Control Synthesis of Continuous-Time T-S Fuzzy Systems With Local Nonlinear Models

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***Abstract*—This paper is concerned with the problem of designing fuzzy controllers for a class of nonlinear dynamic systems. The considered nonlinear systems are described by T-S fuzzy models with nonlinear local models, and the fuzzy models have fewer fuzzy rules than conventional T-S fuzzy models with local linear models. A new fuzzy control scheme with local nonlinear feedbacks is proposed, and the corresponding control synthesis conditions are given in terms of solutions to a set of linear matrix inequalities (LMIs). In contrast to the existing methods for fuzzy control synthesis, the new proposed control design method is based on fewer fuzzy rules and less computational burden. Moreover, the local nonlinear feedback laws in the new fuzzy controllers are also helpful in achieving good control effects. Numerical examples are given to illustrate the effectiveness of the proposed method.**

***Index Terms*—Fuzzy control, guaranteed cost control, H_∞ control, linear matrix inequality (LMI), nonlinear systems, state feedback, Takagi–Sugeno (T-S) fuzzy models.**

I. Introduction

Since more than two decades ago, Takagi–Sugeno (T-S) fuzzy models [1] have attracted wide attention from scientists and engineers, essentially because the well-known fuzzy models can effectively approximate a wide class of nonlinear systems [2]. T-S fuzzy models consist of local linear models, which are smoothly connected together by fuzzy membership functions; then, the control techniques for linear control systems can be applied to T-S fuzzy models. Many systematic approaches for stability analysis and control synthesis for the class of T-S fuzzy models are developed (see the survey paper [3] and the references therein). In particular, based on quadratic Lyapunov functions approaches, control synthesis problems have been well studied in [4]–[11]. Since a common quadratic Lyapunov function is independent of fuzzy membership functions, the results based on a single Lyapunov function might be conservative. Then, parameter-dependent Lyapunov functions (which are also called fuzzy Lyapunov functions) [12]–[15], piecewise Lyapunov functions [16], [17], and k-sample variation Lyapunov functions [18] have been exploited for obtaining less conservative results. In the aforementioned works, the parallel distributed compensation (PDC) control scheme in [2], i.e., the controller shares the same fuzzy rules with the considered fuzzy model, is extensively applied for designing fuzzy controllers. Moreover, a number of alternative control schemes, such as the non-PDC control scheme in [15], switching constant controller gain scheme in [17], switching PDC control scheme in [19], [20], are also developed for designing fuzzy controllers for T-S fuzzy models. It is important to consider not only stability but also some control performance requirements, such as H_∞ performance constraints and guaranteed cost bound constraints. In recent years, the H_∞ control and guaranteed cost control problems are extensively studied, and some linear matrix inequality (LMI)-based design conditions are obtained (see [7], [19], and [21]–[23] for the H_∞ fuzzy control synthesis; [24]–[26] for the guaranteed cost control; [27] for the mixed H_2/H_∞ control; and [20] and [28] for the H_∞ control with quadratic $\mathcal{D}$ stability constraints).

Most of the aforementioned results focus on the stability analysis and synthesis based on T-S fuzzy models with linear local models. However, when a nonlinear system has complex nonlinearities, the constructed T-S fuzzy model will have to consist of a number of fuzzy local models. Then, stability analysis and control synthesis for such a T-S fuzzy model often are very difficult. In general, there are two types of methods to overcome the difficulty. One method is to exploit a good tradeoff between the conservatism and the computational burden by reducing unimportant decision variables [29]. However, the obtained controllers are still with a number of control rules, which may be unfavorable for implementation. The other method is that the original nonlinear model is first simplified as much as possible. Then, a fuzzy model with fewer fuzzy rules is constructed based on the simplified nonlinear model by using a fuzzy local approximation technique [2]. However, the designed control laws based on the fuzzy model may not guarantee the stability of the original nonlinear system. In this paper, a class of T-S fuzzy models with local nonlinear models

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is exploited to describe the considered nonlinear systems. A new fuzzy control scheme with local nonlinear feedbacks is proposed, and the corresponding control synthesis conditions are developed in terms of solutions to a set of LMIs. In contrast to the existing methods for fuzzy control synthesis, the new proposed control design method is based on fewer fuzzy rules and less computational burden. Moreover, the local nonlinear feedback laws in the new fuzzy controllers are also helpful for achieving good control effects. Two numerical examples are given to illustrate the effectiveness of the proposed method.

The rest of this paper is organized as follows: Section II presents the system description and the problem under consideration. In Section III, an LMI-based method for designing H_∞ controllers for the T-S fuzzy systems with local nonlinear models is given. Subsequently, the result is extended to the case for guaranteed cost control. Lastly, a multiobjective control synthesis technique is given. In Section IV, two examples are presented to illustrate the effectiveness of the proposed design methods. Section V concludes this paper.

Notation: For a square matrix E, $\mathtt{He}(E)$ is defined as $\mathtt{He}(E) = E + E^T$. For a two-point x, $y \in R^n$, the convex hull of the two points is $co\{x, y\} = \{\theta_1 x + \theta_2 y : \theta_1 + \theta_2 = 1, \theta_i \geq 0\}$.

II. T-S Fuzzy Model With Local Nonlinear Models

A. Nonlinear System

In this paper, we consider the following nonlinear system:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) + h(x(t))w(t) \\ z(t) &= f_z(x(t)) + g_z(x(t))u(t) + h_z(x(t))w(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $w(t) \in R^{n_w}$ is the disturbance, and $z(t) \in R^{n_z}$ is the controlled output. $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $f_z(\cdot)$, $g_z(\cdot)$, and $h_z(\cdot)$ are nonlinear functions. Assume that nonlinear function $f(x(t))$ can be rewritten as follows:

$$\begin{aligned} f(x(t)) &= f_a(x(t)) + f_b(x(t))\bar{\phi}(t) \\ f_z(x(t)) &= f_{za}(x(t)) + f_{zb}(x(t))\bar{\phi}(t) \end{aligned} \tag{2}$$

where $\bar{\phi}(t) = [\,\bar{\phi}_1(t) \quad \bar{\phi}_2(t) \quad \cdots \quad \bar{\phi}_s(t)\,]^T$, $\bar{\phi}_i(t) \in R$, $1 \leq i \leq s$, are sector-bounded nonlinear functions and satisfy

$$\bar{\phi}_i(t) \in co\{E_{Li}x(t), E_{Ui}x(t)\}, \qquad 1 \leq i \leq s \tag{3}$$

which implies that nonlinear term $\bar{\phi}_i(t)$ is bounded by (3). An example is given as follows:

Consider $x(t) = [\,x_1(t) \quad x_2(t) \quad x_3(t) \quad x_4(t)\,]^T \in R^4$ and $\bar{\phi}(t) = \bar{\phi}_1(t) = \sin(x_3(t)) \in R$, where $x_3 \in [-(\pi/2), (\pi/2)]$. Then, we have that

$$\begin{aligned} \bar{\phi}(t) &\in \mathtt{co}\left\{\frac{2}{\pi}x_3(t), x_3(t)\right\} \\ &= \mathtt{co}\left\{\left[0 \quad 0 \quad \tfrac{2}{\pi} \quad 0\right]x(t), [0 \quad 0 \quad 1 \quad 0]x(t)\right\} \end{aligned}$$

which implies that $E_{L1} = [\,0 \quad 0 \quad (2/\pi) \quad 0\,]$ and $E_{U1} = [\,0 \quad 0 \quad 1 \quad 0\,]$. If $\bar{\phi}_i(t)$ satisfies (3), then a particular property can be obtained as follows:

Lemma 1: If (3) holds, then

$$\left(E_{Li}x(t) - \bar{\phi}_i(t)\right)\left(E_{Ui}x(t) - \bar{\phi}_i(t)\right) \leq 0. \tag{4}$$

Proof: The proof is easily obtained and omitted. ∎

In order to get a convex condition for designing fuzzy controllers, some transformations about nonlinear term $\bar{\phi}_i(t)$ will be performed. Let $\phi_i(t) = \bar{\phi}_i(t) - E_{Li}x(t)$, and substitute $\phi_i(t) + E_{Li}x(t)$ for $\bar{\phi}_i(t)$ in (37); then, we can obtain

$$-\phi_i(t)\left(E_i x(t) - \phi_i(t)\right) \leq 0 \tag{5}$$

with $E_i = E_{Ui} - E_{Li}$. Denote $E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_s \end{bmatrix}$.

Substituting $\phi_i(t) + E_{Li}x(t)$ for $\bar{\phi}_i(t)$ in (2); then, it yields

$$\begin{aligned} f(x(k)) &= \bar{f}_a(x(t)) + f_b(x(t))\phi(t) \\ f_z(x(t)) &= \bar{f}_{za}(x(t)) + f_{zb}(x(t))\phi(t) \end{aligned}$$

where

$$\begin{aligned} \bar{f}_a(x(t)) &= f_a(x(t)) + f_b(x(t))E_L x(t) \\ \bar{f}_{za}(x(t)) &= f_{za}(x(t)) + f_{zb}(x(t))E_L x(t) \end{aligned}$$

$$E_L = \begin{bmatrix} E_{L1} \\ E_{L2} \\ \vdots \\ E_{Ls} \end{bmatrix}.$$

Then, system (1) can be rewritten as follows:

$$\begin{aligned} \dot{x}(t) &= \bar{f}_a(x(t)) + g(x(t))u(t) \\ &\quad + h(x(t))w(t) + f_b(x(t))\phi(t) \\ z(t) &= \bar{f}_{za}(x(t)) + g_z(x(t))u(t) \\ &\quad + h_z(x(t))w(t) + f_{zb}(x(t))\phi(t) \end{aligned} \tag{6}$$

where $\phi(t) = [\,\phi_1(t) \quad \phi_2(t) \quad \cdots \quad \phi_s(t)\,]^T$, and $\phi_i(t) = \bar{\phi}_i(t) - E_{Li}x(t) \in R$, $1 \leq i \leq s$, satisfying (5).

In the next section, nonlinear system (6) [which is equivalent to nonlinear system (1)] will be modeled as a T-S fuzzy model with local nonlinear models.

B. T-S Fuzzy Modeling

In dynamic equation (6), nonlinear terms $\bar{f}_a(x(t))$, $h(x(t))$, $g(x(t))$, $f_b(x(t))$, $\bar{f}_{za}(x(t))$, $h_z(x(t))$, and $g_z(x(t))$, $f_{zb}(x(t))$

can be described by the fuzzy techniques in [2] and [4]; then, we can obtain the following T-S fuzzy model:

Plant Rule i :

IF $v_1(t)$ is Γ_{i1} and $v_2(t)$ is $\Gamma_{i2}, \cdots, v_p(t)$ is Γ_{ip}

$$\text{THEN} \quad \dot{x}(t) = A_i x(t) + B_{1i} w(t) + B_{2i} u(t) + G_i \phi(t)$$
$$z(t) = C_{1i} x(t) + D_{1i} w(t) + D_{2i} u(t) + G_{zi} \phi(t)$$

where $i = 1, \ldots, r$. r is the number of IF–THEN rules, $v(t) = [v_1(t) \quad v_2(t) \quad \cdots \quad v_p(t)]^T \in R^{p\times 1}$ are the premise variables, and Γ_{ij} are the fuzzy sets. By using the fuzzy inference method with a singleton fuzzifier and product inference and center average defuzzifiers, the final T-S fuzzy model is obtained as follows:

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} \mathtt{w}_i(v(t))(A_i x(t) + B_{1i} w(t) + B_{2i} u(t) + G_i \phi(t))}{\sum_{i=1}^{r} \mathtt{w}_i(v(t))}$$
$$z(t) = \frac{\sum_{i=1}^{r} \mathtt{w}_i(v(t))(C_{1i} x(t) + D_{1i} w(t) + D_{2i} u(t) + G_{zi} \phi(t))}{\sum_{i=1}^{r} \mathtt{w}_i(v(t))} \tag{7}$$

where $\mathtt{w}_i(v(t)) = \prod_{j=1}^{p} \eta_{ij}(v_j(t))$. $\eta_{ij}(v_j(t))$ is the grade of membership of $v_j(t)$ in Γ_{ij}, where it is assumed that $\sum_{i=1}^{r} \mathtt{w}_i(v(t)) > 0$, $\mathtt{w}_i(v(t)) \geq 0, i = 1, 2, \ldots, r$. Denote $\alpha_i(v(t)) = (\mathtt{w}_i(v(t)))/(\sum_{i=1}^{r} \mathtt{w}_i(v(t)))$; then

$$0 \leq \alpha_i\left(v(t)\right) \leq 1 \quad \text{and} \quad \sum_{i=1}^{r} \alpha_i\left(v(t)\right) = 1 \tag{8}$$

where $\alpha_i(v(t))$ is said to be normalized membership functions.

Let $\alpha(v(t)) = [\alpha_1(v(t)), \alpha_2(v(t)), \ldots, \alpha_r(v(t))]^T$, and denote $\alpha(v(t))$ as α for a brief description. Furthermore, (7) can be rewritten as follows:

$$\dot{x}(t) = A(\alpha)x(t) + B_1(\alpha)w(t) + B_2(\alpha)u(t) + G(\alpha)\phi(t)$$
$$z(t) = C_1(\alpha)x(t) + D_1(\alpha)w(t) + D_2(\alpha)u(t) + G_z(\alpha)\phi(t) \tag{9}$$

with

$$A(\alpha) = \sum_{i=1}^{r} \alpha_i(t) A_i \quad B_1(\alpha) = \sum_{i=1}^{r} \alpha_i(t) B_{1i}$$
$$B_2(\alpha) = \sum_{i=1}^{r} \alpha_i(t) B_{2i} \quad G(\alpha) = \sum_{i=1}^{r} \alpha_i(t) G_i$$
$$C_1(\alpha) = \sum_{i=1}^{r} \alpha_i(t) C_{1i} \quad D_1(\alpha) = \sum_{i=1}^{r} \alpha_i(t) D_{1i}$$
$$D_2(\alpha) = \sum_{i=1}^{r} \alpha_i(t) D_{2i} \quad G_z(\alpha) = \sum_{i=1}^{r} \alpha_i(t) G_{zi}.$$

In fact, $A(\alpha)x(t)$, $B_1(\alpha)$, $B_2(\alpha)$, $G(\alpha)$, $C_1(\alpha)x(t)$, $D_1(\alpha)$, $D_2(\alpha)$, and $G_z(\alpha)$ in fuzzy model (9) are the new descriptions of $\bar{f}_a(x(t))$, $h(x(t))$, $g(x(t))$, $f_b(x(t))$, $\bar{f}_{za}(x(t))$, $h_z(x(t))$, $g_z(x(t))$, and $f_{zb}(x(t))$ in (6) by fuzzy membership functions, respectively. The following lemma will be useful in the sequel.

Lemma 2 [5]: If the following conditions hold:

$$\begin{array}{ll} \mathcal{M}_{ii} < 0, & 1 \leq i \leq r \\ \frac{1}{r-1}\mathcal{M}_{ii} + \frac{1}{2}(\mathcal{M}_{ij} + \mathcal{M}_{ji}) < 0, & 1 \leq i \neq j \leq r \end{array}$$

then the following inequality holds:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \mathcal{M}_{ij} < 0$$

where α_i, $1 \leq i \leq r$, satisfy $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^{r} \alpha_i = 1$.

C. Fuzzy Control Scheme

Here, the following fuzzy control scheme is exploited for the T-S fuzzy system (9):

Control Rule i :

IF $v_1(t)$ is Γ_{i1} and $v_2(t)$ is $\Gamma_{i2}, \cdots, v_p(t)$ is Γ_{ip}

THEN $\quad u(t) = K_{ai} x(t) + K_{bi} \phi(t)$.

By using the fuzzy inference method with a singleton fuzzifier and product inference and center average defuzzifiers, the final control output can be obtained as follows:

$$u(t) = K_a(\alpha) x(t) + K_b(\alpha) \phi(t) \tag{10}$$

where

$$K_a(\alpha) = \sum_{i=1}^{r} \alpha_i\left(v(t)\right) K_{ai} \quad K_b(\alpha) = \sum_{i=1}^{r} \alpha_i\left(v(t)\right) K_{bi}.$$

Remark 1: Note that a nonlinear feedback control law is used for each control rule, which is different from the conventional PDC control scheme [2] for the T-S fuzzy systems with linear local models, where only a linear feedback for each control rule is used.

From (9) and (10), the closed-loop fuzzy system can be obtained as follows:

$$\begin{aligned} \dot{x}(t) &= \left(A(\alpha) + B_2(\alpha) K_a(\alpha)\right) x(t) + B_1(\alpha) w(t) \\ &\quad + \left(G(\alpha) + B_2(\alpha) K_b(\alpha)\right) \phi(t) \\ z(t) &= \left(C_1(\alpha) + D_2(\alpha) K_a(\alpha)\right) x(t) + D_1(\alpha) w(t) \\ &\quad + \left(G_z(\alpha) + D_2(\alpha) K_b(\alpha)\right) \phi(t). \end{aligned} \tag{11}$$

In this paper, we consider the problems of H_∞ control and guaranteed cost control for the T-S fuzzy system (9). The considered H_∞ control problem is described as follows:

For a prescribed H_∞ performance bound $\gamma > 0$, design a fuzzy state feedback control law $u(t)$ such that the closed-loop system (11) is asymptotically stable with

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt. \qquad (12)$$

Equation (12) characterizes the effect of the disturbance on the regulated output. By minimizing the H_∞ performance bound γ, the effect of the disturbance on the regulated output can be attenuated. However, the H_∞ performance constraint is not enough by itself to achieve a desired control performance. The guaranteed cost control (or H_2 optimal control) is more appealing for control engineers to achieve a desired control performance via a proper choice of weighting matrices [27]. In this paper, we also consider the problem of quadratic guaranteed cost control of the T-S fuzzy system (9). The guaranteed cost control aims at stabilizing the system while maintaining an adequate level of performance represented by a quadratic cost function [26]

$$J = \int_0^\infty \left(x^T(t)R_x x(t) + u^T(t)R_u u(t)\right) dt \qquad (13)$$

where $R_x = R_x^T > 0$ and $R_u = R_u^T > 0$ are given matrices. Associated with function (13), the fuzzy guaranteed cost is defined as follows:

Definition 1 [24]: Consider system (11). If there exists a fuzzy control law $u(t)$ and a scalar $J_0 > 0$ such that closed-loop system (11) is asymptotically stable and the closed-loop value of cost function (13) satisfies $J \le J_0$, then J_0 is said to be a guaranteed cost, and control law $u(t)$ is said to be a guaranteed cost control law $u(t)$ for system (9). In order to give clear comparisons between the new proposed control synthesis methods in this paper and the existing ones, some existing results are recalled. Consider a T-S fuzzy model with linear local models

$$\begin{aligned}\dot{x}(t) &= \bar{A}(\bar{\alpha})x(t) + \bar{B}_1(\bar{\alpha})w(t) + \bar{B}_2(\bar{\alpha})u(t)\\ z(t) &= \bar{C}_1(\bar{\alpha})x(t) + \bar{D}_1(\bar{\alpha})w(t) + \bar{D}_2(\bar{\alpha})u(t)\end{aligned} \qquad (14)$$

where $\bar{\alpha} = [\,\bar{\alpha}_1(t) \quad \bar{\alpha}_2(t) \quad \cdots \quad \bar{\alpha}_{\bar{r}}(t)\,]^T$, $\bar{\alpha}_i(t)$ are membership functions, and

$$\bar{A}(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{A}_i \quad \bar{B}_1(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{B}_{1i}$$

$$\bar{B}_2(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{B}_{2i} \quad \bar{C}_1(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{C}_{1i}$$

$$\bar{D}_1(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{D}_{1i} \quad \bar{D}_2(\bar{\alpha}) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)\bar{D}_{2i}$$

The corresponding fuzzy PDC controller is given as follows:

$$u(t) = \sum_{i=1}^{\bar{r}} \bar{\alpha}_i(t)K_i x(t) \qquad (15)$$

where K_i are the parameters to be designed.

By using the techniques proposed in [7], we have the following lemma for designing the gain matrices K_i in (15).

Lemma 3:

1) For a given $\gamma > 0$, if there exist matrices $Q = Q^T > 0$, L_i $(1 \le i \le \bar{r})$, and $X_{ij} = X_{ji}^T$ $(1 \le i, j \le \bar{r})$ satisfying the following LMIs:

$$\Phi_{ii} \le X_{ii}, \qquad 1 \le i \le \bar{r} \qquad (16a)$$

$$\Phi_{ij} + \Phi_{ji} \le X_{ij} + X_{ij}^T, \qquad 1 \le i \ne j \le \bar{r} \qquad (16b)$$

$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1\bar{r}}\\ X_{21} & X_{22} & \cdots & X_{2\bar{r}}\\ \vdots & \vdots & \ddots & \vdots\\ X_{\bar{r}1} & X_{\bar{r}2} & \cdots & X_{\bar{r}\bar{r}} \end{bmatrix} < 0 \qquad (16c)$$

where

$$\Phi_{ij} = \begin{bmatrix} \mathtt{He}(\bar{A}_i Q + \bar{B}_{2i}L_j) & * & *\\ \bar{B}_{1i}^T & -\gamma^2 I & *\\ \bar{C}_{1i}Q + \bar{D}_{2i}L_j & \bar{D}_{1i} & -I \end{bmatrix}$$

then fuzzy system (14) is asymptotically stable with H_∞ norm less than or equal to γ via controller (15) with

$$K_i = L_i Q^{-1}, \qquad 1 \le i \le \bar{r}. \qquad (17)$$

2) If there exist matrix variables $Q = Q^T$, L_i $(1 \le i \le \bar{r})$, $\bar{X}_{ij} = \bar{X}_{ji}^T$, $(1 \le i, j \le \bar{r})$, and Z satisfying the following LMIs:

$$\begin{bmatrix} Z & I\\ I & Q \end{bmatrix} > 0 \qquad (18a)$$

$$\bar{\Phi}_{ii} < \bar{X}_{ii}, \qquad 1 \le i \le \bar{r} \qquad (18b)$$

$$\bar{\Phi}_{ij} + \bar{\Phi}_{ji} < \bar{X}_{ij} + \bar{X}_{ij}^T, \qquad 1 \le i \ne= j \le \bar{r} \qquad (18c)$$

$$\begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \cdots & \bar{X}_{1\bar{r}}\\ \bar{X}_{21} & \bar{X}_{22} & \cdots & \bar{X}_{2\bar{r}}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{X}_{\bar{r}1} & \bar{X}_{\bar{r}2} & \cdots & \bar{X}_{\bar{r}\bar{r}} \end{bmatrix} < 0 \qquad (18d)$$

where

$$\bar{\Phi}_{ij} \begin{bmatrix} \mathtt{He}(\bar{A}_i Q + \bar{B}_{2i}L_j) & * & *\\ L_i & -R_u^{-1} & *\\ Q & 0 & -R_x^{-1} \end{bmatrix}$$

then fuzzy system (14) is asymptotically stable via controller (15) with (17). In this case, cost function (13) satisfies

$$J \le x^T(0)Zx(0) \qquad (19)$$

where $x(0)$ is the initial state.

3) For a given $\gamma > 0$, if there exist matrix variables $Q = Q^T$, L_i $(1 \le i \le \bar{r})$, $X_{ij} = X_{ji}^T$, $\bar{X}_{ij} = \bar{X}_{ji}^T$ $(1 \le i, j \le \bar{r})$, and Z satisfying (16) and (18), then fuzzy system (14) is asymptotically stable via state feedback controller (15) with (17). In this case, H_∞ and H_2 norms are less than and equal to γ, respectively, and cost function (13) satisfies (19).

Proof: The proof is easily obtained from the techniques in [7] and omitted. ■

Assume that the number of premise variables in system (14) is p and denote the collection of all fuzzy rules as a rank-p tensor $I_1 \otimes I_2 \otimes \cdots \otimes I_p$, where $I_\ell = \{1, 2, \ldots, n_\ell\}$ and n_ℓ is the number of fuzzy partition on the ℓth premise variable. The number i of the ith rule can be viewed as an element of the tensor $I_1 \otimes I_2 \otimes \cdots \otimes I_p$ and denoted as $i_1 i_2 \cdots i_p$. For example, $\bar{A}_i$ and Φ_{ij} can be rewritten as $\bar{A}_{i_1 i_2 \cdots i_p}$ and $\Phi_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p}$, where $i_1 i_2 \cdots i_p$ is an element of rank-p tensor $I_1 \otimes I_2 \otimes \cdots \otimes I_p$, and $i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p$ is an element of rank-$2p$ tensor $I_1 \otimes I_2 \otimes \cdots \otimes I_p \otimes I_1 \otimes I_2 \otimes \cdots \otimes I_p$. By the tensor description, a controller design method based on a tensor-product fuzzy system (which can be viewed as a T-S fuzzy system) is given as follows:

Lemma 4 [11]: For a given $\gamma > 0$, if there exist matrices $Q = Q^T > 0$, L_i $(1 \le i \le \bar{r})$, $X^{[0]}_{i_1 i_2 \cdots i_p i_1 j_2 \cdots j_p} = (X^{[0]}_{i_1 i_2 \cdots j_p j_1 j_2 \cdots i_p})^T \in R^{n_\Phi \times n_\Phi}$ $X^{[1]}_{i_1 i_2 \cdots i_{p-1} j_1 j_2 \cdots j_{p-1}} = (X^{[1]}_{i_1 i_2 \cdots j_{p-1} j_1 j_2 \cdots i_{p-1}})^T \in R^{(n_\Phi \cdot n_p) \times (n_\Phi \cdot n_p)}, \ldots, X^{[p-2]}_{i_1 i_2 j_1 j_2} = (X^{[p-2]}_{i_1 j_2 j_1 i_2})^T \in R^{(n_\Phi \cdot n_p \cdots n_3) \times (n_\Phi \cdot n_p \cdots n_3)}$, and $X^{[p-1]}_{i_1 j_1} = (X^{[p-1]}_{j_1 i_1})^T \in R^{(n_\Phi \cdot n_p \cdots n_2) \times (n_\Phi \cdot n_p \cdots n_2)}$ $(1 \le i_l \le n_l, 1 \le l \le p)$ satisfying the following LMIs:

$$
\begin{aligned}
&\Phi_{i_1 i_2 \cdots i_p i_1 i_2 \cdots i_p} \le X^{[0]}_{i_1 i_2 \cdots i_p i_1 i_2 \cdots i_p},\\
&\qquad 1 \le i_l \le n_l, 1 \le l \le p\\
&\Phi_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p} + \Phi_{i_1 i_2 \cdots j_p j_1 j_2 \cdots i_p}\\
&\quad \le X^{[0]}_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p} + \left(X^{[0]}_{i_1 i_2 \cdots i_p j_1 j_2 \cdots j_p}\right)^T,\\
&\qquad 1 \le i_l \le n_l, 1 \le l \le p-1, 1 \le i_p < j_p \le n_p\\
&Y^{[1]}_{i_1 i_2 \cdots i_{p-1} i_1 i_2 \cdots i_{p-1}} \le X^{[1]}_{i_1 i_2 \cdots i_{p-1} i_1 i_2 \cdots i_{p-1}},\\
&\qquad 1 \le i_l \le n_l, 1 \le l \le p-1\\
&Y^{[1]}_{i_1 i_2 \cdots i_{p-1} j_1 j_2 \cdots j_{p-1}} + Y^{[1]}_{i_1 i_2 \cdots j_{p-1} j_1 j_2 \cdots i_{p-1}}\\
&\quad \le X^{[1]}_{i_1 i_2 \cdots i_{p-1} j_1 j_2 \cdots j_{p-1}} + \left(X^{[1]}_{i_1 i_2 \cdots i_{p-1} j_1 j_2 \cdots j_{p-1}}\right)^T,\\
&\qquad 1 \le i_l \le n_l, 1 \le l \le p-2, 1 \le i_{p-1} < j_{p-1} \le n_{p-1}\\
&\qquad \vdots\\
&Y^{[p-1]}_{i_1 i_1} \le X^{[p-1]}_{i_1 i_1}, \qquad 1 \le i_1 \le n_1\\
&Y^{[p-1]}_{i_1 j_1} + Y^{[p-1]}_{j_1 i_1} \le X^{[p-1]}_{i_1 j_1} + \left(X^{[p-1]}_{i_1 j_1}\right)^T,\\
&\qquad 1 \le i_1 < j_1 \le n_1\\
&Y^{[p]} \le 0
\end{aligned}
$$

where the expression shown at the bottom of the page holds, then fuzzy system (14) is asymptotically stable with H_∞ norm less than or equal to γ via controller (15) with

$$K_{i_1 i_2 \cdots i_p} = L_{i_1 i_2 \cdots i_p} Q^{-1}, \qquad 1 \le i_l \le n_l; \quad 1 \le l \le p.$$

Remark 2: Lemma 4 is a concrete description of a recursive application of the tensor production method in [11] (Theorem 2 in [11]) for designing fuzzy controllers. Here, the method is reviewed in order to give a clear comparison with that presented in this paper.

III. Main Result

In this section, first, an LMI-based condition for designing H_∞ controllers is given. Then, the result is extended to the guaranteed cost control case. Finally, a mixed control synthesis method is derived.

Theorem 1: For a prescribed $\gamma > 0$, if there exist matrices $Q = Q^T > 0$, L_{ai} and L_{bi} $(1 \le i \le r)$, and Λ with

$$\Lambda = \operatorname{diag}[\,\lambda_1 \quad \cdots \quad \lambda_s\,]_{s \times s}$$

such that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j \Xi_{ij} < 0 \tag{20}$$

where

$$\Xi_{ij} = \begin{bmatrix} \mathtt{He}(A_i Q + B_{2i} L_{aj}) & * & * & * \\ \Lambda G_i^T + L_{bj}^T B_{2i}^T + EQ & -2\Lambda & * & * \\ B_{1i}^T & 0 & -\gamma^2 I & * \\ C_{1i} Q + D_{2i} L_{aj} & G_{zi} \Lambda + D_{2i} L_{bj} & D_{1i} & -I \end{bmatrix} \tag{21}$$

then fuzzy system (9) is asymptotically stable with H_∞ norm less than or equal to γ via controller (10) with

$$K_{ai} = L_{ai} Q^{-1} \quad K_{bi} = L_{bi} \Lambda^{-1}, \qquad 1 \le i \le r. \tag{22}$$

Proof: From block (2, 2) in (21) and considering (20), then we have $\Lambda > 0$. Λ is invertible, which implies that (22) is proper, and from (22), we can obtain

$$L_{ai} = K_{ai} Q \quad L_{bi} = K_{bi} \Lambda, \qquad 1 \le i \le r.$$

Substituting $K_{ai} Q$ and $K_{bi} \Lambda$ for L_{ai} and L_{bi} $(1 \le i \le r)$ in (21), and pre- and postmultiplying (20) by

$$
\begin{aligned}
&Y^{[m]}_{i_1 i_2 \cdots i_{p-m} j_1 j_2 \cdots j_{p-m}}\\
&= \begin{bmatrix}
X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 1 j_1 j_2 \cdots j_{p-m} 1} & X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 1 j_1 j_2 \cdots j_{p-m} 2} & \cdots & X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 1 j_1 j_2 \cdots j_{p-m} n_{p-m+1}} \\
X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 2 j_1 j_2 \cdots j_{p-m} 1} & X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 2 j_1 j_2 \cdots j_{p-m} 2} & \cdots & X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} 2 j_1 j_2 \cdots j_{p-m} n_{p-m+1}} \\
\vdots & \vdots & \ddots & \vdots \\
X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} n_{p-m+1} j_1 j_2 \cdots j_{p-m} 1} & \cdots & \cdots & X^{[m-1]}_{i_1 i_2 \cdots i_{p-m} n_{p-m+1} j_1 j_2 \cdots j_{p-m} n_{p-m+1}}
\end{bmatrix}, \qquad 1 \le m \le p
\end{aligned}
$$

diag$[Q^{-1} \quad \Lambda^{-1} \quad I \quad I]$ and its transpose, then it follows that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\hat{\Xi}_{ij} < 0 \tag{23}$$

where

$$\Xi_{ij} = \begin{bmatrix} \mathtt{He}(PA_i+PB_{2i}K_{aj}) & * & * & * \\ G_i^TP+K_{bj}^TB_{2i}^TP+\Lambda^{-1}E & -2\Lambda^{-1} & * & * \\ B_{1i}^TP & 0 & -\gamma^2 I & * \\ C_{1i}+D_{2i}K_{aj} & G_{zi}+D_{2i}K_{bj} & D_{1i} & -I \end{bmatrix} \tag{24}$$

and $P = Q^{-1}$.

Equation (23) is shown at the bottom of the page. Applying Schur complement to it then yields

$$\begin{bmatrix} \mathtt{He}\left(PA(\alpha)+PB_2(\alpha)K_a(\alpha)\right) & * & * \\ G^T(\alpha)P+K_b^T(\alpha)B_2^T(\alpha)P & 0 & * \\ B_1^T(\alpha)P & 0 & -\gamma^2 I \end{bmatrix}$$
$$+\begin{bmatrix} 0 & * & * \\ \Lambda^{-1}E & -2\Lambda^{-1} & * \\ 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} C_1^T(\alpha)+K_a^T(\alpha)D_2^T(\alpha) \\ G_z^T(\alpha)+K_b^T(\alpha)D_2^T(\alpha) \\ D_1^T(\alpha) \end{bmatrix}$$
$$\times\left[C_1(\alpha)+D_2(\alpha)K_a(\alpha)G_z(\alpha)+D_2(\alpha)K_b(\alpha)D_1(\alpha)\right]<0.$$

Pre- and post-multiplying the preceding inequality by $[x^T(t) \quad \phi^T(t) \quad w^T(t)] \neq 0$ and its transpose, we can then obtain

$$\begin{aligned} &2x^T(t)\left(PA(\alpha)+PB_2(\alpha)K_a(\alpha)\right)x(t)+2x^T(t) \\ &\times\left(PG(\alpha)+PB_2(\alpha)K_b(\alpha)\right)\phi(t)-\gamma^2w^T(t)w(t) \\ &+2x^T(t)PB_1(\alpha)w(t)+z^T(t)z(t) \\ &+2\left(\phi^T(t)\Lambda^{-1}Ex(t)-\phi^T(t)\Lambda^{-1}\phi(t)\right) \\ &\quad=2x^T(t)P\dot{x}(t)+z^T(t)z(t)-\gamma^2w^T(t)w(t) \\ &\quad\quad+2\left(\phi^T(t)\Lambda^{-1}Ex(t)-\phi^T(t)\Lambda^{-1}\phi(t)\right)<0. \end{aligned} \tag{25}$$

Considering $\Lambda^{-1} = \mathrm{diag}[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_s]^{-1} = \mathrm{diag}[\lambda_1^{-1} \quad \lambda_2^{-1} \quad \cdots \quad \lambda_s^{-1}]$, then

$$\begin{aligned} &\phi^T(t)\Lambda^{-1}Ex(t)-\phi^T(t)\Lambda^{-1}\phi(t) \\ &\quad=\left[\lambda_1^{-1}\phi_1(t) \quad \lambda_2^{-1}\phi_2(t) \quad \cdots \quad \lambda_s^{-1}\phi_s(t)\right]\begin{bmatrix} E_1x(t) \\ E_2x(t) \\ \vdots \\ E_sx(t) \end{bmatrix} \\ &\quad\quad-\sum_{i=1}^{s}\lambda_i^{-1}\phi_i^2(t) \\ &\quad=\sum_{i=1}^{s}\lambda_i^{-1}\left(\phi_i(t)E_ix(t)-\phi_i^2(t)\right) \\ &\quad=\sum_{i=1}^{s}\lambda_i^{-1}\left(\phi_i(t)\left(E_ix(t)-\phi_i(t)\right)\right). \end{aligned} \tag{26}$$

Since $\phi_i(t) \in \mathtt{co}\{0, E_ix(t)\}$, $1 \le i \le r$, then, from Lemma 1, we have that

$$\phi_i(t)\left(E_ix(t)-\phi_i(t)\right) \ge 0, \qquad \text{for } 1 \le i \le s.$$

Combining it and (26) then yields

$$\phi^T(t)\Lambda^{-1}Ex(t)-\phi^T(t)\Lambda^{-1}\phi(t) \ge 0.$$

From the preceding inequality and (25), then it follows that

$$2x^T(t)P\dot{x}(t)+z^T(t)z(t)-\gamma^2w^T(t)w(t) < 0. \tag{27}$$

Because $Q > 0$, $P = Q^{-1} > 0$; we then choose Lyapunov function candidate

$$V(t) = x^T(t)Px(t).$$

Considering (27), it then follows that

$$\dot{V}(t)+z^T(t)z(t)-\gamma^2w^T(t)w(t) < 0 \tag{28}$$

which implies that $\dot{V}(t) < 0$ in the disturbance-free case; then, the systems are asymptotically stable.

Integrating both sides of (28) yields

$$\begin{aligned} &\int_0^\infty \dot{V}(t)dt+\int_0^\infty\left(z^T(t)z(t)-\gamma^2w^T(t)w(t)\right)dt \\ &\quad=V(\infty)-V(0)+\int_0^\infty\left(z^T(t)z(t)-\gamma^2w^T(t)w(t)\right)dt<0. \end{aligned}$$

Under zero initial condition, combining the preceding inequality and $V(\infty) \ge 0$, we can then obtain that

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2\int_0^\infty w^T(t)w(t)dt$$

which implies that the H_∞ norm of the closed-loop system (11) is less than or equal to γ. Thus, the proof is complete. ■

Remark 3: Since $\alpha(t)$ appears in the closed-loop system, the parameterized LMI (PLMI) of Theorem 1 may be nonlinear in $\alpha(t)$. Hence, it needs to be checked for all the values of $\alpha(t)$, which is equivalent to solving an infinite number of LMIs. Thus, it is impossible to directly solve the PLMI. There have

$$\begin{bmatrix} \mathtt{He}\left(PA(\alpha)+PB_2(\alpha)K_a(\alpha)\right) & * & * & * \\ G^T(\alpha)P+K_b^T(\alpha)B_2^T(\alpha)P+\Lambda^{-1}E & -2\Lambda^{-1} & * & * \\ B_1^T(\alpha)P & 0 & -\gamma^2 I & * \\ C_1(\alpha)+D_2(\alpha)K_a(\alpha) & G_z(\alpha)+D_2(\alpha)K_b(\alpha) & D_1(\alpha) & -I \end{bmatrix} < 0$$

TABLE I
NUMBER OF DECISION VARIABLES $\mathcal{D}$ AND LINES $\mathcal{L}$

	$\mathcal{D}$	$\mathcal{L}$
Theorem 2	$1+s+\frac{1}{2}n(n+1)+m(n+s)r$ [1]	$n+r^2n_\Xi$
Corollary 1	$1+s+m(n+s)r+\frac{1}{2}\Big(n(n+1)+n_\Xi(n_\Xi+1)r+r(r-1)n_\Xi^2\Big)$	$n+2rn_\Xi+\frac{1}{2}r(r-1)n_\Xi$
Lemma 3 (i)	$1+mn\bar{r}+\frac{1}{2}\Big(n(n+1)+n_\Phi(n_\Phi+1)\bar{r}+\bar{r}(\bar{r}-1)n_\Phi^2\Big)$	$n+2\bar{r}n_\Phi+\frac{1}{2}\bar{r}(\bar{r}-1)n_\Phi$
Lemma 4	$1+\frac{1}{2}n(n+1)+mn\prod_{i=1}^{p}n_i+\frac{1}{2}n_\Phi(\prod_{i=1}^{p}n_i)\times\Big[\sum_{m=0}^{p-1}\Big((\prod_{i=0}^{m}n_i)\big[n_\Phi(\prod_{i=m+1}^{p}n_i)+1\big]\Big)\Big]$	$n+\frac{1}{2}n_\Phi(\prod_{i=1}^{p}n_i)\times\Big[\sum_{m=0}^{p}\Big((n_m+1)(\prod_{i=-1}^{m-1}n_i)\Big)\Big]$

[1] n_Ξ, n_Φ denote the number of the row of matrices Ξ_{11} and Φ_{11}, respectively. n, m, s are respectively the dimensions of the state vector $x(t)$, input vector $u(t)$, and $\phi(t)$. $n_{-1}=n_0=1$ is defined for obtaining a compact expression. r is the number of IF-THEN rules in the fuzzy model with local nonlinear model (7). $\bar{r}=\prod_{i=1}^{p}n_i$ is the number of IF-THEN rules in the conventional T-S fuzzy model (14). n_i is the number of fuzzy partition on i-th premise variable of the conventional T-S fuzzy model (14). p is the number of the premise variables of the conventional T-S fuzzy model (14).

been many approaches [2], [5]–[11] for converting the PLMI into a finite number of LMIs. In particular, by multidimensional fuzzy summations or tensor-production descriptions, [10] and [11] give a recursive procedure to design fuzzy controllers. Then, relaxed conditions can be obtained. Furthermore, [10] presents a sufficient and necessary condition by applications of Polya's theorem. However, the algorithms with a recursive procedure in [10] and [11] lead to a heavy computational burden. The method in [5] has much fewer variables involved and is much more efficient in computation than other approaches. Here, we apply the technique given in [5] to fulfill the task. The resulting LMI-based design condition is given by the following theorem. Moreover, Corollary 1 is obtained by applications of the technique in [7] to Theorem 1, in order to make an impartial comparison with the method in [11], where the technique of [7] is used.

Theorem 2: For a prescribed $\gamma>0$, if there exist matrices $Q=Q^T>0$, L_{ai}, and L_{bi} $(0\le i\le r)$ satisfying

$$\Xi_{ii}<0,\qquad 1\le i\le r \tag{29}$$
$$\frac{1}{r-1}\Xi_{ii}+\frac{1}{2}(\Xi_{ij}+\Xi_{ji})<0,\qquad 1\le i\ne j\le r \tag{30}$$

where Ξ_{ij}, $1\le i,j\le r$, is the same as that in (21). Then, fuzzy system (9) is asymptotically stable with H_∞ norm less than or equal to γ via controller (10) with (22).

Proof: From Theorem 1 and Lemma 2, the proof is easily obtained and omitted. ■

Based on Theorem 1 and the technique in [7], we can also obtain the following corollary.

Corollary 1: For a prescribed $\gamma>0$, if there exist matrices $Q=Q^T>0$, L_{ai} and L_{bi} $(0\le i\le r)$, and $X_{ij}=X_{ji}^T$ $(1\le i,j\le r)$ satisfying

$$\Xi_{ii}\le X_{ii},\qquad 1\le i\le r \tag{31}$$
$$\Xi_{ij}+\Xi_{ji}\le X_{ij}+X_{ij}^T,\qquad 1\le i\ne j\le r \tag{32}$$
$$\begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1r}\\ X_{21} & X_{22} & \cdots & X_{2r}\\ \vdots & \vdots & \ddots & \vdots\\ X_{r1} & X_{r2} & \cdots & X_{rr}\end{bmatrix}<0 \tag{33}$$

where Ξ_{ij}, $1\le i,j\le r$, is the same as that in (21). Then, fuzzy system (9) is asymptotically stable with H_∞ norm less than or equal to γ via controller (10) with (22).

Based on the condition of Theorem 2 or Corollary 1, the H_∞ norm bound constraint γ can be minimized by solving the following optimization problem:

$$\begin{aligned}\texttt{OP1}: &\ \textbf{Minimize}: \gamma\\ &\ \textbf{subjectto}: (29) \text{ and } (30)\ (\text{or } (31)-(33)).\end{aligned}$$

Remark 4: In Theorem 2, a new approach is presented for designing fuzzy controllers for nonlinear systems. The method can effectively reduce the rules of fuzzy controllers, which are significant for implementation in engineering. Moreover, the new technique has less computational burden than the existing ones. In general, the numerical complexity of LMI conditions is closely related to the number of lines $\mathcal{L}$ and decision variables $\mathcal{D}$ in the LMIs to be solved, and LMI conditions can be solved in polynomial time with complexity proportional to $\mathcal{C}=\mathcal{D}^3\mathcal{L}$ [30]. The number of variables and the number of lines in Lemma 3 (property 1), Lemma 4, and Theorem 2 are shown in Table I. It can be seen from Table I that the number of fuzzy rules has a dominate influence on numerical complexity, when there are many fuzzy rules. Because the new controller design technique is based on a new T-S fuzzy model with fewer fuzzy rules, it can effectively lighten the computational burden. The more concrete comparisons will be given in Section IV.

In the following, we extend the result of Theorem 2 to the guaranteed cost control case.

Theorem 3: If there exist matrices $Q=Q^T>0$, L_{ai} and L_{bi} $(1\le i\le r)$, and Λ with

$$\Lambda=\mathrm{diag}[\lambda_1\quad\cdots\quad\lambda_s]_{s\times s}$$

such that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\Psi_{ij}<0 \tag{34}$$

where

$$\Psi_{ij} = \begin{bmatrix} \mathtt{He}(A_iQ + B_{2i}L_{aj}) & * & * & * \\ \Lambda G_i^T + L_{bj}^T B_{2i}^T + EQ & -2\Lambda & * & * \\ L_{ai} & L_{bi} & -R_u^{-1} & * \\ Q & 0 & 0 & -R_x^{-1} \end{bmatrix} \quad (35)$$

then system (9), with controller (10) and its gains defined by (22), is asymptotically stable, and cost function (13) satisfies

$$J \le x^T(0)Q^{-1}x(0) \quad (36)$$

where $x(0)$ is the initial state.

Proof: From block (2, 2) in (35) and considering (34), then we have $\Lambda > 0$, which implies that Λ is invertible. From (22), we can obtain $L_{ai} = K_{ai}Q$ and $L_{bi} = K_{bi}\Lambda$, $1 \le i \le r$. Substituting $K_{ai}Q$ and $K_{bi}\Lambda$ for L_{ai} and L_{bi}, $1 \le i \le r$, in (35) and pre- and postmultiplying (34) by $\mathrm{diag}[Q^{-1} \quad \Lambda^{-1} \quad I \quad I]$ and its transpose, then it follows that

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j\hat{\Psi}_{ij} < 0 \quad (37)$$

where

$$\Psi_{ij} = \begin{bmatrix} \mathtt{He}(PA_i + PB_{2i}K_{aj}) & * & * & * \\ G_i^T P + K_{bj}^T B_{2i}^T P + \Lambda^{-1}E & -2\Lambda^{-1} & * & * \\ K_{ai} & K_{bi} & -R_u^{-1} & * \\ I & 0 & 0 & -R_x^{-1} \end{bmatrix}$$

and $P = Q^{-1}$.

Equation (37) is rewritten as follows:

$$\begin{bmatrix} \mathtt{He}\,(PA(\alpha) + PB_2(\alpha)K_a(\alpha)) & * & * & * \\ G^T(\alpha)P + K_b^T(\alpha)B_2^T(\alpha)P + \Lambda^{-1}E & -2\Lambda^{-1} & * & * \\ K_a(\alpha) & K_b(\alpha) & -R_u^{-1} & * \\ I & 0 & 0 & -R_x^{-1} \end{bmatrix} < 0$$

Applying Schur complement to it then yields

$$\begin{bmatrix} \mathtt{He}\,(PA(\alpha) + PB_2(\alpha)K_a(\alpha)) + R_x & * \\ G^T(\alpha)P + K_b^T(\alpha)B_2^T(\alpha)P & 0 \end{bmatrix} + \begin{bmatrix} 0 & * \\ \Lambda^{-1}E & -2\Lambda^{-1} \end{bmatrix} + \begin{bmatrix} K_a^T(\alpha) \\ K_b^T(\alpha) \end{bmatrix} R_u\,[K_a(\alpha) \quad K_b(\alpha)] < 0.$$

Pre- and postmultiplying the preceding inequality by $\left[\begin{smallmatrix} x(t) \\ \phi(t) \end{smallmatrix}\right]^T \neq 0$ and its transpose, we can then obtain

$$\begin{aligned} &2x^T(t)\,(PA(\alpha) + PB_2(\alpha)K_a(\alpha))\,x(t) + x^T(t)R_x x(t) \\ &\quad + 2x^T(t)\,(PG(\alpha) + PB_2(\alpha)K_b(\alpha))\,\phi(t) \\ &\quad + 2\left(\phi^T(t)\Lambda^{-1}Ex(t) - \phi^T(t)\Lambda^{-1}\phi(t)\right) \\ &\quad + u^T(t)R_u u(t) < 0. \end{aligned} \quad (38)$$

Considering $\Lambda^{-1} = \mathrm{diag}[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_s]^{-1} = \mathrm{diag}[\lambda_1^{-1} \quad \lambda_2^{-1} \quad \cdots \quad \lambda_s^{-1}]$, then

$$\phi^T(t)\Lambda^{-1}Ex(t) - \phi^T(t)\Lambda^{-1}\phi(t) = \sum_{i=1}^{s} \lambda_i^{-1}\left(\phi_i(t)\left(E_i x(t) - \phi_i(t)\right)\right). \quad (39)$$

Since $\phi_i(t) \in \mathtt{co}\{0, E_i x(t)\}$, $1 \le i \le r$, then, from Lemma 1, we have

$$\phi_i(t)\left(E_i x(t) - \phi_i(t)\right) \ge 0, \qquad \text{for } 1 \le i \le s.$$

Combining it and (39) then yields

$$\phi^T(t)\Lambda^{-1}Ex(t) - \phi^T(t)\Lambda^{-1}\phi(t) \ge 0.$$

From the preceding inequality and (38), then it follows that

$$\begin{aligned} &2x^T(t)\,(PA(\alpha) + PB_2(\alpha)K_a(\alpha))\,x(t) + x^T(t)R_x x(t) + 2x^T(t) \\ &\quad \times (PG(\alpha) + PB_2(\alpha)K_b(\alpha))\,\phi(t) + u^T(t)R_u u(t) < 0. \end{aligned} \quad (40)$$

Because $Q > 0$, $P = Q^{-1} > 0$; we then choose Lyapunov function candidate

$$V(t) = x^T(t)Px(t).$$

From (40), we have

$$\dot{V}(t) + x^T(t)R_x x(t) + u^T(t)R_u u(t) < 0. \quad (41)$$

Considering $R_x > 0$, $R_u > 0$, and (41), it follows that $\dot{V}(t) < 0$. Then, system (9) with controller (10) is asymptotically stable.

Moreover, integrating both sides of (41) yields

$$\int_0^\infty \left(x^T(t)R_x x(t) + u^T(t)R_u u(t)\right) dt \le V(0) = x^T(0)Px(0)$$

i.e.,

$$J \le V(0) = x^T(0)Px(0) = x^T(0)Q^{-1}x(0) \quad (42)$$

where J is the same as that in (13). Thus, the proof is complete. ■

Note that the condition in Theorem 3 is dependent on $\alpha_i(v(t))$. By applying Lemma 2 to Theorem 3, we can obtain the corresponding LMI-based condition as follows:

Theorem 4: If there exist matrices $Q = Q^T > 0$, L_{ai} and L_{bi} $(0 \le i \le r)$, and Λ with

$$\Lambda = \mathrm{diag}[\lambda_1 \quad \cdots \quad \lambda_s]_{s\times s}$$

satisfying

$$\Psi_{ii} < 0, \qquad 1 \le i \le r \quad (43)$$

$$\frac{1}{r-1}\Psi_{ii} + \frac{1}{2}(\Psi_{ij} + \Psi_{ji}) < 0, \qquad 1 \le i \neq j \le r \quad (44)$$

where Ψ_{ij}, $1 \le i,\ j \le r$, is the same as that in (35). Then, system (9) is asymptotically stable via controller (10) with gains (22). In this case, cost function (13) satisfies (36).

Proof: From Theorem 3 and Lemma 2, the proof is easily obtained and omitted. ■

The upper bound of cost function J in Theorem 4 depends on the initial condition $x(0)$ [see (36)]. To avoid this dependence on $x(0)$, we consider the following assumption about $x(0)$:

$$\mathcal{E}\{x(0)\} = 0 \qquad \mathcal{E}\left\{x(0)x^T(0)\right\} = I \quad (45)$$

where $\mathcal{E}$ denotes the expectation operator. Then

$$\mathcal{E}\left\{x(0)Q^{-1}x^T(0)\right\} = \mathtt{trace}\{Q^{-1}\}. \quad (46)$$

To minimize $\mathtt{trace}\{Q^{-1}\}$, an auxiliary variable Z is introduced as an upper bound on Q^{-1} as follows:

$$\begin{bmatrix} Z & I \\ I & Q \end{bmatrix} > 0. \tag{47}$$

Furthermore, we can minimize cost function J by solving the following optimization problem:

$$\mathtt{OP2}: \textbf{Minimize}: \mathtt{trace}\{Z\}$$
$$\textbf{subject to}: (43), (44), \text{ and } (47).$$

Based on the results in Theorems 2 and 4, a sufficient condition for designing controllers for guaranteeing the asymptotical stability of closed-loop system (11) with the H_∞ norm and cost function constraints can be obtained as follows:

Theorem 5: For a prescribed $\gamma > 0$, if there exist matrices $Q = Q^T > 0$, L_{ai} and L_{bi}, $0 \le i \le r$, and Λ with

$$\Lambda = \mathrm{diag}[\lambda_1 \quad \cdots \quad \lambda_s]_{s\times s}$$

satisfying (29), (30), (43), and (44) Then, system (9) via controller (10) with gains (22) is asymptotically stable with an H_∞ norm less than or equal to γ and cost function (13) satisfying (36).

Based on Theorem 5, an optimization algorithm for minimizing cost function J under an H_∞ performance bound constraint can be given as follows:

$$\mathtt{OP3}: \text{For a given } \gamma > 0, \quad \textbf{Minimize}: \mathtt{trace}\{Z\}$$
$$\textbf{subject to}: (29), (30), (43), (44), \text{ and } (47).$$

IV. Example

In this section, two examples are given to show the effectiveness of the new techniques. Example 1 is presented to illustrate that the new technique can design fuzzy controllers with fewer control rules and less or slightly more conservatism. However, in contrast to the existing methods, the new proposed method can still achieve good control effects using local nonlinear feedbacks. In Example 2, the multiple objective control synthesis technique based on optimization algorithm OP3 is applied to a plate-and-beam system. In Example 2, it is shown that the proposed method can effectively lighten computational burden than the conventional methods, where a heavy computational burden leads to hardly designing controllers.

Example 1: Consider the nonlinear system shown at the bottom of the page, where x_1 and x_2 are the states of the nonlinear system with $x_1 \in \{-(\pi/2), (\pi/2)\}$, u is the control input, and w is the disturbance with

$$w(t) = \begin{cases} 1, & 1 \le t \le 3 \\ 0, & \text{others.} \end{cases}$$

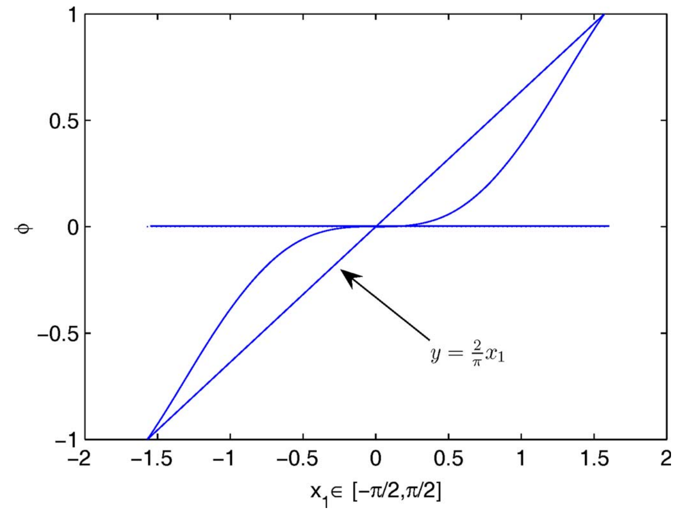

Fig. 1. Bound of $\phi(t)$.

Let

$$\bar{f}_a(x(t)) = \begin{bmatrix} -2.6x_1 - 2x_2 - 2.4x_1\sin^2(x_2) - x_2\sin^2(x_2) \\ x_1\cos^2(x_2) - 4x_2 + 8x_2\sin^2(x_2) \end{bmatrix}$$
$$g(x(t)) = \begin{bmatrix} 0 \\ 3\sin^2(x_2) - 1 \end{bmatrix}$$
$$\phi(t) = (1 - \cos(x_1))\sin(x_1)$$
$$h(x(t)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$f_b(x(t)) = \begin{bmatrix} 0 \\ 6\sin^2(x_2) - 2 \end{bmatrix}.$$

The nonlinear system can be rewritten in the form of (6), with $\phi(t)$ satisfying $-\phi(t)((2/\pi)x_1(t) - \phi(t)) \le 0$ (see Fig. 1). Furthermore, the nonlinear system can be represented by the following two-rule T-S fuzzy model with local nonlinear models:

Model 1:

$$\dot{x} = A(\alpha)x(t) + B_1(\alpha)w(t) + B_2(\alpha)u(t) + G(\alpha)\phi(t)$$

where

$$A(\alpha) = \sum_{i=1}^{2}\alpha_i A_i \quad B_1(\alpha) = \sum_{i=1}^{2}\alpha_i B_{1i}$$
$$B_2(\alpha) = \sum_{i=1}^{2}\alpha_i B_{2i} \quad G(\alpha) = \sum_{i=1}^{2}\alpha_i G_i$$
$$A_1 = \begin{bmatrix} -5 & -3 \\ 0 & 4 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2.6 & -2 \\ 1 & -4 \end{bmatrix} \quad B_{11} = B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$B_{21} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad G_1 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sin^2(x_2) - 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} -2.6x_1 - 2x_2 - 2.4x_1\sin^2(x_2) - x_2\sin^2(x_2) \\ x_1\cos^2(x_2) - 4x_2 + 8x_2\sin^2(x_2) + \left(6\sin^2(x_2) - 2\right)(1 - \cos(x_1))\sin(x_1) \end{bmatrix}$$

and $\alpha_1(t) = \sin^2(x_2(t))$ and $\alpha_2(t) = \cos^2(x_2(t))$ are the membership functions.

On the other hand, by using the method in [2], a four-rule fuzzy model can be constructed as follows:

Model 2:

$$\dot{x} = \bar{A}(\eta)x(t) + \bar{B}_1(\eta)w(t) + \bar{B}_2(\eta)u(t)$$

where

$$\bar{A}(\eta) = \sum_{i=1}^{4}\eta_i A_i \qquad \bar{B}_1(\alpha) = \sum_{i=1}^{4}\eta_i B_{1i}$$

$$\bar{B}_2(\eta) = \sum_{i=1}^{4}\eta_i B_{2i} \qquad \bar{A}_1 = \begin{bmatrix} -5 & -3 \\ 4\times\frac{2}{\pi} & 4 \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} -5 & -3 \\ 0 & 4 \end{bmatrix} \qquad \bar{A}_3 = \begin{bmatrix} -2.6 & -2 \\ 1-2\times\frac{2}{\pi} & -4 \end{bmatrix}$$

$$\bar{A}_4 = \begin{bmatrix} -2.6 & -2 \\ 1 & -4 \end{bmatrix} \qquad \bar{B}_{11} = \bar{B}_{12} = \bar{B}_{13} = \bar{B}_{14} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{B}_{21} = \bar{B}_{22} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \bar{B}_{23} = \bar{B}_{24} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$\beta_1(t) = \begin{cases} \frac{(1-\cos(x_1(t)))\sin(x_1(t))/x_1(t)}{\frac{2}{\pi}}, & x_1(t)\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right] \\ 0, & x_1(t)=0 \end{cases}$$

and $\beta_2(t) = 1-\beta_1(t)$, $\eta_1(t) = \alpha_1\beta_1(t)$, $\eta_2(t) = \alpha_1\beta_2(t)$, $\eta_3(t) = \alpha_2\beta_1(t)$, and $\eta_4(t) = \alpha_2\beta_2(t)$ are the membership functions. Assume that the controlled output of the nonlinear system is

$$z(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

Note that Corollary 1 and Lemmas 3 and 4 apply the technique of [7] for obtaining controller design conditions. The comparisons among these methods are necessary. By these methods, the following results can be obtained.

Applying Corollary 1 to Model 1

$$K_{a1} = [-0.8496 \quad -6.1880] \quad K_{a2} = [0.7843 \quad 0.4261]$$
$$K_{b1} = -1.3346 \qquad K_{b2} = -1.3340$$
$$\gamma_{\text{opt}} = 1.7637 \qquad cputime = 0.2031 \text{ s.}$$

Applying Lemma 3 (property 1) to Model 2

$$K_1 = [-0.3232 \quad -4.3670] \quad K_2 = [-0.5221 \quad -4.4362]$$
$$K_3 = [0.1375 \quad -1.2759] \quad K_4 = [-0.1343 \quad -1.3478]$$
$$\gamma_{\text{opt}} = 2.3667 \qquad cputime = 1.7188 \text{ s.}$$

TABLE II
H_∞ PERFORMANCE AND *cputime*

	Corollary 1	Lemma 3 (i)	Lemma 4
H_∞ performance bound γ	1.7637	2.3667	1.7637
cputime (s)	0.2031	1.7188	10.9688
Numerical complexity $\mathcal{C}=\mathcal{D}^3\mathcal{L}$	$116^3\cdot 37$	$312^3\cdot 86$	$624^3\cdot 134$

Applying Lemma 4 to Model 2

$$K_{11} = [0.1378 \quad -3.5790] \quad K_{12} = [-0.9109 \quad -3.5543]$$
$$K_{21} = [0.4360 \quad -1.8277] \quad K_{22} = [-0.4885 \quad -1.7805]$$
$$\gamma_{\text{opt}} = 1.7637 \qquad cputime = 10.9688 \text{ s.}$$

Both the obtained H_∞ performance bounds and the used *cputime*'s are also given in Table II, which shows that the new approach can give less conservative results than Lemma 3 (property 1), and the same as conservative results by Lemma 4. In particular, it can also be seen that the new method lightens the computational burden from the *cputime* and the corresponding numerical complexity index $\mathcal{C}=\mathcal{D}^3\mathcal{L}$ (referring to Remark 4). It should be pointed that the new technique might give more conservative results than the existing ones in some cases; however, the advantage of the new technique consists of designing a fuzzy controller with fewer rules and less computational burden, which is significant for implementation in engineering. This fact will further be illustrated by designing a fuzzy controller for a ball-and-plate system in the next example. Assuming the initial condition $x(0)=0$, the response curve of $z_1(t)$ and the ratio of $\int_0^t z^T(s)z(s)ds / \int_0^t w^T(s)w(s)ds$ are shown in Figs. 2 and 3. From the simulations, it can be seen that the new method can achieve better system responses using local nonlinear feedbacks.

Example 2: In this example, the ball-and-plate system [31] is considered. The state-space description of the ball-and-plate system is given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \begin{bmatrix} x_2 \\ B\left(x_1x_4^2 + x_4x_5x_8 - g\sin x_3\right) \\ x_4 \\ 0 \\ x_6 \\ B\left(x_5x_8^2 + x_1x_4x_8 - g\sin x_7\right) \\ x_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$z = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad u_1 \quad u_2]^T \tag{48}$$

where $B = 0.7143$, and $g = 9.81$ m/s^2.

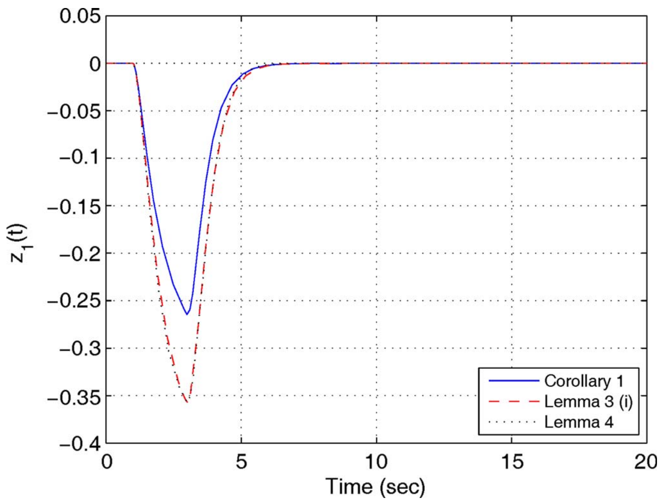

Fig. 2. Response curve of state $z_1(t)$ under the initial condition $x^T(0) = [0 \quad 0]$.

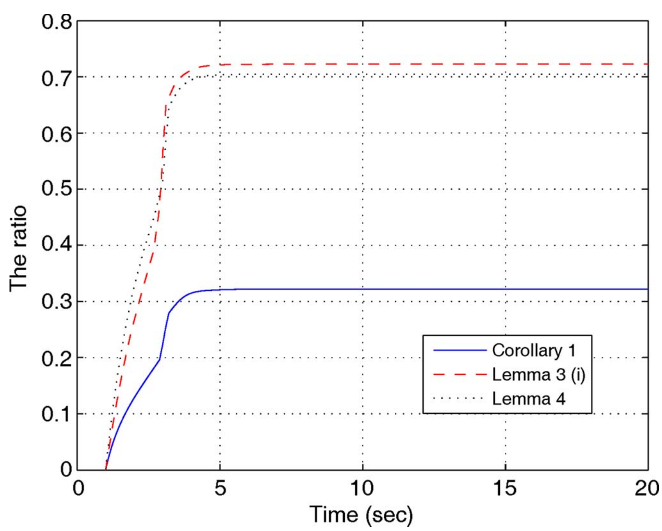

Fig. 3. Ratio of $\int_{s=0}^{t} z^T(s)z(s)ds / \int_{s=0}^{t} w^T(s)w(s)ds$.

There are six nonlinear terms $x_1x_4^2$, $x_4x_5x_8$, $\sin x_3$, $x_5x_8^2$, $x_1x_4x_8$, and $\sin x_7$ in (48). Assume that x_3, $x_7 \in [-(\pi/2), (\pi/2)]$, $x_1x_4 \in [-d_1, d_1]$, $x_5x_8 \in [-d_2, d_2]$, $x_4x_5 \in [-d_3, d_3]$, $x_1x_8 \in [-d_4, d_4]$, and $d_i = 4$ $(1 \le i \le 4)$. Letting $\bar{\phi}_1(t) = \sin x_3$, $\bar{\phi}_2(t) = \sin x_7$, $\bar{\phi}_3(t) = x_4x_5x_8$, and $\bar{\phi}_4(t) = x_1x_4x_8$, then we have

$$\bar{\phi}_1(t) \in co\left\{\frac{2}{\pi}x_3, x_3\right\} \quad \bar{\phi}_2(t) \in co\left\{\frac{2}{\pi}x_7, x_7\right\}$$
$$\bar{\phi}_3(t) \in co\{-d_3x_8, d_3x_8\} \quad \bar{\phi}_4(t) \in co\{-d_4x_4, d_4x_4\}$$
$$-d_1\|x_4\| \le x_1x_4^2 \le d_1\|x_4\| \quad -d_2\|x_8\| \le x_5x_8^2 \le d_2\|x_8\|.$$

Furthermore, let $\phi_1(t) = \sin x_3 - (2/\pi)x_3$, $\phi_2(t) = \sin x_7 - (2/\pi)x_7$, $\phi_3(t) = x_4x_5x_8 + d_3x_8$, and $\phi_4(t) = x_1x_4x_8 + d_4x_4$. Then, it follows that $\phi_1(t) \in co\{0, ((\pi-2)/\pi)x_3\}$, $\phi_2(t) \in co\{0, ((\pi-2)/\pi)x_7\}$, $\phi_3(t) \in co\{0, 2d_3x_8\}$, and $\phi_4(t) \in co\{0, 2d_4x_4\}$.

Therefore, the nonlinear system can be rewritten as the following form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = \begin{bmatrix} x_2 \\ B\left(x_1x_4^2 - d_3x_8 - \frac{2}{\pi}gx_3\right) \\ x_4 \\ 0 \\ x_6 \\ B\left(x_5x_8^2 - d_4x_4 - \frac{2}{\pi}gx_7\right) \\ x_8 \\ 0 \end{bmatrix} + B_1\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + B_2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + G\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}$$

$$z = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad u_1 \quad u_2]^T$$

where

$$B_1 = B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -Bg & 0 & B & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -Bg & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we construct a T-S fuzzy model to represent the ball-and-plate system from the preceding dynamic equation, and a four-rule fuzzy model is obtained.

Plant Rule 1:

IF x_1x_4 is Γ_{11} and x_5x_8 is Γ_{12}

THEN $\dot{x}(t) = A_1x(t) + B_1w(t) + B_2u(t) + G\phi(t)$.

Plant Rule 2:

IF x_1x_4 is Γ_{21} and x_5x_8 is Γ_{12}

THEN $\dot{x}(t) = A_2x(t) + B_1w(t) + B_2u(t) + G\phi(t)$.

Plant Rule 3:

IF x_1x_4 is Γ_{11} and x_5x_8 is Γ_{22}

THEN $\dot{x}(t) = A_2x(t) + B_1w(t) + B_2u(t) + G\phi(t)$.

Plant Rule 4:

IF x_1x_4 is Γ_{21} and x_5x_8 is Γ_{22}

THEN $\dot{x}(t) = A_2x(t) + B_1w(t) + B_2u(t) + G\phi(t)$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\pi}Bg & Bd_1 & 0 & 0 & 0 & -Bd_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Bd_4 & 0 & 0 & -\frac{2}{\pi}Bg & Bd_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\pi}Bg & -Bd_1 & 0 & 0 & 0 & -Bd_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Bd_4 & 0 & 0 & -\frac{2}{\pi}Bg & Bd_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\pi}Bg & Bd_1 & 0 & 0 & 0 & -Bd_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Bd_4 & 0 & 0 & -\frac{2}{\pi}Bg & -Bd_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\pi}Bg & -Bd_1 & 0 & 0 & 0 & -Bd_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Bd_4 & 0 & 0 & -\frac{2}{\pi}Bg & -Bd_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma_{11}(x_1x_4) = \frac{x_1x_4 + d_1}{2d_1} \quad \Gamma_{21}(x_1x_4) = \frac{-x_1x_4 + d_1}{2d_1}$$

$$\Gamma_{12}(x_5x_8) = \frac{x_5x_8 + d_2}{2d_2} \quad \Gamma_{22}(x_5x_8) = \frac{-x_5x_8 + d_2}{2d_2}.$$

If the modeling method in [2] is used for the ball-and-plate system, then a T-S fuzzy model with $2^6 = 64$ rules will be obtained. Because the computational burden of the methods in [7] and [11] are heavy for such a fuzzy model, the concrete computations do not be executed, but their numerical complexities, which can reflect the computation burden, are given in Table III. By applying Theorem 2 to the example, the obtained optimal H_∞ performance index is $\gamma = 1$, but the controller gains are too big to be used. In order to obtain the gains with proper magnitudes, we consider minimizing quadratic cost function J in (13) with the weighting matrices $R_x = \mathrm{diag}[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$, $R_u = \mathrm{diag}[0.1 \quad 0.1]$ under a given H_∞ performance bound constraint $\gamma = 2$. This is a multiobjective control design problem, which can be solved using Theorem 5. The results obtained by Theorem 5 are shown at the top of the next page. Assume initial state $x(0) = [1 \quad 0 \quad 0.2 \quad 0 \quad 1 \quad 0 \quad -0.3 \quad 0]^T$ and disturbance

$$w_1(t) = \begin{cases} 5, & 7 \le t \le 8 \\ 0, & \text{others} \end{cases} \quad w_2(t) = \begin{cases} 10, & 5 \le t \le 6 \\ 0, & \text{others} \end{cases}.$$

A simulation is done by controller (10) with local feedback gain (49), and the trajectories of the states are shown in Figs. 4 and 5. It can be seen from Figs. 4 and 5 that the ball-and-plate system by controller (10) with local feedback gain (49) is asymptotically stable.

TABLE III
NUMBER OF DECISION VARIABLES $\mathcal{D}$ AND LINES $\mathcal{L}$

	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{D}^3\mathcal{L}$
Theorem 2	137	392	$1Re$ [2]
Corollary 1	4793	344	$3.7578 \times 10^4 Re$
Lemma 3 (i)	820901	60800	$3.3368 \times 10^{13} Re$
Lemma 4	4956581	122248	$1.4769 \times 10^{16} Re$

[2] $Re = 1.0080 \times 10^9$

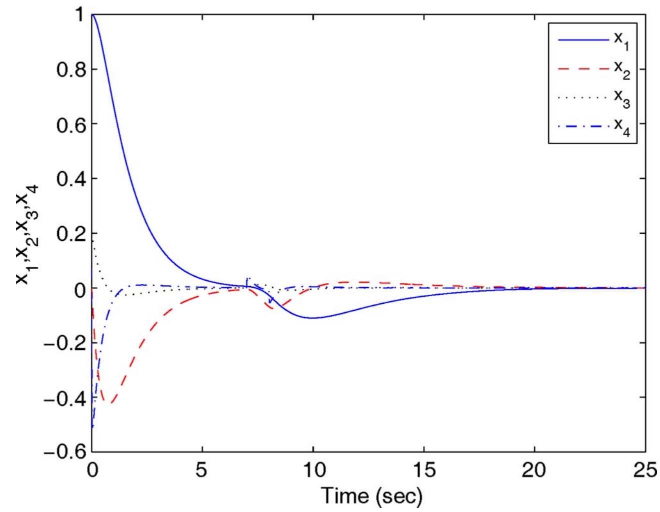

Fig. 4. Trajectory of $x_i(t)$, $1 \le i \le 4$.

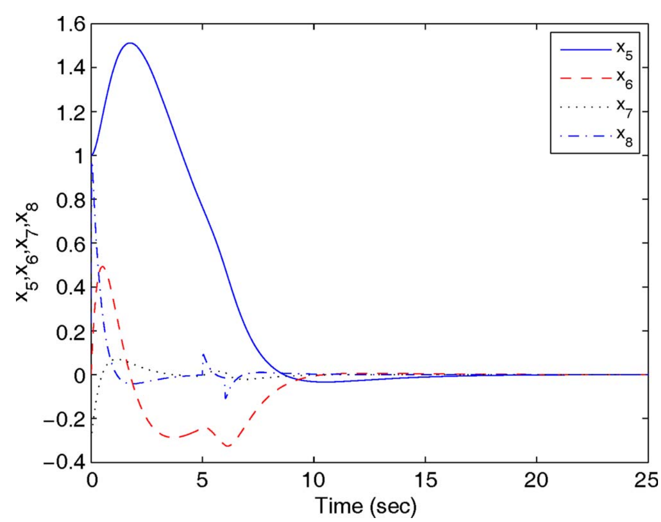

Fig. 5. Trajectory of $x_i(t)$, $5 \le i \le 8$.

$$\begin{aligned}
K_{a1} &= \begin{bmatrix} 8.2853 & 36.0951 & -309.4584 & -96.9387 & 0.0007 & 0.0028 & -0.0208 & -0.0003 \\ 0.0007 & 0.0028 & -0.0207 & -0.0004 & 8.2856 & 36.0964 & -309.4691 & -96.9396 \end{bmatrix} \\
K_{a2} &= \begin{bmatrix} 8.3330 & 36.2556 & -310.6423 & -97.4196 & 0.0007 & 0.0029 & -0.0209 & -0.0004 \\ 0.0007 & 0.0028 & -0.0207 & -0.0004 & 8.3372 & 36.2725 & -310.7666 & -97.5061 \end{bmatrix} \\
K_{a3} &= \begin{bmatrix} 8.3369 & 36.2711 & -310.7557 & -97.5054 & 0.0007 & 0.0028 & -0.0208 & -0.0005 \\ 0.0007 & 0.0029 & -0.0209 & -0.0003 & 8.3333 & 36.2570 & -310.6532 & -97.4204 \end{bmatrix} \\
K_{a4} &= \begin{bmatrix} 8.3272 & 36.2793 & -310.8039 & -97.6595 & 0.0007 & 0.0029 & -0.0209 & -0.0004 \\ 0.0007 & 0.0028 & -0.0208 & -0.0004 & 8.3275 & 36.2807 & -310.8145 & -97.6601 \end{bmatrix} \\
K_{b1} &= \begin{bmatrix} 0.0004 & 0.0001 & -0.0000 & -0.0022 \\ 0.0000 & 0.0003 & -0.0022 & 0.0000 \end{bmatrix} \\
K_{b2} &= \begin{bmatrix} 0.0003 & -0.0001 & -0.0000 & -0.0022 \\ -0.0000 & 0.0002 & -0.0022 & 0.0000 \end{bmatrix} \\
K_{b3} &= \begin{bmatrix} 0.0004 & -0.0000 & 0.0000 & -0.0022 \\ -0.0000 & 0.0004 & -0.0022 & -0.0000 \end{bmatrix} \\
K_{b4} &= \begin{bmatrix} 0.0004 & 0.0000 & -0.0000 & -0.0022 \\ 0.0000 & 0.0005 & -0.0022 & 0.0000 \end{bmatrix}
\end{aligned}$$

$$\mathtt{trace}\{Z\} = 2.8270 \times 10^4 \quad cputime = 486.7500 \text{ s} \tag{49}$$

V. Conclusion

In this paper, the problem of designing fuzzy controllers for a class of nonlinear dynamic systems has been studied. The considered nonlinear systems are described by T-S fuzzy models with local nonlinear models, and the fuzzy models have fewer fuzzy rules than conventional T-S fuzzy models with linear local models. A new fuzzy control scheme with local nonlinear feedbacks is proposed, and the corresponding control synthesis conditions are given in terms of solutions to a set of LMIs. In contrast to existing methods for fuzzy control synthesis, the new proposed control design method is based on fewer fuzzy rules and less computational burden. Moreover, the local nonlinear feedback laws in the new fuzzy controllers are also helpful in achieving good control effects. Two numerical examples have been given to illustrate the effectiveness of the proposed method.

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