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TOWARDS A BETTER UNDERSTANDING OF MODELING FEASIBILITY ROBUSTNESS IN ENGINEERING DESIGN

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ABSTRACT

In robust design, it is important not only to achieve robust design objectives but also to maintain the robustness of design feasibility under the effect of variations (or uncertainties). However, the evaluation of feasibility robustness is often a computationally intensive process. Simplified approaches in existing robust design applications may lead to either over-conservative or infeasible design solutions. In this paper, several feasibility-modeling techniques for robust optimization are examined. These methods are classified into two categories: methods that require probability and statistical analyses (i.e., the probabilistic feasibility formulation and the moment matching method) and methods do not require probability and statistical analyses (i.e., the worst case analysis, the corner space evaluation, and the variation pattern method). Using illustrative examples, the effectiveness of each method is compared in terms of its efficiency and accuracy. Constructive recommendations are made to employ different techniques for modeling feasibility robustness under different circumstances. Under the framework of probabilistic robust optimization, we propose to use a most probable point (*MPP*) based importance sampling method, a method rooted in the field of reliability analysis, for evaluating the feasibility robustness. The advantages of this approach are discussed. Though our discussions are centered on robust design, the principles presented are also applicable for general probabilistic optimization problems. The practical significance of this work also lies in the development of efficient feasibility evaluation methods that can support quality engineering practice, such as the Six Sigma approach that is being widely used in American industry.

NOMENCLATURE

b	width of beam cross-section
cdf	cumulative distribution function
$F.$	a cdf
$f.$	a pdf
g	a constraint function
h	height of beam cross-section
I	indicator function
J	number of constraints
L	length of the beam
MPP	Most Probable Point
m	number of parameters
n	number of variables
P	probability
p	vector of design parameters
P_0	desired probability of constraint satisfaction
pdf	probability density function
Q	external force on the beam
R	allowable stress of the beam
S_{max}	maximum tensile stress of beam
s	cross-section area of beam
T	tolerance space
x	vector of design variables
x_l	lower bound of x
x_u	upper bound of x
Y	vector of random design variables and parameters
U	vector of basic variables in standard normal space
U^*	*MPP* in standard normal space
v	importance density function
W	corner space

F	cdf of standard normal distribution
b	safety index
m	mean value
s	standard deviation

1. INTRODUCTION

Deterministic optimization techniques have been successfully applied to a large number of engineering design problems. However, it is generally recognized that there always exist uncertainties in any engineering systems due to variations in design conditions, such as loading, material properties, physical dimensions of parts, and operating conditions. With the introduction of the integrated product and process development (IPPD) paradigm, manufacturing variations could be considered as another contributing source of uncertainty in the product design stage. Deterministic approaches do not consider the impact of such variations and as a result, the design solution may be very sensitive to the variations. Moreover, deterministic optimization lacks the ability to achieve specified levels of constraint satisfaction (such as under reliability considerations). Therefore, a design based on the deterministic factor of safety may be under-designed (infeasible) or over-conservative.

Robust design, originally proposed by G. Taguchi (Taguchi, 1993), is a probabilistic-based design method for improving the quality of a product through minimizing the effect of the causes of variation without eliminating the causes (Phadke, 1989). Although Taguchi's robust design principle has been widely accepted, the methods Taguchi offers have received much criticism (Chen et al. 1996a), including the limitation of not being able to consider design constraints. In recent years, the advancement of robust design methods in the design community has produced nonlinear programming based robust design methods that can be used in a variety of applications (Otto and Antonsson, 1991; Parkinson et al., 1993; Sundaresan et al., 1993; Cagan and Williams, 1993; Eggert and Mayne, 1993; Chen et al.1996a; Su and Renaud, 1997). With the introduction of the nonlinear programming framework to robust design, both the robustness of design objectives as well as the robustness of design constraints can be considered. It is generally recognized that the robustness of a design objective can be achieved by simultaneously "optimizing the mean performance" and "minimizing the performance variance". Modeling the tradeoff between these two aspects has been widely studied in the literature (Sudaresan, et al., 1993; Bras and Mistree, 1995; Chen, et al., 1996a; Iyer and Krishnamurty, 1998). In recent developments, a multiobjective mathematical programming approach has been proposed (Chen, et al., 1998) to overcome the limitations of Taguchi's signal-to-noise ratio approach and the simplistic weighted-sum method. In general, objective robustness is an issue related to how to better model a designer's preference structure when making tradeoffs between the mean and variance attributes.

No matter what objective expression we use to achieve the robustness of product performance, it is even more critical to maintain the design feasibility under variations (uncertainties). For example, for a key structural component, satisfying strictly its strength constraint (or reliability) subject to random parameters is more important than achieving the robustness of the design objective, e.g., minimizing the weight. This raises the question: *how can we describe the design feasibility under the effect of variations to maintain the feasibility robustness*? Moreover, as we will discuss later in details, depending on the formulation, the evaluation of feasibility robustness could become a very complicated and time-consuming process. This leads to another question: *what kind of constraint model should we adopt to ensure the accuracy in evaluating levels of constraint satisfaction with an acceptable computational efficiency*?

Although alternative approaches, such as the probabilistic feasibility analysis (Eggert 1991), the moment matching method (including the use of Taylor expansion) (Parkinson, et al., 1993), the worst case analysis (Parkinson, et al., 1993; Sundaresan, et al., 1995), the method of corner space evaluation (Sundaresan, et al., 1993), and the variation patterns method (Yu and Ishii, 1998), have been proposed to model feasibility robustness, it is not clear the effectiveness of each individual method in terms of its efficiency and accuracy. Koch et al. (1998) compared three methods (Taylor expansion, design of experiments (DOE)-based Monte Carlo simulation, and Taguchi's product array) for predicting performance variance. However, their study focused on only the evaluation of performance variance rather than the overall level of constraint satisfaction. Due to the lack of guidelines in the area of evaluating feasibility robustness, simplistic approaches such as the first order Taylor expansion and the worst case analysis are often used in existing applications.

Our aim in this paper is to conduct an in depth analysis of the existing feasibility-modeling techniques in robust design and compare these methods using illustrative examples. We will show that, although some of these approaches are easy to use, they may lead to either over-conservative or infeasible design solutions in robust design applications. Constructive recommendations are made to employ different techniques for modeling feasibility robustness under different circumstances. To improve the accuracy and efficiency in evaluating the probability of constraint satisfaction, we propose to use a most probable point (*MPP*) based importance sampling method, a method rooted in the field of reliability analysis, for evaluating the feasibility robustness. The advantages of this approach and the directions of future improvement are discussed.

This paper is organized as follows. In Section 2, the existing methods for feasibility modeling in robust design are analyzed. The feasibility analysis and the comparison of these methods are discussed in detail by illustrative examples in Section 3. In Section 4, a most probable point (*MPP*) based importance sampling method is introduced and the relevant issues of utilizing it for the evaluation of feasibility robustness are discussed. Section 5 is the closure of this paper.

2. EXISTING APPROACHES FOR MODELING FEASIBILITY ROBOUSTNESS

2.1 Objective Robustness and Feasibility Robustness

Before reviewing the existing approaches for modeling feasibility robustness, we first explain the roles of two major robustness issues involved in robust design problems: *objective robustness* and *feasibility robustness*. We consider an engineering design problem stated using the conventional optimization model in Eqn. (2.1):

$$\begin{aligned} &\text{minimize } F(x, p) \\ &\text{subject to } \mathrm{g}_{\mathrm{j}}(x, p) \geq 0, \qquad \mathrm{j} = 1, 2, ..., \mathrm{J} \qquad (2.1) \\ &x_l \leq x \leq x_u, \end{aligned}$$

where $x = [x_1, \cdots x_n]^T$ is a vector of design variables and $p = [p_1, \cdots p_m]^T$ is a vector of design parameters whose values are fixed as a part of the problem specifications. In robust optimization, both design variables and design parameters could be the contributing sources of design variations. Consequently, the system performance *F(x, p)* is a random function. Both its mean value $m_F(x, p)$ and variance $s_F^2(x, p)$ are expected to be minimized. The general form of the objective can be expressed as

$$\min \quad [m_F(x, p), s_F(x, p)] \qquad (2.2)$$

In a deterministic optimization as shown in Eqn. (2.1), those design points that satisfy all the constraint equations define the feasible region. This is a go or no-go problem, either yes or no, and the limit-state of feasibility or unfeasibility is distinguished. In robust design, however, the problem needs to be converted into a consideration of the degree of feasibility between yes or no. According to Parkinson et al. (1993), a design is described to have "feasibility robustness", *if it can be characterized by a definable probability, set by designers, to remain feasible relative to the nominal constraint boundaries as it undergoes variations.* It is obvious that, compared with the deterministic feasible region, the size of the feasible region will be reduced under the robustness consideration. In addition, based on the above definition, we note that the degree of feasibility can be defined by the desired level of probability chosen by the decision maker. In the following sections, several existing feasibility modeling methods are analyzed. These methods are classified into two categories: methods that require probability and statistical analyses and those do not require such analysis.

2.2 Methods Requiring Probability and Statistical Analyses

The Probabilistic Feasibility Formulation

Under the definition of feasibility robustness in Section 2.1, feasibility in robust design can be considered as the probability of events, that constraints are satisfied, should be greater than the user specified probability. This will ensure that the desired degree of constraint satisfaction is achieved exactly so as to avoid over-designed or under-designed situations. A general probabilistic feasibility formulation can be expressed as follows:

$$P[g_j(x, p) \geq 0] \geq P_{oj} \qquad j = 1, \cdots, J \qquad (2.3)$$

where P_{oj} is the desired probability for satisfying constraint *j*. If the distributions of all the variables *x* and parameters *p* are known, the probability *P* in Eqn. (2.3) can be obtained accurately by the following integral:

$$P[g_j(x, p) \geq 0] = \int_{g_j(x,p)\geq 0} f_{xp}(x, p) dx dp \qquad (2.4)$$

where $f_{xp}(x, p)$ is the joint probability density function (pdf) of *x* and *p*.

Practically, it is very difficult or even impossible to get an analytical solution of the above equation because of the multi-dimensional integration and the complicated integral region. Only if the distribution of $g_j(x, p)$ is known, the probability can then be simplified into the following one-dimensional integral:

$$P[g_j(x, p) \geq 0] = \int_0^\infty f_{gj}(g_j) dg_j \qquad (2.5)$$

where $f_{gj}(g_j)$ is the pdf of $g_j(x, p)$. For several typical variable distributions (for example, Normal and Lognomal), when used for simple constraint functions and low-dimensional problems, the analytical expression of the probability can be derived (Eggert, 1991).

In the case that the analytical method is not applicable, simulation-based approaches, such as Monte Carlo simulations, are often used to obtain a more accurate estimation of the probability. The estimation of the probability is expressed as:

$$P[g_j(x,p) \ge 0] = \int_{all \quad x,p} I[g_j(x,p)] f_{xp}(x,p) dxdp$$

$$= \frac{1}{N} \sum_{i=1}^{N} I[g_j(x_i, p_i)] \quad (2.6)$$

where N is the simulation size, x_i and p_i are samples of x and p, and $I(\cdot)$ is an indicator function defined as

$$I[g_j(x,p)] = \begin{cases} 1 & if \quad g_j(x,p) \ge 0 \\ 0 & otherwise \end{cases} \quad (2.7)$$

Compared with other approximate methods, simulation methods are flexible for any types of distributions and any forms of constraint functions. Neglecting the algorithmic error caused by simulation, if a sufficient number of simulations are used, simulation methods often result in solutions with a high accuracy. However, when the desired level of probability for constraint satisfaction is very high (approaching to 1.0), the computational burden may not be affordable.

The Moment Matching Formulation

To reduce the computational burden associated with the probabilistic feasibility evaluation, simplistic approaches are widely used in the literature. One of these approaches is the moment matching method (Parkinson et al. 1993). The title of this method comes from the fact that it uses the first and second moments (mean and variance) of statistical distributions. With this approach, $g_j(x,p)$ is assumed to be normally distributed. The probability of the event $g_j(x,p) \ge 0$ becomes:

$$P[g_j(x,p) \ge 0] = \Phi(\frac{m_{gj}}{s_{gj}}) \quad (2.8)$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of a standard normal distribution, m_{gj} and s_{gj} are the mean value and the standard deviation of $g_j(x,p)$, respectively. The constraint can then be written as:

$$m_{gj} - k_j s_{gj} \ge 0 \quad (2.9)$$

where $k_j = \Phi^{-1}(P_{0j})$ and $\Phi^{-1}(\cdot)$ is the inverse function of the cdf of a standard normal distribution. For example, $k_j = 2$ stands for $P_{0j} = 0.9772$ and $k_j = 3$ means $P_{0j} = 0.9987$.

Several methods could be used to evaluate m_{gj} and s_{gj}. A simplistic approach is to use Taylor series approximations of the constraint function $g_j(x,p)$ at the mean values of x and p. The mean value and the variance of $g_j(x,p)$ are estimated as

$$m_{gj} = g_j(m_x, m_p) \quad (2.10a)$$

$$s_{gj}^2 = \sum_{i=1}^{n} \left[\frac{\partial g_j}{\partial x_i}\right]_{m_x m_p}^2 s_{xi}^2 + \sum_{i=1}^{m} \left[\frac{\partial g_j}{\partial p_i}\right]_{m_x m_p}^2 s_{pi}^2 \quad (2.10b)$$

Based on Eqns. (2.9) and (2.10), the feasibility formulation can be expressed as:

$$g_j(m_x, m_p) - k_j \sqrt{\sum_{i=1}^{n} \left[\frac{\partial g_j}{\partial x_i}\right]_{m_x m_p}^2 s_{xi}^2 + \sum_{i=1}^{m} \left[\frac{\partial g_j}{\partial p_i}\right]_{m_x m_p}^2 s_{pi}^2} \ge 0 \quad (2.11)$$

2.3 The Methods Not Requiring Probability and Statistical Analyses

The Worst Case Analysis

Worst case analysis is another simplistic approach to the evaluation of feasibility robustness in robust design. It is applicable to general robust design problems including those in which the distributions of random variables are not given. The worst case analysis (Parkinson et al., 1993) assumes that all fluctuations may occur simultaneously in the worst possible combinations. The effect of variations on a constraint function is estimated from a first order Taylor's series as follows:

$$\Delta g_j(x,p) = \sum_{i=1}^{n} \left|\frac{\partial g_j}{\partial x_i} \Delta x_i\right| + \sum_{i=1}^{m} \left|\frac{\partial g_j}{\partial p_i} \Delta p_i\right| \quad (2.12)$$

By subtracting $\Delta g_j(x,p)$ from $g_j(x,p)$ to maintain the feasibility, the constraint becomes:

$$g_j(m_x, m_p) - (\sum_{i=1}^{n} \left|\frac{\partial g_j}{\partial x_i} \Delta x_i\right|_{m_x m_p} + \sum_{i=1}^{m} \left|\frac{\partial g_j}{\partial p_i} \Delta p_i\right|_{m_x m_p}) \ge 0 \quad (2.13)$$

In most cases, the worst case analysis is almost always conservative because it is unlikely that the worst cases of variable or parameter deviations will simultaneously occur. On the other hand, the estimation using Taylor expansion is not as accurate as identifying the extreme conditions such as the minimum and maximum of the performance within the given intervals of variations. However, due to its simplification, worst case analysis is used widely in robust optimization applications.

The Corner Space Evaluation

Following the similar idea of the worst case analysis, Sundaresan et al. (1995) presented the method of corner space evaluation. Identical to the worst case analysis, their method does not require the descriptions of the distributions of random variables. What is different is that, with their approach, the variations on design variables are not transmitted into constraint functions as the way in the worst case analysis.

Assume that the design variables have nominal values x and a tolerance Δx. The *tolerance space* (T) is defined as a set of points close to the target design point where each point represents a possible combination of design variables due to uncertainties in each variable:

$$T(x_t) = \{x_t : |x_t - x| \le \Delta x\} \quad (2.14)$$

The corner space (W) consists only of corner vertices of a tolerance space:

$$W(x_t) = \{x_t : |x_t - x| = \Delta x\} \quad (2.15)$$

To maintain the design feasibility, the nominal value x should be inside the feasible region. This can be achieved by keeping the corner space always touching the original constraint (expressed by x and p) boundary. Fig. 1 shows the feasibility of a two- dimension problem with this approach.
With this approach, the constraint can be stated as:

$$Min\{g_j(x), \quad \forall x \in W(x)\} \ge 0 \quad j = 1, \cdots J \quad (2.16)$$

If the distributions of variables of interest are known, the tolerance Δx can be determined by a prescribed confidential level. For example, for a normally distributed random variable, the tolerance can be chosen as three standard deviations under the confidential coefficient of 99.87%. This method does not require the calculation of the partial differential of the constraint function, so it is very easy to use. However, the overall probability of constraint satisfaction is not evaluated as the result of this procedure even though the tolerance is under confidential consideration.

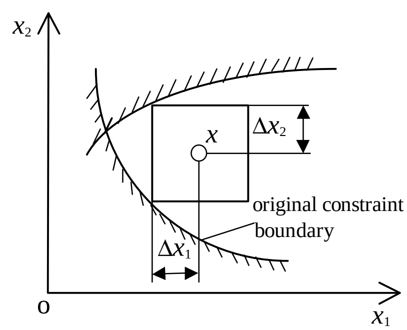

Figure 1. Feasibility under the Corner Space Evaluation Method

The Variation Patterns Formulation

In the same category of the corner space evaluation, Yu and Ishii (1998) presented an improved method named Manufacturing Variation Patterns (*MVP*) analysis based on the consideration that the manufacturing errors may be correlated with each other, for example the correlation among dimensional errors in typical manufacturing processes. Since the approach is not restricted to manufacturing related problems only, a general title "variation patterns formulation" is given. With their approach, *MVP(1-a)* denotes the space of possible variable combinations at the confidence coefficient of 1-α, where α indicates the probability of design variable distribution outside the variation pattern. The shape of the pattern is determined by the variable distributions and the size of the pattern is determined by the confidence coefficient. For example, for the problem with two normally distributed dependant variables, the shape of the pattern is an ellipsoid as shown in Fig. 2.

Under this concept, the constraint is formulated as:

$$g_j(x, p) \ge 0, \quad \forall x \in MVP(1-a), \quad j = 1, \cdots J \quad (2.17)$$

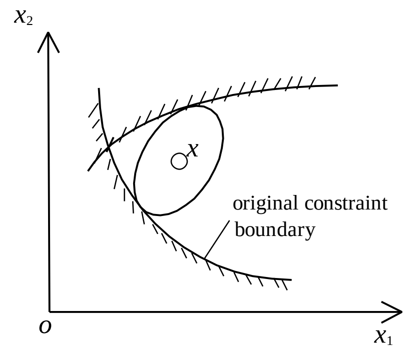

Figure 2. Variation Pattern Analysis Method

It is obvious that the process of searching for the robust design solution is quite complicated if the shape of the pattern is irregular. According to Yu and Ishii (1998), the details of the application procedure of this method still await future investigation.

3. A COMPARISON OF ALTERANTIVE TECHNIQUES

In this section, the feasibility-modeling techniques analyzed in Sections 2.2 and 2.3 for robust optimization are compared using two illustrative examples. One example only illustrates the differences in feasibility evaluation when using different approaches. The other used to illustrate the impact on both feasibility evaluation and the final robust design solution. Constructive recommendations are made to employ different techniques for modeling feasibility robustness under different circumstances.

3.1 A Mathematical Example

In this example, we consider a simple linear constraint that involves only two design variables x_1 and x_2, both are normally distributed and represented as $x_1 \sim N(m_1, s_1)$ and $x_2 \sim N(m_2, s_2)$, where $s_1 = c_1 m_1$, $c_1 = 0.2$ and $s_2 = 0.25$. Note that the standard deviation of x_1 is considered as a function of its mean in this case. The original constraint function is given as

$$g(x) = x_1 - x_2 \quad (3.1)$$

The design variables in optimization are the mean values m_1 and m_2 of x_1 and x_2.

Deterministic situation

When not considering uncertainties, the constraint function can be expressed as

$$g(x) = m_1 - m_2 \quad (3.2)$$

The constraint curve is a line through the origin (see Fig. 3).

The Probabilistic Feasibility Formulation

Based on the discussion in Section 2.2, the probabilistic feasibility formulation in Eqn. (2.3) is represented here by Eqn. (3.3), given the desired probability of constraint satisfaction P_0 as 99.98%, i.e.

$$P[g(x) \geq 0] = P[x_1 - x_2 \geq 0] \geq P_0 = 0.9998 \quad (3.3)$$

Since $g(x)$ is the difference of two normal variables, $g(x)$ is also normally distributed with its mean value being $m_1 - m_2$ and its variance being $s_1^2 + s_2^2$, the probability can be further calculated as:

$$P[g(x) \geq 0] = \Phi(\frac{m_1 - m_2}{\sqrt{s_1^2 + s_2^2}}) \quad (3.4)$$

Eqn. (3.4) is an exact expression of the achievable probability of constraint satisfaction.

Based on Eqn. (3.4), Eqn. (3.3) can be written as the following:

$$m_1 - m_2 - \Phi^{-1}(P_0)\sqrt{c_1^2 m_1^2 + s_2^2} \geq 0 \quad (3.5)$$

From Fig. 3, it is noted that the probabilistic feasibility has resulted in a reduced feasible region compared to the deterministic constraint.

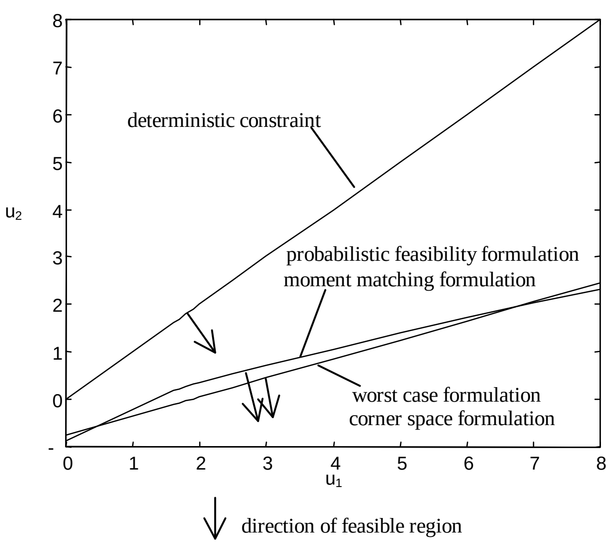

Figure 3. Comparisons of Feasibility Curves

The Moment Matching Formulation

The moment matching formulation in Eqn. (2.11) can be expressed for the example as:

$$m_1 - m_2 - \Phi^{-1}(P_0)\sqrt{c_1^2 m_1^2 + s_2^2} \geq 0 \quad (3.6)$$

Eqn. (3.6) is the same as the probabilistic feasibility formulation in Eqn. (3.5). From this example, we can see that if the constraint function is normally distributed under the effect of variations, the moment matching formulation can give an exact estimation of the level of constraint satisfaction just as that from the probabilistic feasibility formulation.

The Worst Case Formulation

Based on the discussion of the worst case formulation in Section 2.4, for $\Delta x_1 = 3s_1$ and $\Delta x_2 = 3s_2$, the constraint formulation in Eqn. (2.13) can be expressed as:

$$m_1(1 - 3c_1) - m_2 - 3s_2 \geq 0 \quad (3.7)$$

From Fig. 3, we note that the use of this constraint formulation is over-conservative over the majority of the design space. However, the problem becomes under-constrained either near the origin or when m_1 is bigger than about 6.7 where the probability of constraint satisfaction is less than the expected probability. We can conclude from this example that though the worst case analysis is widely considered as a conservative approach for modeling feasibility robustness, we should use it with caution because the violation of constraints is still possible over certain design regions.

The Corner Space Formulation

Based on the introduction of this method in Section 2.5, we could set $\Delta x_1 = 3s_1$ and $\Delta x_2 = 3s_2$, which indicate that the confidential coefficient is 99.87%. By keeping the rectangle with dimensions of $2\Delta x_1 \times 2\Delta x_2$ touching the deterministic constraint curve, we obtain the locus of the centroid of the rectangle which strands for the position of the constraint limit. For the special linear function $g(x) = x_1 - x_2$, the constraint curve obtained by the corner space formulation is the same as the one from the worst case formulation, due to the fact that the first-order Taylor expansion under the worst case approach provides an accurate evaluation for a linear function. However, the confidence coefficient chosen for the variables and parameters cannot be used to estimate the probability of constraint satisfaction. The discrepancies between the two values are representatively presented in Table 1 for a set of k values at the point where $m_1 = 5$.

Table 1 Discrepancies between the Confidential Level of Parameters and the Probability of Constraint Satisfaction

Number of Standard deviation k ($\Delta x = ks$)	Confidential Level	Probability of constraint satisfaction
1	0.8413	0.8874
2	0.9772	0.9924
3	0.9987	0.9999
3.5	0.9998	0.999989

3.2 Design of a Cantilever Beam

In this section, an engineering design problem is used to further illustrate the differences between the existing approaches for modeling feasibility robustness and their impacts on final robust design solutions. The cantilever beam in Fig. 4 is designed against yielding due to bending stress while the cross-sectional area is desired to be kept as minimum. Five random variables are involved in this problem, including two design variables: $x=[x_1,x_2]^T=[b,h]^T$ and three design parameters $p=[p_1,p_2,p_3]^T=[R,Q,L]^T$. b and h are the dimensions of the cross-section, L is the length of the beam, and b, h, and L are all normally distributed. Q is the external force with an extreme value distribution, and R is the allowable stress of the beam with a Weibull distribution.

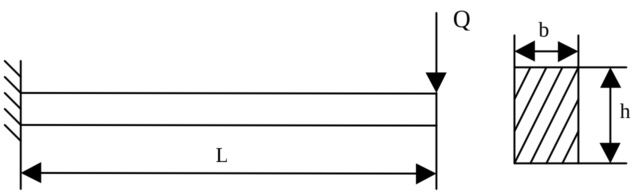

Figure 4. Cantilever Beam

Distribution parameters for each random variable are described in Table 2. In robust design, the variables to be determined are the mean values (m_{x1} and m_{x2}) *of b* and *h.*

Table 2 Distributions of Random Variable

Name	Symbol	Mean Value	Standard Deviation	Distribution Type
R	p_1	200Mpa	200Mpa	Two-parameter Weibull
Q	p_2	20KN	2KN	Extreme Value Distribution
L	p_3	0.2m	1.0mm	Normal
b	x_1	μ_{x1}	0.05mm	Normal
H	x_2	μ_{x2}	0.05mm	Normal

The maximum tensile stress is calculated as

$$S_{\max}=\frac{QLh/2}{I}=\frac{QLh/2}{bh^3/12}=\frac{6QL}{bh^2} \tag{3.8}$$

The strength requirement can then be defined by the following constraint:

$$g(x,p)=R-\frac{6QL}{bh^2}=p_1-\frac{6p_2p_3}{x_1x_2^2}\geq 0 \tag{3.9}$$

If not considering the uncertainties, all the random variables/parameters can be presented by their mean values and the constraint can then be formulated in terms of the mean values of variables/parameters as follows:

$$g(x,p)=m_R-\frac{6m_Qm_L}{m_{x1}m_{x2}^2}\geq 0 \tag{3.10}$$

If the prescribed probability of this constraint satisfaction P_0 is 99.95%, the Probabilistic Feasibility Formulation can be written as:

$$P[p_1-\frac{6p_2p_3}{x_1x_2^2}\geq 0]\geq P_0=99.95\% \tag{3.11}$$

By keeping the probability at 99.95% and varying the combinations of m_{x1} and m_{x2}, we obtain the position of the constraint curve (see Fig. 5). The feasibility direction is also indicated on the Fig. 5.

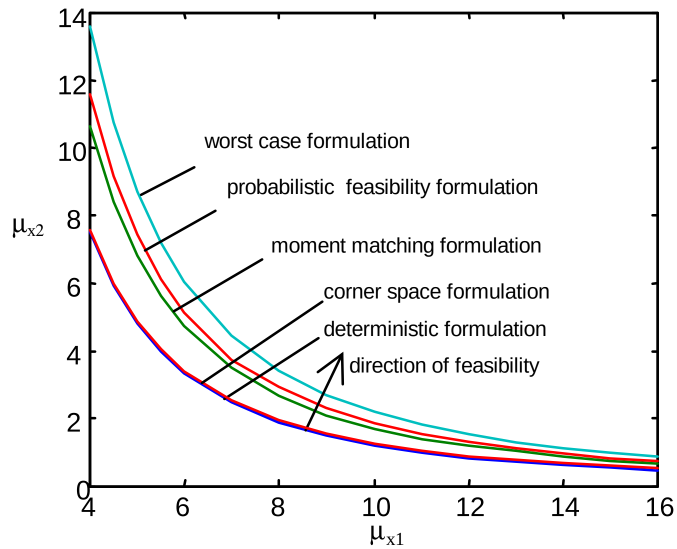

Figure 5. Comparisons of Feasibility Analyses

The constraint curve obtained based on the moment matching formulation shown in Eqn. (2.11) is also plotted on the same figure. It is noted that its location is different from that obtained from the accurate probability feasibility analysis. This is true due to the nonlinearality of the constraint function which follows a nonnormal distribution. Therefore for this particular problem, the moment matching method provides an under-constrained formulation.

When using the worst case analysis by assuming $\Delta x=3s_X$ and $\Delta p=3s_P$, we find from Fig. 5 that the formulation generates conservative results, especially over the design region where m_{x2} is large and m_{x1} is small. As for the corner space formulation and the variation pattern formulation, because the variations of the design variable x_1 and x_2 are very small, the obtained constraint curves (the curve of variation pattern not shown in Fig. 5 are very close to the deterministic ones. This indicates that the feasibility robustness evaluated by these two methods is not reliable for this particular example.

In terms of the objective of keeping the cross-sectional area the minimum, the cross-sectional area can be expressed as

$$s = x_1 x_2 \quad (3.12)$$

In robust design, the objective robustness is achieved by minimizing both the mean value and the variance of the cross-sectional area:

$$m_s = m_{x1} m_{x2} \quad (3.13)$$

$$s_s^2 = m_{x1}^2 s_{x2}^2 + m_{x2}^2 s_{x1}^2 + s_{x1}^2 s_{x2}^2 \quad (3.14)$$

For feasibility robustness, we expect that the strength constraint should be satisfied exactly with the probability of satisfaction of 99.95% and the ratio of *h/b* should be less than 2. Because the first constraint is very important, we consider it as a critical constraint (with high priority) and formulate it using the probabilistic feasibility formulation (see Eqn. 3.16). For the constraint on ratio relationship, the probability of constraint satisfaction is not so strict and so we use the nominal of *b* and *h* to express the constraint function (see Eqn. 3.17). In this problem, we need to decide the mean value (m_{x1} and m_{x2}) of *b* and *h*. The robust optimization model can be stated as:

Find: mean value (m_{x1} and m_{x2}) of *b* and *h*

$$\min \quad F(x, p) = w_1 m_S / m_s^* + w_2 s_S / s_s^* \quad (3.15)$$

$$\text{s.t.} \quad P[p_1 - \frac{6 p_2 p_3}{x_1 x_2^2} \ge 0] \ge P_0 = 99.95\% \quad (3.16)$$

$$2 - \frac{m_{x2}}{m_{x1}} \ge 0 \quad (3.17)$$

where m_s^* and s_s^* are the best achievable optimal solution of m_s and s_s, respectively. Here we use the weighting factor method to formulate the multiple objective function. For the purpose of illustration, we use weighting factors w_1=w_2=0.5. The optimal solutions are shown in Table 3 and Fig. 6. Other solutions with different formulations of the first constraint are also provided for comparison. The solution from the probabilistic feasibility formulation is obtained by the *MPP*-based sampling method (details see Section 4).

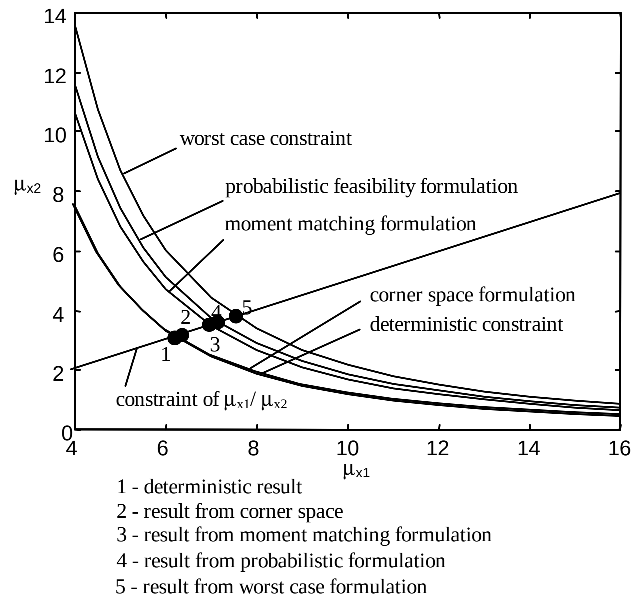

Figure 6. Results of the Beam Example

Table 3 Solutions of the Beam Example

Formulation method	(m_{x1},m_{x2}) (mm)	Probability of strength constraint satisfaction	Mean value of cross-sectional area m_S (mm^2)	Standard deviation of cross-sectional area s_s (mm)
Probabilistic feasibility formulation	(39.25,68.71)	0.9995	2596.87	3.9565
Moment matching formulation	(34.91,69.81)	0.9852	2437.07	3.9026
Worst case formulation	(37.88,75.77)	0.99993	2870.17	4.2356
Corner space formulation	(31.22, 62.43)	0.5433	1949.06	3.4901
Conventional deterministic optimization	(31.07, 62.14)	0.5360	1930.69	3.4737

Even though deterministic optimization generates the lowest values of mean and variance of the cross-sectional area, its feasibility is the worst with only 53.6% possibility of constraint satisfaction. The worst case formulation obtains the most conservative result with the probability greater than the specified limit. On the other hand, the moment matching method obtains a solution with its actual probability of constraint satisfaction equal to 0.9852, which is less than the specified limit. From the viewpoint of reliability, this means the beam will have a higher risk of failure than expected and this may lead to safety problems. The corner space formulation gives a solution that is very close to that from the deterministic formulation with slightly higher probability of constraint satisfaction. This is due to the fact that with the corner space formulation, we can only consider the deviations of design variables (tolerances) (x_1, x_2) but cannot introduce the deviations of design parameters (p_1, p_2, p_3).

3.3 A Summary of Comparisons

Based on the two example problems presented in Sections 3.1 and 3.2, the features of various existing methods for modeling feasibility robustness are summarized and compared in Table 4. We have considered various attributes in this comparison, such as whether the constraint function requires statistical evaluation, whether the description of uncertainty distribution has to be given, how the performance distributions are described, and whether the calculation of partial differential of the function is needed, etc. The number of function evaluations required, the capacity and accuracy of each method are also summarized. In summary, if neglecting the computational effort, the probabilistic feasibility formulation is the ideal method to describe the feasibility robustness that can ensure the solution achieve an accurate level of constraint satisfaction. For simple constraint functions, adopting this formulation will lead to a quick solution. However, in general, it is very time-consuming and difficult to evaluate the probability of constraint satisfaction. If the calculation cost is more concerned by designers, the alternative formulations, such as the moment matching formulation should be considered. Moment matching formulation provides an accurate estimation of the probability when the constraint function is linear and the variables are normally distributed or when the functions are nonlinear but normally distributed. The moment matching formulation is much more computationally efficient compared to the probabilistic feasibility formulation. On the other hand, we should pay attention to the fact that when using different mathematical structures for the same constraint function, different results may be obtained by the moment matching method due to the differences exist in the first-order Taylor's expansion (Chen and Weng, 1998). The methods in the category of "not requiring probability and statistical analyses" suit better the problems in which the distributions of variables and parameters are not available. The worst case formulation is a good selection under this situation. Though the worst case analysis is widely considered as a conservative approach for modeling feasibility robustness, we should use it with caution since the violation of the constraint is still possible over certain design regions.

To avoid statistical analysis or the evaluation of partial differential of constraint functions, the methods of Corner Space Formulation and Variation Pattern can be adopted. The accuracy of these methods depends on whether the constraint function is monotonic with respect to all design variables in the tolerance space and whether the tolerance of design variables are the only source of variation. One limitation is that these two methods do not provide the information on the probability (level) of constraint satisfaction.

4. AN EMERGING METHOD FOR PROBABILISTIC FEASIBILITY EVALUATION

From the preceding discussions, we note that the probabilistic feasibility formulation is the ideal method to describe the feasibility robustness, as it can ensure the solution achieve an accurate level of constraint satisfaction when the distributions of parameter variations can be described by designers. However, due to the reasons explained earlier, probabilistic feasibility formulation could be a very difficult as well as time-consuming task, especially when this becomes a part of an iterative optimization process. The issue becomes: *how can we develop an affordable probabilistic feasible evaluation technique so that the probabilistic robust optimization framework can be used more widely in robust design practices?*

We propose to introduce a most probable point (*MPP*) based importance sampling method into the process of evaluating the feasibility robustness. The *MPP* method was originally developed in the field of reliability analysis (Wu, 1990) and has caused more and more attention in recent implementations of probabilistic optimization (Maglaras**,** et al., 1996). We find the same principle can be applied to evaluate the feasibility robustness in robust design problems. The advantages of this approach are discussed here.

Table 4 Comparisons of Feasibility Modeling Techniques

	Probabilistic Feasibility Formulation	**Moment Matching Formulation**	**Worst Case Analysis**	**Corner Space Evaluation**	**Variation Pattern**
Require statistical evaluation of constraint function	Yes	Yes	No	No	No
Description of uncertainty distribution	Necessary	Not necessary	Not necessary	Not necessary	Necessary
Description of constraint performance distribution	Yes	Only mean value and standard deviation	Extreme values	Extreme values	Extreme values
Deal with correlation	Yes	No	No	No	Yes
Calculation of partial differential of function	May or may not	Yes	Yes	No	No
Number of constraint function evaluation (*N*)	Methods includes *MPP* searching and simulation, etc. In general, *N* is very large.	Evaluation includes mean and variance (function differentiation). $N=m+n+1$.	Evaluation includes mean and variance (function differentiation). $N=m+n+1$.	Evaluation includes calculating function values at the "corners". $N=2^n$.	Evaluation involves searching the tangent point of $MVP(1-a)$ with the original constraint boundary. *N* depends on the shape of the variation pattern.
Capability and accuracy	Gives exact probability estimation; Solve complicated problems; Difficult to get analytical solution; Needs great computational effort especially when simulations are involved.	Gives exact probability estimation for normally distributed functions; Provides approximations for other problems; The accuracy of result is sensitive to the mathematical structure of the constraint.	Simple to use; Low estimation accuracy; In most cases, gives over- conservative results.	Simple to use; Calculation amount increases with variable dimension increasing. Doesn't provide the probability (level) of constraint satisfaction.	More accurate than the corner space method; Complicated to use in the process of optimization; Doesn't provide the probability (level) of constraint satisfaction.

For simplicity, we call both the random design variables and random parameters as basic random variables and use the vector $Y=[y_1,\cdots y_k]^T$ to denote them. The constraint can then be written as

$$g(Y)\ge 0\,. \qquad (4.1)$$

We assume $y_i\,(i=1,\cdots,k)$ are mutually independent and their probability density functions are $f_i(y_i)$ and their cumulative distribution functions are $F_i(y_i)$. Two steps are followed to calculate the probability of $P[g(Y)\ge 0]$. The first step is to search the so-called most probably point (*MPP*), and then in the second step, probability is calculated by the importance sampling around *MPP*.

The *MPP* method uses the properties of standard normal space. The basic random variables *Y* are transformed into standard, uncorrelated, and normal variables $U=[u_1,\cdots u_k]^T$. The transformation is given by Rosenblatt transformation (Rosenblatt, 1952) as

$$u_i=\Phi^{-1}[F(y_i)] \qquad (4.2)$$

Eqn. (4.1) can now be rewritten as

$$g(U)\ge 0\,. \qquad (4.3)$$

In the transformed *U* space, the *MPP* is defined as the minimum distance point, which is the point in the *U* space that has the highest probability of producing the value of constraint function *g*(*U*) (Hasofer and Lind,1974, Wu, 1990). The minimum distance β is called as safety factor (Hasofer and Lind, 1974, Wu, 1990). If the constraint function *g*(*Y*) is linear in terms of the normally distributed random variables *U*, the accurate probability of constraint satisfaction is given by the equation:

$$P[g(Y)\ge 0]=\Phi(b) \qquad (4.4)$$

If the constraint function *g*(*Y*) is nonlinear or random variables *Y* are not normally distributed, a good approximation can still be obtained by the above equation, provided that the magnitude of the principal curvatures of the constraint surface at the *MPP* is not too large. Different techniques can be used to search the *MPP*, such as using optimization, advance mean value (*AMV*) (Wu, 1990), sampling-based MPP search (Wu, 1998), etc. In this paper, a set of non-sampling based *MPP* search technique, such as sensitivity analysis, modified searching direction, and *MPP* locus tracking, are used to ensure the robustness and efficiency of the search.

After *MPP* is obtained, samples are picked around the *MPP* to evaluate the probability of constraint satisfaction by importance sampling method. An importance-sampling density, $v_Y(Y)$, is introduced into the Monte Carlo estimation equation (2.5) to obtain

$$P[g(x,p) \geq 0] = \int_{all\ \ Y} I[g(Y)] \frac{f_Y(Y)}{v_Y(Y)} v_Y(Y) dY \qquad (4.5)$$

A Monte Carlo algorithm to evaluate the integral in Eqn. (4.5) would be to sample a series of Y_i from $v_Y(Y)$ and to estimate the probability through

$$P[g(x,p) \geq 0] = \frac{1}{N} \sum_{i=1}^{N} I[g(Y_i)] \frac{f_Y(Y_i)}{v_Y(Y_i)} \qquad (4.6)$$

We execute importance sampling in the standard normal space *U* and chose the importance-sampling density as the standard normal distribution with its mean value shifted to the *MPP* U^* (Ang, et al., 1992). This gives a good estimation of the probability with a small number of simulations. The concept is illustrated in Fig. 7. We can see that about half of the samples will fall into either the unfeasible region or the feasible region. The evaluation efficiency can be significantly improved by this way. The probabilistic feasibility analysis of the beam design example in Section 3.2 is implemented by this *MPP*-based importance sampling method.

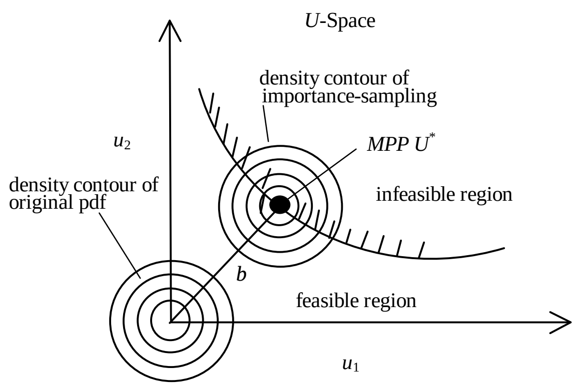

Figure 7. Importance Sampling

As the evaluation of probabilistic feasibility is a part of a robust optimization process, we believe measures need to be taken into account on how to use this approach more effectively in the solution process of optimization.

To reduce the computational effort of evaluating the constraint function $g(Y)$, we suggest not to provide an accurate evaluation of probability of constraint evaluation at each iteration of optimization, but to use the sampling method only when it is necessary. For example, if the safety factor b obtained in the *MPP* searching step is too far away (either much larger or much smaller) from the one corresponding to the desired probability, $\Phi(b)$ will be used to approximate the probability.

In the process of sampling, we suggest to determine the number of simulations by a prescribed error with a certain confidence level. The system will keep tracking the number of samples that fall into the feasible region and computing the simulation error due to randomness (Law and Kelton, 1982). If the error is less than the acceptable error defined by designers under a certain confidential level, the sampling process will stop and the probability will be estimated.

To keep the stability of convergence in an optimization process and to ensure the repeatability of solutions, except for using enough number of simulations determined by the prescribed error, we also suggest using the same "seed" number to generate random variables.

5. CLOSURE

In robust design, it is important not only to achieve the robust objective performance but also to maintain the robustness of design feasibility. In this paper, we discussed how to define the robustness of design feasibility under the effect of variations. By providing analytical interpretations and using illustrative examples, the features of various existing methods for modeling feasibility robustness are compared from different aspects. We illustrate that, although some of these approaches are easy to use, they may lead to either over-conservative or infeasible design solutions in robust design applications. The summary of comparisons is provided in Section 3.3 and will not repeat here. We expect that they could serve as guidelines for choosing the right technique under different circumstances.

It is our belief that the probabilistic feasibility formulation is the ideal method to describe the feasibility robustness and to ensure the solution achieve an accurate level of constraint satisfaction. To improve the efficiency of using this formulation, we propose to use a most probable point (*MPP*) based importance sampling method, a technique rooted in reliability analysis, for evaluating the feasibility robustness. Though our discussions have been centered on robust design, the principles presented are generally applicable for any probabilistic optimization problems. The practical significance of this work also lies in the development of efficient feasibility evaluation methods that can support quality engineering practice, such as the Six Sigma approach that is being widely used in American industry.

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