{\sum_{l=1}^{L} B_{l,K_l}}{T} + \frac{R_1}{T} = \frac{R_1^{(\epsilon)}}{T},$$

where $R_1^{(\epsilon)} = LM_1 + 2\sum_{l=1}^{L} B_{l,K_l} + \sum_{l=1}^{L} B_{l,K_l} + R_1$.

Appendix F
Proof of Proposition 4

(1) Proof of the first part of Proposition 4: For any $\delta \in (0, \max_{\mathbf{r}\in\mathcal{R}}\|\mathbf{r}-\mathbf{r}^*\|)$, we can easily find a $\mathbf{r}^{(\delta)} = (r_l^{(\delta)})_{l=1}^{L} \in \mathcal{R}$ satisfying the following conditions:

(i) $\|\mathbf{r}^{(\delta)} - \mathbf{r}^*\| = \delta$;
(ii) $\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)} \notin \mathcal{G}^{(\delta)}$,

where

$$\mathcal{G}^{(\delta)} \triangleq \left\{\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) S_l : \mathbf{S} = (S_l)_{l=1}^{L} \in \mathcal{R} \text{ and } S_l \in \mathcal{F}_l, \forall l\right\}.$$

Due to the discrete structure of scheduling rates $\mathbf{S}$, without loss of generality, we assume $\mathcal{G}^{(\delta)}$ has $K^{(\delta)}$ elements. Let $\mathcal{G}^{(\delta)} = \{a_1^{(\delta)}, a_2^{(\delta)}, ..., a_{K^{(\delta)}}^{(\delta)}\}$ and assume $\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)} \in (a_i^{(\delta)}, a_{i+1}^{(\delta)})$ for some $i = 1, 2, ..., K^{(\delta)} - 1$.

Consider any policy in $\mathcal{P}$ with parameter ϵ. By Assumption 2, $D_l^{(\epsilon)}[t] \in \mathcal{F}_l, \forall t \ge 1, \forall l$, for any departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t] = (D_l^{(\epsilon)}[t])_l\}_{t\ge 1}$. Next, we show that there exists a

$$c^{(\delta)} \in \left(0, \min\left\{\frac{\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)} - a_i^{(\delta)}}{2}, \frac{a_{i+1}^{(\delta)} - \sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)}}{2}\right\}\right), \quad (51)$$

and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that (21) holds. Let $f_{\mathbf{y}}(\mathbf{r}) \triangleq \nabla U(\mathbf{r})^{\mathrm{T}}(\mathbf{y}-\mathbf{r}) = \sum_{l=1}^{L} U_l'(r_l)(y_l - r_l)$, where $\mathbf{r}, \mathbf{y} \in \mathcal{R}$ and T means transpose operation. Then, we have

$$|f_{\mathbf{y}}(\mathbf{r}) - f_{\mathbf{y}}(\mathbf{r}^*)| \overset{(a)}{=} \left|\nabla f_{\mathbf{y}}(\mathbf{z})^{\mathrm{T}}(\mathbf{r}-\mathbf{r}^*)\right| \overset{(b)}{\le} \|\nabla f_{\mathbf{y}}(\mathbf{z})\| \|\mathbf{r}-\mathbf{r}^*\|, \quad (52)$$

where the step (a) follows from the Mean Value Theorem, where $\mathbf{z}$ lies between $\mathbf{r}$ and $\mathbf{r}^*$; (b) follows from the Cauchy Schwartz's inequality. Since $\frac{\partial f_{\mathbf{y}}}{\partial r_l} = U_l''(r_l)(y_l - r_l) - U_l'(r_l)$, we have

$$\left|\frac{\partial f_{\mathbf{y}}}{\partial r_l}\right| \le |U_l''(r_l)|(y_l + r_l) + |U_l'(r_l)| \le 2\beta_{\max} b_{l,K_l} + h_{\max}.$$

Thus, we have

$$\|\nabla f_{\mathbf{y}}(\mathbf{z})\| \le H \triangleq \sqrt{L}(2\beta_{\max} \max_{l\in\mathcal{L}} b_{l,K_l} + h_{\max}). \quad (53)$$

Hence, we have

$$|f_{\mathbf{y}}(\mathbf{r}) - f_{\mathbf{y}}(\mathbf{r}^*)| \le H\|\mathbf{r}-\mathbf{r}^*\|, \qquad \forall \mathbf{y}, \mathbf{r} \in \mathcal{R}. \quad (54)$$

By setting $\mathbf{r} = \mathbf{r}^{(\delta)}$, we have

$$\left|f_{\mathbf{y}}(\mathbf{r}^{(\delta)}) - f_{\mathbf{y}}(\mathbf{r}^*)\right| \le H\delta, \qquad \forall \mathbf{y} \in \mathcal{R}, \quad (55)$$

which implies that

$$f_{\mathbf{y}}(\mathbf{r}^{(\delta)}) \le f_{\mathbf{y}}(\mathbf{r}^*) + H\delta, \qquad \forall \mathbf{y} \in \mathcal{R}. \quad (56)$$

By the first order optimality condition, we have

$$f_{\mathbf{y}}(\mathbf{r}^*) = \nabla U(\mathbf{r}^*)^{\mathrm{T}}(\mathbf{y}-\mathbf{r}^*) \le 0, \qquad \forall \mathbf{y} \in \mathcal{R}. \quad (57)$$

Thus, we have $f_{\mathbf{y}}(\mathbf{r}^{(\delta)}) \le H\delta, \forall \mathbf{y} \in \mathcal{R}$. By setting $\mathbf{y} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{D}^{(\epsilon)}[t] \in \mathcal{R}$, we have

$$\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) \left(\frac{1}{T}\sum_{t=1}^{T} D_l^{(\epsilon)}[t] - r_l^{(\delta)}\right) \le H\delta. \quad (58)$$

Since $\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)} \notin \mathcal{G}^{(\delta)}$, by Lemma 1, there exists a $c^{(\delta)}$ satisfying (51) and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that

$$\left|\frac{1}{T_k}\sum_{t=1}^{T_k}\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) D_l^{(\epsilon)}[t] - \sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)}\right| \ge \frac{c^{(\delta)}}{T_k}. \quad (59)$$

Thus, if $H\delta < \frac{c^{(\delta)}}{T_k}$, that is, $T_k < \frac{c^{(\delta)}}{H\delta}$, then, we have

$$\frac{1}{T_k}\sum_{t=1}^{T_k}\sum_{l=1}^{L} U_l'(r_l^{(\delta)}) D_l^{(\epsilon)}[t] - \sum_{l=1}^{L} U_l'(r_l^{(\delta)}) r_l^{(\delta)} \le -\frac{c^{(\delta)}}{T_k}. \quad (60)$$

By using the concavity of the utility function U, we have

$$U\left(\frac{1}{T_k}\sum_{t=1}^{T_k}\mathbf{D}^{(\epsilon)}[t]\right) \le U(\mathbf{r}^{(\delta)}) + \nabla U(\mathbf{r}^{(\delta)})^{\mathrm{T}}\left(\frac{1}{T_k}\sum_{t=1}^{T_k}\mathbf{D}^{(\epsilon)}[t] - \mathbf{r}^{(\delta)}\right) \le U(\mathbf{r}^{(\delta)}) - \frac{c^{(\delta)}}{T_k}. \quad (61)$$

In addition, we have

$$\begin{aligned}U(\mathbf{r}^{(\delta)}) &\overset{(a)}{=} U(\mathbf{r}^*) + \nabla U(\mathbf{r}^*)^{\mathrm{T}}(\mathbf{r}^{(\delta)} - \mathbf{r}^*) \\ &\quad + \frac{1}{2}(\mathbf{r}^{(\delta)} - \mathbf{r}^*)^{\mathrm{T}}\nabla^2 U(\mathbf{z})(\mathbf{r}^{(\delta)} - \mathbf{r}^*) \\ &\overset{(b)}{\leq} U(\mathbf{r}^*) - \frac{1}{2}\beta_{\min}\sqrt{L}\|\mathbf{r}^{(\delta)} - \mathbf{r}^*\|^2 \\ &= U(\mathbf{r}^*) - \frac{1}{2}\beta_{\min}\sqrt{L}\delta^2, \end{aligned} \tag{62}$$

where (a) follows the Mean Value Theorem, where $\mathbf{z}$ is between $\mathbf{r}^*$ and $\mathbf{r}^{(\delta)}$; (b) uses the first order optimality condition and $\|\nabla^2 U(\mathbf{z})\| \geq \sqrt{L}\beta_{\min}$. By substituting (62) into (61), we have inequality (21).

(2) Proof of the second part of Proposition 4: We will show the following claim:

Claim 2: If we further have

$$\sum_{l=1}^{L} U_l'(r_l^*)r_l^* \notin \mathcal{H},$$

i.e., $\sum_{l=1}^{L} U_l'(r_l^*)r_l^* \in (\sum_{l=1}^{L} U_l'(r_l^*)y_l, \sum_{l=1}^{L} U_l'(r_l^*)z_l)$, where $\mathbf{y} = (y_l)_{l=1}^L, \mathbf{z} = (z_l)_{l=1}^L \in \mathcal{R}$ and $y_l, z_l \in \mathcal{F}_l, \forall l \in \mathcal{L}$, then, for any

$$\begin{aligned}\delta < \min\Bigg\{&\frac{1}{2G_1}\min_{y',z'\in\mathcal{G}^{(0)}:y'\neq z'}|y'-z'|, \\ &\frac{1}{G_1+G_2}\left(\sum_{l=1}^{L} U_l'(r_l^*)r_l^* - \sum_{l=1}^{L} U_l'(r_l^*)y_l\right), \\ &\frac{1}{G_1+G_2}\left(\sum_{l=1}^{L} U_l'(r_l^*)z_l - \sum_{l=1}^{L} U_l'(r_l^*)r_l^*\right)\Bigg\},\end{aligned} \tag{63}$$

there exists

$$\begin{aligned}c_1{}^{(\delta)} = &\frac{1}{4}\min\Bigg\{\sum_{l=1}^{L} U_l'(r_l^*)r_l^* - \sum_{l=1}^{L} U_l'(r_l^*)y_l - (G_1+G_2)\delta, \\ &\sum_{l=1}^{L} U_l'(r_l^*)z_l - \sum_{l=1}^{L} U_l'(r_l^*)r_l^* - (G_1+G_2)\delta\Bigg\}\end{aligned} \tag{64}$$

and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that (21) holds, where $G_1 \triangleq \sqrt{L}\beta_{\max}\max_{l\in\mathcal{L}} b_{l,K_l}$ and $G_2 \triangleq \sqrt{L}(\beta_{\max}\max_{l\in\mathcal{L}} b_{l,K_l} + h_{\max})$. ◇

When δ is sufficiently small (e.g., where δ satisfies (63)), we take $c^{(\delta)} = c_1^{(\delta)}$ and thus we have the desired result. Next, we prove this claim to complete the proof. By using similar technique in showing inequality (54), we have

$$\left|\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)} - \sum_{l=1}^{L} U_l'(r_l^*)r_l^*\right| \leq G_2\delta \tag{65}$$

$$\left|\sum_{l=1}^{L} U_l'(r_l^{(\delta)})y_l - \sum_{l=1}^{L} U_l'(r_l^*)y_l\right| \leq G_1\delta \tag{66}$$

$$\left|\sum_{l=1}^{L} U_l'(r_l^{(\delta)})z_l - \sum_{l=1}^{L} U_l'(r_l^*)z_l\right| \leq G_1\delta. \tag{67}$$

Since δ satisfies (63), the relationship between $\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)}$ and $\sum_{l=1}^{L} U_l'(r_l^*)r_l^*$ is shown in Figure 11.

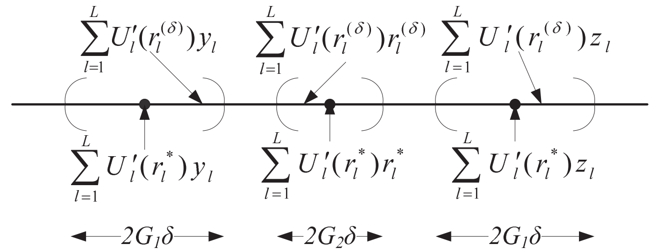


Fig. 11: The relationship between $\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)}$ and $\sum_{l=1}^{L} U_l'(r_l^*)r_l^*$.

Thus, we have

$$\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)} \in \left(\sum_{l=1}^{L} U_l'(r_l^{(\delta)})y_l, \sum_{l=1}^{L} U_l'(r_l^{(\delta)})z_l\right). \tag{68}$$

By statement (1), for any departure rate vector sequence $\{\mathbf{D}^{(\epsilon)}[t]\}_{t\geq1}$, there exists a $c^{(\delta)}$ satisfying

$$\begin{aligned}c^{(\delta)} < \min\Bigg\{&\frac{\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)} - \sum_{l=1}^{L} U_l'(r_l^{(\delta)})y_l}{2}, \\ &\frac{\sum_{l=1}^{L} U_l'(r_l^{(\delta)})z_l - \sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)}}{2}\Bigg\},\end{aligned} \tag{69}$$

and a positive integer-valued sequence $\{T_k\}_{k=1}^{\infty}$ such that (21) holds. By using inequality (65), (66) and (67), we have

$$\begin{aligned}&\min\Bigg\{\frac{\sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)} - \sum_{l=1}^{L} U_l'(r_l^{(\delta)})y_l}{2}, \\ &\quad\frac{\sum_{l=1}^{L} U_l'(r_l^{(\delta)})z_l - \sum_{l=1}^{L} U_l'(r_l^{(\delta)})r_l^{(\delta)}}{2}\Bigg\} \\ &\geq \min\Bigg\{\frac{\sum_{l=1}^{L} U_l'(r_l^*)r_l^* - \sum_{l=1}^{L} U_l'(r_l^*)y_l - (G_1+G_2)\delta}{2}, \\ &\quad\frac{\sum_{l=1}^{L} U_l'(r_l^*)y_l - \sum_{l=1}^{L} U_l'(r_l^*)r_l^* - (G_1+G_2)\delta}{2}\Bigg\}.\end{aligned} \tag{70}$$

Thus, we can take $c_1^{(\delta)}$ as in (64) that satisfies (51).

Appendix G
Proof of Proposition 5

$$\begin{aligned}U\left(\overline{\mathbf{d}}^{(\epsilon)}[T]\right) &\overset{(a)}{\geq} U(\overline{\mathbf{x}}^{(\epsilon)}) + \nabla U\left(\overline{\mathbf{d}}^{(\epsilon)}[T]\right)^T\left(\overline{\mathbf{x}}^{(\epsilon)} - \overline{\mathbf{d}}^{(\epsilon)}[T]\right) \\ &\overset{(b)}{\geq} U(\overline{\mathbf{x}}^{(\epsilon)}) - \left\|\nabla U\left(\overline{\mathbf{d}}^{(\epsilon)}[T]\right)\right\|\left\|\overline{\mathbf{d}}^{(\epsilon)}[T] - \overline{\mathbf{x}}^{(\epsilon)}\right\| \\ &\overset{(c)}{\geq} U(\overline{\mathbf{x}}^{(\epsilon)}) - \sqrt{L}h_{\max}\left\|\overline{\mathbf{d}}^{(\epsilon)}[T] - \overline{\mathbf{x}}^{(\epsilon)}\right\|,\end{aligned} \tag{71}$$

where the step (a) follows from the definition of concavity; (b) follows from Cauchy-Schwartz's inequality; (c) follows from Assumption 3.

By using similar line of argument in Proposition 2, it is not hard to show that

$$\phi(\overline{\mathbf{d}}^{(\epsilon)}[T], \overline{\mathbf{x}}^{(\epsilon)}) = \left\|\overline{\mathbf{d}}^{(\epsilon)}[T] - \overline{\mathbf{x}}^{(\epsilon)}\right\| \leq \frac{R_2^{(\epsilon)}}{T}, \tag{72}$$

where $R_2^{(\epsilon)}$ is some positive constant. Thus, we have

$$U\left(\overline{\mathbf{d}}^{(\epsilon)}[T]\right) \geq U(\overline{\mathbf{x}}^{(\epsilon)}) - \frac{\sqrt{L}h_{\max}R_2^{(\epsilon)}}{T}. \tag{73}$$

Since $\|\overline{\mathbf{x}}^{(\epsilon)} - \mathbf{r}^*\| \leq \rho^{(\epsilon)}$, by the Mean Value Theorem, we have

$$\begin{aligned}|U(\overline{\mathbf{x}}^{(\epsilon)}) - U(\mathbf{r}^*)| &= |\nabla U(\mathbf{z})^{\mathrm{T}}(\overline{\mathbf{x}}^{(\epsilon)} - \mathbf{r}^*)| \\ &\leq \|\nabla U(\mathbf{z})\| \|\overline{\mathbf{x}}^{(\epsilon)} - \mathbf{r}^*\| \\ &\leq h_{\max}\sqrt{L}\rho^{(\epsilon)}, \end{aligned} \tag{74}$$

where $\mathbf{z}$ is between $\overline{\mathbf{x}}^{(\epsilon)}$ and $\mathbf{r}^*$. Thus, we have

$$U(\overline{\mathbf{x}}^{(\epsilon)}) \geq U(\mathbf{r}^*) - h_{\max}\sqrt{L}\rho^{(\epsilon)}, \tag{75}$$

By substituting (75) into (73), we have the desired result.

Appendix H
Proof of Proposition 7

Before analyzing the convergence speed of the Dual Algorithm, we need to establish the boundedness of queue length for all links.

Lemma 4: For Dual Algorithm with parameter $\epsilon > 0$, the queue lengths for all links are bounded all the time, i.e.,

$$Q_l^{(\epsilon)}[t] \leq G^{(\epsilon)}, \forall l, t, \tag{76}$$

where $G^{(\epsilon)} \triangleq \sqrt{W} + \frac{h_{\max}}{\epsilon}$ and $W \triangleq \left(\frac{\beta_{\max}}{\epsilon} + 2\right) LM^2 + \left(\frac{3\beta_{\max}}{\epsilon} + 2\right) L \max_{l\in\mathcal{L}} b_{l,K_l}^2$.

Proof: See Appendix I for the proof. ■

We are ready to analyze the convergence speed of the Dual Algorithm in terms of its utility benefit. By Assumption 2, there is no unused service in the system and thus we have

$$\begin{aligned}
&U(\overline{\mathbf{d}}^{(\epsilon)}[T]) = U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{S}^{(\epsilon)}[t]\right)\\
&\overset{(a)}{\geq} U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right)\\
&\quad - \nabla U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{S}^{(\epsilon)}[t]\right)^{\mathrm{T}} \frac{\sum_{t=1}^{T}(\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t])}{T}\\
&\overset{(b)}{\geq} U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right)\\
&\quad - \frac{1}{T}\left\|\nabla U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{S}^{(\epsilon)}[t]\right)\right\| \left\|\sum_{t=1}^{T}(\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t])\right\|\\
&\overset{(c)}{\geq} U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right) - \frac{h_{\max}\sqrt{L}}{T}\|\mathbf{Q}^{(\epsilon)}[T+1] - \mathbf{Q}^{(\epsilon)}[1]\|\\
&\overset{(d)}{\geq} U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right) - \frac{h_{\max}\sqrt{L}}{T}\left(\|\mathbf{Q}^{(\epsilon)}[1]\| + \|\mathbf{Q}^{(\epsilon)}[T+1]\|\right)\\
&\overset{(e)}{\geq} U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right) - \frac{h_{\max}\sqrt{L}}{T}\left(\|\mathbf{Q}^{(\epsilon)}[1]\| + G^{(\epsilon)}\sqrt{L}\right),
\end{aligned} \tag{77}$$

where the step (a) follows from the concavity of utility function; (b) follows from the Cauchy-Schwarz inequality; (c) uses Assumption 3 and the queue evolution (27); (d) follows from the triangle inequality; (e) follows from the Lemma 4.

Next, we give the lower bound for $U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right)$.

$$\begin{aligned}
&\frac{1}{\epsilon}U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right)\\
&\overset{(a)}{\geq} \frac{1}{\epsilon}\frac{1}{T}\sum_{t=1}^{T}U\left(\mathbf{x}^{(\epsilon)}[t]\right)\\
&= \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\epsilon}U(\mathbf{x}^{(\epsilon)}[t]) - \sum_{l=1}^{L}Q_l^{(\epsilon)}[t]x_l^{(\epsilon)}[t]\right)\\
&+ \frac{1}{T}\sum_{t=1}^{T}\sum_{l=1}^{L}Q_l^{(\epsilon)}[t]x_l^{(\epsilon)}[t]\\
&\overset{(b)}{\geq} \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\epsilon}U(\mathbf{r}^*) - \sum_{l=1}^{L}Q_l^{(\epsilon)}[t]r_l^*\right) + \frac{1}{T}\sum_{t=1}^{T}\sum_{l=1}^{L}Q_l^{(\epsilon)}[t]x_l^{(\epsilon)}[t]\\
&= \frac{1}{\epsilon}U(\mathbf{r}^*) + \frac{1}{T}\sum_{t=1}^{T}\sum_{l=1}^{L}Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t] - r_l^*),
\end{aligned} \tag{78}$$

where the step (a) follows from the Jensen's inequality; (b) follows from equation (25). Hence, we have

$$U\left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}^{(\epsilon)}[t]\right) \geq U(\mathbf{r}^*) + \frac{\epsilon}{T}\sum_{t=1}^{T}\sum_{l=1}^{L}Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t] - r_l^*). \tag{79}$$

Next, let's consider $\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-r_l^*)$.

$$\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-r_l^*)$$
$$=\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])+\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*). \tag{80}$$

For $\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)$, we have

$$\begin{aligned}&\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)\\ &=\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)\mathbb{1}_{\{Q_l^{(\epsilon)}[t]\geq b_{l,K_l}\}}\\ &+\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)\mathbb{1}_{\{Q_l^{(\epsilon)}[t]< b_{l,K_l}\}}\\ &\overset{(a)}{\geq}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)\mathbb{1}_{\{Q_l^{(\epsilon)}[t]< b_{l,K_l}\}}\\ &\geq-\sum_{l=1}^{L} Q_l^{(\epsilon)}[t]r_l^*\mathbb{1}_{\{Q_l^{(\epsilon)}[t]< b_{l,K_l}\}}\\ &\geq -L\max_{l\in\mathcal{L}} b_{l,K_l}^2,\end{aligned} \tag{81}$$

where the step (a) follows from equation (26). Thus, we have

$$\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](S_l^{(\epsilon)}[t]-r_l^*)\geq -TL\max_{l\in\mathcal{L}} b_{l,K_l}^2. \tag{82}$$

Next, we consider $\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])$. By using the queue length evolution (27), we have

$$\|\mathbf{Q}^{(\epsilon)}[t+1]\|^2=\|\mathbf{Q}^{(\epsilon)}[t]+\mathbf{x}^{(\epsilon)}[t]-\mathbf{S}^{(\epsilon)}[t]\|^2$$
$$=\|\mathbf{Q}^{(\epsilon)}[t]\|^2+2\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])+\|\mathbf{x}^{(\epsilon)}[t]-\mathbf{S}^{(\epsilon)}[t]\|^2.$$

Thus, we have

$$\begin{aligned}&\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])\\ &=\frac{1}{2}\left(\|\mathbf{Q}^{(\epsilon)}[t+1]\|^2-\|\mathbf{Q}^{(\epsilon)}[t]\|^2\right)-\frac{1}{2}\|\mathbf{x}^{(\epsilon)}[t]-\mathbf{S}^{(\epsilon)}[t]\|^2\\ &\geq\frac{1}{2}\left(\|\mathbf{Q}^{(\epsilon)}[t+1]\|^2-\|\mathbf{Q}^{(\epsilon)}[t]\|^2\right)-\frac{L}{2}(M^2+\max_{l\in\mathcal{L}} b_{l,K_l}^2),\end{aligned} \tag{83}$$

where we use the fact that $\|\mathbf{x}^{(\epsilon)}[t]-\mathbf{S}^{(\epsilon)}[t]\|^2=\sum_{l=1}^{L}(x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])^2 \leq \sum_{l=1}^{L}((x_l^{(\epsilon)}[t])^2+(S_l^{(\epsilon)}[t])^2)\leq L(M^2+\max_{l\in\mathcal{L}} b_{l,K_l}^2)$. Hence, we have

$$\begin{aligned}&\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])\\ &\geq\frac{1}{2}\left(\|\mathbf{Q}^{(\epsilon)}[T+1]\|^2-\|\mathbf{Q}^{(\epsilon)}[1]\|^2\right)-\frac{TL}{2}(M^2+\max_{l\in\mathcal{L}} b_{l,K_l}^2)\\ &\geq-\frac{1}{2}\|\mathbf{Q}^{(\epsilon)}[1]\|^2-\frac{TL}{2}(M^2+\max_{l\in\mathcal{L}} b_{l,K_l}^2).\end{aligned} \tag{84}$$

Thus, by substituting (82) and (84) into (80), we have

$$\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-r_l^*)\geq-\frac{1}{2}\|\mathbf{Q}^{(\epsilon)}[1]\|^2-\frac{TL}{2}(M^2+3\max_{l\in\mathcal{L}} b_{l,K_l}^2).$$

Hence, we have

$$\begin{aligned}&\frac{\epsilon}{T}\sum_{t=1}^{T}\sum_{l=1}^{L} Q_l^{(\epsilon)}[t](x_l^{(\epsilon)}[t]-r_l^*)\\ &\geq-\frac{\epsilon}{2T}\|\mathbf{Q}^{(\epsilon)}[1]\|^2-\frac{\epsilon L}{2}(M^2+3\max_{l\in\mathcal{L}} b_{l,K_l}^2).\end{aligned} \tag{85}$$

Thus, by combining (77), (79) and (85), we have the desired result.

Appendix I
Proof for Lemma 4

Definition 7: (**Invariant Pair:**)
The pair $(\mathbf{Q}^{*(\epsilon)},\mathbf{r}^{*(\epsilon)})$ forms an invariant pair if they satisfy the following conditions:

$$U_l'(r_l^{*(\epsilon)})=\epsilon Q_l^{*(\epsilon)}, \tag{86}$$
$$\mathbf{r}^{*(\epsilon)}\in\underset{\eta\in\mathcal{R}}{\operatorname{argmax}}\sum_{l=1}^{L} Q_l^{*(\epsilon)}\eta_l. \tag{87}$$

By using similar technique as in [4], we can show the existence and uniqueness of $(\mathbf{Q}^{*(\epsilon)},\mathbf{r}^{*(\epsilon)})$. In addition, $\mathbf{r}^{*(\epsilon)}$ is the same for all ϵ and thus $\mathbf{r}^{*(\epsilon)}=\mathbf{r}^*$.

Choose Lyapunov function $V_2(\mathbf{Q}^{(\epsilon)}[t])=\frac{1}{2}\sum_{l=1}^{L}(Q_l^{(\epsilon)}[t]-Q_l^{*(\epsilon)})^2$. By using similar technique as in [4], we have

$$\begin{aligned}\Delta V_2\triangleq&V_2(\mathbf{Q}^{(\epsilon)}[t+1])-V_2(\mathbf{Q}^{(\epsilon)}[t])\\ \leq&-\frac{2\epsilon}{\beta_{\max}}V_2(\mathbf{Q}^{(\epsilon)}[t])+W_1,\end{aligned} \tag{88}$$

where $W_1\triangleq\frac{1}{2}\sum_{l=1}^{L}(x_l^{(\epsilon)}[t]-S_l^{(\epsilon)}[t])^2+L\max_{l\in\mathcal{L}} b_{l,K_l}^2$. Thus, if $V_2(\mathbf{Q}^{(\epsilon)}[t])>\frac{W_1\beta_{\max}}{2\epsilon}$, then, $V_2(\mathbf{Q}^{(\epsilon)}[t+1])<V_2(\mathbf{Q}^{(\epsilon)}[t])$; otherwise, $V_2(\mathbf{Q}^{(\epsilon)}[t+1])$ may be greater than or equal to

$V_2(\mathbf{Q}^{(\epsilon)}[t])$. Thus, if $V_2(\mathbf{Q}^{(\epsilon)}[t]) = \frac{W_1\beta_{\max}}{2\epsilon}$, then

$$\begin{aligned}
2V_2(\mathbf{Q}^{(\epsilon)}[t+1]) &= \sum_{l=1}^{L}(Q_l^{(\epsilon)}[t+1] - Q_l^{*(\epsilon)})^2 \\
&\overset{(a)}{=} \sum_{l=1}^{L}(Q_l^{(\epsilon)}[t] + x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)})^2 \\
&= \sum_{l=1}^{L}(Q_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)})^2 + \sum_{l=1}^{L}(x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t])^2 \\
&\quad + 2\sum_{l=1}^{L}(Q_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)})(x_l^{(\epsilon)}[t] - S_l^{(\epsilon)}[t]) \\
&\overset{(b)}{\leq} \|\mathbf{Q}^{(\epsilon)}[t] - \mathbf{Q}^{*(\epsilon)}\|^2 + \|\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t]\|^2 \\
&\quad + 2\|\mathbf{Q}^{(\epsilon)}[t] - \mathbf{Q}^{*(\epsilon)}\|\|\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t]\| \\
&\leq 2\|\mathbf{Q}^{(\epsilon)}[t] - \mathbf{Q}^{*(\epsilon)}\|^2 + 2\|\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t]\|^2, \qquad (89)
\end{aligned}$$

where (a) uses the fact that there is always no unused service by Assumption 2; (b) follows from Cauchy-Schwarz inequality. Since

$$\|\mathbf{Q}^{(\epsilon)}[t] - \mathbf{Q}^{*(\epsilon)}\|^2 = 2V_2(\mathbf{Q}^{(\epsilon)}[t]) = \frac{W_1\beta_{\max}}{\epsilon} \qquad (90)$$

and

$$\|\mathbf{x}^{(\epsilon)}[t] - \mathbf{S}^{(\epsilon)}[t]\|^2 \leq L(M^2 + \max_{l\in\mathcal{L}} b_{l,K_l}^2), \qquad (91)$$

we have

$$2V_2(\mathbf{Q}^{(\epsilon)}[t+1]) \leq 2\left(\frac{W_1\beta_{\max}}{\epsilon} + L(M^2 + \max_{l\in\mathcal{L}} b_{l,K_l}^2)\right) \triangleq W_2.$$

Thus, we have

$$2V_2(\mathbf{Q}^{(\epsilon)}[t]) \leq W_2, \forall t. \qquad (92)$$

Now, we can give an upper bound for queue lengths at each time. Indeed, we have

$$\begin{aligned}
Q_l^{(\epsilon)}[t] = Q_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)} + Q_l^{*(\epsilon)} &\leq |Q_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)}| + Q_l^{*(\epsilon)} \\
&\leq \sqrt{W_2} + \frac{h_{\max}}{\epsilon}, \forall l, t,
\end{aligned}$$

where the last step follows from $|Q_l^{(\epsilon)}[t] - Q_l^{*(\epsilon)}| \leq \sqrt{2V_2(\mathbf{Q}^{(\epsilon)}[t])} \leq \sqrt{W_2}$ and $Q_l^{*(\epsilon)} = \frac{U_l'(r_l^*)}{\epsilon} \leq \frac{h_{\max}}{\epsilon}$. Since $W_1 \leq \frac{1}{2}L(M^2 + \max_{l\in\mathcal{L}} b_{l,K_l}^2) + L\max_{l\in\mathcal{L}} b_{l,K_l}^2$, we have

$$W_2 \leq \left(\frac{\beta_{\max}}{\epsilon} + 2\right)LM^2 + \left(\frac{3\beta_{\max}}{\epsilon} + 2\right)L\max_{l\in\mathcal{L}} b_{l,K_l}^2 \triangleq W. \qquad (93)$$

Thus, we have

$$Q_l^{(\epsilon)}[t] \leq \sqrt{W} + \frac{h_{\max}}{\epsilon} \triangleq G^{(\epsilon)} \qquad (94)$$