# The Concept of a Linguistic Variable and its Application to Approximate Reasoning–III*

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## 1. LINGUISTIC PROBABILITIES AND AVERAGES OVER FUZZY SETS

In the classical approach to probability theory, an *event*, $A$, is defined as a member of a $\sigma$-field, $\mathscr{A}$, of subsets of a sample space $\Omega$. Thus, if $P$ is a normed measure over a measurable space $(\Omega, \mathscr{A})$, the probability of $A$ is defined as $P(A)$, the measure of $A$, and is a number in the interval [0, 1].

There are many real-world problems in which one or more of the basic assumptions which are implicit in the above definition are violated. First, the event, $A$, is frequently ill-defined, as in the question, "What is the probability that it will be a *warm day* tomorrow?" In this instance, the event *warm day* is a *fuzzy event* in the sense that there is no sharp dividing line between its occurrence and nonoccurrence. As shown in [48], such an event may be characterized as a fuzzy subset, $A$, of the sample space $\Omega$, with $\mu_A$, the membership function of $A$, being a measurable function.

Second, even if $A$ is a well-defined nonfuzzy event, its probability, $P(A)$, may be ill-defined. For example, in response to the question, "What is the probability that the Dow Jones average of stock prices will be higher in a month from now?" it would be patently unreasonable to give an unequivocal numerical answer, e.g., 0.7. In this instance, a vague response like "quite probable," would be much more commensurate with our lack of understanding of the dynamics of stock prices, and hence a more realistic–if less precise–characterization of the probability in question.

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The limitations imposed by the assumption that $A$ is well-defined may be removed, at least in part, by allowing $A$ to be a fuzzy event, as was done in [48]. Another and perhaps more important step that can be taken to widen the applicability of probability theory to ill-defined problems is to allow $P$ to be a linguistic variable in the sense defined in Part II, Sec. 3. In what follows, we shall outline a way in which this can be done and explore some of the elementary consequences of allowing $P$ to be a linguistic variable.

*LINGUISTIC PROBABILITIES*

To simplify our exposition, we shall assume that the object of our concern is a variable, $X$, whose universe of discourse, $U$, is a finite set

$$U = u_1 + u_2 + \cdots + u_n. \tag{1.1}$$

Furthermore, we assume that the restriction imposed by $X$ coincides with $U$. Thus, any point in $U$ can be assigned as a value to $X$.

With each $u_i$, $i = 1, \ldots, n$, we associate a *linguistic probability*, $\mathscr{P}_i$, which is a Boolean linguistic variable in the sense of Part II, Definition 2.2, with $p_i$, $0 \leqslant p_i \leqslant 1$, representing the base variable for $\mathscr{P}_i$. For concreteness, we shall assume that $V$, the universe of discourse associated with $\mathscr{P}_i$, is either the unit interval [0, 1] or the finite set

$$V = 0 + 0.1 + \cdots + 0.9 + 1. \tag{1.2}$$

Using $\mathscr{P}$ as a generic name for the $\mathscr{P}_i$, the term-set for $\mathscr{P}$ will typically be the following.

$$\begin{aligned} T(\mathscr{P}) = {} & \textit{likely} + \textit{not likely} + \textit{unlikely} + \textit{very likely} + \textit{more or less likely} \\ & + \textit{very unlikely} + \cdots \\ & + \textit{probable} + \textit{improbable} + \textit{very probable} + \ldots \\ & + \textit{neither very probable nor very improbable} + \cdots \\ & + \textit{close to } 0 + \textit{close to } 0.1 + \cdots + \textit{close to } 1 + \cdots \\ & + \textit{very close to } 0 + \textit{very close to } 0.1 + \cdots, \end{aligned} \tag{1.3}$$

in which *likely, probable* and *close to* play the role of primary terms.

The shape of the membership function of *likely* will be assumed to be like that of *true* [see Part II, Eq. (3.2)], with *not likely* and *unlikely* defined by

$$\mu_{\textit{not likely}}(p) = 1 - \mu_{\textit{likely}}(p), \tag{1.4}$$

and

$$\mu_{unlikely}(p) = \mu_{likely}(1 - p), \tag{1.5}$$

where $p$ is a generic name for the $p_i$.

EXAMPLE 1.1. A graphic example of the meaning attached to the terms *likely, not likely, very likely* and *unlikely* is shown in Fig. 1. In numerical terms, if the primary term *likely* is defined as

$$likely = 0.5/0.6 + 0.7/0.7 + 0.9/0.8 + 1/0.9 + 1/1, \tag{1.6}$$

then

$$not\ likely = 1/(0 + 0.1 + 0.2 + 0.3 + 0.4 + 0.5) + 0.5/0.6 + 0.3/0.7 + 0.1/0.8, \tag{1.7}$$

$$unlikely = 1/0 + 1/0.1 + 0.9/0.2 + 0.7/0.3 + 0.5/0.4 \tag{1.8}$$

and

$$very\ likely = 0.25/0.6 + 0.49/0.7 + 0.81/0.8 + 1/0.9 + 1/1. \tag{1.9}$$

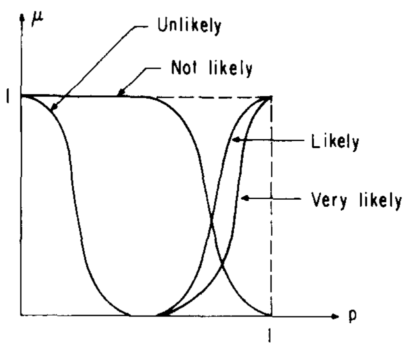


Fig. 1. Compatibility functions of *likely, not likely, unlikely* and *very likely*.

The term *probable* will be assumed to be more or less synonymous with *likely*. The term *close to* α, where α is a point in [0, 1], will be abbreviated as $\vec{\alpha}$ or, alternatively, as "α",[1] suggesting that α is a "best example" of the fuzzy set "α". In this sense, then,

[1]The symbol "α" will be employed in place of $\vec{\alpha}$ when the constraints imposed by typesetting dictate its use.

$$likely \triangleq close\ to\ 1 \triangleq \text{“1”}, \tag{1.10}$$

$$unlikely \triangleq close\ to\ 0 \triangleq \text{“0”}, \tag{1.11}$$

and

$$close\ to\ 0.8 \triangleq \text{“0.8”} = 0.6/0.7 + 1/0.8 + 0.6/0.9, \tag{1.12}$$

from which it follows that

$$\begin{aligned} very\ close\ to\ 0.8 &= very\ \text{“0.8”} \\ &= (\text{“0.8”})^2 \qquad \text{[in the sense of Part II, Eq. (2.38)]} \\ &= 0.36/0.7 + 1/0.8 + 0.36/0.9. \end{aligned}$$

A particular term in $T(\mathscr{P})$ will be denoted by $T_j$, or $T_{ji}$ in case a double subscript notation is needed. Thus, if $T_4$ = *very likely,* then $T_{43}$ would indicate that *very likely* is assigned as a value to the linguistic variable $\mathscr{P}_3$.

The $n$-ary linguistic variable $(\mathscr{P}_1, \ldots, \mathscr{P}_n)$ constitutes a *linguistic probability assignment list* associated with $X$. A variable $X$ which is associated with a linguistic probability assignment list will be referred to as a *linguistic random variable.* By analogy with linguistic truth-value distributions [see Part II, Eq. (3.74)], a collection of probability assignment lists will be referred to as a *linguistic probability distribution.*

The assignment of a probability-value $T_j$ to $P_i$ may be expressed as

$$P_i = T_j, \tag{1.13}$$

where $P_i$ is used in a dual role as a generic name for the fuzzy variables which comprise $\mathscr{P}_i$. For example, we may write

$$\begin{aligned} P_3 &= T_4 \\ &= very\ likely \end{aligned} \tag{1.14}$$

in which case *very likely* will be identified as $T_{43}$ (i.e., $T_4$ assigned to $P_3$).

An important characteristic of the linguistic probabilities $P_1, \ldots, P_n$ is that they are $\beta$-interactive in the sense of Part II, Definition 3.2. The interaction between the $P_i$ is a consequence of the constraint ($+ \triangleq$ arithmetic sum)

$$p_1 + p_2 + \cdots + p_n = 1, \tag{1.15}$$

in which the $p_i$ are the base variables (i.e., numerical probabilities) associated with the $P_i$.

More concretely, let $R(p_1 + \cdots + p_n = 1)$ denote the nonfuzzy $n$-ary relation in $[0, 1] \times \cdots \times [0, 1]$ representing (1.15). Furthermore, let $R(P_i)$ denote the restriction on the values of $p_i$. Then the restriction imposed by the $n$-ary fuzzy variable $(P_1, \ldots, P_n)$ may be expressed as

$$R(P_1, \ldots, P_n) = R(P_1) \times \cdots \times R(P_n) \cap R(p_1 + \cdots + p_n = 1) \quad (1.16)$$

which implies that, apart from the constraint imposed by (1.15), the fuzzy variables $P_1, \ldots, P_n$ are noninteractive.

EXAMPLE 1.2. Suppose that

$$\begin{aligned} P_1 &= \textit{likely} \\ &= 0.5/0.8 + 0.8/0.9 + 1/1 \end{aligned} \quad (1.17)$$

and

$$\begin{aligned} P_2 &= \textit{unlikely} \\ &= 1/0 + 0.8/0.1 + 0.5/0.2. \end{aligned} \quad (1.18)$$

Then

$$\begin{aligned} R(P_1) \times R(P_2) &= \textit{likely} \times \textit{unlikely} \\ &= (0.5/0.8 + 0.8/0.9 + 1/1) \times (1/0 + 0.8/0.1 + 0.5/0.2) \\ &= 0.5/(0.8,0) + 0.8/(0.9,0) + 1/(1,0) \\ &\quad + 0.5/(0.8,0.1) + 0.8/(0.9,0.1) + 0.8/(1,0.1) \\ &\quad + 0.5/(0.8,0.2) + 0.5/(0.9,0.2) + 0.5/(1,0.2). \end{aligned} \quad (1.19)$$

As for $R(p_1 + \cdots + p_n = 1)$, it can be expressed as

$$R(p_1 + p_2 = 1) = \sum_k 1/(k, 1\text{-}k), \qquad k = 0, 0.1, \ldots, 0.9, 1, \quad (1.20)$$

and forming the intersection of (1.19) and (1.20), we obtain

$$R(P_1,P_2) = 1/(1,0) + 0.8/(0.9,1) + 0.5/(0.8,0.2) \quad (1.21)$$

as the expression for the restriction imposed by $(P_1,P_2)$. Obviously, $R(P_1,P_2)$ comprises those terms in $R(P_1) \times R(P_2)$ which satisfy the constraint (1.15).

REMARK 1.1. It should be observed that $R(P_1, P_2)$ as expressed by (1.21) is a normal restriction [see Part I, Eq. (3.23)]. This will be the case, more generally, when the $P_i$ are of the form

$$P_i = \text{“}q_i\text{”}, \qquad i = 1, \ldots, n \tag{1.22}$$

and $q_1 + \cdots + q_n = 1$. Note that in Example 1.2, we have

$$P_1 = \text{“1”}, \tag{1.23}$$

$$P_2 = \text{“0”} \tag{1.24}$$

and

$$1 + 0 = 1. \tag{1.25}$$

*COMPUTATION WITH LINGUISTIC PROBABILITIES*

In many of the applications of probability theory, e.g., in the calculation of means, variances, etc., one encounters linear combinations of the form ($+ \triangleq$ arithmetic sum)

$$z = a_1 p_1 + \cdots + a_n p_n, \tag{1.26}$$

where the $a_i$ are real numbers and the $p_i$ are probability-values in [0, 1]. Computation of the value of $z$ given the $a_i$ and the $p_i$ presents no difficulties when the $p_i$ are points in [0, 1]. It becomes, however, a nontrivial problem when the probabilities in question are linguistic in nature, that is, when

$$Z = a_1 P_1 + \cdots + a_n P_n, \tag{1.27}$$

where the $P_i$ represent linguistic probabilities with names such as *likely, unlikely, very likely, close to* $\alpha$, etc. Correspondingly, $Z$ is not a real number—as it is in (1.26)—but a fuzzy subset of the real line $W \triangleq (-\infty, \infty)$, with the membership function of $Z$ being a function of those of the $P_i$.

Assuming that the fuzzy variables $P_1, \ldots, P_n$ are noninteractive [apart from the constraint expressed by (1.15)], the restriction imposed by $(P_1, \ldots, P_n)$ assumes the form [see (1.16)]

$$R(P_1, \ldots, P_n) = R(P_1) \times \cdots \times R(P_n) \cap R(p_1 + \cdots + p_n = 1). \tag{1.28}$$

Let $\mu(p_1, \ldots, p_n)$ be the membership function of $R(P_1, \ldots, P_n)$, and let $\mu_i(p_i)$ be that of $R(P_i)$, $i = 1, \ldots, n$. Then, by applying the extension principle

[Part I, Eq. (3.90)] to (1.26), we can express $Z$ as a fuzzy set ($+ \triangleq$ arithmetic sum)

$$Z = \int_W \mu(p_1, \ldots, p_n)/(a_1 p_1 + \cdots + a_n p_n), \tag{1.29}$$

which in view of (1.28) may be written as

$$Z = \int_W \mu_1(p_1) \wedge \cdots \wedge \mu_n(p_n)/(a_1 p_1 + \cdots + a_n p_n) \tag{1.30}$$

with the understanding that the $p_i$ in (1.30) are subject to the constraint

$$p_1 + \cdots + p_n = 1. \tag{1.31}$$

In this way, we can express a linear combination of linguistic probability-values as a fuzzy subset of the real line.

The expression for $Z$ may be cast into other forms which may be more convenient for computational purposes. Thus, let $\mu(z)$ denote the membership function of $Z$, with $z \in W$. Then (1.30) implies that

$$\mu(z) = \vee_{p_1, \ldots, p_n} \mu_1(p_1) \wedge \cdots \wedge \mu_n(p_n), \tag{1.32}$$

subject to the constraints

$$z = a_1 p_1 + \cdots + a_n p_n, \tag{1.33}$$

$$p_1 + \cdots + p_n = 1. \tag{1.34}$$

In this form, the computation of $Z$ reduces to the solution of a nonlinear programming problem with linear constraints. In more explicit terms, this problem may be expressed as: Maximize $z$ subject to the constraints ($+ \triangleq$ arithmetic sum)

$$\begin{aligned} &\mu_1(p_1) \geqslant z, \\ &\ldots \\ &\mu_n(p_n) \geqslant z, \\ &z = a_1 p_1 + \cdots + a_n p_n, \\ &p_1 + \cdots + p_n = 1. \end{aligned} \tag{1.35}$$

EXAMPLE 1.3. As a very simple illustration, assume that

$$P_1 = likely \tag{1.36}$$

and

$$P_2 = unlikely, \tag{1.37}$$

where

$$likely = \int_0^1 \mu_{likely}(p)/p \tag{1.38}$$

and

$$unlikely = \neg\, likely \tag{1.39}$$

Thus [see (1.5)]

$$\mu_{unlikely}(p) = \mu_{likely}(1-p), \qquad 0 \leqslant p \leqslant 1. \tag{1.40}$$

Suppose that we wish to compute the expectation ($+ \triangleq$ arithmetic sum)

$$Z = a_1\ likely + a_2\ unlikely. \tag{1.41}$$

Using (1.32), we have

$$\mu(z) = \vee_{p_1, p_2}\, \mu_{likely}(p_1) \wedge \mu_{unlikely}(p_2), \tag{1.42}$$

subject to the constraints

$$z = a_1 p_1 + a_2 p_2, \tag{1.43}$$

$$p_1 + p_2 = 1.$$

Now in view of (1.40), if $p_1 + p_2 = 1$, then

$$\mu_{likely}(p_1) = \mu_{unlikely}(p_2), \tag{1.44}$$

and hence (1.42) reduces to

$$\mu(z) = \mu_{likely}(p_1), \qquad (1.45)$$

$$z = a_1 p_1 + a_2(1 - p_1),$$

or, more explicitly,

$$\mu(z) = \mu_{likely}\left(\frac{z - a_2}{a_1 - a_2}\right). \qquad (1.46)$$

This result implies that the fuzziness in our knowledge of the probability $p_1$ induces a corresponding fuzziness in the expectation of [see Fig. 2]

$$z = a_1 p_1 + a_2 p_2.$$

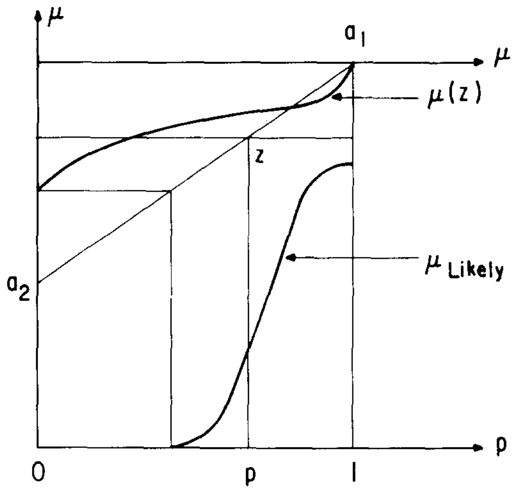


Fig. 2. Computation of the linguistic value of $a_1 p_1 + a_2 p_2$.

If the universe of probability-values is assumed to be $V = 0 + 0.1 + \cdots + 0.9 + 1$, then the expression for $Z$ can be obtained more directly by using the extension principle in the form given in Part I, Eq. (3.97). As an illustration, assume that

$$P_1 = \text{“0.3”} = 0.8/0.2 + 1/0.3 + 0.6/0.4, \qquad (1.47)$$

$$P_2 = \text{“0.7”} = 0.8/0.6 + 1/0.7 + 0.6/0.8, \qquad (1.48)$$

and ($\oplus \stackrel{\Delta}{=}$ arithmetic sum)

$$Z = a_1 P_1 \oplus a_2 P_2, \tag{1.49}$$

where the symbol $\oplus$ is used to avoid confusion with the union.

On substituting (1.47) and (1.48) in (1.49), we obtain

$$\begin{aligned} Z &= a_1(0.8/0.2 + 1/0.3 + 0.6/0.4) \oplus a_2(0.8/0.6 + 1/0.7 + 0.6/0.8) \\ &= (0.8/0.2a_1 + 1/0.3a_1 + 0.6/0.4a_1) \oplus (0.8/0.6a_2 + 1/0.7a_2 + 0.6/0.8a_2). \end{aligned} \tag{1.50}$$

In expanding the right-hand side of (1.50), we have to take into account the constraint $p_1 + p_2 = 1$, which means that a term of the form

$$\mu_1/p_1 a_1 \oplus \mu_2/p_2 a_2 \tag{1.51}$$

evaluates to

$$\begin{aligned} \mu_1/p_1 a_1 \oplus \mu_2/p_2 a_2 &= \mu_1 \wedge \mu_2/(p_1 a_1 \oplus p_2 a_2) \qquad \text{if } p_1 + p_2 = 1 \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{1.52}$$

In this way, we obtain

$$Z = 1/(0.3a_1 \oplus 0.7a_2) + 0.6/(0.2a_1 \oplus 0.8a_2) + 0.6/(0.4a_1 \oplus 0.6a_2), \tag{1.53}$$

which expresses $Z$ as a fuzzy subset of the real line $W = (-\infty, \infty)$.

*AVERAGES OVER FUZZY SETS*

Our point of departure in the foregoing discussion was the assumption that with each point $u_i$ of a finite[2] universe of discourse $U$ is associated a linguistic probability-value $P_i$ which is a component of a linguistic probability distribution $(\mathscr{P}_1, \ldots, \mathscr{P}_n)$.

In this context, a fuzzy subset, $A$, of $U$ plays the role of a *fuzzy event*. Let $\mu_A(u_i)$ be the grade of membership of $u_i$ in $A$. Then, if the $P_i$ are conventional numerical probabilities, $p_i$, $0 \leqslant p_i \leqslant 1$, then the probability of $A$, $P(A)$, is defined as (see [48]; $+ \triangleq$ arithmetic sum)

[2]The assumption that $U$ is a finite set is made solely for the purpose of simplifying our exposition. More generally, $U$ can be a countable set or a continuum.

$$P(A) = \mu_A(u_1)p_1 + \cdots + \mu_A(u_n)p_n. \tag{1.54}$$

It is natural to extend this definition to linguistic probabilities by defining the linguistic probability[3] of $A$ as

$$P(A) = \mu_A(u_1)P_1 + \cdots + \mu_A(u_n)P_n \tag{1.55}$$

with the understanding that the right-hand side of (1.55) is a linear form in the sense of (1.27). In connection with (1.55), it should be noted that the constraint

$$p_1 + \cdots + p_n = 1 \tag{1.56}$$

on the underlying probabilities, together with the fact that

$$0 \leqslant \mu_A(u_i) \leqslant 1, \qquad i = 1, \ldots, n,$$

insures that $P(A)$ is a fuzzy subset of $[0, 1]$.

EXAMPLE 1.4. As a very simple illustration, assume that

$$U = a + b + c, \tag{1.57}$$

$$A = 0.4a + b + 0.8c, \tag{1.58}$$

$$P_a = \text{“0.3”} = 0.6/0.2 + 1/0.3 + 0.6/0.4, \tag{1.59}$$

$$P_b = \text{“0.6”} = 0.6/0.5 + 1/0.6 + 0.6/0.7, \tag{1.60}$$

$$P_c = \text{“0.1”} = 0.6/0 + 1/0.1 + 0.6/0.2. \tag{1.61}$$

Then ($\oplus \overset{\Delta}{=}$ arithmetic sum)

$$P(A) = 0.4(0.6/0.2 + 1/0.3 + 0.6/0.4) \oplus (0.6/0.5 + 1/0.6 + 0.6/0.7) \oplus 0.8\,(0.6/0 + 1/0.1 + 0.6/0.2), \tag{1.62}$$

subject to the constraint

$$p_1 + p_2 + p_3 = 1. \tag{1.63}$$

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[3]It should be noted that the computation of the right-hand side of (1.55) defines $P(A)$ as a fuzzy subset of $[0, 1]$. In general, a linguistic approximation would be needed to express $P(A)$ as a linguistic probability-value.

Picking those terms in (1.62) which satisfy (1.63), we obtain

$$\begin{aligned} P(A) = {} & 0.6/(0.4 \times 0.2 \oplus 0.6 \oplus 0.8 \times 0.2) \\ & + 0.6/(0.4 \times 0.2 \oplus 0.7 \oplus 0.8 \times 0.1) \\ & + 0.6/(0.4 \times 0.3 \oplus 0.5 \oplus 0.8 \times 0.2) \\ & + 1/(0.4 \times 0.3 \oplus 0.6 \oplus 0.8 \times 0.1) \\ & + 0.6/(0.4 \times 0.3 \oplus 0.7) \\ & + 0.6/(0.4 \times 0.4 \oplus 0.5 \oplus 0.8 \times 0.1) \\ & + 0.6/(0.4 \times 0.4 \oplus 0.6), \end{aligned} \tag{1.64}$$

which reduces to

$$P(A) = 0.6/(0.84 + 0.86 + 0.78 + 0.82 + 0.74) + 1/0.8, \tag{1.65}$$

and which may be roughly approximated as

$$P(A) = \text{“}0.8\text{”}. \tag{1.66}$$

The linguistic probability of a fuzzy event as expressed by (1.55) may be viewed as a particular instance of a more general concept, namely, the *linguistic average* or, equivalently, the *linguistic expectation* of a function (defined on $U$) over a fuzzy subset of $U$. More specifically, let $f$ be a real-valued function defined on $U$; let $A$ be a fuzzy subset of $U$; and let $P_1, \ldots, P_n$ be the linguistic probabilities associated with $u_1, \ldots, u_n$, respectively. Then, the *linguistic average of $f$ over $A$* is denoted by $Av(f;A)$ and is defined by ($+ \triangleq$ arithmetic sum)

$$\mathrm{Av}(f;A) = f(u_1)\mu_A(u_1)P_1 + \cdots + f(u_n)\mu_A(u_n)P_n. \tag{1.67}$$

A concrete example of (1.67) is the following. Assume that individuals named $u_1, \ldots, u_n$ are chosen with linguistic probabilities $P_1, \ldots, P_n$, with $P_i$ being a restriction on $p_i$, $i = 1, \ldots, n$. Suppose that $u_i$ is fined an amount $f(u_i)$, which is scaled down in proportion to the grade of membership of $u_i$ in a class $A$. Then, the linguistic average (expected) amount of the fine will be expressed by (1.67).

COMMENT 1.1. Note that (1.67) is basically a linear combination of the form (1.27) with

$$a_i = f(u_i)\mu_A(u_i). \tag{1.68}$$

Thus, to evaluate (1.67), we can employ the technique described earlier for the computation of linear forms in linguistic probabilities. In particular, it should be noted that, in the special case where $f(u_i) = 1$, the right-hand side of (1.67) becomes

$$\mu_A(u_1)P_1 + \cdots + \mu_A(u_n)P_n, \tag{1.69}$$

and $\mathrm{Av}(f;A)$ reduces to $P(A)$.

In addition to subsuming the expression for $P(A)$, the expression for $\mathrm{Av}(f;A)$ subsumes as special cases other types of averages which occur in various applications. Among them there are two that may be regarded as degenerate forms of (1.67) and which are encountered in many problems of practical interest. In what follows, we shall dwell briefly on these averages and, for convenience in exposition, will state their definitions in the form of answers to questions.

QUESTION 1.1. What is the number of elements in a given fuzzy set $A$? Clearly, this question is not well posed, since in the case of a fuzzy set the dividing line between membership and nonmembership is not sharp. Nevertheless, the concept of the *power* of a fuzzy set [49], which is defined as

$$|A| \triangleq \sum_i \mu_A(u_i), \tag{1.70}$$

appears to be a natural generalization of that of the number of elements in $A$.

As an illustration of $|A|$, suppose that $U$ is the universe of residents in a city, and $A$ is the fuzzy set of the unemployed in that city. If $\mu_A(u_i)$ is interpreted as the grade of membership of an individual, $u_i$, in the class of the unemployed [e.g., $\mu_A(u_i) = 0.5$ if $u_i$ is working half-time and is looking for a full-time job], then $|A|$ may be interpreted as the number of *full-time equivalent* unemployed.

QUESTION 1.2. Suppose that $f$ is a real-valued function defined on $U$. What is the average value of $f$ over a fuzzy subset, $A$, of $U$?

Using the same notation as in (1.67), let $\mathrm{Av}(f;A)$ denote the average value of $f$ over $A$. If $A$ were nonfuzzy, $\mathrm{Av}(f;A)$ would be expressed by

$$\mathrm{Av}(f;A) = \frac{\Sigma_{u_i \in A} f(u_i)}{|A|}, \tag{1.71}$$

where $\Sigma_{u_i \in A}$ is the summation over those $u_i$ which are in $A$, and $|A|$ is the number of the $u_i$ which are in $A$. To extend (1.71) to fuzzy sets, we note that (1.71) may be rewritten as

$$\mathrm{Av}(f;A) = \frac{\Sigma_{u_i \in U} f(u_i)\mu_A(u_i)}{\Sigma_{u_i \in U} \mu_A(u_i)}, \tag{1.72}$$

where $\mu_A$ is the characteristic function of $A$. Then, we adopt (1.72) as the definition of $\mathrm{Av}(f;A)$ for a *fuzzy* $A$ by interpreting $\mu_A(u_i)$ as the grade of membership of $u_i$ in $A$. In this way, we arrive at an expression for $\mathrm{Av}(f;A)$ which may be viewed as a special case of (1.67).

As an illustration of (1.72), suppose that $U$ is the universe of residents in a city and $A$ is the fuzzy subset of residents who are *young*. Furthermore, assume that $f(u_i)$ represents the income of $u_i$. Then, the average income of young residents in the city would be expressed by (1.72).

COMMENT 1.2. Since the expression for $|A|$ is a linear form in the $\mu_A(u_i)$, the power of a fuzzy set of type 2 (see Part I, Definition 3.1) can readily be computed by employing the technique which we had used earlier to compute $P(A)$. In the case of $\mathrm{Av}(f;A)$, however, we are dealing with a ratio of linear forms, and hence the computation of $\mathrm{Av}(f;A)$ for fuzzy sets of type 2 presents a more difficult problem.

In the foregoing discussion, our very limited objective was to indicate that the concept of a linguistic variable provides a basis for defining linguistic probabilities and, in conjunction with the extension principle, may be applied to the computation of linear forms in such probabilities. We shall not dwell further on this subject and, in what follows, will turn our attention to a basic rule of inference in fuzzy logic.

## 2. COMPOSITIONAL RULE OF INFERENCE AND APPROXIMATE REASONING

The basic rule of inference in traditional logic is the *modus ponens,* according to which we can infer the truth of a proposition $B$ from the truth of $A$ and the implication $A \Rightarrow B$. For example, if $A$ is identified with "John is in a hospital," and $B$ with "John is ill," then if it is true that "John is in a hospital," it is also true that "John is ill."

In much of human reasoning, however, *modus ponens* is employed in an approximate rather than exact form. Thus, typically, we know that $A$ is true and that $A^* \Rightarrow B$, where $A^*$ is, in some sense, an approximation to $B$. Then, from $A$ and $A^* \Rightarrow B$ we may infer that $B$ is approximately true.

In what follows, we shall outline a way of formalizing approximate reasoning based on the concepts introduced in the preceding sections. However, in a departure from traditional logic, our main tool will not be the *modus ponens,* but a so-called *compositional rule of inference* of which *modus ponens* forms a very special case.

*COMPOSITIONAL RULE OF INFERENCE*

The compositional rule of inference is merely a generalization of the following familiar procedure. Referring to Fig. 3, suppose that we have a curve $y = f(x)$ and are given $x = a$. Then from $y = f(x)$ and $x = a$, we can infer $y \triangleq b = f(a)$.

Next, let us generalize the above process by assuming that $a$ is an interval and $f(x)$ is an interval-valued function such as shown in Fig. 4. In this instance, to find the interval $y \triangleq b$ which corresponds to the interval $a$, we first construct a cylindrical set, $\bar{a}$, with base $a$ [see Part I, Eq. (3.58)] and find its intersection, $I$, with the interval-valued curve. Then we project the intersection on the $OY$ axis, yielding the desired $y$ as the interval $b$.

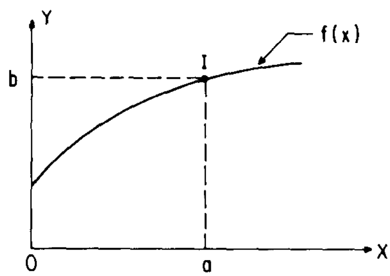


Fig. 3. Infering $y = b$ from $x = a$ and $y = f(x)$.

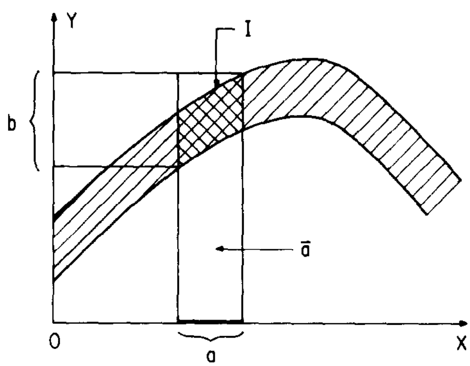


Fig. 4. Illustration of the compositional rule of inference in the case of interval-valued variables.

Going one step further in our chain of generalizations, assume that $A$ is a fuzzy subset of the $OX$ axis and $F$ is a fuzzy relation from $OX$ to $OY$. Again, forming a cylindrical fuzzy set $\bar{A}$ with base $A$ and intersecting it with the fuzzy relation $F$ (see Fig. 5), we obtain a fuzzy set $\bar{A} \cap F$ which is the analog of the point of intersection $I$ in Fig. 3. Then, projecting this set on $OY$, we obtain $y$ as a fuzzy subset of $OY$. In this way, from $y = f(x)$ and $x \overset{\Delta}{=} A$ (fuzzy subset of $OX$), we infer $y$ as a fuzzy subset, $B$, of $OY$.

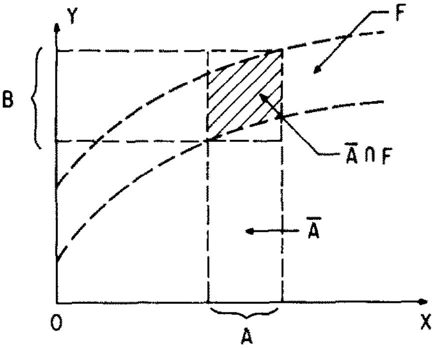


Fig. 5. Illustration of the compositional rule of inference for fuzzy variables.

More specifically, let $\mu_A$, $\mu_{\bar{A}}$, $\mu_F$ and $\mu_B$ denote the membership functions of $A$, $\bar{A}$, $F$ and $B$, respectively. Then, by the definition of $\bar{A}$ [see Part I, Eq. (3.58)]

$$\mu_{\bar{A}}(x, y) = \mu_A(x), \tag{2.1}$$

and consequently

$$\begin{aligned} \mu_{\bar{A} \cap F}(x,y) &= \mu_{\bar{A}}(x,y) \wedge \mu_F(x,y) \\ &= \mu_A(x) \wedge \mu_F(x,y). \end{aligned} \tag{2.2}$$

Projecting $\bar{A} \cap F$ on the $OY$ axis, we obtain from (2.2) and from Eq. (3.57) of Part I

$$\mu_B(y) = \vee_x \mu_A(x) \wedge \mu_F(x,y) \tag{2.3}$$

as the expression for the membership function of the projection (shadow) of $\bar{A} \cap F$ on $OY$. Comparing this expression with the definition of the composition of $A$ and $F$ [see Part I, Eq. (3.55)], we see that $B$ may be represented as

$$B = A \circ F, \tag{2.4}$$

where $\circ$ denotes the operation of composition. As stated in Part I, Sec. 3, this operation reduces to the max-min matrix product when $A$ and $F$ have finite supports.

EXAMPLE 2.1. Suppose that $A$ and $F$ are defined by

$$A = 0.2/1 + 1/2 + 0.3/3 \tag{2.5}$$

and

$$\begin{aligned} F = 0.8/(1,1) + 0.9/(1,2) + 0.2/(1,3) \\ + 0.6/(2,1) + 1/(2,2) + 0.4/(2,3) \\ + 0.5/(3,1) + 0.8/(3,2) + 1/(3,3). \end{aligned} \tag{2.6}$$

Expressing $A$ and $F$ in terms of their relation matrices and forming the matrix product (2.4), we obtain

$$\overset{A}{[0.2 \quad 1 \quad 0.3]} \circ \overset{F}{\begin{bmatrix} 0.8 & 0.9 & 0.2 \\ 0.6 & 1 & 0.4 \\ 0.5 & 0.8 & 1 \end{bmatrix}} = \overset{B}{[0.6 \quad 1 \quad 0.4]} \ . \tag{2.7}$$

The foregoing comments and examples serve to motivate the following rule of inference.

RULE 2.1. Let $U$ and $V$ be two universes of discourse with base variables $u$ and $v$, respectively. Let $R(u)$, $R(u, v)$ and $R(v)$ denote restrictions on $u$, $(u, v)$ and $v$, respectively, with the understanding that $R(u)$, $R(u, v)$ and $R(v)$ are fuzzy relations in $U$, $U \times V$ and $V$. Let $A$ and $F$ denote particular fuzzy subsets of $U$ and $U \times V$. Then the *compositional rule of inference* asserts that the solution of the *relational* assignment equations

$$R(u) = A \tag{2.8}$$

and

$$R(u, v) = F \tag{2.9}$$

is given by

$$R(v) = A \circ F, \tag{2.10}$$

where $A \circ F$ is the composition of $A$ and $F$. In this sense, we can *infer* $R(v) = A \circ F$ from $R(u) = A$ and $R(u, v) = F$.

As a simple illustration of the use of this rule, assume that

$$U = V = 1 + 2 + 3 + 4, \tag{2.11}$$

$$A = \textit{small} = 1/1 + 0.6/2 + 0.2/3 \tag{2.12}$$

and

$$\begin{aligned} F &= \textit{approximately equal} \\ &= 1/(1,1) + 1/(2,2) + 1/(3,3) + 1/(4,4) \\ &\quad + 0.5/[(1,2) + (2,1) + (2,3) + (3,2) + (3,4) + (4,3)]. \end{aligned} \tag{2.13}$$

In other words, $A$ is unary fuzzy relation in $U$ named *small* and $F$ is a binary fuzzy relation in $U \times V$ named *approximately equal.*

The relational assignment equations in this case read

$$R(u) = \textit{small}, \tag{2.14}$$

$$R(u, v) = \textit{approximately equal} \tag{2.15}$$

and hence

$$\begin{aligned} R(v) &= \textit{small} \circ \textit{approximately equal} \\ &= [1 \quad 0.6 \quad 0.2 \quad 0] \circ \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \\ &= [1 \quad 0.6 \quad 0.5 \quad 0.2] \end{aligned} \tag{2.16}$$

which may be approximated by the linguistic term

$$R(v) = \textit{more or less small} \tag{2.17}$$

if *more or less* is defined as a fuzzifier [see Part I, Eq. (3.48)], with

$$K(1) = 1/1 + 0.7/2,$$
$$K(2) = 1/2 + 0.7/3,$$
$$K(3) = 1/3 + 0.7/4,$$
$$K(4) = 1/4. \tag{2.18}$$

Note that the application of this fuzzifier to $R(u)$ yields

$$[1 \quad 0.7 \quad 0.42 \quad 0.14] \tag{2.19}$$

as an approximation to $[1 \quad 0.6 \quad 0.5 \quad 0.2]$.

In summary, then, by using the compositional rule of inference, we have infered from $R(u)$ = *small* and $R(u, v)$ = *approximately equal*

$$R(v) = [1 \quad 0.6 \quad 0.5 \quad 0.2] \qquad \text{exactly} \tag{2.20}$$

and

$$R(v) = \textit{more or less small} \qquad \text{as a linguistic approximation.} \tag{2.21}$$

Stated in English, this approximate inference may be expressed as

| | |
|---|---|
| $u$ is small | premiss |
| $u$ and $v$ are approximately equal | premiss |
| $v$ is more or less small | approximate conclusion. (2.22) |

The general idea behind the method sketched above is the following. Each fact or a premiss is translated into a relational assignment equation involving one or more restrictions on the base variables. These equations are solved for the desired restrictions by the use of the composition of fuzzy relations. The solutions to the equations then represent deductions from the given set of premisses.

### MODUS PONENS *AS A SPECIAL CASE OF THE COMPOSITIONAL RULE OF INFERENCE*

As we shall see in what follows, *modus ponens* may be viewed as a special case of the compositional rule of inference. To establish this connection, we shall first extend the notion of material implication from propositional variables to fuzzy sets.

In traditional logic, the material implication $\Rightarrow$ is defined as a logical connective for propositional variables. Thus, if $A$ and $B$ are propositional variables, the truth table for $A \Rightarrow B$ or, equivalently, IF $A$ THEN $B$, is defined by Table 1 (see Part II, Table 2).

TABLE 1

| $A$ \ $B$ | $T$ | $F$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ |

In much of human discourse, however, the expression IF $A$ THEN $B$ is used in situations in which $A$ and $B$ are fuzzy sets (or fuzzy predicates) rather than propositional variables. For example, in the case of the statement IF John is *ill* THEN John is *cranky*, which may be abbreviated as *ill* $\Rightarrow$ *cranky*, *ill* and *cranky* are, in effect, names of fuzzy sets. The same is true of the statement IF apple is *red* THEN apple is *ripe*, where *red* and *ripe* play the role of fuzzy sets.

To extend the notion of material implication to fuzzy sets, let $U$ and $V$ be two possibly different universes of discourse and let $A$, $B$ and $C$ be fuzzy subsets of $U$, $V$ and $V$, respectively. First we shall define the meaning of the expression IF $A$ THEN $B$ ELSE $C$, and then we shall define IF $A$ THEN $B$ as a special case of IF $A$ THEN $B$ ELSE $C$.

DEFINITION 2.1. The expression IF $A$ THEN $B$ ELSE $C$ is a binary fuzzy relation in $U \times V$ defined by

$$\text{IF } A \text{ THEN } B \text{ ELSE } C = A \times B + \neg A \times C. \qquad (2.23)$$

That is, if $A$, $B$ and $C$ are unary fuzzy relations in $U$, $V$ and $V$, then IF $A$ THEN $B$ ELSE $C$ is a binary fuzzy relation in $U \times V$ which is the union of the Cartesian product of $A$ and $B$ [see Part I, Eq. (3.45)] and the Cartesian product of the negation of $A$ and $C$.

Now IF $A$ THEN $B$ may be viewed as a special case of IF $A$ THEN $B$ ELSE $C$ which results when $C$ is allowed to be the entire universe $V$. Thus

$$\begin{aligned}\text{IF } A \text{ THEN } B &\triangleq \text{IF } A \text{ THEN } B \text{ ELSE } V\\ &= A \times B + \neg A \times V. \qquad (2.24)\end{aligned}$$

In effect, this amounts to interpreting IF $A$ THEN $B$ as IF $A$ THEN $B$ ELSE *don't care.*[4]

It is helpful to observe that in terms of the relation matrices of $A$, $B$ and $C$, (2.23) may be expressed as the sum of dyadic products involving $A$ and $B$ (and $\neg A$ and $C$) as column and row matrices, respectively. Thus,

$$\text{IF } A \text{ THEN } B \text{ ELSE } C = \begin{bmatrix} A \end{bmatrix} [\quad B \quad] + \neg \begin{bmatrix} A \end{bmatrix} [\quad C \quad]. \tag{2.25}$$

EXAMPLE 2.2. As a simple illustration of (2.23) and (2.24), assume that

$$U = V = 1 + 2 + 3, \tag{2.26}$$

$$A = \mathit{small} = 1/1 + 0.4/2, \tag{2.27}$$

$$B = \mathit{large} = 0.4/2 + 1/3, \tag{2.28}$$

$$C = \mathit{not\ large} = 1/1 + 0.6/2. \tag{2.29}$$

Then

$$\begin{aligned} \text{IF } A \text{ THEN } B \text{ ELSE } C &= (1/1 + 0.4/2) \times (0.4/2 + 1/3) + (0.6/2 + 1/3) \times (1/1 \\ &\quad + 0.6/2) \\ &= 0.4/(1,2) + 1/(1,3) + 0.6/(2,1) + 0.6/(2,2) \\ &\quad + 0.4/(2,3) + 1/(3,1) + 0.6/(3,2). \end{aligned} \tag{2.30}$$

which, represented as a relation matrix, reads

$$\text{IF } A \text{ THEN } B \text{ ELSE } C = \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.4 \\ 1 & 0.6 & 0 \end{bmatrix} \tag{2.31}$$

Similarly

$$\begin{aligned} \text{IF } A \text{ THEN } B &= (1/1 + 0.4/2) \times (0.4/2 + 1/3) + (0.6/2 + 1/3) \times (1/1 + 1/2 + 1/3) \\ &= 0.4/(1,2) + 1/(1,3) + 0.6/(2,1) + 0.6/(2,2) \\ &\quad + 0.6/(2,3) + 1/(3,1) + 1/(3,2) + 1/(3,3), \end{aligned}$$

[4] An alternative interpretation that is consistent with Lukasiewicz's definition of implication [46] is expressed by IF $A$ THEN $B \triangleq \neg(A \times V) \oplus (U \times B)$, where the operation $\oplus$ (bounded-sum) is defined for fuzzy sets $P$, $Q$ by $\mu_{P \oplus Q} \triangleq 1 \wedge (\mu_P + \mu_Q)$, with + denoting the arithmetic sum. More generally, IF $A$ THEN $B$ ELSE $C$ $\triangleq [\neg(A \times V) \oplus (U \times B)] \cap [(A \times V) \oplus (U \times C)]$.

or equivalently

$$\text{IF } A \text{ THEN } B = \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.6 \\ 1 & 1 & 1 \end{bmatrix} \tag{2.32}$$

COMMENT 2.1. It should be noted that in defining IF $A$ THEN $B$ by (2.24) we are tacitly assuming that $A$ and $B$ are noninteractive in the sense that there is no joint constraint involving the base variables $u$ and $v$. This would not be the case in the nonfuzzy statement IF $u \in A$ THEN $u \in B$, which may be expressed as IF $u \in A$ THEN $v \in B$, subject to the constraint $u = v$. Denoting this constraint by $R(u = v)$, the relation representing the statement in question would be

$$\text{IF } u \in A \text{ THEN } u \in B \stackrel{\Delta}{=} (A \times B + \neg A \times V) \cap [R(u = v)]. \tag{2.33}$$

REMARK 2.1. In defining $A \Rightarrow B$, we assumed that IF $A$ THEN $B$ is a special case of IF $A$ THEN $B$ ELSE $C$ resulting from setting $C = V$. If we set $C$ equal to $\theta$ (empty set) rather than $V$, the right-hand side of (2.23) reduces to the Cartesian product $A \times B$—which may be interpreted as $A$ COUPLED WITH $B$ (rather than $A$ ENTAILS $B$). Thus, by definition,

$$A \text{ COUPLED WITH } B \stackrel{\Delta}{=} A \times B, \tag{2.34}$$

and hence

$$A \Rightarrow B \stackrel{\Delta}{=} A \text{ COUPLED WITH } B \text{ plus } \neg A \text{ COUPLED WITH } V. \tag{2.35}$$

More generally, an expression of the form

$$A_1 \times B_1 + \cdots + A_n \times B_n \tag{2.36}$$

would be expressed in words as

$$A_1 \text{ COUPLED WITH } B_1 \text{ plus} \ldots \text{plus } A_n \text{ COUPLED WITH } B_n. \tag{2.37}$$

It should be noted that expressions such as (2.37) may be employed to represent a fuzzy graph as a union of fuzzy points (see Fig. 6). For example, a fuzzy graph $G$ may be represented as

$$G = \text{“}u_1\text{”} \times \text{“}v_1\text{”} + \text{“}u_2\text{”} \times \text{“}v_2\text{”} + \cdots + \text{“}u_n\text{”} \times \text{“}v_n\text{”}, \tag{2.38}$$

where the $u_i$ and $v_i$ are points in $U$ and $V$, respectively, and “$u_i$” and “$v_i$”, $i = 1, \ldots, n$, represent fuzzy sets named *close to* $u_i$ and *close to* $v_i$ [see (1.12)].

COMMENT 2.2. The connection between (2.24) and the conventional definition of material implication becomes clearer by noting that

$$\neg A \times B \subset \neg A \times V \tag{2.39}$$

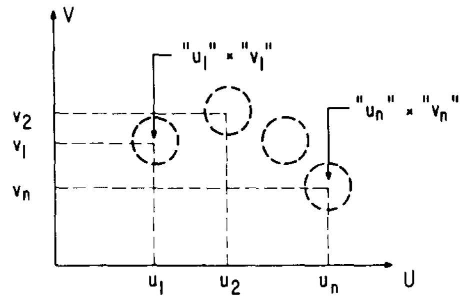


Fig. 6. Representation of a fuzzy graph as a union of fuzzy points.

and hence that (2.24) may be rewritten as

$$\text{IF } A \text{ THEN } B = A \times B + \neg A \times B + \neg A \times V$$

$$= (A + \neg A) \times B + \neg A \times V. \tag{2.40}$$

Now, if $A$ is a nonfuzzy subset of $U$, then

$$A + \neg A = U, \tag{2.41}$$

and hence IF $A$ THEN $B$ reduces to

$$\text{IF } A \text{ THEN } B = U \times B + \neg A \times V, \tag{2.42}$$

which is similar in form to the familiar expression for $A \Rightarrow B$ in the case of propositional variables, namely

$$A \Rightarrow B \equiv \neg A \vee B. \tag{2.43}$$

Turning to the connection between *modus ponens* and the compositional rule of inference, we first define a *generalized modus ponens* as follows.

DEFINITION 2.2. Let $A_1$, $A_2$ and $B$ be fuzzy subsets of $U$, $U$ and $V$, respectively. Assume that $A_1$ is assigned to the restriction $R(u)$, and the relation $A_2 \Rightarrow B$ [defined by Eq. (3.24) of Part I] is assigned to the restriction $R(u, v)$. Thus

$$R(u) = A_1, \tag{2.44}$$

$$R(u, v) = A_2 \Rightarrow B. \tag{2.45}$$

As was shown earlier, these relational assignment equations may be solved for the restriction on $v$, yielding

$$R(v) = A_1 \circ (A_2 \Rightarrow B). \tag{2.46}$$

An expression for this conclusion in the form

$$A_1 \qquad \text{premiss} \tag{2.47}$$

$$A_2 \Rightarrow B \qquad \text{implication} \tag{2.48}$$

$$\overline{A_1 \circ (A_2 \Rightarrow B)} \qquad \text{conclusion} \tag{2.49}$$

constitutes the statement of the *generalized modus ponens.*[5]

COMMENT 2.3. The above statement differs from the traditional *modus ponens* in two respects: First, $A_1$, $A_2$ and $B$ are allowed to be fuzzy sets, and second, $A_1$ need not be identical with $A_2$. To check on what happens when $A_1 = A_2 = A$ and $A$ is nonfuzzy, we substitute the expression for $A_2 \Rightarrow B$ in (2.46), yielding

$$\begin{aligned} A \circ (A \Rightarrow B) &= A \circ (A \times B + \neg A \times V) \\ &= A_r A_c B_r + A_r (\neg A_c) V_r, \end{aligned} \tag{2.50}$$

where $r$ and $c$ stand for *row* and *column,* respectively; $A_r$ and $A_c$ denote the relation matrices for $A$ expressed as a row matrix and a column matrix, respectively; and the matrix product is understood to be taken in the max-min sense.

Now, since $A$ is nonfuzzy,

$$A_r (\neg A_c) = 0, \tag{2.51}$$

and so long as $A$ is normal [see Part I, Eq. (3.23)]

$$A_r A_c = 1. \tag{2.52}$$

Consequently

$$A \circ (A \Rightarrow B) = B, \tag{2.53}$$

which agrees with the conclusion yieled by *modus ponens.*

EXAMPLE 2.3. As a simple illustration of (2.49), assume that

$$U = V = 1 + 2 + 3, \tag{2.54}$$

[5]The generalized *modus ponens* as defined here is unrelated to probabilistic rules of inference. A discussion of such rules and related issues may be found in [50].

$$A_2 = small = 1/1 + 0.4/2, \tag{2.55}$$

$$A_1 = more\ or\ less\ small = 1/1 + 0.4/2 + 0.2/3 \tag{2.56}$$

and

$$B = large = 0.4/2 + 1/3. \tag{2.57}$$

Then (see (2.32))

$$small \Rightarrow large = \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.6 \\ 1 & 1 & 1 \end{bmatrix} \tag{2.58}$$

and

$$more\ or\ less\ small \circ (small \Rightarrow large) = [1 \quad 0.4 \quad 0.2] \circ \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= [0.4 \quad 0.4 \quad 1], \tag{2.59}$$

which may be roughly approximated as *more or less large.* Thus, in the case under consideration, the *generalized modus ponens* yields

| | | |
|---|---|---|
| $u$ is *more or less small* | premiss | |
| IF $u$ is *small* THEN $v$ is *large* | implication | (2.60) |
| $v$ is *more or less large* | approximate conclusion | |

COMMENT 2.4. Because of the way in which $A \Rightarrow B$ is defined, namely,

$$A \Rightarrow B = A \times B + \neg A \times V,$$

the grade of membership of a point $(u, v)$ will be high in $A \Rightarrow B$ if the grade of membership of $u$ is low in $A$. This gives rise to an overlap between the terms $A \times B$ and $\neg A \times V$ when $A$ is fuzzy, with the result that [see (2.50)], the inference drawn from $A$ and $A \Rightarrow B$ is not $B$ but[6]

[6]We assume that $A$ is normal, so that $A_r A_c = 1$.

$$A \circ (A \Rightarrow B) = B + A \circ (\neg A \times V), \tag{2.61}$$

where the difference term $A \circ (\neg A \times V)$ represents the effect of the overlap.

To avoid this phenomenon it may be necessary to define $A \Rightarrow B$ in a way that differentiates between the numerical truth-values in [0, 1] and the truth-value *unknown* [see Part II, Eq. (3.52)]. Also, it should be noted that for $A$ COUPLED WITH $B$ [see (2.34)], we do have

$$A \circ (A \text{ COUPLED WITH } B) = B \tag{2.62}$$

so long as $A$ is a normal fuzzy set.

*FUZZY THEOREMS*

By a fuzzy theorem or an assertion we mean a statement, generally of the form IF $A$ THEN $B$, whose truth-value is *true* in an approximate sense and which can be inferred from a set of axioms by the use of approximate reasoning, e.g., by repeated application of the generalized *modus ponens* or similar rules.

As an informal illustration of the concept of a fuzzy theorem, let us consider the theorem in elementary geometry which asserts that if $M_1$, $M_2$ and $M_3$ are the midpoints of the sides of a triangle (see Fig. 7), then the lines $AM_1$, $BM_2$ and $CM_3$ intersect at a point.

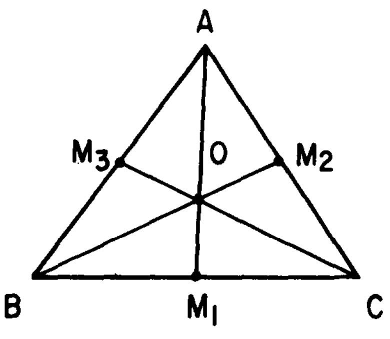


Fig. 7. An elementary theorem in geometry.

FUZZY THEOREM 2.1. *Let AB, BC and CA be approximate straight lines which form an approximate equilateral triangle with vertices A, B, C (see Fig. 8). Let* $M_1$, $M_2$ and $M_3$ *be approximate midpoints of the sides BC, CA and AB, respectively. Then the approximate straight lines* $AM_1$, $BM_2$ *and* $CM_3$ *form an approximate triangle* $T_1 T_2 T_3$ *which is more or less (more or less small) in relation to ABC.*

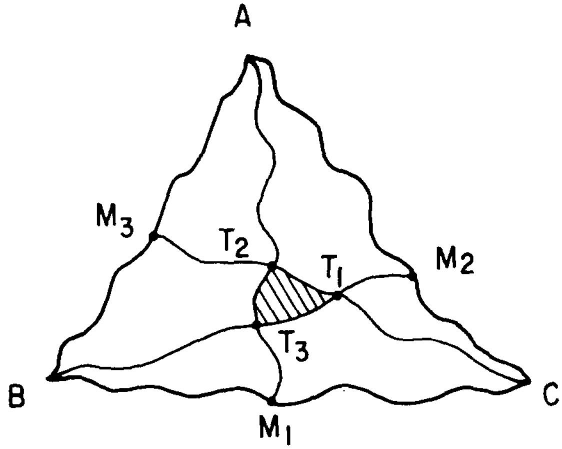


Fig. 8. A fuzzy theorem in geometry.

Before we can proceed to "prove" this fuzzy theorem, we must make more specific the sense in which the terms approximate straight line, approximate midpoint, etc. should be understood. To this end, let us agree that by an *approximate straight line AB* we mean a curve passing through $A$ and $B$ such that the distance of any point on the curve from the straight line $AB$ is small in relation to the length of $AB$. With reference to Fig. 9, this implies that we are assigning a linguistic value *small* to the distance $d$, with the understanding that $d$ is interpreted as a fuzzy variable.

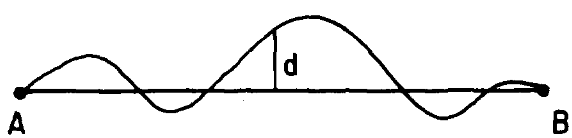


Fig. 9. Definition of *approximately straight line.*

Let $(AB)^0$ denote the straight line $AB$. Then, by an *approximate midpoint* of $AB$ we mean a point on $AB$ whose distance from $M_1^0$, the midpoint of $(AB)^0$, is *small.*

Turning to the statement of the fuzzy theorem, let $O$ be the intersection of the straight lines $(AM_1^0)^0$ and $(BM_2^0)^0$ (Fig. 10). Since $M_1$ is assumed to be an approximate midpoint of $BC$, the distance of $M_1$ from $M_1^0$ is *small.* Consequently, the distance of any point on $(AM_1)^0$ from $(AM_1^0)^0$ is *small.* Furthermore, since the distance of any point on $AM_1$ from $(AM_1)^0$ is *small,* it follows that the distance of any point on $AM_1$ from $(AM_1^0)^0$ is *more or less small.*

The same argument applies to the distance of points on $BM_2$ from $(BM_2^0)^0$. Then, taking into consideration that the angle between $(AM_1)^0$ and $(BM_2)^0$ is approximately 120°, the distance between an intersection of $AM_1$ and $BM_2$ and $O$ is *(more or less)*$^2$ *small* [that is, *more or less* (*more or less small*)]. From this it follows that the distance of any vertex of the triangle $T_1 T_2 T_3$ from $O$ is (*more or less*)$^2$ *small*. It is in this sense that the triangle $T_1 T_2 T_3$ is (*more or less*)$^2$ *small* in relation to *ABC*.

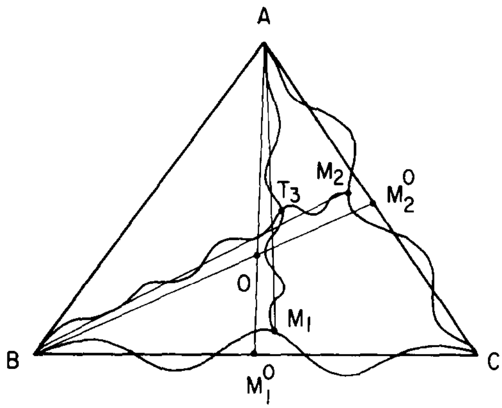


Fig. 10. Illustration of an approximate proof of the fuzzy theorem.

The reasoning used above is both approximate and qualitative in nature. It uses as its point of departure the fact that $AM_1$, $BM_2$ and $CM_3$ intersect at $O$, and employs what, in effect, are qualitative continuity arguments. Clearly, the "proof" would be longer and more involved if we had to start from the basic axioms of Euclidean geometry rather than the nonfuzzy theorem which served as our point of departure.

At this point, what we can say about fuzzy theorems is highly preliminary and incomplete in nature. Nonetheless, it appears to be an intriguing area for further study and eventually may prove to be of use in various types of ill-defined decision processes.

*GRAPHICAL REPRESENTATION BY FUZZY FLOWCHARTS*

As pointed out in [7], in the representation and execution of fuzzy algorithms it is frequently very convenient to employ flowcharts for the purpose of defining relations between variables and assigning values to them.

In what follows, we shall not concern ourselves with the many complex issues arising in the representation and execution of fuzzy algorithms. Thus, our limited objective is merely to clarify the role played by the decision boxes which are associated with fuzzy rather than nonfuzzy predicates by relating their function to the assignment of restrictions on base variables.

In the conventional flowchart, a decision box such as $A$ in Fig. 11 represents a unary[7] predicate, $A(x)$. Thus, transfer from point 1 to point 2 signifies that $A(x)$ is *true*, while transfer from 1 to 3 signifies that $A(x)$ is *false*.

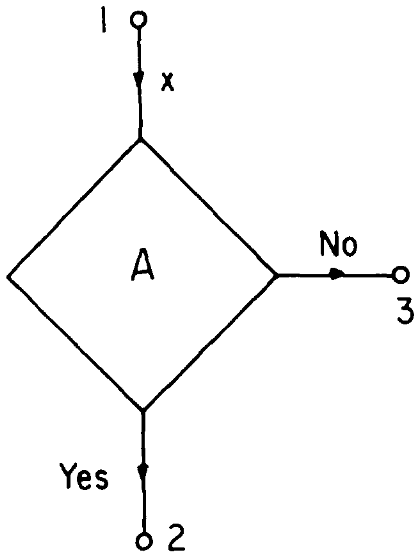


Fig. 11. A fuzzy decision box.

The concepts introduced in the preceding sections provide us with a basis for extending the notion of a decision box to fuzzy sets (or predicates). Specfically, with reference to Fig. 11, suppose that $A$ is a fuzzy subset of $U$, and the question associated with the decision box is: "Is $x$ $A$?" as in "Is $x$ *small*?" where $x$ is a generic name for the input variable. Flowcharts containing decision boxes of this type will be referred to as *fuzzy flowcharts*.

If the answer is simply YES, we assign $A$ to the restriction on $x$. That is, we set

$$R(x) = A \tag{2.63}$$

and transfer $x$ from 1 to 2.

---

[7]For simplicity, we shall not consider decision boxes having more than one input and two outputs.

On the other hand, if the answer is NO, we set

$$R(x) = \neg A \qquad (2.64)$$

and transfer $x$ from 1 to 3.

As an illustration, if $A \triangleq$ *small*, then (2.63) would read

$$R(x) = small. \qquad (2.65)$$

If the answer is YES/$\mu$, where $0 \leqslant \mu \leqslant 1$, then we transfer $x$ to 2 with the conclusion that the grade of membership of $x$ in $A$ is $\mu$. We also transfer $x$ to 3 with the conclusion that the grade of membership of $x$ in $\neg A$ is $1 - \mu$.

If the grade of membership, $\mu$, is linguistic rather than numerical, we represent it as a linguistic truth-value. Typically, then, the answer would have the form YES/*true* or YES/*very true* or YES/*more or less true*, etc. As before, we conclude that the grade of membership of $x$ in $A$ is $\mu$, where $\mu$ is a linguistic truth-value, and transfer $x$ to 3 with the conclusion that the grade of membership of $x$ in $\neg A$ is $1 - \mu$.

If we have a chain of decision boxes as in Fig. 12, a succession of YES answers would transfer $x$ from 1 to $n + 1$ and would result in the assignment to $R(x)$ of the intersection of $A_1, \ldots, A_n$. Thus,

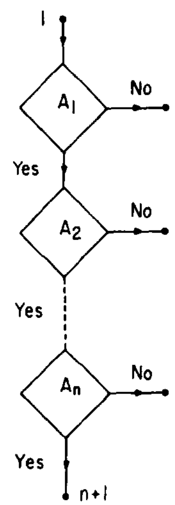


Fig. 12. A tandem combination of decision boxes.

$$R(x) = A_1 \cap \cdots \cap A_n, \tag{2.66}$$

where $\cap$ denotes the intersection of fuzzy sets. (See also Fig. 13.)

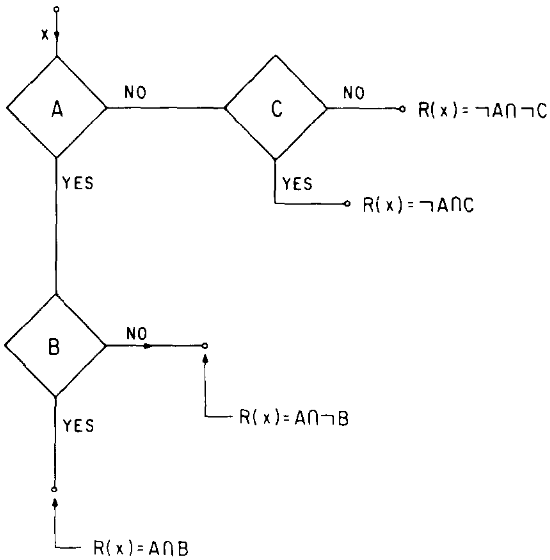


Fig. 13. Restrictions associated with various exits from a fuzzy flowchart.

As a simple illustration, suppose that $x$ = John, $A_1$ = *tall* and $A_2$ = *fat*. Then, if the response to the question "Is John *tall*?" is YES, and the response to "Is John *fat*?" is YES, the restriction imposed by John is expressed by

$$R(\text{John}) = tall \cap fat. \tag{2.67}$$

It should be noted that "John" is actually the name of a binary linguistic variable with two components named *Height* and *Weight*. Thus (2.67) is equivalent to the assignment equations

$$Height = tall \tag{2.68}$$

and

$$Weight = fat. \tag{2.69}$$

As implied by (2.66), a tandem connection of decision boxes represents the intersection of the fuzzy sets (or, equivalently, the conjunction of the fuzzy predicates) associated with them. In the case of nonfuzzy sets, their union may be realized by the scheme shown in Fig. 14. In this arrangement of decision boxes, it is clear that transfer from 1 to 2 implies that

$$R(u) = A + \neg A \cap B, \tag{2.70}$$

and since

$$A \cap B \subset A, \tag{2.71}$$

it follows that (2.70) may be rewritten as

$$\begin{aligned} R(u) &= A + A \cap B + \neg A \cap B \\ &= A + (A + \neg A) \cap B \\ &= A + B, \end{aligned} \tag{2.72}$$

since

$$A + \neg A = U \tag{2.73}$$

and

$$U \cap B = B. \tag{2.74}$$

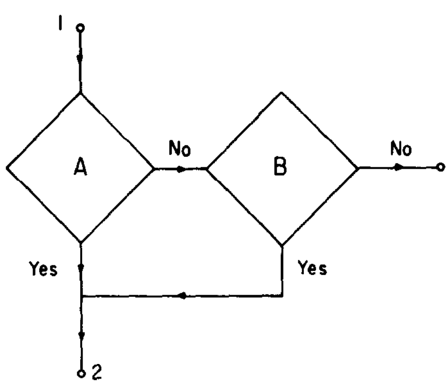


Fig. 14. A graphical representation of the disjunction of fuzzy predicates.

The same scheme would not yield the union of fuzzy sets, since the identity

$$A + \neg A = U \tag{2.75}$$

does not hold exactly if $A$ is fuzzy. Nevertheless, we can agree to interpret the arrangement of decision boxes in Fig. 14 as one that represents the union of $A$ and $B$. In this way, we can remain on the familiar ground of flowcharts involving nonfuzzy decision boxes. The flowchart shown in Fig. 16 below illustrates the use of this convention in the definition of *Hippie.*

The conventions described above may be used to represent in a graphical form the assignment of a linguistic value to a linguistic variable. Of particular use in this connection is a tandem combination of decision boxes which represent a series of *bracketing* questions which are intended to narrow down the range of possible values of a variable. As an illustration, suppose that $x$ = John and (see Fig. 15)

$$\begin{aligned} A_1 &= \textit{tall}, \\ A_2 &= \textit{very tall}, \\ A_3 &= \textit{very very tall}, \\ A_4 &= \textit{extremely tall}. \end{aligned} \tag{2.76}$$

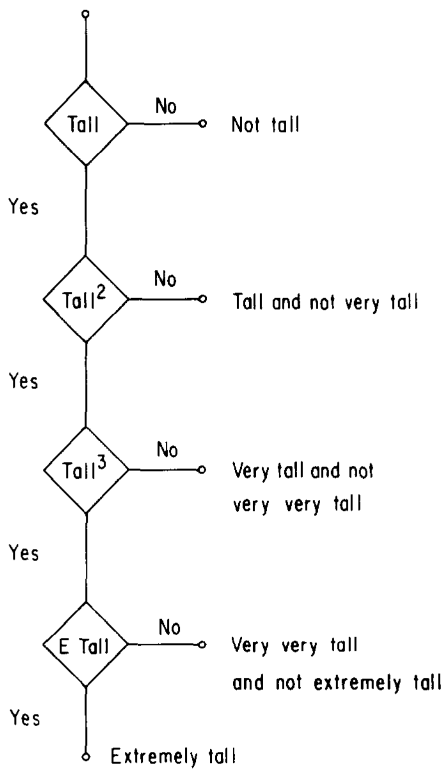


Fig. 15. Use of a tandem combination of decision boxes for purposes of bracketing.

If the answer to the first question is YES, we have

$$R(x) = \textit{tall}. \tag{2.77}$$

If the answer to the second question is YES and to the third question is NO, then

$$R(\text{John}) = \textit{very tall and not very very tall}, \tag{2.78}$$

which brackets the height of John between *very tall* and *not very very tall.*

By providing a mechanism—as in bracketing—for assigning linguistic values in stages rather than in one step, fuzzy flowcharts can be very helpful in the representation of algorithmic definitions of fuzzy concepts. The basic idea in this instance is to define a complex or a new fuzzy concept in terms of simpler or more familiar ones. Since a fuzzy concept may be viewed as a name for a fuzzy set, what is involved in this approach is, in effect, the decomposition of a fuzzy set into a combination of simpler fuzzy sets.

As an illustration, suppose that we wish to define the term *Hippie,* which may be viewed as a name of a fuzzy subset of the universe of humans. To this end, we employ the fuzzy flowchart[8] shown in Fig. 16. In essense, this flowchart defines the fuzzy set *Hippie* in terms of the fuzzy sets labeled *Long Hair, Bald, Shaved, Job* and *Drugs.* More specifically, it defines the fuzzy set *Hippie* as (+ $\triangleq$ union)

$$\textit{Hippie} = (\textit{Long Hair} + \textit{Bald} + \textit{Shaved}) \cap \textit{Drugs} \cap \neg\, \textit{Job} \tag{2.79}$$

Suppose that we pose the following questions and receive the indicated answers.

| | |
|---|---|
| Does $x$ have *Long Hair* | YES |
| Does $x$ have a *Job*? | NO |
| Does $x$ take *Drugs*? | YES |

Then we assign to $x$ the restriction

$$R(x) = \textit{Long Hair} \cap \neg\, \textit{Job} \cap \textit{Drugs},$$

[8]It should be understood, of course, that this highly oversimplified definition is used merely as an illustration and has no pretense at being accurate, complete or realistic.

and since it is contained in the right-hand side of (2.79), we conclude that $x$ is a *Hippie.*

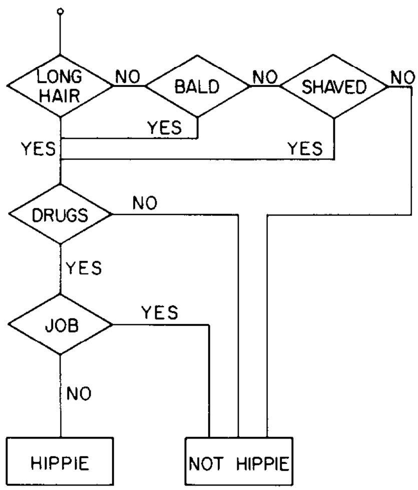


Fig. 16. Algorithmic definition of *Hippie* presented in the form of a fuzzy flowchart.

By modifying the fuzzy sets entering into the definition of *Hippie* through the use of hedges such as *very, more or less, extremely,* etc., and by allowing the answers to be of the form YES/$\mu$ or NO/$\mu$, where $\mu$ is a numerical or linguistic truth-value, the definition of *Hippie* can be adjusted to fit more closely our conception of what we want to define. Furthermore, we may use a soft *and* (see Part I, Comment 3.1) to allow some trade-offs between the characteristics which define a hippie. And, finally, we may allow our decision boxes to have multiple inputs and multiple outputs. In this way, a concept such as *Hippie* can be defined as completely as one may desire in terms of a set of constituent concepts each of which, in turn, may be defined algorithmically. In essence, then, in employing a fuzzy flowchart to define a fuzzy concept such as *Hippie,* we are decomposing a statement of the general form

$$\nu(u \text{ is : linguistic value of a Boolean linguistic variable } \mathscr{X}) = \text{linguistic value of a Boolean linguistic truth-variable } \mathscr{T} \quad (2.80)$$

into truth-value assignments of the same form, but involving simpler or more familiar variables on the left-hand side of (2.80).

*CONCLUDING REMARKS*

In this as well as in the preceding sections, our main concern has centered on the development of a conceptual framework for what may be called a *linguistic approach* to the analysis of complex or ill-defined systems and decision processes. The substantive differences between this approach and the conventional quantitative techniques of system analysis raise many issues and problems which are novel in nature and hence require a great deal of additional study and experimentation. This is true, in particular, of some of the basic aspects of the concept of a linguistic variable on which we have dwelt only briefly in our exposition, namely: linguistic approximation, representation of linguistic hedges, nonnumerical base variables, $\lambda$- and $\beta$-interaction, fuzzy theorems, linguistic probability distributions, fuzzy flowcharts and others.

Although the linguistic approach is orthogonal to what have become the prevailing attitudes in scientific research, it may well prove to be a step in the right direction, that is, in the direction of lesser preoccupation with exact quantitative analyses and greater acceptance of the pervasiveness of imprecision in much of human thinking and perception. It is our belief that, by accepting this reality rather than assuming that the opposite is the case, we are likely to make more real progress in the understanding of the behavior of humanistic systems than is possible within the confines of traditional methods.

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