Topics: Computational complexity.

Resource-Bounded Dense Genericity, Stochasticity and Weak Randomness

Extended Abstract

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Abstract

We introduce dense $t(n)$-genericity which is a refinement of the genericity concept of Ambos-Spies, Fleichhack and Huwig [3] and which in addition controls the frequency with which a condition is met. We show that this concept coincides with the resource-bounded version of Curch's stochasticity [7]. By uniformly describing these concepts and weaker notions of stochasticity introduced by Wilber [24] and Ko [13] in terms of prediction functions, we clarify the relations among these resource-bounded stochasticity concepts. Moreover, we give descriptions of these concepts in the framework of Lutz's resource bounded measure theory [16] based on martingales: We show that $t(n)$-stochasticity coincides with a weak notion of $t(n)$-randomness based on so called simple martingales but that it is strictly weaker than $t(n)$-randomness in the sense of Lutz.

1 Introduction

Over the last years resource bounded versions of Baire category and Lebesgue measure have been introduced in complexity theory. These concepts allow a quantitative analysis of the structural properties of complexity classes. In most cases the concepts were introduced for deterministic time classes, where in general the $t(n)$-time bounded concepts correspond to the class $DTIME(t(2^n))$. In particular, polynomial time bounded versions of these concepts have been used to analyse the structure of the class E of the deterministic exponential time sets.

Many applications of category and measure can be reduced to questions about the typical sets for these concepts, i.e., the generic sets in case of Baire category and the random sets in the case of Lebesgue measure. These typical sets have all properties which are shared by a large class of sets, i.e., by a comeager respectively measure-1 class (in the corresponding resource-bounded sense).

Resource-bounded genericity concepts have been introduced by Ambos-Spies, Fleischhack and Huwig ([2, 3]), Lutz [16], Fenner ([9, 10]), Ambos-Spies [1], and others. Resource-bounded randomness concepts can be found e.g. in Wilber [24], Ko [13] and Lutz [16]. While an attempt to clarify the relations among the various genericity notions has been recently made by Ambos-Spies in [1], it seems that the relations among the different resource-bounded randomness notions have not yet been explored systematically, though some isolated results have been obtained.

As also shown in [1], as in the classical case, most of the genericity concepts are incompatible with the randomness concepts in the resource bounded case too. There is a notable exception, however: the genericity concept of Ambos-Spies, Fleischhack and Huwig [3] is compatible with Lutz's randomness [16]. In fact, Ambos-Spies, Neis and Terwijn [4] have used this type of genericity to get new measure results and simpler proofs of certain older measure results. As observed by them too, however, genericity cannot control the density of a set whereas random sets are exponentially dense. More generally, in case of genericity certain events which may happen infinitely often are forced to actually happen infinitely often. Beyond this, however, genericity cannot determine the relative frequency with which the events will happen, i.e., which

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fractions of the chances are actually realized. In contrast, for a random set, the distribution of the events (captured by the measure concept) will be determined.

These observations have motivated our investigations here. We address the question, whether the gap between $t(n)$-genericity (in the sense of [3]) and $t(n)$-randomness (in the sense of [16]) can be bridged by considering an extension of the former which in addition controls the frequency. The answer we obtain is negative, but we hope that our investigations help to better understand the relations among some of the resource-bounded randomness notions in the literature. We show that our new dense $t(n)$-genericity concept coincides with the resource-bounded version of some other, weaker, randomness concept, namely that of Church [7]. According to the classification of randomness concepts by Kolmogorov (see [14]), Church's randomness, which is based on the distribution of the 0s and 1s in effectively chosen subsequences, is a stochasticity notion, while Lutz's concept, a resource-bounded version of Schnorr's randomness concept based on martingales [21], is a notion of typicalness. Our equivalence proof is based on the characterization of genericity and stochasticity in terms of prediction functions. By giving characterizations of other, weaker, resource-bounded stochasticity notions in terms of prediction functions too, we clarify the relations among these concepts, which originally were defined in quite different terms, and we prove some new separation results for these notions.

Finally we compare stochasticity and typicalness. We show that the expressive power of prediction functions is that of so-called simple martingales. So, $t(n)$-stochasticity, hence dense $t(n)$-genericty, coincides with weak $t(n)$-randomness, where $t(n)$-randomness is defined by simple martingales. We also show, however, that general martingales are more powerful, i.e., that there are weakly $t(n)$-random sets which are not $t(n)$-random.

The outline of the paper is as follows. In Section 2 we review the $t(n)$-genericity concept of Ambos-Spies et al. [3] and introduce its dense counterpart. Section 3 is devoted to the equivalence of dense-$t(n)$-genericty and $t(n)$-stochasticity and a classification of stochasticity concepts. Finally, in Section 4, we derive from Lutz's $t(n)$-randomness the new, weaker concept of weak-$t(n)$-randomness, and show the equivalence of this concept with stochasticity.

We close this section by introducing some notation. Let ω be the set of natural numbers and let $\{0,1\}^*$ be the set of (finite) binary strings. For a string x, $x(m)$ denotes the $(m+1)$th bit in x, i.e., $x = x(0)...x(n-1)$, where $n = |x|$ is the length of x. λ is the empty string. We identify strings with numbers by letting n be the $(n+1)$th string under the canonical ordering. Note that $|n| \approx \log(n)$. Lower case letters $\cdots, k, l, m, n, \cdots, x, y, z$ from the middle and the end of the alphabet will denote numbers and strings. The letters i and j are reserved for elements of $\{0,1\}$, and lower case Greek letters denote nonnegative real numbers.

A set of strings is called a problem or shortly a set, while sets of sets are called classes. Capital letters denote sets, $\|A\|$ denotes the cardinality of A. We identify a set with its infinite characteristic string, i.e., $n \in A$ iff $A(n) = 1$ and $n \notin A$ iff $A(n) = 0$, so that $\{0,1\}^\omega$, the set of infinite binary sequences, is identified with the power class of $\{0,1\}^*$. We let $A\upharpoonright n$ denote the initial segment $A(0)...A(n-1) \in \{0,1\}^*$ of A of length n. I.e., interpreted as a set, $A\upharpoonright n = \{x : x < n \;\&\; x \in A\}$.

We will use strings in two different meanings: as elements of sets and as finite initial segments of sets. In an attempt to avoid confusion, usually we will write $X\upharpoonright x$ for strings intended to denote initial segments. Then $X\upharpoonright x$ denotes a string of length x and, for $y < x$, $X(y)$ or $(X\upharpoonright x)(y)$ will denote the $(y+1)$th bit of $X\upharpoonright x$. Also note the difference in the length of an initial segment $A\upharpoonright x$ and the length of its bound x : $2^{|x|} - 1 \leq |A\upharpoonright x| \leq 2^{|x|+1} - 1$. Since, as mentioned before, many of the genericity and randomness concepts we discuss in this paper are based on functions defined on initial segments, this will be responsible for the fact that the $DTIME(t(n))$ bounded concepts will correspond to the class $DTIME(t(2^n))$.

Throughout this paper, $t(n)$ is an arbitrary recursive function such that $t(n)$ is nondecreasing and $t(n) > n$ for all n. Finally, for a partial function f, we let $f(x)\downarrow$ ($f(x)\uparrow$) denote that $f(x)$ is (un)defined.

2 Genericity and Dense Genericity

Ambos-Spies, Fleischhack and Huwig [3] introduced resource bounded genericity notions corresponding to finite-extension diagonalization arguments in which every single diagonalization step requires only a one-string extension. The properties enforced by the single diagonalization steps are formalized by conditions in these concepts.

Definition 2.1 *A* condition C *is a set* $C \subseteq \{0,1\}^*$*. A* $t(n)$-condition *is a condition* $C \in DTIME(t(n))$*. A condition* C *is* dense along *a set* A *if there are infinitely many* x *such that* $(A \restriction x)i \in C$ *for some* $i \in \{0,1\}$*. A set* A meets *a condition* C *if* $A \restriction x \in C$ *for some* x*.* A *is* $t(n)$-generic *if* A *meets every* $t(n)$*-condition which is dense along* A*.*

Intuitively, a condition C is dense along a set A if in the inductive definition of A there are infinitely many chances to extend $A \restriction x$ to $A \restriction (x+1)$ in such a way that $A \restriction (x+1)$ will force the property encoded by C for A. So a $t(n)$-generic set A has all properties which can be encoded by $DTIME(t(n))$-conditions and which can be forced infinitely often along the construction of A. In the following lemma we give an example.

Lemma 2.2 *[3] Let* A *be* $t(n)$*-generic, then* A *is* P*-bi-immune.*

Proof. By symmetry, it suffices to show that A is P-immune. So let $B \in P$ be infinite. Define a condition C by $C = \{(X \restriction x)0 \ : \ x \in B\}$. Then, by $B \in P$, C is an n- (hence $t(n)$-) condition and, by infinity of B, C is dense along all sets. Morever, by definition, no superset of B meets C. So by $t(n)$-genericity, A meets C whence B is not contained in A. □

As observed already in [3], a $t(n)$-generic set A meets a condition C which is dense along A not just once but infinitely often.

Lemma 2.3 *[3] Let* A *be* $t(n)$*-generic and let* C *be a* $t(n)$*-condition which is dense along* A*. There are infinitely many strings* x *such that* $A \restriction x \in C$*.*

For analyzing the frequency with which a set meets conditions which are dense along it, it is convenient to consider only proper conditions.

Definition 2.4 *A condition* C *is* proper *if for every string* x*,* $x0 \notin C$ *or* $x1 \notin C$*. A set* A meets (avoids) *a proper condition at* x *if* $(A \restriction x)A(x) \in C$ $((A \restriction x)(1 - A(x)) \in C)$*.*

As the following observation shows, in the definition of $t(n)$-genericity, it suffices to consider proper conditons.

Lemma 2.5 *Let* C *be a* $t(n)$*-condition. Then* C' *defined by*

$$C' = \{x0 \ : \ x0 \in C\} \cup \{x1 \ : \ x1 \in C \ \& \ x0 \notin C\}, \tag{1}$$

is a proper $t(n)$*-condition such that, for any* x*,*

$$\exists i \in \{0,1\} \ (xi \in C) \iff \exists i \in \{0,1\} \ (xi \in C'),$$

and $x \in C'$ *implies that* $x \in C$*. Hence, for any set* A*,* C *is dense along* A *iff* C' *is dense along* A*, and* A *meets* C *if* A *meets* C'*.*

Proof. Straightforward. □

Lemma 2.6 *For any set* A*, the following are equivalent.*

1. *A is $t(n)$-generic.*

2. *A meets every proper $t(n)$-codition C which is dense along A.*

3. *A infinitely often meets every proper $t(n)$-codition C which is dense along A.*

Proof. $1 \Rightarrow 3$ holds by Lemma 2.3, $3 \Rightarrow 2$ is obvious, and $2 \Rightarrow 1$ holds by Lemma 2.5. □

Moreover, we can replace "meets" by "avoids" in 2 and 3 of Lemma 2.6.

Lemma 2.7 *Let C be a proper $t(n)$-condition. Then, for $\hat{C}$ defined by*

$$\hat{C} = \{x(1-i) \ : \ xi \in C\}, \tag{2}$$

$\hat{C}$ is a proper $t(n)$-condition and, for any set A and any string x, A meets (avoids) $\hat{C}$ at x iff A avoids (meets) C at x.

Proof. Straightforward. □

So a $t(n)$-generic set infinitely often meets and infinitely often avoids every proper $t(n)$-condition which is dense along A. As the following theorem shows, however, we cannot say anything about the relative frequency of these events.

Theorem 2.8 *Let C be a proper $t(n)$-condition and let f be an unbounded, nondecreasing recursive function. There is a $t(n)$-generic set A such that $\|\{y < n \ : \ A \text{ meets } C \text{ at } y\}\| \leq f(n)$ for all n.*

We omit the proof of the theorem which is similar to the proofs of related results on the possible densities of $t(n)$-generic sets in [3] and [4].

Corollary 2.9 *[3] There is a sparse $t(n)$-generic set.*

Proof. Apply Theorem 2.8 to $C = \{x1 \ : \ x \in \{0,1\}^*\}$ and $f(n) = \log n$. □

As the above results show, $t(n)$-genericity can force events to happen infinitely often but it cannot control the frequency distribution of these events. To overcome this shortcoming we introduce the following strengthening of $t(n)$-genericity.

Definition 2.10 *A set A* meets *a condition C* densely *if*

$$\lim_n \frac{\|\{y < n \ : \ A\restriction(y+1) \in C\}\|}{\|\{y < n \ : \ \exists i \ ((A\restriction y)i \in C)\}\|} = \frac{1}{2}. \tag{3}$$

A set A is densely $t(n)$-generic *if A meets densely every proper $t(n)$-condition which is dense along A.*

Note that A meets a proper $t(n)$-condition C densely if and only if

$$\lim_n \frac{\|\{y < n \ : \ A \text{ meets } C \text{ at } y\}\|}{\|\{y < n \ : \ A \text{ avoids } C \text{ at } y\}\|} = 1,$$

i.e., iff the frequency of A meeting and avoiding C is the same. The following lemma gives a characterization of dense genericity in terms of arbitrary (not necessarily proper) conditions.

Lemma 2.11 *For any set A, the following are equivalent.*

1. *A is densely $t(n)$-generic.*

2. *For any $t(n)$-condition C which is dense along A,*

$$\liminf_n \frac{\|\{y < n \ : \ A\restriction(y+1) \in C\}\|}{\|\{y < n \ : \ \exists i \ ((A\restriction y)i \in C)\}\|} \geq \frac{1}{2}. \tag{4}$$

The following theorem demonstrates the additional power of dense genericity compared with genericity.

Theorem 2.12 *Let A be densely $t(n)$-generic. Then A is exponentially dense, i.e., there exits a real $\varepsilon > 0$ such that $\|A^{\leq n}\| \geq 2^{n^\varepsilon}$ for almost all n.*

Corollary 2.13 *There is a $t(n)$-generic set which is not densely n-generic, whence not densely $t(n)$-generic.*

Proof. By Corollary 2.9 and by Theorem 2.12. □

3 Resource Bounded Stochasticity

The first notion of randomness based on formal computability was introduced by Church [7] in 1940. Church called an infinite 0-1-sequence A a random sequence, if, for every infinite subsequence S of A selected by a recursive rule, the numbers of occurences of 0s and 1s in the sequences are asymptotically the same. Following Uspenskii et al. [22] we call randomness in the sense of Church stochasticity.

For a formal definition of stochasticity, we first formalize the notion of a selective rule.

Definition 3.1 *A* selection function f *is a total recursive function* $f : \{0,1\}^* \to \{0,1\}$. *A selection function* f *is* dense along A *if* $f(A\restriction x) = 1$ *for infinitely many* x.

By interpreting A as the infinite 0-1-sequence $A(0)A(1)A(2)\cdots$, a selection function f selects the subsequence $A(x_0)A(x_1)A(x_2)\cdots$ of A where $x_0 < x_1 < x_2 < \cdots$ are the strings x such that $f(A\restriction x) = 1$. In particular, f selects an infinite subsequence S of A iff f is dense along A. So Church's stochasticity concept can be defined as follows

Definition 3.2 *(Church [7]) A set A is* stochastic *if, for every selection function f which is dense along A and for $i \in \{0,1\}$,*

$$\lim_n \frac{\|\{y < n \;:\; f(A\restriction y) = 1 \;\&\; A(y) = i\}\|}{\|\{y < n \;:\; f(A\restriction y) = 1\}\|} = \frac{1}{2}. \tag{5}$$

Di Paola [8] studied subrecursive versions of Church stochasticity corresponding to the Ritchie and Grzegorczyk hierarchies. Here we will consider $t(n)$-time bounded Church stochasticity corresponding to $DTIME(t(2^n))$.

Definition 3.3 *A* $t(n)$-selection function f *is a selection function* $f \in DTIME(t(n))$. *A set A is* $t(n)$-stochastic *if, for every $t(n)$-selection function f which is dense along A and for $i \in \{0,1\}$, (5) holds.*

To show that $t(n)$-stochasticity and dense $t(n)$-genericity coincide, we characterize these concepts in terms of prediction functions. A prediction function f is a procedure which, given a finite initial segment of a 0-1-sequence, predicts the value of the next member of the sequence. We will show that a sequence A is stochastic (densely generic) iff, for every partial prediction function which makes infinitely many predictions along A, the number of the correct and incorrect predictions is asymptotically the same.

Definition 3.4 *A* prediction function f *is a partial function* $f : \{0,1\}^* \to \{0,1\}$. *A* $t(n)$-prediction function f *is a prediction function f such that $f \in DTIME(t(n))$ and domain$(f) \in DTIME(t(n))$. A prediction function f is* dense along A *if $f(A\restriction x)$ is defined for infinitely many x. A* meets (avoids) f *at x if $f(A\restriction x)$ is defined and $f(A\restriction x) = A(x)$ ($f(A\restriction x) = 1 - A(x)$). A* meets f densely *if*

$$\lim_n \frac{\|\{y < n \;:\; f(A\restriction y) = A(y)\}\|}{\|\{y < n \;:\; f(A\restriction y)\downarrow\}\|} = \frac{1}{2}. \tag{6}$$

Note that (6) can be rephrased by

$$\lim_n \frac{\|\{y < n \;:\; A \text{ meets } f \text{ at } y\}\|}{\|\{y < n \;:\; A \text{ avoids } f \text{ at } y\}\|} = 1. \tag{7}$$

Theorem 3.5 *For any set A, the following are equivalent.*

1. *A is densely $t(n)$-generic.*
2. *A is $t(n)$-stochastic.*
3. *A meets densely every $t(n)$-prediction function which is dense along A.*

Proof. We prove the implications $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$.

1⇒ 3. Assume that A is densely $t(n)$-generic and let f be a $t(n)$-prediction function which is dense along A. Define a proper $t(n)$-condition C_f by

$$C_f = \{(X \restriction x) f(X \restriction x) : f(X \restriction x) \downarrow\}.$$

Then C_f is dense along A, whence, by $t(n)$-genericity of A, (3) holds for C_f (in place of C). By definition of C_f, this is equivalent to (6), whence A meets f densely.

3⇒2. Assume that A meets densely every $t(n)$-prediction function which is dense along A, and let f be any $t(n)$-selection function which is dense along A. To show that (5) holds, fix $i \in \{0,1\}$ and define the $t(n)$-prediction function f' by letting $f'(X \restriction x) = i$ if $f(X \restriction x) = 1$ and by letting $f'(X \restriction x)$ be undefined otherwise. Then f' is dense along A, whence, by assumption, (6) holds for f' (in place of f). But this is equivalent to (5) (for f).

2⇒1. Assume that A is $t(n)$-stochastic and let C be a proper $t(n)$-condition which is dense along A. We have to show that (3) holds. Define $t(n)$-selection functions $f_i, i = 0, 1$, by letting $f_i(X \restriction x) = 1$ for $(X \restriction x)i \in C$ and letting $f_i(X \restriction x) = 0$ otherwise. Then

$$\{y : A \restriction (y+1) \in C\} = \bigcup_{i=0,1} \{y : f_i(A \restriction y) = 1 \ \&\ A(y) = i\}$$

and

$$\{y : \exists i \in \{0,1\} \ ((A \restriction y) i \in C)\} = \bigcup_{i=0,1} \{y : f_i(A \restriction y) = 1\}$$

where the unions of the right hand sides of the equations are disjoint. Hence

$$\text{(8)} \qquad \frac{\|\{y < n : A \restriction (y+1) \in C\}\|}{\|\{y < n : \exists i \in \{0,1\} \ ((A \restriction y) i \in C)\}\|} = \frac{\sum_{i=0,1} \|\{y < n : f_i(A \restriction y) = 1 \ \&\ A(y) = i\}\|}{\sum_{i=0,1} \|\{y < n : f_i(A \restriction y) = 1\}\|}.$$

Now distinguish the following two cases.

First assume that, for some $i \in \{0,1\}$, f_i is not dense along A. Then f_{1-i} is dense along A and $\{y : f_i(A \restriction y) = 1\}$ is finite. So, by (8),

$$\frac{\|\{y < n : A \restriction (y+1) \in C\}\|}{\|\{y < n : \exists i \in \{0,1\} \ ((A \restriction y) i \in C)\}\|} = \frac{\|\{y < n : f_{1-i}(A \restriction y) = 1 \ \&\ A(y) = 1-i\}\|}{\|\{y < n : f_{1-i}(A \restriction y) = 1\}\|}$$

in the limit and, by $t(n)$-stochasticity of A, the right hand side has limit $\frac{1}{2}$. So (3) holds.

Otherwise, f_0 and f_1 are dense along A whence, by $t(n)$-stochasticity, (5) holds for f_i in place of f $(i = 0, 1)$. It follows that the right hand side of (8) has limit $\frac{1}{2}$, whence (3) holds in this case too. □

In the remainder of this section we shortly discuss some other, weaker resource-bounded stochasticity concepts. We will characterize these concepts by different types of prediction functions thereby clarifying the relations among these notions. The first concept we will consider, was introduced by Ko in [13] and was defined in terms of prediction functions already.

Definition 3.6 *(Ko [13]) A set A is* Ko-$t(n)$-stochastic *if, for every total $t(n)$-prediction function f, A meets f densely, i.e.,*

$$\text{(9)} \qquad \lim_n \frac{\|\{y < n : f(A \restriction y) = A(y)\}\|}{n} = \frac{1}{2}.$$

Lemma 3.7 *Every $t(n)$-stochastic set is Ko-$t(n)$-stochastic.*

Proof. Since, obviously, a total prediction function is dense along all sets, this follows from Theorem 3.5 immediately. □

A still older notion of stochasticity can be found in [24].

Definition 3.8 *(Wilber [24]) A set A is Wilber-$t(n)$-stochastic if, for every set $B \in DTIME(t(n))$,*

$$\lim_n \frac{\|\{y < n : A(y) = B(y)\}\|}{n} = \frac{1}{2}. \tag{10}$$

To relate Wilber's notion to the other stochasticity concepts, we need the following property of prediction functions.

Definition 3.9 *A prediction function f is* oblivious *if, for all strings x and y with $|x| = |y|$, $f(x)$ is defined if and only if $f(y)$ is defined, and, if $f(x)$ is defined, then $f(x) = f(y)$.*

Intuitively, for an oblivious prediction function f, the predicted value $A(y)$ of a set A does not depend on the previously seen values $A{\upharpoonright}y$ of A, so that f makes the same predictions for all sets.

Lemma 3.10 *For any set A the following are equivalent.*

1. *A is Wilber-$t(2^n)$-stochastic.*
2. *A meets densely every total oblivious $t(n)$-prediction function.*

Definition 3.11 *A set A is* weakly *$t(n)$-stochastic if, for every infinite set $B \in DTIME(t(n))$,*

$$\lim_n \frac{\|A \cap B{\upharpoonright}n\|}{\|B{\upharpoonright}n\|} = \frac{1}{2}. \tag{11}$$

Loveland [15] called a set A which satisfies (11) *unbiased* with respect to B. Also note that weak $t(n)$-stochasticity may be viewed as dense $DTIME(t(n))$-bi-immunity: If A is weakly $t(n)$-stochastic and $B \in DTIME(t(n))$ is infinite then $A \cap B$ and $\overline{A} \cap B$ are infinite and, moreover, $\|\{y < n : y \in A \cap B\}\|$ and $\|\{y < n : y \in \overline{A} \cap B\}\|$ grow at the same rate.

Lemma 3.12 *For any set A, the following are equivalent.*

1. *A is weakly $t(2^n)$-stochastic.*
2. *A meets densely every oblivious $t(n)$-prediction function which is dense along A.*

The relation between stochasticity and weak-stochasticity can be further illustrated by the following characterization of $t(n)$-stochasticity in the style of Definition 3.11.

Lemma 3.13 *A set A is $t(n)$-stochastic iff, for every infinite set $B \in DTIME^{<A}(t(2^n))$, (11) holds. Here $B \in DTIME^{<A}(t(2^n))$ means that there is a $t(2^n)$-time bounded deterministic oracle Turing machine M such that $B(x) = M(A{\upharpoonright}x; x)$ for all x.*

The above characterizations of the stochasticity concepts in terms of prediction functions imply the following relations:

$$\begin{array}{ccccc} & & \text{Ko-}t(n)\text{-stochastic} & & \\ & \nearrow & & \searrow & \\ t(n)\text{-stochastic} & & & & \text{Wilber-}t(2^n)\text{-stochastic} \\ & \searrow & & \nearrow & \\ & & \text{weakly-}t(2^n)\text{-stochastic} & & \end{array} \tag{12}$$

To show that no other implications hold, we analyze some closure properties of the stochasticity concepts. We first extend an observation of Huynh [11] on Wilber-stochasticity to Ko-stochasticity.

Theorem 3.14 *Let A be Ko-$t(n)$-stochastic where $t(n) \geq n^2$ and let $B \in P$ be sparse. Then $A \cup B$ is Ko-$t(n)$-stochastic too.*

Corollary 3.15 *There is a Ko-$t(n)$-stochastic set which is not weakly-n-stochastic.*

Proof. Let $B = \{0\}^*$. By Theorem 3.14 there is a Ko-$t(n)$-stochastic set A which contains B. So A is not $DTIME(n)$-immune. As observed above, however, every weakly-n-stochastic set is $DTIME(n)$-bi-immune. □

Theorem 3.16 *Let A be weakly-$t(n)$-stochastic. Then $\tilde{A} = \{2n, 2n+1 : 2n+1 \in A\}$ is weakly $t(n)$-stochastic too.*

Corollary 3.17 *There is a weakly $t(n)$-stochastic set which is not Ko-n-stochastic.*

Proof. By Theorem 3.16, it suffices to show that there is no set A such that $\tilde{A} = \{2n, 2n+1 : 2n+1 \in A\}$ is Ko-n-stochastic. So fix A. We have to give a total n-prediction function f such that $\tilde{A}$ does not meet f densely.

Define total n-prediction functions f_0 and f_1 by letting $f_i(X{\restriction}2n) = i$ and $f_i(X{\restriction}2n+1) = 1 - X(2n)$. Then, for all $n \in \omega$ and for $i \in \{0,1\}$, $f_i(\tilde{A}{\restriction}2n+1) \neq \tilde{A}(2n+1) = \tilde{A}(2n)$. Since $f_0(\tilde{A}{\restriction}2n) \neq f_1(\tilde{A}{\restriction}2n)$ for all n, it follows that $\tilde{A}$ cannot meet both f_0 and f_1 densely. □

Lutz and Mayordomo [18] have introduced another stochasticity notion which resembles our weak-stochasticity notion here but does not coincide with it. As one can easily check, a set A is weakly $t(n)$-stochastic iff there are sets $B, C \in DTIME(t(n))$ such that B is infinite and

$$\lim_n \frac{\|(A \Delta C) \cap B{\restriction}n\|}{\|B{\restriction}n\|} = \frac{1}{2}.$$

Now Lutz and Mayordomo's stochasticity notion is more liberal in allowing a linear advice for computing C but also more restrictive in requiring that B is exponentially dense. By the latter, a variant of the proof of lemma 3.14 shows that, in contrast to weak stochasticity, Lutz and Mayordomo stochasticity does not imply bi-immunity.

We should remark that many results in this section have some parallels in the theory of genericity. There prediction functions are usually viewed as extension functions. Corresponding to Theorem 3.5, Ambos-Spies [1] has shown that a set A is $t(n)$-generic iff A meets every $t(n)$-prediction function which is dense along A. As shown in [1] too, total $t(n)$-prediction functions yield an almost trivial genericity concept (cf Lemma 6.6 of [1]), while in [3], it was implicitly shown that genericity for oblivious $t(n)$-prediction functions coincides with $DTIME(t(2^n))$-bi-immunity. Finally, a characterization of $t(n)$-genericity corresponding to Lemma 3.13 was given by Balcazar and Mayordomo in [6].

4 Randomness and Weak Randomness

Schnorr [20, 21] defined a randomness concept based on computable martingales, where martingale is a betting strategy. In classical measure theory, a class has measure 0 iff there is a martingale which succeeds on all members of the class. Schnorr calls a set random if no recursive martingale effectively succeeds on it. Lutz [17] introduced a resource-bounded version of this concept by imposing time or space bounds on the martingales.

Definition 4.1 *[17] A* martingale *is a function $d : \{0,1\}^* \to Q_+$, where Q_+ is the set of nonnegative rationals, such that*

(13) $$\forall x \in \{0,1\}^* \ (d(x0) + d(x1) = 2d(x)).$$

A martingale d succeeds *on a set A if $\limsup_n d(A{\restriction}n) = \infty$. A martingale d is a $t(n)$-*martingale *if $d \in DTIME(t(n))$. A set A is $t(n)$-*random *if no $t(n)$-martingale succeeds on A.*

In the following, $S^\infty[d]$ denotes the class of the sets on which the martingale d succeeds. Originally, Lutz used $t(n)$-time computable approximations of real valued martingales, but the above, technically simpler, definition is equivalent to his concept. This was already shown by Schnorr [21] for the recursive case and, as observed independently in [5], [12] and [19], this equivalence holds in the resource bounded case too.

The existence of recursive $t(n)$-random sets was proved by Schnorr [21]. Later Lutz [17] obtained some existence result for p-randomness, and Ambos-Spies et al. [5] have proved the following general existence result for $t(n)$-random sets.

Theorem 4.2 *([5]) There is a $t(n)$-random set in $DTIME(t'(2^n))$, where $t'(n) = n^2 t(n) \log t(n)$.*

The following definition will allow us to give a martingale characterization of stochasticity, hence dense genericity.

Definition 4.3 *A martingale d is* simple *if there is a rational number $\alpha \in (0,1)$ such that*

$$\forall x \in \{0,1\}^* \ \forall i \in \{0,1\} \ (d(xi) \in \{d(x), (1+\alpha)d(x), (1-\alpha)d(x)\}). \tag{14}$$

A martingale d is almost simple *if there is a finite set $F = \{\alpha_0, \cdots, \alpha_m\}$ of rational numbers $\alpha_k \in (0,1)$ such that*

$$\forall x \in \{0,1\}^* \ \forall i \in \{0,1\} \ \exists k \in \{0, \cdots, m\} \ (d(xi) \in \{d(x), (1+\alpha_k)d(x), (1-\alpha_k)d(x)\}). \tag{15}$$

A set A is weakly $t(n)$-random *if there is no simple $t(n)$-martingale which succeeds on A.*

Lemma 4.4 *Every $t(n)$-random set is weakly $t(n)$-random.*

Before we show that weak randomness and stochasticity coincide, we observe that weak randomness can be defined by almost simple martingales too.

Theorem 4.5 *For any set A, the following are equivalent.*

1. *A is weakly $t(n)$-random.*
2. *There is no almost simple $t(n)$-martingale which succeeds on A.*

The nontrivial implication of the theorem follows from the next lemma immediately.

Lemma 4.6 *Let d be an almost simple $t(n)$-martingale. There are finitely many simple $t(n)$-martingales $d_0, \cdots, d_m$ such that*

$$S^\infty[d] \subseteq \bigcup_{k=0}^{m} S^\infty[d_k].$$

We now state our main theorem.

Theorem 4.7 *For any set A, the following are equivalent.*

1. *A is densely $t(n)$-generic.*
2. *A is $t(n)$-stochastic.*
3. *A is weakly $t(n)$-random.*

By Theorem 3.5, for a proof of Theorem 4.7 it suffices to show the equivalence of the prediction function and simple martingale concepts. Since the weak stochasticity notions in Section 3 could be characterized by special types of prediction functions, we first define corresponding restrictions for martingales. Then the equivalence proof will also yield martingale characterizations of these stochasticity concepts.

Definition 4.8 *A martingale d is* strict, *if, for all $x \in \{0,1\}^*$ and $i \in \{0,1\}$, $d(xi) \neq d(x)$. A martingale d is* oblivious *if, for all numbers n there is a rational α_n such that $d(x0) = \alpha_n d(x)$ for all strings x of length n.*

Note that $d((A\upharpoonright x)i) = d(A\upharpoonright x)$, i.e., $d((A\upharpoonright x)0) = d((A\upharpoonright x)1) = d(A\upharpoonright x)$ by (13) expresses that the strategy d does not bet on $A(x)$. For prediction functions, this corresponds to making no prediction for $A(x)$. So, as we will show below, strictness of a martingale corresponds to totality of a prediction function.

Lemma 4.9 *For any set A, the following are equivalent.*

1. *A meets densely every (total, oblivious, total and oblivious) $t(n)$-prediction function which is dense along A.*

2. *No simple (strict, oblivious, strict and oblivious) $t(n)$-martingale succeeds on A.*

Proof. *1*$\Rightarrow$*2.* For a contradiction, assume that *1* holds but the simple $t(n)$-martingale d succeeds on A. Fix a rational number $\alpha \in (0,1)$ such that (14) holds. Define a $t(n)$-prediction function f by letting $f(X\upharpoonright x) = i$ if $d((X\upharpoonright x)i) = (1+\alpha)d(X\upharpoonright x)$ and by letting $f(X\upharpoonright x)$ be undefined if $d((X\upharpoonright x)0) = d(X\upharpoonright x)$. Then f is total and oblivious if d is strict and oblivious, respectively. Morever, since d succeeds on A,

$$\limsup_n \frac{\|\{y < n : d((A\upharpoonright y)A(y)) = (1+\alpha)d(A\upharpoonright y)\}\|}{\|\{y < n : d((A\upharpoonright y)A(y)) = (1-\alpha)d(A\upharpoonright y)\}\|} > 1.$$

So, by definition of f,

$$\limsup_n \frac{\|\{y < n : A \text{ meets } f \text{ at } y\}\|}{\|\{y < n : A \text{ avoids } f \text{ at } y\}\|} > 1,$$

whence (7) fails. So A does not meet f densely, contrary to assumption.

2$\Rightarrow$*1.* For a contradiction assume that *2* holds but there is a $t(n)$-prediction funtion f such that f is dense along A but A does not meet f densely. Then (7) fails. So, by symmetry, w.l.o.g. we may assume that there is a rational number $\varepsilon \in (0,1)$ such that

$$\liminf_n \frac{\|\{y < n : f(A\upharpoonright y) = A(y)\}\|}{\|\{y < n : f(A\upharpoonright y) = 1 - A(y)\}\|} < 1 - \varepsilon. \tag{16}$$

Fix such an ε and let $p(n) = \|\{y < n : f(A\upharpoonright y) = A(y)\}\|$ and $q(n) = \|\{y < n : f(A\upharpoonright y) = 1 - A(y)\}\|$. Note that for $\alpha \in (0,1)$,

$$\lim_{\alpha \to 0} -\frac{\log(1+\alpha)}{\log(1-\alpha)} = 1$$

whence we may choose a rational number $\alpha \in (0,1)$ such that

$$-\frac{\log(1+\alpha)}{\log(1-\alpha)} > 1 - \frac{\varepsilon}{2}$$

whence

$$(1+\alpha) \geq \left(\frac{1}{1-\alpha}\right)^{1-\frac{\varepsilon}{2}}. \tag{17}$$

Define a $t(n)$-martingale d by letting $d(\lambda) = 1$ and

$$d(xi) = \begin{cases} (1-\alpha)d(x) & \text{if } f(x) = i, \\ (1+\alpha)d(x) & \text{if } f(x) = 1 - i, \\ d(x) & \text{if } f(x)\uparrow. \end{cases}$$

Then d is strict and oblivious if f is total and oblivious, respectively. Moreover,

$$\begin{aligned} d(A\upharpoonright n) &= (1-\alpha)^{p(n)}(1+\alpha)^{q(n)} \\ &\geq (1-\alpha)^{p(n)}\left(\tfrac{1}{1-\alpha}\right)^{(1-\frac{\varepsilon}{2})q(n)} \quad \text{by (17)} \\ &= \left(\tfrac{1}{1-\alpha}\right)^{(1-\frac{\varepsilon}{2})q(n)-p(n)}. \end{aligned}$$

Note that, by (16), $p(n) < (1-\varepsilon)q(n)$ for infinitely many n, whence it follows that

$$d(A\upharpoonright n) \geq \left(\frac{1}{1-\alpha}\right)^{\frac{\varepsilon}{2}q(n)} \quad i.o..$$

So d succeeds on A, contrary to assumption. □

Now Theorem 4.7 is immediate by Lemma 4.9 and Theorem 3.5. Moreover, we obtain the following martingale characterizations of the weak stochasticity notions.

Theorem 4.10 **(a)** *A set A is Ko-$t(n)$-stochastic iff no strict simple $t(n)$-martingale succeeds on A.*

(b) *A set A is weakly-$t(2^n)$-stochastic iff no oblivious simple martingale succeeds on A.*

(c) *A set A is Wilber-$t(2^n)$-stochastic iff no simple $t(n)$-martingale which is strict and oblivious succeeds on A.*

We close with the observation that randomness is strictly stronger than weak randomness.

Theorem 4.11 *For $t(n) \geq n$, there is a weakly $t(n)$-random set which is not $t(n)$-random.*

Proof (idea). In [23] Ville constructed a stochastic set A such that for all n

$$0 \leq \frac{1}{2} - \frac{1}{n}\|A\upharpoonright n\| \leq \frac{\log\log\log n}{2n}. \tag{18}$$

By Theorem 4.7, A is weakly $t(n)$-random. On the other hand, however, we can show that no $t(n)$-random set A satisfies (18). □

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