# THE $cl$-core OF AN IDEAL

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ABSTRACT. We expand the notion of core to $cl$-core for Nakayama closures $cl$. In the characteristic $p > 0$ setting, when $cl$ is the tight closure, we give some examples of ideals when the core and the $*$-core differ. Moreover, we show that the $*\text{-core}(I) = \text{core}(I)$, if $I$ is an ideal in a one-dimensional domain with infinite residue field or if $I$ is an ideal generated by a system of parameters in any Noetherian ring. More generally, we show the same result in a local Cohen–Macaulay normal domain with perfect infinite residue field, if the analytic spread, $\ell$, is equal to the $*$-spread and $I$ is $G_\ell$ and weakly-$(\ell-1)$-residually $S_2$. This last is dependent on our result that generalizes the notion of general reductions to general minimal $*$-reductions. We also determine that the $*$-core of a tightly closed ideal in certain one-dimensional semigroup rings is tightly closed and therefore integrally closed.

## 1. Introduction

The core of an ideal, the intersection of all reductions of the ideal, was introduced by Rees and Sally in [RS] in the 80's. Then over a decade past before Huneke and Swanson [HS1] analyzed the core of ideals in 2-dimensional regular local rings. Then a stream of papers came out within a decade by Corso, Polini and Ulrich [CPU1], [CPU2], [PU], Hyry and Smith [HyS1], [HyS2] and Huneke and Trung [HT] expanding the understanding and computability of core. As it is the intersection of reductions, in general it lies deep within the ideal. In fact, the core is related to the *Briançon-Skoda Theorem* [LS]: Let $(R,\mathfrak{m})$ be a regular local ring, then $\overline{I^d} \subseteq J$ for any reduction $J$ of $I$. Hence, $\overline{I^d} \subseteq \text{core}(I)$. A very slick proof of the Briançon-Skoda Theorem was given in characteristic $p > 0$, using tight closure, [HH, Theorem 5.4]. We would like to expand the notion of core to other closure operations; in particular, Nakayama closure operations. Epstein defined the notion of Nakayama closure as follows:

**Definition 1.1.** ([Ep])A closure operation $cl$, defined on a Noetherian local ring $(R,\mathfrak{m})$ is a Nakayama closure if for all ideals $J \subset I \subset (J+\mathfrak{m}I)^{cl}$, then $I \subset J^{cl}$.

Note that integral closure, tight closure and Frobenius closure are examples of Nakayama closures, [Ep, Proposition 2.1]. Recall, that both the tight closure and the Frobenius closure are characteristic $p > 0$ notions. It may be that we can formulate

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some Briançon-Skoda like Theorems for other cores such as tight closure using our new definition of *cl*-core.

Epstein's main reason for the definition of Nakayama closure was to expand the notion of reduction and spread to these other closure operations. With a well defined notion of reduction and spread, we can easily extend the notion of core to these other closure operations. In general, the *cl*-cores will not lie as deep in the ideal as the core itself. This will follow from the fact that the partial ordering of closure operations leads to a reverse partial ordering on the *cl*-cores. Our hope in studying these *cl*-cores is that tight closure methods may be used to compute the core in situations where the core and the $*$-core agree.

In Section 2, we provide some background information about the core and tight closure theory, along with a review of some central theorems that are used in this article. In Section 3, we review *cl*-reductions of ideals. We also discuss the *cl*-spread of an ideal and define both the *cl*-deviation and the second *cl*-deviation in terms of the *cl*-spread. We also introduce the notion of *cl*-core. In Section 4, we show different instances when the core and the $*$-core agree. Our main result, Theorem 4.5, shows that we can form general $*$-reductions. This allows us to show in particular that if $(R, \mathfrak{m})$ is Gorenstein normal domain of positive characteristic with infinite perfect field with the test ideal equal to $\mathfrak{m}$ and $I$ is an $\mathfrak{m}$-primary tightly closed ideal then $*$-core$(I)$ =core$(I)$. Also, when $(R, \mathfrak{m})$ is a Cohen–Macaulay normal domain with perfect infinite residue field of prime characteristic and $I$ is an ideal which is $G_\ell$ and weakly residually $(\ell-1)$-$S_2$ with $\ell^*(I) = \ell(I) = \ell$ then core$(I)$ = $*$-core$(I)$. In Section 5, we discuss when the $*$-core is tightly closed in some one-dimensional semigroup rings. In Section 6, we give some examples of the $*$-core of an ideal and in each case we compare the core with the $*$-core.

## 2. Background

In this section we recall some notions that we will use extensively in this article and also recall some results that we use.

**Definition 2.1.** Let $R$ be a Noetherian local ring of prime characteristic $p > 0$. We denote positive powers of $p$ by $q$ and the set of elements of $R$ which are not contained in the union of minimal primes by $R^o$. Then

**(a)** For any ideal $I \subset R$, $I^{[q]}$ is the ideal generated by the $q$th powers of elements in $I$.

**(b)** We say an element $x \in R$ is in the *tight closure*, $I^*$, of $I$ if there exists a $c \in R^o$, such that $cx^q \in I^{[q]}$ for all large $q$.

**(c)** We say an element $x \in R$ is in the *Frobenius closure* $I^F$ of $I$ if $x^q \in I^{[q]}$ for all large $q$.

Finding the tight closure of an ideal would be hard without test elements and test ideals. A *test element* is an element $c \in R$ which is not in any minimal prime and

$cI^* \subset I$ for all $I \subset R$. Note that $c \in \bigcap_{I \subset R} (I : I^*)$. Since the intersection of ideals is an ideal we call the ideal $\tau = \bigcap_{I \subset R} (I : I^*)$ *the test ideal*, i.e the ideal generated by all the test elements. We say that $I$ is a *parameter ideal* if $I$ is generated by part of a system of parameters. In a Gorenstein isolated singularity, the following theorem of Smith [Sm] gives a nice way to compute the tight closure of a parameter ideal using the test ideal.

**Theorem 2.2.** ([Sm, Lemma 3.6, Proposition 4.5]) *Let $(R, \mathfrak{m})$ be a Gorenstein isolated singularity with $\mathfrak{m}$-primary test ideal $\tau$. Then for any system of parameters $x_1, x_2, \ldots, x_d$,*

$$(x_1, x_2, \ldots, x_d) : \tau = (x_1, x_2, \ldots, x_d)^*.$$

Related concepts are parameter test elements and parameter test ideals. A *parameter test element* is an element $c \in R$ which is not in any minimal prime and $cI^* \subset I$ for all parameter ideals $I \subset R$. Note that $c \in \bigcap_{I \subset R} (I : I^*)$. Let $P(R)$ be the set of parameter ideals in $R$. We call $\tau_{par} = \bigcap_{I \in P(R)} (I : I^*)$ the parameter test ideal. It is known in a Gorenstein ring that $\tau = \tau_{par}$. We can relax the Gorenstein assumption from the above theorem and we obtain:

**Theorem 2.3.** ([Va1]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay isolated singularity with $\mathfrak{m}$-primary parameter test ideal $\tau_{par}$. For any system of parameters $x_1, x_2, \ldots, x_d$,*

$$(x_1, x_2, \ldots, x_d) : \tau_{par} = (x_1, x_2, \ldots, x_d)^*.$$

Note, if the parameter test ideal is known to be $\mathfrak{m}$, even in a Cohen–Macaulay ring, the test ideal will also be $\mathfrak{m}$.

Another result that we will use repeatedly is the following due to Aberbach:

**Proposition 2.4.** ([Ab, Proposition 2.4]) *Let $(R, \mathfrak{m})$ be an excellent, analytically irreducible local ring of characteristic $p$, let $I$ be an ideal, and let $f \in R$. Assume that $f \notin I^*$. Then there exists $q_0 = p^{e_0}$ such that for all $q \geq q_0$ we have $I^{[q]} : f^q \subset \mathfrak{m}^{[q/q_0]}$.*

Notice that later on we will be assuming that $R$ is an excellent normal local ring, which implies that $R$ is analytically irreducible, since the completion of an excellent normal ring is again normal, and thus a domain. Hence one may use Proposition 2.4.

Let $R$ be a Noetherian ring and $I$ an $R$-ideal. We say that $J \subset I$ is a *reduction* of $I$ if $I^{n+1} = JI^n$ for some nonnegative integer $n$. Northcott and Rees introduced this notion in [NR] in order to study multiplicities. The ideal $I$ and it's reduction $J$ have the same multiplicity and thus one would want to shift the attention from $I$ to such a simpler ideal $J$. If $R$ is a Noetherian local ring with infinite residue field then $I$ has infinitely many reductions ([NR]). A reduction $J$ of $I$ is called minimal if it is minimal with respect to inclusion. To facilitate this lack of uniqueness for minimal reductions, Rees and Sally introduced the core of an ideal:

**Definition 2.5.** ([RS])Let $R$ be a Noetherian local ring with infinite residue field. Let $I$ be an $R$-ideal. Then $\operatorname{core}(I) = \bigcap_{J \subset I} J$ where $J$ is a reduction of $I$.

This could be further refined by taking the intersection over all minimal reductions. There has been a significant effort by several authors to find efficient ways of computing this infinite intersection. One result in particular is of special interest to us.

**Theorem 2.6.** ( [CPU1, Theorem 4.5] ) *Let $R$ be a local Cohen–Macaulay ring with infinite residue field and $I$ an $R$-ideal of analytic spread $\ell$. Assume that $I$ is $G_\ell$ and weakly $(\ell - 1)$-residually $S_2$. Then $\operatorname{core}(I) = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_t$ for $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ general $\ell$-generated ideals in $I$ which are reductions of $I$ and for some $t$ finite integer.*

We now explain the conditions in the statement of Theorem 2.6.

The *analytic spread* of $I$, $\ell(I)$, is the Krull dimension of the special fiber ring of, $\mathcal{F}(I) := \bigoplus_{i \geq 0} I^i/\mathfrak{m}I^i$, of $I$. It is well known that if $R$ is a Noetherian local ring with infinite residue field then any minimal reduction $J$ of $I$ has the same minimal number of generators, namely $\mu(J) = \ell(I)$, [NR]. It is straightforward to see that in general $\operatorname{ht} I \leq \ell(I) \leq \dim R$.

Following the definitions given in [CEU] we say that an ideal $I$ satisfies the *property* $G_s$ if for every prime ideal $\mathfrak{p}$ containing $I$ with $\dim R_\mathfrak{p} \leq s - 1$, the minimal number of generators, $\mu(I_\mathfrak{p})$, of $I_\mathfrak{p}$ is at most $\dim R_\mathfrak{p}$. A proper ideal $K$ is called an *$s$-residual intersection* of $I$ if there exists an $s$-generated ideal $\mathfrak{a} \subset I$ so that $K = \mathfrak{a} : I$ and height of $K$ is at least $s \geq g = \operatorname{ht} I$. If the height of $I + K$ is at least $s + 1$, then $K$ is said to be a *geometric $s$-residual intersection* of $I$. If $R/K$ is Cohen–Macaulay for every $i$-residual intersection (geometric $i$-residual intersection) $K$ of $I$ and every $i \leq s$ then $I$ satisfies $AN_s$ ($AN_s^-$). An ideal $I$ is called *$s$-residually $S_2$* (*weakly $s$-residually $S_2$*) if $R/K$ satisfies Serre's condition $S_2$ for every $i$-residual intersection (geometric $i$-residual intersection) $K$ of $I$ and every $i \leq s$.

**Remark 2.7.** Let $(R, \mathfrak{m})$ be a local Noetherian ring and $I$ an $R$-ideal. Let $g = \operatorname{ht} I$ The condition $G_s$ is not difficult to be satisfied. If $I$ is an $\mathfrak{m}$-primary ideal or in general an equimultiple ideal, i.e. $\ell = \ell(I) = \operatorname{ht} I$, then $I$ satisfies $G_\ell$ automatically.

If $(R, m)$ is a local Cohen–Macaulay ring of dimension $d$ and $I$ an $R$-ideal satisfying $G_s$, then $I$ is universally $s$-residually $S_2$ in the following cases:

**(a)** $R$ is Gorenstein, and the local cohomology modules $H_\mathfrak{m}^{d-g-j}(R/I^j)$ vanish for all $1 \leq j \leq s - g + 1$, or equivalently, $Ext_R^{g+j}(R/I^j, R) = 0$ for all $1 \leq j \leq s - g + 1$ ([CEU, Theorem 4.1 and 4.3]).

**(b)** $R$ is Gorenstein, depth $R/I^j \geq \dim R/I - j + 1$ for all $1 \leq j \leq s - g + 1$ ([U, Theorem 2.9(a)]).

**(c)** $I$ has sliding depth ([HVV, Theorem 3.3]).

Notice that condition (b) implies (a) and the property $AN_s$ by [U, Theorem 2.9(a)]. Also the conditions (b) and (c) are satisfied by *strongly Cohen–Macaulay* ideals, i.e. ideals whose Koszul homology modules are Cohen–Macaulay. If $I$ is a Cohen–Macaulay almost complete intersection or a Cohen–Macaulay deviation two ideal of a Gorenstein ring [AH, p. 259] then $I$ is a strongly Cohen–Macaulay. Furthermore, if $I$ is in the linkage class of a complete intersection [Hu1, Theorem 1.11] then $I$ is again a strongly Cohen–Macaulay ideal. Standard examples include perfect ideals of height two and perfect Gorenstein ideals of height three.

## 3. $cl$-Reductions and the definition of $cl$-core

Recall that $J \subset I$ is a reduction of an ideal $I$ if $JI^n = I^{n+1}$. If $J$ is a reduction of $I$, then $J \subset I \subset \overline{J}$. Epstein defines a $cl$-reduction of an ideal $I$ to be an ideal $J \subset I \subset J^{cl}$. If $cl$ is a Nakayama closure we have the following Lemma:

**Lemma 3.1.** ([Ep, Lemma 2.2]) *If $cl$ is a Nakayama closure, then for any $cl$-reduction $J$ of $I$, there is a minimal $cl$-reduction $K$ of $I$ contained in $J$. Moroever, in this situation any minimal generating set of $K$ extends to a minimal generating set of $J$.*

This Lemma shows in particular that minimal $cl$-reductions exist. Following the idea in Definition 2.5 we now define the $cl$-core.

**Definition 3.2.** Let $(R, m)$ be a Noetherian local ring and $cl$ a closure defined on $R$. The $cl$-core of an ideal $I$, $cl\text{-core}(I) = \bigcap_{J \subset I} J$ where $J$ is a $cl$-reduction of $I$.

Recall, an ideal is basic if it does not have any nontrivial reductions. We will say that an ideal is $cl$-basic if it does not have any nontrivial $cl$-reductions. Clearly if $I$ is a basic ideal $\text{core}(I) = I$. If $I$ is a $cl$-basic ideal then $cl\text{-core}(I) = I$. Note that we can restrict the intersection to the minimal $cl$-reductions of $I$. In [Va2], the second author has discussed the partial ordering on the set of closure operations of a ring defined as follows: If $cl_1$ and $cl_2$ are closure operations we say that $cl_1 \leq cl_2$ if and only if $I^{cl_1} \subset I^{cl_2}$.

**Lemma 3.3.** *Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal. If $J_1$ is a minimal $cl_1$-reduction of $I$ then there exists a minimal $cl_2$-reduction $J_2$ of $I$ with $J_2 \subset J_1$.*

*Proof.* Notice that $J_1 \subset I \subset J_1^{cl_1}$, as $J_1$ is a $cl_1$-reduction of $I$. Since $cl_1 \leq cl_2$ then $K^{cl_1} \subset K^{cl_2}$ for all ideals $K \subset R$. Hence $J_1^{cl_1} \subset J_1^{cl_2}$ and $J_1 \subset I \subset J_1^{cl_1} \subset J_1^{cl_2}$. So $J_1$ is a $cl_2$-reduction of $I$ also. Now by Lemma 3.1, there is a minimal reduction of $I$ contained in $J_1$.

□

One consequence of Lemma 3.3 is the following:

**Proposition 3.4.** *Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal. $cl_2$-core$(I) \subset cl_1$-core$(I)$.*

*Proof.* We know that $cl_1\text{-core}(I) = \bigcap_{J_1 \subset I} J_1$ where $J_1$ is a $cl_1$-reduction of $I$. Now for every $J_1$, a $cl_1$-reduction of $I$ there exists a minimal $cl_2$-reduction, $J_2$ contained in $J_1$ by Lemma 3.3. Clearly, $cl_2\text{-core}(I) \subset \bigcap_{J_2 \subset J_1 \subset I} J_2 \subset \bigcap_{J_1 \subset I} J_1$ where $J_2$ are minimal $cl_2$ reductions of $J_1$. ☐

Note that $I^F \subseteq I^* \subseteq \overline{I}$. The first inclusion is clear as $x \in I^F$ if $x^q \in I^{[q]}$ for all $q >> 0$ implies that $cx^q \in I^{[q]}$ for some $c \in R^o$. The second inclusion holds, by [HH, Theorem 5.2]. In particular, we have the following corollary regarding the Frobenius or $F$-core, the $*$-core and the core, which is a $cl$-core where $cl$ is the integral closure.

**Corollary 3.5.** *Let $R$ be an excellent analytically irreducible local domain of characteristic $p$ then* $\text{core}(I) \subset *\text{-core}(I) \subset F\text{-core}(I)$ *for all ideals $I$ in $R$.*

Mimicking the following Proposition in [HS2, Proposition 17.8.9] we see:

**Corollary 3.6.** *Let $R$ be a Noetherian local ring, then $\sqrt{I} = \sqrt{cl\text{-core}(I)}$ for any $cl \leq^-$. In particular, if $R$ is an excellent analytically irreducible local domain of characteristic $p$, $\sqrt{I} = \sqrt{*\text{-core}(I)} = \sqrt{F\text{-core}(I)}$ for all ideals $I$ in $R$.*

To better understand these minimal $cl$-reductions, Epstein mimicked Vraciu's definition of $*$-independence in [Vr1] to define $cl$-independence. The elements $x_1, \ldots, x_n$ are said to be $cl$-independent if $x_i \notin (x_1, \ldots, \widehat{x_i}, \ldots, x_n)^{cl}$, for all $1 \leq i \leq n$. Then he further refines the notion to that of strong $cl$-independence. An ideal is strongly $cl$-independent if every minimal set of generators is $cl$-independent. Epstein then showed in [Ep, Proposition 2.3] that when $cl$ is a Nakayama closure, $J$ is a minimal $cl$-reduction of $I$ if and only if $J$ is a strongly $cl$-independent ideal.

In a Noetherian local ring of characteristic $p$ Vraciu [Vr1] defined the special tight closure, $I^{*sp}$, to be the elements $x \in R$ such that $x \in (\mathfrak{m}I^{[q_0]})^*$ for some $q_0$. Huneke and Vraciu show in [Vr1, Proposition 4.2] that $I^{*sp} \cap I = \mathfrak{m}I$ if $I$ is generated by $*$-independent elements. Note that the minimal $*$-reductions of $I$ are generated by $*$-independent elements. Epstein showed in [Ep, Lemma 3.4] that $I^{*sp} = J^{*sp}$ for all $*$-reductions of $I$.

An ideal $I$ is said to have $cl$-spread, $\ell^{cl}(I)$, if all minimal $cl$ reductions have the same size generating sets. As with the analytic spread, Epstein proves that if $J$ is a minimal $cl$-reduction then $\mu(J) = \ell^{cl}(I)$. He also goes on to prove [Ep, Theorem 5.1] that the $*$-spread is well defined over an excellent analytically irreducible local domain of characteristic $p$. Now if the $cl_1$ and the $cl_2$ spread are defined for $I$, we have:

**Proposition 3.7.** *Let $cl_1$ be a closure operation and $cl_2$ be Nakayama closure operation defined on a Noetherian ring $R$ with $cl_1 \leq cl_2$. Let $I$ be an ideal with well-defined $cl_1$- and $cl_2$-spread then $\ell^{cl_1}(I) \geq \ell^{cl_2}(I)$.*

*Proof.* Let $J_1$ be a $cl_1$–minimal reduction of $I$. Then $\mu(J_1) = \ell^{cl_1}(I)$ ([Ep, Proposition 2.4]). Also $J_1 \subset I \subset J_1^{cl_1} \subset J_1^{cl_2}$, since $cl_1 \leq cl_2$. Therefore $J_1$ is also a $cl_2$ reduction of $I$ (not necessarily minimal). Hence $\mu(J_1) \geq \ell^{cl_2}(I)$ and equality holds if and only if $J_1$ is a minimal $cl_2$ reduction of $I$, according to [Ep, Proposition 2.4]. Hence $\ell^{cl_1}(I) = \mu(J_1) \geq \ell^{cl_2}(I)$. □

In particular, we have the following corollary regarding the Frobenius or $F$-spread, the $*$-spread and the spread of an ideal:

**Corollary 3.8.** *Let $R$ be an excellent analytically irreducible local domain of characteristic $p$ then $\ell(I) \leq \ell^*(I) \leq \ell^F(I)$ for all ideals $I$ in $R$.*

The spread is bounded by the dimension of the ring, but in principle, the $cl$-spreads can grow arbitrarily large. The $cl$-spread of an ideal $I$ is however bounded by the minimal number of generators of $I$, $\mu(I)$.

There are two invariants of a ring related to the spread: the analytic deviation and the second analytic deviation. Recall that in a Noetherian ring, the *analytic deviation* of an ideal $I$ is $ad(I) = \ell(I) - \operatorname{ht} I$. Note that $I$ is equimultiple if $ad(I) = 0$. The *second analytic deviation* of $I$ is $ad_2(I) = \mu(I) - \ell(I)$. We make the following definitions with respect to the $cl$-spread of an ideal $I$.

**Definition 3.9.** The $cl$-deviation of an ideal $I$ in a Noetherian ring is $cld(I) = \ell^{cl}(I) - \operatorname{ht} I$. The second $cl$-deviation of $I$ is $cld_2(I) = \mu(I) - \ell^{cl}(I)$.

**Remark 3.10.** The following are straightforward from the definition above.

**(a)** Note that every $m$-primary ideal $I$ is equimultiple, i.e., $ad(I) = 0$. In general, for $cl \leq^-$, $cld(I) \geq 0$.

**(b)** Note that in a Cohen–Macaulay ring, if $I$ is generated by a system of parameters then $I$ is equimultiple and we have $cld(I) = 0$.

**(c)** Since $\ell(I) \leq \ell^{cl}(I)$, then $cld_2(I) \leq ad_2(I)$.

Note if $I$ is $cl$-closed, then $\ell^{cl}(I) = \mu(I)$. If $I$ is a basic ideal (i.e. $^-$-basic) and $cl \leq^-$, then $\ell^{cl}(I) = \ell(I)$. We would like to know how the core$(I)$ and the $cl$-core$(I)$ are related when $\ell(I) = \ell^{cl}(I)$.

## 4. When *-core and core agree

First we record some straightforward cases when the core and the $*$-core agree. Since an ideal generated by a system of parameters is both basic and $*$-basic we obtain the following:

**Proposition 4.1.** *Let $(R, m)$ be a Noetherian ring of characteristic $p$ and $I$ be an ideal generated by a system of parameters, then $*$-core$(I) = $ core$(I)$.*

*Proof.* A system of parameters is basic and $*$-basic, hence, the only reduction (and $*$-reduction) of $I$ is $I$. Hence, $*$-core$(I) = $ core$(I) = I$. □

Note, when $I$ is generated by a system of parameters, we may have $I^* \subsetneq \overline{I}$, but the core and the $*$-core are equal.

In a one-dimensional domain with infinite residue field, the integral closure and tight closure of any ideal agree [Hu2, Example 1.6.2]. Hence, we have the following:

**Proposition 4.2.** *Let $(R, m)$ be a one-dimensional domain of characteristic $p$ with infinite residue field, then $*$-core$(I) =$ core$(I)$ for all $I \subset R$.*

*Proof.* If $I = 0$ then the assertion is clear. Suppose then that $I \neq 0$ then $\ell(I) = 1$. By [Hu2, Example 1.6.2] it is known that for principal ideals $\overline{(x)} = (x)^*$ and also that $I^* = (x)^*$, for some $x \in R$. Then every minimal reduction and hence minimal $*$-reduction of $I$ is principal. Therefore we obtain equality of the core and the $*$-core. □

We would like to show that in an excellent normal local ring the core and the $*$-core agree for ideals of second $*$-deviation 1. Note that if $(R, \mathfrak{m})$ is a Gorenstein local isolated singularity of characteristic $p > 0$ with test ideal equal to the maximal ideal and $I$ is an ideal generated by part of a system of parameters, then $*d_2(I) = 1$ by Theorem 2.2(since the tight closure is the socle in this case).

To show that the core and the $*$-core agree for ideals with $*d_2(I) = 1$, we will consider general minimal reductions. Recall:

**Definition 4.3.** Let $R$ be a Noetherian local ring with infinite residue field $k$. Let $I = (f_1, \ldots, f_m)$ be an $R$-ideal and let $t$ be a fixed positive integer. We say that $b_1, \ldots, b_t$ are $t$ general elements in $I$ if there exists a dense open subset $U$ of $\mathbb{A}_k^{tm}$ such that for $1 \leq j \leq m$, we have $b_i = \sum_{j=1}^{m} \lambda_{ij} f_j$, where $\underline{\underline{\lambda}} = [\lambda_{ij}]_{ij} \in \mathbb{A}_R^{tm}$, $\overline{\underline{\underline{\lambda}}} \in U$ vary in $U$ and $\overline{\underline{\underline{\lambda}}}$ is the image of $\underline{\underline{\lambda}}$ in $\mathbb{A}_k^{tm}$. The ideal $J$ is called a general minimal reduction of $I$ if $J$ is a reduction of $I$ generated by $\ell(I)$ general elements.

The next two Theorems show that general minimal $*$-reductions exist.

**Theorem 4.4.** *Let $R$ be an excellent normal local ring of characteristic $p > 0$ with perfect infinite residue field. Let $I$ be an ideal with $*d_2(I) = 1$. Then any ideal generated by $\ell^*(I)$ general elements of $I$ is a minimal $*$-reduction of $I$.*

*Proof.* Let $\ell^*(I) = s$. Note that for some $*$-independent elements $f_1, \ldots, f_s \in I$, $I^* = J^*$ where $J = (f_1, \ldots, f_s)$. Thus $J$ is a minimal $*$–reduction of $I$. By [Ep, Lemma 2.2] we know that this generating set of $J$ can be extended to a generating set of $I$. In other words, $I = (f_1, \cdots, f_s, f_{s+1})$.

Let $T = R[X_{ij}]$ where $1 \leq i \leq s$ and $1 \leq j \leq s+1$. Let $a_i = \sum_{j=1}^{s+1} X_{ij} f_j$ for $1 \leq i \leq s$ and consider the $T$–ideal $\widetilde{J} = (a_1, \cdots, a_s)$. Write $\underline{\underline{X}}$ for $[X_{ij}]_{ij}$.

Consider the $R$–homomorphism $\pi_{\underline{\underline{\lambda}}} : T \to R$ that sends $\underline{\underline{X}}$ to $\underline{\underline{\lambda}}$, where $\underline{\underline{\lambda}} \in \mathbb{A}_R^{s(s+1)}$. Notice that for $\underline{\underline{\lambda_0}} = [\delta_{ij}]$ one has $\pi_{\underline{\underline{\lambda_0}}}(\widetilde{J}) = J$.

Let $\mathfrak{m}$ denote the maximal ideal of $R$ and $k = R/\mathfrak{m}$ be the residue field of $R$. We need to find a dense open set $U \subset \mathbb{A}_k^{s(s+1)}$, such that $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is also a $*$-reduction for $\underline{\underline{\overline{\lambda}}} \in U$. Let $\overline{\lambda_{ij}}$ be the image of $\lambda_{ij}$ in $R/\mathfrak{m}$. Then the generators of the $R/\mathfrak{m}$ vector space $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})/\mathfrak{m}\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ are $\overline{u_i} = \sum_{j=1}^{s+1} \overline{\lambda_{ij}} f_j$.

Define $L = [\overline{\lambda_{ij}}]_{ij}$ to be the matrix defined by the coefficients of the $a_i$, $i = 1, \dots, s$. $L$ is a $s \times (s+1)$ matrix with coefficients in $k = R/\mathfrak{m}$. Suppose $L_s$ is the $s \times s$ submatrix of $L$ obtained by omitting the last column. We define the open set $U \subset \mathbb{A}_k^{s(s+1)}$ to be set of $L$'s satisfying $\det(L_s) \neq 0$. Since $\{L | \det(L_s) = 0\}$ is closed, then clearly $U$ is open. To see that $U$ is dense, suppose $U'$ is an open set containing a point $\underline{\underline{\overline{\mu}}} \notin U$. We need to see that $U \cap U'$ is not empty. Let $M$ be the $s \times (s+1)$ matrix with coefficients in $k = R/\mathfrak{m}$ representing $\underline{\underline{\overline{\mu}}}$. Suppose $M_s$ is the $s \times s$ submatrix of $M$ obtained by omitting the last column. Since $\underline{\underline{\overline{\mu}}} \notin U$ then $\det(M_s) = 0$. Since the set of $\underline{\underline{\overline{\mu}}}$ with $\det(M_s) = 0$ forms an ideal in $\mathbb{A}_k^{s(s+1)}$, then if $U'$ is open, then $U'$ has to contain elements $\underline{\underline{\overline{\lambda}}}$ with $\det(L_s) \neq 0$. Hence, $U \cap U' \neq \varnothing$ and $U$ is dense.

Since for any $\underline{\underline{\overline{\lambda}}} \in U$, $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is a general reduction with $\det(L_s) \neq 0$, then $V = \pi_{\underline{\underline{\lambda}}}(\widetilde{J})/\mathfrak{m}\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is a $s$-dimensional $k = R/\mathfrak{m}$-vector space with basis $\overline{a_1}, \dots, \overline{a_s}$. Row reducing $\overline{L}$, we obtain the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 1 & 0 & \cdots & 0 & \beta_2 \\ 0 & 0 & 1 & \cdots & 0 & \beta_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_s \end{pmatrix}$$

where the $\beta_i \in k$. This implies that an alternate basis for $V$ is $\{f_1 + \beta_1 f_{s+1}, \dots, f_s + \beta_s f_{s+1}\}$. Let $J_{\text{gen}} = (f_1 + \beta_1 f_{s+1}, \dots, f_s + \beta_s f_{s+1}) = \pi_{\underline{\underline{\lambda}}}(\widetilde{J})$.

Case 1: Suppose that for all $1 \leq i \leq s$ we have that $\beta_i \in \mathfrak{m}$. Let $K = J_{\text{gen}} + \mathfrak{m}I$. Then we claim that $K = J + \mathfrak{m}I$. To see this consider a generator $\alpha$ for $K$. Then $\alpha = f_i + \beta_i f_{s+1} + \delta$, where $\delta \in \mathfrak{m}I$. But as $\beta_i \in \mathfrak{m}$ and $f_i \in J$ then $\alpha \in J + \mathfrak{m}I$. Now let $\alpha'$ be a generator of $J + \mathfrak{m}I$. Then $\alpha' = f_i + \delta'$, where $\delta' \in \mathfrak{m}I$. Since $\beta_i \in \mathfrak{m}$ then $\delta' - \beta_i f_{s+1} \in \mathfrak{m}I$. Hence $\alpha' = f_i + \beta_i f_{s+1} + (\delta' - \beta_i f_{s+1}) \in J_{\text{gen}} + \mathfrak{m}I = K$.

Next we claim that $(J + \mathfrak{m}I)^* = J^*$. Notice that $J \subset J + \mathfrak{m}I \subset I$. Taking the tight closure we obtain $J^* \subset (J + \mathfrak{m}I)^* \subset I^* = J^*$. Thus, $(J + \mathfrak{m}I)^* = J^*$. Overall we have the following inclusions:

$$J_{\text{gen}} \subset I \subset (J + \mathfrak{m}I)^* = (J_{\text{gen}} + \mathfrak{m}I)^*$$

Now by [Ep, Proposition 2.1] we have that $I^* \subset J^*_{\text{gen}}$.

Case 2: If $\beta_i \notin \mathfrak{m}$ for some $i$ then without loss of generality we may assume that $i = s$ and $\beta_s = 1$. Then $J_{\text{gen}} = (f_1 + \beta_1 f_{s+1}, \dots, f_s + f_{s+1})$. Hence $f_1 - \beta_1 f_s \in J_{\text{gen}}$. Let $f'_1 = f_1 - \beta_1 f_s$ and replace $f_1$ with $f'_1$. Continuing this way we may assume that $J_{\text{gen}} = (f_1, \dots, f_{s-1}, f_s + f_{s+1})$. Suppose that $f_{s+1} \notin (f_1, \dots, f_{s-1}, f_s + f_{s+1})^*$.

Since $f_{s+1} \in I \subset J^*$, then we may take $c \in R^0$ such that $cf^q_{s+1} \in J^{[q]}$ for every $q = p^e$. Hence $cf^q_{s+1} = \sum_{i=1}^s r_{iq} f^q_i$. Then $cf^q_{s+1} + r_{sq} f^q_{s+1} = \sum_{i=1}^s r_{iq} f^q_i + r_{sq} f^q_{s+1} = \sum_{i=1}^{s-1} r_{iq} f^q_i + r_{sq}(f^q_s + f^q_{s+1})$. Let $c_q = c + r_{sq}$. Then $c_q f^q_{s+1} = \sum_{i=1}^{s-1} r_{iq} f^q_i + r_{sq}(f^q_s + f^q_{s+1})$ and in particular $c_q f^q_{s+1} \in (f_1, \dots, f_{s-1}, f_s + f_{s+1})^{[q]}$. Therefore by [Ab, Proposition 2.4] $c_q \in \mathfrak{m}^{[N(q)]}$. For every integer $N$ there exists a $q_N$ such that $r_{sq_N} \in \mathfrak{m}^N$. Fix $N_0$ such that $c \notin \mathfrak{m}^{N_0}$. It then follows that for every $N \geq N_0$ we have $c + r_{sq_N} \in \mathfrak{m}^N$. Since $c \notin \mathfrak{m}^N$ then $r_{sq_N} \notin \mathfrak{m}^N$ for all $N \geq N_0$.

Notice that $cf^q_{s+1} - \sum_{i=1}^{s-1} r_{iq} f^q_i = r_{sq} f^q_s$. Thus by [Ab, Proposition 2.4] we have that $f_s \in (f_1, \dots, f_{s-1}, f_{s+1})^*$ since $r_{sq_N} \notin \mathfrak{m}^N$. Therefore $(f_1, \dots, f_{s-1}, f_{s+1})^* = I^*$ and thus $(f_1, \dots, f_{s-1}, f_{s+1})$ is a minimal $*$-reduction of $I$.

By [HV, Theorem 2.1] we have that $J^* = J + J^{*sp}$. Also by [Ep, Lemma 3.4] since $J \subset I$ and $J^* = I^*$ then $J^{*sp} = I^{*sp}$. Therefore $I^* = J + I^{*sp}$. Since $f_{s+1} \in I^*$ and $f_{s+1} \notin J$ then $f_{s+1} \in I^{*sp}$. Therefore $(f_1, \dots, f_{s-1}, f_{s+1})$ is not a minimal $*$-reduction of $I$, by [Vr2, Proposition 1.12(b)], which is a contradiction. Therefore $f_{s+1} \in J^*_{\text{gen}}$ and thus $J^*_{\text{gen}} = I^*$.

□

We are able to generalize Theorem 4.4 and relax the condition on $*d_2(I)$.

**Theorem 4.5.** *Let $R$ be an excellent normal local ring of characteristic $p > 0$ with perfect infinite residue field. Let $I$ be an ideal. Then any ideal generated by $\ell^*(I)$ general elements of $I$ is a minimal $*$-reduction of $I$.*

*Proof.* Let $\ell^*(I) = s$. Then there exists $*$-independent elements $f_1, \dots, f_s \in I$ such that $I^* = (f_1, \dots, f_s)^*$. Let $J = (f_1, \dots, f_s)$. Hence $J$ is a minimal $*$-reduction of $I$. By [Ep, Lemma 2.2] we know that this generating set of $J$ can be extended to a generating set of $I$. In other words, $I = (f_1, \cdots, f_s, f_{s+1}, f_{s+2}, \dots, f_{s+n})$.

As above we form an ideal generated by general elements and we may assume that $J_{\text{gen}} = (f_1 + \beta_{11} f_{s+1} + \ldots + \beta_{1n} f_{s+n}, \dots, f_s + \beta_{s1} f_{s+1} + \ldots + \beta_{sn} f_{s+n})$. Let $\mathfrak{m}$ denote the maximal ideal of $R$ and $k = R/\mathfrak{m}$ be the residue field of $R$.

Case 1: Suppose that for all $1 \leq i \leq s$ and for all $1 \leq j \leq n$ we have that $\beta_{ij} \in \mathfrak{m}$. Let $K = J_{\text{gen}} + \mathfrak{m}I$. Then we claim that $K = J + \mathfrak{m}I$. To see this consider a generator $\alpha$ for $K$. Then $\alpha = f_i + \beta_{i1} f_{s+1} + \ldots + \beta_{in} f_{s+n} + \delta$, where $\delta \in \mathfrak{m}I$. But as $\beta_{ij} \in \mathfrak{m}$ and $f_i \in J$ then $\alpha \in J + \mathfrak{m}I$. Now let $\alpha'$ be a generator of $J + \mathfrak{m}I$. Then $\alpha' = f_i + \delta'$, where $\delta' \in \mathfrak{m}I$. Since $\beta_{ij} \in \mathfrak{m}$ for all $1 \leq i \leq s$ and for all $1 \leq j \leq n$ then $\delta' - \beta_{i1} f_{s+1} + \ldots + \beta_{in} f_{s+n} \in \mathfrak{m}I$. Hence $\alpha' = f_i + \beta_{i1} f_{s+1} + \ldots + \beta_{in} f_{s+n} + (\delta' - \beta_{i1} f_{s+1} + \ldots + \beta_{in} f_{s+n}) \in J_{\text{gen}} + \mathfrak{m}I = K$.

Next we claim that $(J+\mathfrak{m}I)^* = J^*$. Notice that $J \subset J+\mathfrak{m}I \subset I$. Taking the tight closure we obtain $J^* \subset (J+\mathfrak{m}I)^* \subset I^* = J^*$. Thus, $(J+\mathfrak{m}I)^* = J^*$. Overall we have the following inclusions:

$$J_{\text{gen}} \subset I \subset (J+\mathfrak{m}I)^* = (J_{\text{gen}}+\mathfrak{m}I)^*$$

Now by [Ep, Proposition 2.1] we have that $I^* \subset J^*_{\text{gen}}$.

Case 2: Suppose $\beta_{ij} \notin \mathfrak{m}$ for some $1 \leq i \leq s$ and $1 \leq j \leq n$. Without loss of generality we may assume that $i = s$ and $j = n$ and that $\beta_{sn} = 1$. Hence $J_{\text{gen}} = (f_1 + \beta_{11} f_{s+1} + \ldots + \beta_{1n} f_{s+n}, \ldots, f_s + \beta_{s1} f_{s+1} + \ldots + f_{s+n})$.

We will proceed by induction on $n = {*d_2}(I) = \mu(I) - \ell^*(I)$. If $n = 0$ there is nothing to show and if $n = 1$ then Theorem 4.4 gives the result. So we assume that $n > 1$ and for the results holds for any ideal $I'$ with ${*d_2}(I') = n-1$.

Let $g_i = f_i + \beta_{i1} f_{s+1} + \ldots + \beta_{in} f_{s+n}$. Then $g_i - \beta_{in} g_s = (f_i - \beta_{in} f_s) + \sum_{j=1}^{n-1} (\beta_{ij} - \beta_{in}\beta_{sj}) f_{s+j} \in J_{\text{gen}}$. Notice that $f'_i = f_i - \beta_{in} f_s \in J$ and let $\beta'_{ij} = \beta_{ij} - \beta_{in}\beta_{sj}$. Therefore, we can replace $f_i$ with $f'_i$ and $\beta_{ij}$ with $\beta'_{ij}$ to assume that $J_{\text{gen}} = (f_1 + \beta_{11} f_{s+1} + \ldots + \beta_{1(n-1)} f_{s+n-1}, \ldots, f_s + \beta_{(s-1)1} f_{s+1} + \ldots + \beta_{(s-1)(n-1)} f_{s+n-1}, f_s + \beta_{s1} f_{s+1} + \ldots + f_{s+n})$.

Let $h_i = f_i + \beta_{i1} f_{s+1} + \ldots + \beta_{i(n-1)} f_{s+n-1}$. Then $J_{\text{gen}} = (h_1, \ldots, h_{s-1}, h_s + f_{s+n})$. Let $L = (h_1, \ldots, h_s)$ and $J_1 = (f_1, \ldots, f_s, f_{s+1}, \ldots, f_{s+n-1})$.

Since $g_i$ is a general element for all $1 \leq i \leq s$ then there exists $U \subset \mathbb{A}_k^{s(s+n)}$ a dense open such that the image of $\beta_i = [0, \ldots, 0, 1, 0, \ldots, \beta_{i1}, \ldots, \beta_{i(n-1)}, \beta_{in}]$ varies in $U$. Consider the natural projection map $\pi : \mathbb{A}_k^{s(s+n)} \to \mathbb{A}_k^{s(s+n-1)}$ such that $\pi((\underline{a_1}, \ldots, \underline{a_{s+n-1}}, \underline{a_{s+n}})) = (\underline{a_1}, \ldots, \underline{a_{s+n-1}})$ and $\underline{a_i} \in \mathbb{A}_k^s$. Let $W = \pi(U)$. As $U$ is dense and open then $U \neq \emptyset$ and thus $W \neq \emptyset$ and $W$ is also open, since $\pi$ is an open map. Therefore $W$ is a dense open set. As $\beta_i$ is allowed to vary in $U$ then $\pi(\beta_i)$ varies in $W$ and thus $h_i$ is also a general element.

Notice that $J \subset J_1 \subset I^*$ and thus $J^* = J_1^*$. Hence $\ell^*(J_1) \leq s$. Therefore ${*d_2}(J_1) \leq n-1$ and hence by our inductive hypothesis $L$ is a $*$-reduction of $J_1$ and $L^* = J_1^* = J^* = I^*$. We are claiming that $J^*_{\text{gen}} = L^* = I^*$. It is enough to show that $f_{s+n} \in J^*_{\text{gen}}$. Suppose that $f_{s+n} \notin J^*_{\text{gen}}$. Then as in the proof of Theorem 4.4 we obtain that $h_s \in (h_1, \ldots, h_{s-1}, f_{s+n})^*$. By [HV, Theorem 2.1] we have that $L^* = L + L^{*sp}$. Also by [Ep, Lemma 3.4] since $L \subset I$ and $L^* = I^*$ then $L^{*sp} = I^{*sp} = J^{*sp}$. Therefore $I^* = L + I^{*sp}$. Since $f_{s+n} \in I^*$ and $f_{s+n} \notin L$ then $f_{s+n} \in I^{*sp}$. Therefore $(h_1, \ldots, h_{s-1}, f_{s+n})$ is not a minimal $*$-reduction of $I$, by [Vr2, Proposition 1.12(b)], which is a contradiction. Hence $f_{s+n} \in J^*_{\text{gen}}$ and $J^*_{\text{gen}} = L^* = I^*$.

□

**Corollary 4.6.** *Let $(R, \mathfrak{m})$ be a Gorenstein isolated singularity of characteristic $p > 0$ with perfect infinite residue field. Suppose that the test ideal of $R$ is $\mathfrak{m}$. Let $I$ be a tightly closed ideal and suppose that $I = J^*$ where $J$ is a parameter ideal minimally generated by $s$ elements. Then any ideal generated by $s$ general elements of $I$ is a minimal $*$-reduction of $I$.*

*Proof.* Suppose $J = (f_1, \ldots, f_s)$ is a parameter ideal. Then $J$ is a minimal $*$-reduction of $I = J^* = (J : m)$, where the last equality is obtained by Theorem 2.2. Since $R$ is Gorenstein then the socle $(J : \mathfrak{m})/J$ is a one dimensional vector space. Hence $I = (f_1, \ldots, f_s, f_{s+1})$, where $f_{s+1} \notin J$. Therefore $\mu(I) = s + 1$ and $*d_2(I) = 1$. Thus by Theorem 4.4, any ideal generated by $s$ general elements is a minimal $*$-reduction of $I$. □

**Corollary 4.7.** *Let $(R, \mathfrak{m})$ be a $d$-dimensional Gorenstein isolated singularity of characteristic $p > 0$ with perfect infinite residue field. Suppose that the test ideal of $R$ is $\mathfrak{m}$. Let $I$ be an $\mathfrak{m}$-primary tightly closed ideal and suppose that $I = J^*$ where $J$ is a parameter ideal. Then* $\operatorname{core}(I) = *\text{-core}(I)$.

*Proof.* Since $I$ is $\mathfrak{m}$-primary then $\ell(I) = \ell^*(I) = d$. By Corollary 4.6 any ideal generated by $d$ general elements is a general minimal $*$-reduction. Notice that these minimal general $*$-reductions are also minimal reductions of $I$, since $\ell(I) = d$.

Also, since $I$ is $\mathfrak{m}$-primary then by Theorem 2.6 ([CPU1, Theorem 4.5]) we have that $\operatorname{core}(I)$ is a finite intersection of general minimal reductions. Since each general minimal reduction is also a minimal $*$-reduction then $*\text{-core}(I) \subset \operatorname{core}(I)$. On the other hand $\operatorname{core}(I) \subset *\text{-core}(I)$, by Corollary 3.5.

□

**Theorem 4.8.** *Let $R$ be a local Cohen–Macaulay normal domain of characteristic $p > 0$ with perfect infinite residue field. Let $I$ be an ideal with $\ell^*(I) = \ell(I) = s$. We further assume that $I$ satisfies $G_s$ and is weakly $(s-1)$-residually $S_2$. Then* $\operatorname{core}(I) = *\text{-core}(I)$.

*Proof.* We know that $\operatorname{core}(I) \subset *\text{-core}(I)$ by Corollary 3.5. According to Theorem 2.6 the core is a finite intersection of general minimal reductions. Since every general minimal reduction is a minimal $*$-reduction by Theorem 4.5, we obtain the opposite inclusion. □

## 5. The $*$-core in complete one dimensional semigroup rings

In Proposition 4.1, we saw that the core and the $*$-core agree for all ideals in a one dimensional domain of characteristic $p > 0$ with infinite residue field . In Huneke and Swanson's paper [HS2], one of the first questions that they ask is: if $I$ is integrally closed, is $\operatorname{core}(I)$ integrally closed? They settle this question in the setting of a two-dimensional regular local ring. Corso, Polini and Ulrich in [CPU2, Theorem 2.11] showed that if $R$ is a local Cohen–Macaulay normal ring with infinite residue field then $\operatorname{core}(I)$ is integrally closed, when $I$ is a normal ideal of positive height, universally weakly $(\ell - 1)$-residually $S_2$ and satisfies $G_\ell$ and $AN^-_{\ell-1}$, where $\ell = \ell(I)$. A related question is: if $I$ is tightly closed, is $*\text{-core}(I)$ tightly closed? We will consider this question now for complete one-dimensional semigroup rings with test ideal equal to the maximal ideal. The second author showed the following:

**Theorem 5.1.** ([Va1]) *Let $(R, \mathfrak{m})$ be a one-dimensional domain. The test ideal of $R$ is equal to the conductor, i.e. $\tau = \mathfrak{c} = \{c \in R | \phi(1) = c, \phi \in Hom_R(\overline{R}, R)\}$.*

Note, in a one-dimensional local semigroup ring, the semigroup is a sub-semigroup of $N_0$. For each sub-semigroup $S$ of $N_0$, there is a smallest $m$ such that for all $i \geq m$, $i \in S$. The conductor of such a one dimensional semigroup ring is $\mathfrak{c} = < t^m, t^{m+1}, t^{m+2}, \dots >$, [Ei, Exercise 21.11]. Hence, the test ideal in a one-dimensional semigroup ring is the maximal ideal, if the conductor is the maximal ideal. This can only happen if the semigroup has the form $\{n + i \mid i \geq 0\}$ for some $n \geq 0$. Hence, if $R$ is complete the ring is of the form $R = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$. As in [Va2, Proposition 4.1], we will show that the principal ideals are of the form $(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$:

**Proposition 5.2.** *Each nonzero nonunit principal ideal of $R = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ can be expressed in the form $(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$, $a_i \in k$, $m \geq n$.*

*Proof.* Suppose $0 \neq f \in R$. Thus, after multiplying by a nonzero element of $k$, $f = t^m + a_1 t^{m+1} + a_2 t^{m+2} + \cdots$ for $m \geq n$. We will show that $t^r \in (f)$ for $r \geq m+n$. Hence, $t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1} \in (f)$. Similarly, $t^r \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$ for $r \geq m + n$. Hence, $f \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$.

Let $g \in k[[t]]$. Note that $t^{r-m} g \in k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$. Hence, if $g$ is a unit in $k[[t]]$, then $t^{r-m} g^{-1} \in k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ also. In $k[[t]]$,

$$f = t^m (1 + a_1 t + a_2 t^2 + \cdots) = t^m g.$$

Note that $t^{r-m} g^{-1} f = t^r$. Similarly $t^r \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$. Since $f - (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}) = a_n t^{2n} + a_{n+1} t^{2n+1} + \cdots$ which is an element of $(f) \bigcap (t^m + a_1 t^{n+1} + \cdots + a_{n-1} t^{m+n-1})$, we see that $(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}) = (f)$. Hence, all principal ideals of $k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ have the form

$$(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}).$$

□

**Proposition 5.3.** *Let $k$ be an infinite field of characteristic $p > 0$. Any tightly closed ideal in $R = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ is of the form $(t^m, t^{m+1}, \dots, t^{m+n-1})$ for some $m \geq n$.*

*Proof.* Suppose $I$ is a tightly closed ideal in $R$. Since $R$ is a one dimensional domain, there is a principal ideal $(f) \in I$, with $(f)^* = I$. By Proposition 5.2,

$$(f) = (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$$

for some $m \geq n$ and $a_i \in k$. Using Theorem 2.2 and the arguments we followed in Proposition 5.2, $I = (f)^* = (f) : \mathfrak{m} = (t^m, t^{m+1}, \dots, t^{m+n-1})$. □

**Proposition 5.4.** *Let $R = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$ with $k$ an infinite field of characteristic $p > 0$. If $I \subset R$ is tightly closed, then $*$-core$(I)$ is tightly closed.*

*Proof.* If $I = (0)$, then clearly $*$-core$(I) = (0)$ and thus the assertion is clear. Since $R$ is a one-dimensional domain then core$(I) = *$-core$(I)$. If $I$ is basic then $I$ is also $*$-basic and again the assertion is clear. So suppose $I$ is not basic, nonzero and tightly closed. Then $I = (t^m, t^{m+1}, \ldots, t^{m+n-1})$ for some $m \geq n$, by Proposition 5.3. Since $I$ is non-zero then $I$ is $\mathfrak{m}$-primary, where $\mathfrak{m}$ is the maximal ideal of $R$. Hence by Theorem 2.6 we have that core$(I) = \bigcap_{i=1}^{s} (f_i)$, for some positive integer $s$ and $(f_i)$ general minimal reductions of $I$ for all

$1 \leq i \leq s$. Let $(f_i)$ be such a general minimal reduction. Then $(f_i) = (t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for some $a_{ij} \in k$, since $f_i$ is a general element in $I$. As in the proof of Proposition 5.2, we see that $t^r \in (t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$ for all $r \geq m+n$. Hence, $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset (f_i)$ for all $i$ and thus $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset *$-core$(I)$.

On the other hand let $g \in *$-core$(I)$. Hence $g \in \bigcap_{i=1}^{s} (f_i)$. It is clear that $(g) \neq (f_i)$ for some $i$. Then $g = a(t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for some $a \in R$ and $a_{ij} \in k$. If $a$ is a unit then $(g) = (f_i)$, which is a contradiction. Hence we may assume that $a$ is not a unit. Thus $a = \beta_1 t^n + \beta_2 t^{n+1} + \ldots$ and $g = \gamma_0 t^{m+n} + \gamma_1 t^{m+n+1} + \ldots \gamma_{n-1} t^{m+2n-1} + t^n(\gamma_n t^{m+n} + \gamma_{n+1} t^{m+n+1} + \ldots \gamma_{2n-1} t^{m+2n-1}) + \ldots$. Therefore $g \in (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ and thus $*$-core$(I) \subset (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$. Finally notice that $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ is a tightly closed ideal and $*$-core$(I) = (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$.

□

Note, since the core$(I) = *$-core$(I)$ by Proposition 4.1 and the tight closure of an ideal agrees with the integral closure in a one dimensional domain with infinite residue field, we obtain:

**Corollary 5.5.** *Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ with $k$ an infinite field of characteristic $p > 0$. If $I \subset R$ is integrally closed, then* core$(I)$ *is integrally closed.*

**Remark 5.6.** The question of whether the core of an integrally closed ideal is integrally closed as well was first addressed by Huneke and Swanson, [HS1]. They answer this question positively when the ring is a 2–dimensional regular ring, [HS1, Corollary 3.12]. This question was also addressed by several other authors later, (see [CPU2, Theorem 2.11, Corollary 3.7], [PU, Corollary 4.6], and [HyS1, Proposition 5.5.3]).

We note here that Corollary 5.5 is not covered by any of the results mentioned above. In [CPU2, Corollary 3.7] and [PU, Corollary 4.6] it is required that the ring $R$ is Gorenstein. The ring $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ with $k$ an infinite field of characteristic $p > 0$ is not Gorenstein unless $n = 2$. In [CPU1, Theorem 2.11] the Gorenstein condition can be relaxed to Cohen–Macaulay rings, but in addition the Rees algebra of $I$ and $I$ are assumed to be normal and $J : I$ is independent of $J$ for every minimal reduction $J$ of $I$. Notice that the ideal $I$ in Corollary 5.5 is normal and $J : I = \tau = \mathfrak{m}$ is independent of the minimal reduction $J$. However, the Rees algebra of $I$ is not normal, since $R$ is not normal. Finally, in [HyS1, Proposition 5.5.3] it is assumed that the ring $R$ is Cohen–Macaulay, $R$ contains the rational numbers and the Rees algebra of $I$ is Cohen–Macaulay whereas the ring in Corollary 5.5 does not contain the rational numbers.

## 6. Examples

Since, the tight closure of an ideal is much closer to the ideal than the integral closure, we expected to find examples of ideals $I$ where the $*$-core$(I) \supsetneq$core$(I)$. The following example gives a family of rings where $*$-core$(\mathfrak{m}^2) \neq$ core$(\mathfrak{m}^2)$.

**Example 6.1.** Let $R = \mathbb{Z}/p\mathbb{Z}(u, v, w)[[x, y, z]]/(ux^p+vy^p+wz^p)$. Then $R$ is a normal domain, [Ep]. In [VV], Vraciu and the second author computed the test ideal of $R$ to be $\mathfrak{m}^{p-1}$, where $\mathfrak{m}$ is the maximal ideal of $R$.

For $p = 2$, we compute the $*$-core of $\mathfrak{m}^2$. In this case the test ideal is $\mathfrak{m}$. Note that the $*$-spread of $\mathfrak{m}^2$ is 3. For example $J = (y^2, yz, z^2)$ is a minimal $*$-reduction of $\mathfrak{m}^2$. To see this notice that $x^2 = \frac{v}{u}y^2 + \frac{w}{u}z^2 \in J$ and $(xz)^2 = (\frac{v}{u}y^2 + \frac{w}{u}z^2)z^2 = \frac{v}{u}(yz)^2 + \frac{w}{u}(z^2)^2$. Hence $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and thus $\mathfrak{m}^2 = J^*$. On the other hand $I = (y^2, z^2)$ is not a $*$-reduction of $\mathfrak{m}^2$ since $I^* = I : \mathfrak{m} = (y^2, z^2, xyz) \neq \mathfrak{m}^2$. Similarly $(y^2, yz)$ and $(yz, z^2)$ are not $*$-reductions of $\mathfrak{m}^2$ and thus $\ell^*(\mathfrak{m}^2) = 3$.

In addition we note that $(x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2)$,and $(yz, xz, xy)$ are all minimal $*$-reductions of $\mathfrak{m}^2$. Hence $*$-core$(\mathfrak{m}^2) \subset (x^2, xy, y^2) \cap (x^2, xz, z^2) \cap (y^2, yz, z^2) \cap (yz, xz, xy) = (x^2, y^2, z^2, xyz) \cap (yz, xz, xy) = \mathfrak{m}^3$. Note that $\mathfrak{m}^3 = \mathfrak{m}J^* \subset J$ for all $J$ minimal $*$-reductions of $\mathfrak{m}^2$, since $\mathfrak{m}$ is the test ideal. Hence, $*$-core$(\mathfrak{m}^2) = \mathfrak{m}^3$.

For $p \geq 3$, the computation of the $*$-core of $\mathfrak{m}^2$ is as follows: The $*$-spread of $\mathfrak{m}^2$ is 3. Once again, $J = (y^2, yz, z^2)$ is a minimal $*$-reduction of $\mathfrak{m}^2$. Notice that $(x^2)^p = (x^p)^2 = (\frac{v}{u}y^p + \frac{w}{u}z^p)^2 = \frac{v^2}{u^2}(y^2)^p + 2\frac{vw}{u^2}(yz)^p + \frac{w^2}{u^2}(z^2)^p \in J^F$ and $(xz)^p = (\frac{v}{u}y^p + \frac{w}{u}z^p)z^p$. Thus $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and therefore $J^* = \mathfrak{m}^2$. Note that $I = (y^2, z^2)$ is not a $*$-reduction of $\mathfrak{m}^2$ since $I^* = I : \mathfrak{m}^{p-1} = (y^2, z^2, x^{p-1}yz) : \mathfrak{m}^{p-2} = (y^2, z^2, x^{p-1}y, x^{p-1}z, x^{p-2}yz) : \mathfrak{m}^{p-3} = \cdots = (y^2, z^2) + \mathfrak{m}^3 \neq \mathfrak{m}^2$. As the test ideal is $\mathfrak{m}^{p-1}$, we see that $\mathfrak{m}^{p-1}J^* = \mathfrak{m}^{p+1} \subset J$ for all minimal $*$-reductions $J$ of $\mathfrak{m}^2$. Notice also that $(x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2)$, and $(yz, xz, xy)$ are all minimal $*$-reductions of $\mathfrak{m}^2$. Therefore $*$-core$(\mathfrak{m}^2) \subset (x^2, xy, y^2) \cap (x^2, xz, z^2) \cap (y^2, yz, z^2) \cap (yz, xz, xy) = m^{p+1} + (xyz, x^2y^2, x^2z^2, y^2z^2)$. Hence $\mathfrak{m}^{p+1} \subset *$-core$(\mathfrak{m}^2) \subset \mathfrak{m}^{p+1} + (xyz, x^2y^2, x^2z^2, y^2z^2)$.

We also note that the $\ell(\mathfrak{m}^2) = 2$, and that $H = (x^2, yz)$ is a minimal reduction of $\mathfrak{m}^2$ in any characteristic.

If $p = 2$ then the reduction number of $\mathfrak{m}^2$ with respect to $H$ is 1. Since char $k = 2 > 1$ then we may use the formula for the core as in [PU, Theorem 4.5]. Hence core($\mathfrak{m}^2$) $= H^2 : \mathfrak{m}^2 = m^4$, where the last equality follows from calculations using the computer algebra program Macaulay 2, [M2]. Therefore core($\mathfrak{m}^2$) $\subsetneq$ $*$-core($\mathfrak{m}^2$).

If $p = 3$ then the reduction number of $\mathfrak{m}^2$ with respect to $H$ is 2. Since now char $k = 3 > 2$ we may again use the formula as in [PU, Theorem 4.5]. Thus core($\mathfrak{m}^2$) $= H^3 : \mathfrak{m}^4 = \mathfrak{m}^5$, where the last equality is again obtained using the computer algebra program Macaulay 2, [M2]. Notice that since $\mathfrak{m}^4 \subset *$-core($\mathfrak{m}^2$) then core($\mathfrak{m}^2$) $\subsetneq$ $*$-core($\mathfrak{m}^2$) again.

When the spread and the $*$-spread agree, it is not necessarily the case that all reductions of an ideal are $*$-reductions. However, the following example exhibits that even so, the core and the $*$-core agree for the maximal ideal in the following ring. In some sense, the following example prompted us to prove Theorem 4.4, Theorem 4.5 and Theorem 4.8.

**Example 6.2.** Let $R = k[[x, y, z]]/(x^2 - y^3 - z^7)$, where the $k$ is an infinite field and char $k > 7$. Let $\mathfrak{m} = (x, y, z)$ denote the maximal ideal of $R$. We observe first that $\mathfrak{m}$ is the test ideal, [Va1].

We will show that $*$-spread of $\mathfrak{m}$ is 2, $\ell(\mathfrak{m}) = 2$ and core($\mathfrak{m}$) $= \mathfrak{m}^2 = *$-core($\mathfrak{m}$).

First note that $R$ is a 2–dimensional Gorenstein local ring and hence $\ell(\mathfrak{m}) = 2$. Let $J = (y, z)$. Then $J$ is a reduction of $\mathfrak{m}$ with reduction number 1. Since char $k > 1$ then core($\mathfrak{m}$) $= J^2 : \mathfrak{m} = \mathfrak{m}^2$ by [PU, Theorem 4.5]. Notice that this does not agree with the formula in Hyry-Smith [HyS2, Theorem 4.1] or Fouli-Polini-Ulrich [FPU, Theorem 4.4] since $a = 42 - 21 - 14 - 6 = 1$ and core($\mathfrak{m}$) $\neq \mathfrak{m}^{2+a+1} = \mathfrak{m}^4$. The hypothesis that $\mathfrak{m}$ is generated by elements of degree 1 is important in their formula.

On the other hand, $J$ is also a minimal $*$-reduction of $\mathfrak{m}$. Note that $y, z$ form a system of parameter, hence by Theorem 2.2 $(y, z)^* = (y, z) : \mathfrak{m} = (x, y, z) = \mathfrak{m}$. Therefore $\ell^*(\mathfrak{m}) = 2 = \ell(\mathfrak{m})$. We claim that $J_1 = (x + z, y)$ and $J_2 = (x + y, z)$ are also minimal $*$-reductions. Denote $p_n(x, y) = x^n + x^{n-1}y + \ldots + xy^{n-1} + y^n$. Note that if $n$ is odd,

$$x^n + y^n = (x + y)p_{n-1}(x, -y).$$

Now we can see that $(x+z)p_6(x, -z)+y^3 = x^7 +z^7 +x^2 -y^3 +y^3 = x^2(1+x^5)$. Since $(1 + x^5)$ is a unit in $R$, then $x^2 \in (x + z, y)$. Since $x(x + z) = x^2 + xz$ we also observe that $xz \in (x + z, y)$ and similarly, we see that $z^2 \in (x + z, y)$. Hence $\mathfrak{m}^2 \subset (x + z, y)$ and thus $\mathfrak{m} \subset (x + z, y) : \mathfrak{m} = (x + z, y)^* \subset \mathfrak{m}$, i.e. $(x + z, y)^* = \mathfrak{m}$. Using the same argument exchanging $y$ and $z$ and exchanging the powers 3 and 7, we see that $J_2$ is a minimal $*$-reduction of $\mathfrak{m}$.

Let $K$ be a minimal $*$-reduction of $\mathfrak{m}$. Then $\mathfrak{m} = K^*$. For all minimal reductions $K$, $\mathfrak{m}^2 = \mathfrak{m}K^* \subset K$. Thus $\mathfrak{m}^2 \subset *$-core($\mathfrak{m}$). We can easily see that $\mathfrak{m}^2 = J \bigcap J_1 \bigcap J_2$. We can then conclude that $\mathfrak{m}^2$ in fact is $*$-core($\mathfrak{m}$) and core($\mathfrak{m}$) $= *$-core($\mathfrak{m}$).

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