Ordinal MDS-based Localization for Wireless Sensor Networks

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***Abstract*— There are various applications in wireless sensor networks which require knowing the relative or actual position of the sensor nodes. Recently, there have been different localization algorithms proposed in the literature. The algorithms based on classical Multidimensional Scaling (MDS) [1][2] only require 3 or 4 anchor nodes and can provide higher accuracy than some other schemes. In this paper, we propose and analyze another type of MDS (called *ordinal MDS*) for localization in wireless sensor networks. Ordinal MDS differs from classical MDS by that it only requires a monotonicity constraint between the shortest path distance and the Euclidean distance for each pair of nodes. We conduct simulation studies under square and C-shaped topologies with different connectivity levels and number of anchors. Results show that ordinal MDS provides a lower position estimation error than classical MDS.**

I. Introduction

The miniaturization of small devices capable of sensing and communicating with each other has made the possibility of deploying large-scale wireless sensor networks a reality. Sensor networks can be deployed in different scenarios, ranging from military applications to wildlife and environment monitoring. For applications such as event discovery and target tracking, the geographic location of the sensor nodes need to be known. Consider the example where a sensor network is used to detect a fire event in a forest. Once a sensor node has detected that the temperature is higher than a certain threshold, it sends a message to the central authority by relaying through other nodes in a multi-hop manner. The message needs to indicate the location of the node which detected the event. Thus, localization of sensor nodes is important in some applications.

Recently, various localization schemes have been proposed in the literature. These algorithms can be divided into two groups: centralized [1][4] and distributed [6]-[8]. It is generally true that distributed algorithms are more robust and energy efficient than centralized algorithms. In each group, some algorithms assume simple connectivity information between neighboring nodes [1][2] while some others need to gather the ranging information (e.g., estimated distance between two neighboring nodes) [10][11][12] and angle information [13][14]. In order to determine the actual or absolute position of each sensor node, a small fraction of special nodes (called *anchor nodes*) with known positions is necessary.

Localization algorithms based on *classical Multidimensional Scaling* (MDS) [1][2][10] have proven to be robust with respect to both hop-based and range-based implementations. Only 3 or 4 anchor nodes are necessary to determine the absolute locations, in two or three dimensions, respectively. These MDS algorithms achieve a higher accuracy than some other schemes. By using the similar terminology in [1], we use the term MDS-MAP(C) for the classical MDS localization algorithm. The original MDS-MAP(C) is a centralized algorithm. In [2], a distributed MDS algorithm called MDS-MAP(P, C) was proposed where P denotes the use of patching of local maps and C denotes the use of classical MDS. As mentioned in [1][2], further work is required to study the application of other MDS techniques (e.g., probabilistic MDS, ordinal MDS) on localization in sensor networks.

In this paper, we propose the implementation of *ordinal MDS* for localization in sensor networks and compare the performance with classical MDS. We call our proposed scheme MDS-MAP(P, O) where P again denotes the use of patching of local maps and O denotes the use of ordinal MDS. MDS-MAP(P, O) is also a distributed algorithm. The main difference between classical MDS and ordinal MDS is that the former assumes there is a linear equation which relates the shortest path distance and the Euclidean distance between each pair of nodes, the latter simply assumes a monotonicity constraint. That is, for ordinal MDS, given two pairs of nodes (i, j) and (k, l), if the shortest path distance of (i, j) is greater than that of (k, l), then the Euclidean distance of (i, j) is also greater than that of (k, l), and vice versa.

The contributions of this paper are as follows [15]:

1) We present the implementation details of ordinal MDS algorithm for localization in wireless sensor networks.
2) We conduct simulations to study the performance between classical and ordinal MDS by varying the connectivity levels and number of anchors. Under square and C-shaped topologies, results show that MDS-MAP(P, O) has a lower position estimation error than MDS-MAP(P, C).

Our proposed MDS-MAP(P, O) algorithm is essential for future sensor applications which require a high accuracy of nodes' position by using a small number of anchor nodes.

The rest of this paper is organized as follows. The related work is summarized in Section II. The MDS-MAP(P, O) algorithm is described in Section III. Performance comparisons between classical MDS and ordinal MDS algorithms are given in Section IV. Conclusions are given in Section V.

II. Related Work

In this section, we first summarize several recent papers on localization on sensor networks. Survey paper in this area can also be found in [3]. We then review the MDS-MAP(C) and MDS-MAP(P, C) algorithms [2].

In the APIT scheme [8], each node first identifies if it is within a particular triangle formed by a set of anchors within radio range. The position is estimated to be the center of the intersection of all triangles in which the node has identified to be within. In the convex optimization scheme [4], anchors have to be placed near the corners and edges of the network for optimal sufficient performance.

Anchor information propagation methods [6][7][11] require each anchor to broadcast its position to the network. Nodes use this information as well as the distance or hop counts from the anchors to laterate or bound their positions. In a slightly different approach, iterative localization [12] can be used. Nodes with sufficient neighboring anchors can compute their positions. As more nodes obtain their positions, these nodes can also be acted as anchors.

Several direction or angle-based schemes have been proposed. In [13], the original APS scheme [11] is modified to propagate bearings to anchors. Nodes that have at least three bearings to anchors can triangulate their positions. In [14], both range and angle information is used to determine the node's position. The advantage is that only one anchor is needed to obtain an estimate for a node.

In [16], a single mobile anchor is used to localize the system. The mobile anchor node traverses within the network and allows all nodes to compute the location estimate based on at least three neighboring nodes' locations. In [17], multiple mobile anchors are used. A monte carlo localization algorithm for mobile sensor networks was proposed in [18]. In [19], mobile robots and robust extended Kalman filter-based state estimator are used for localization.

The advantage of MDS localization algorithms is the relative low percentage of estimation error while using a small number of anchor nodes. The MDS-MAP(C) scheme is a centralized algorithm. The major steps are as follows [1]: Given the network hop-count or distance information, Dijkstra's algorithm is used to determine the shortest path between each pair of nodes. The results are stored in a distance matrix. The classical MDS algorithm is then applied on the distance matrix to create the global relative map. By using the anchor nodes' positions, the global relative map is transformed into the global absolute map.

The classical (or metric) MDS algorithm assumes that there exists a linear transformation which relates the shortest path distance and the Euclidean distance between each pair of nodes. For each pair of nodes (i, j), if the shortest path distance is denoted by p_{ij} and the Euclidean distance is denoted by d_{ij}, then $d_{ij} = mp_{ij} + c$ for some constants m and c. Classical MDS uses *singular value decomposition* to determine the relative coordinates of the sensor nodes. Simulation results show that in a topology where the nodes are uniformly placed, MDS-MAP(C) has a lower location estimation error when compared with [4] and [6].

The MDS-MAP(P, C) is a distributed localization algorithm [2]. Each node first creates a local map within its two-hop neighbors by using the classical MDS algorithm. Each local map is then refined by using the least-squares minimization. The local maps are then patched or merged to create a global relative map. Finally, by using the anchor nodes' positions, the global relative map is transformed into the global absolute map. Simulation results from [2] show that MDS-MAP(P, C) has a better performance than MDS-MAP(C). In addition, as stated in [2], performance may further be improved if other MDS algorithms (e.g., weighted MDS, probabilistic MDS, ordinal MDS) are used. In the next section, we study the performance of using the ordinal (or non-metric) MDS algorithm for localization.

III. MDS-MAP(P, O) Localization Algorithm

In this section, we describe our proposed MDS-MAP(P, O) localization algorithm. MDS-MAP(P, O) is distributed and can be considered as an extension of MDS-MAP(P, C). The modification is the use of the ordinal MDS (instead of classical MDS) during the estimation phase. The major steps of the MDS-MAP(P, O) algorithm are as follows:

1) Each node first gathers either the distance (for range-based) or hop count (for hop-based) information within its two-hop neighborhood.
2) In each node, the Dijkstra's algorithm is invoked to determine the shortest path between each pair of nodes within the two-hop neighborhood. We use the notation p_{ij} to denote the shortest path distance between nodes i and j.
3) The *ordinal MDS algorithm* is applied to create the relative local map for each node.
4) Each local map is refined by using the least-squares minimization between the calculated Euclidean distance and the measured distance (or hop) between each pair of neighboring nodes.
5) The local maps are then patched (or merged) into a global map by using a predetermined initial starting node's local map and sequentially adding each neighbor that has the largest number of common nodes to the starting node. This map then grows until all nodes have been included.
6) The global absolute map is created by using the anchors' positions and the global relative map.

Assume that the average number of sensor nodes in each two-hop neighborhood is $\overline{M}$, the average number of neighbors is $\overline{K}$, the total number of sensor nodes is N, and total number of anchors is A. In the above MDS-MAP(P, O) algorithm, steps (2) and (4) have a complexity of $O(\overline{M}^3)$. Step (3) has a complexity of $O(\overline{M}^4)$. Steps (5) and (6) have a complexity of $O(\overline{K}^3 N)$ and $O(A^3 + N)$, respectively.

We now describe the *ordinal MDS algorithm* (step (3) above) in detail. The major steps of the *ordinal MDS algorithm* are as follows [20]:

1) Assign arbitrary initial location estimation (x_i^0, y_i^0) for $i \in M$, where M includes all the nodes within the two-hop neighborhood. Specify $\epsilon > 0$ and set $n = 0$.
2) For each $i, j \in M$, compute the Euclidean distance by

$$d_{ij}^n = \sqrt{\left(x_i^n - x_j^n\right)^2 + \left(y_i^n - y_j^n\right)^2} \quad (1)$$

3) By using the matrices $[p_{ij}]$ and $[d_{ij}^n]$, apply monotone regression by using the *pool-adjacent violators (PAV)* algorithm [20] to determine $[\hat{d}_{ij}^n]$. For example, once the p_{ij}'s are ordered from the smallest to the largest, if ($p_{ij} < p_{kl}$) and ($d_{ij}^n > d_{kl}^n$), then

$$\hat{d}_{ij}^n = \hat{d}_{kl}^n = \left(d_{ij}^n + d_{kl}^n\right)/2.$$

Otherwise, $\hat{d}_{ij}^n = d_{ij}^n$ and $\hat{d}_{kl}^n = d_{kl}^n$.
4) Increment n by 1. For $i \in M$, compute the new relative coordinate (x_i^n, y_i^n) for node i by

$$x_i^n = x_i^{n-1} + \frac{\alpha}{|M|-1} \sum_{j \in M, j \neq i} \left(1 - \frac{\hat{d}_{ij}^{n-1}}{d_{ij}^{n-1}}\right)\left(x_j^{n-1} - x_i^{n-1}\right)$$

$$y_i^n = y_i^{n-1} + \frac{\alpha}{|M|-1} \sum_{j \in M, j \neq i} \left(1 - \frac{\hat{d}_{ij}^{n-1}}{d_{ij}^{n-1}}\right)\left(y_j^{n-1} - y_i^{n-1}\right)$$

where $|M|$ denotes the number of sensor nodes within the two-hop neighborhood.
5) For each $i, j \in M$, update the Euclidean distance d_{ij}^n by using equation (1).
6) Use Kruskal's Stress1 test to determine the goodness fit [21][22]:

$$Stress1 = \sqrt{\frac{\sum_{i<j}\left(d_{ij}^n - \hat{d}_{ij}^{n-1}\right)^2}{\sum_{i<j}\left(d_{ij}^n\right)^2}} \quad (2)$$

7) If $Stress1 < \epsilon$, stop. Otherwise, go to Step (3).

In the above algorithm, the first two steps calculate the Euclidean distance from an arbitrary initial configuration. Step (3) determines the *disparities* $\hat{d}_{ij}^n$ by constructing a monotone regression relationship [23] between p_{ij}'s and d_{ij}^n's. Step (4) updates the relative positions. The parameter α is the step width. We use $\alpha = 0.2$ as suggested by Kruskal [24]. Step (5) updates the Euclidean distance. The *Stress1* measure in step (6) determines whether or not the updated values d_{ij}^n fit the given dissimilarities $\hat{d}_{ij}^{n-1}$. Note that other goodness fit tests (e.g., Kruskal's *Stress2*, normalized raw stress, S-Stress) can also be used; however we choose the *Stress1* measure since it is the most common measure used for ordinal MDS. Step (7) determines if the derived configuration's goodness fits are close enough such that the procedure can be terminated.

The MDS-MAP(P, O) algorithm assumes that there is a monotonic relationship between the shortest path distances and the actual Euclidean distances. This assumption may not be valid if the network being considered is sparse and large. However, most of the applications in wireless sensor networks require the networks to be dense (i.e., with a high connectivity or average node degree) in order to provide redundancy and robustness in case of a node's failure. In addition, in our distributed approach, only the nodes within the 2-hop neighborhood are being considered. In this case, the assumption of the monotonic relationship between the shortest path distances and the actual Euclidean distances is valid.

By the iterative nature of the ordinal MDS algorithm in minimizing stress in equation (2), the final solution may not guarantee to be the global minimum [25]. In fact, the ordinal MDS algorithm can have several local minima. However, the use of the anchors in our application of the ordinal MDS algorithm increases the likelihood of reaching the global minimum. This is due to the imposed transformation required to obtain the absolute coordinates for all of the nodes. Another way to further increase the chance of reaching the global minimum is by using the multiple starting configurations approach and retaining the configuration which results in the lowest stress value. However, this approach is inefficient due to the additional computation effort required.

IV. Performance Evaluation and Comparison

The algorithm was simulated in Matlab 7.0 on a 3.06 GHz Pentium IV processor. To implement MDS-MAP(P, O) algorithm, we modified the source codes for MDS-MAP(P, C) [2]. Two different topologies are considered as the sensor network's coverage area. The first one is a uniformly distributed square region. The second one is an irregular C-shaped topology. In both topologies, we vary the *average connectivity levels* (i.e., average number of neighboring sensors) and the number of anchors in the area. The average connectivity level is varied between 9 and 21 by modifying the radio range *R*, within the fixed coverage area. The number of anchors is between four and ten. In each set of simulation run, 50 trials were performed and 95% confidence intervals were plotted. In the ordinal MDS algorithm, we set ϵ to be 10^{-4}. We conducted simulations for both hop-based and range-based scenarios. Due to space limitation, we only present the results for the hop-based scenario in this paper. Results for the range-based scenarios can be found in [15]. In the hop-based scenarios, hop count is used as the distance metric between a pair of nodes. For each node to have a unique position in MDS-MAP(P, O), the hop count values are blurred with noise so that nodes with identical hop count values to neighbors are not co-located.

A. Random Uniform Network Topology

For evaluation of the random uniform deployment, a 10*r* by 10*r* square topology was used, where *r* represents the reference unit length. Anchor nodes are placed randomly within the coverage area, and have the same communication range (i.e., radio range denoted by *R*) as other nodes.

Figure 1 shows the position estimation errors as a function of the average connectivity level by hop-based MDS-MAP(P, C) and MDS-MAP(P, O), respectively, with different numbers of anchors deployed. Results show that MDS-MAP(P, O) outperforms MDS-MAP(P, C) by a 5% lower position estimation

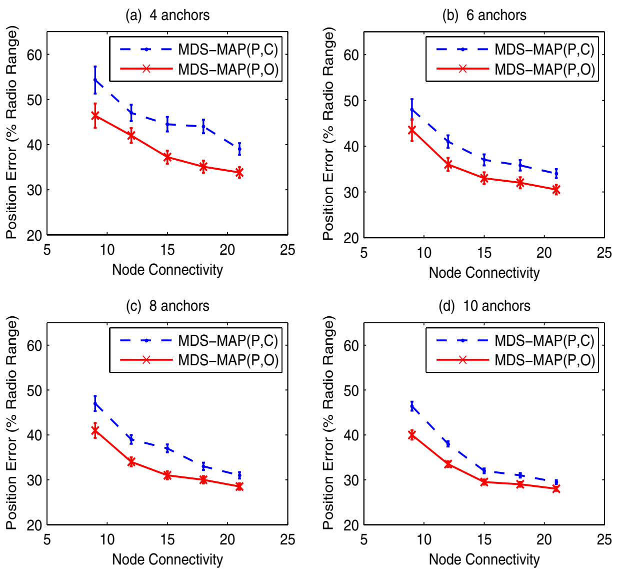

Fig. 1. Hop-based performance between MDS-MAP(P, C) and MDS-MAP(P, O) in a $10r \times 10r$ square topology with 200 nodes.

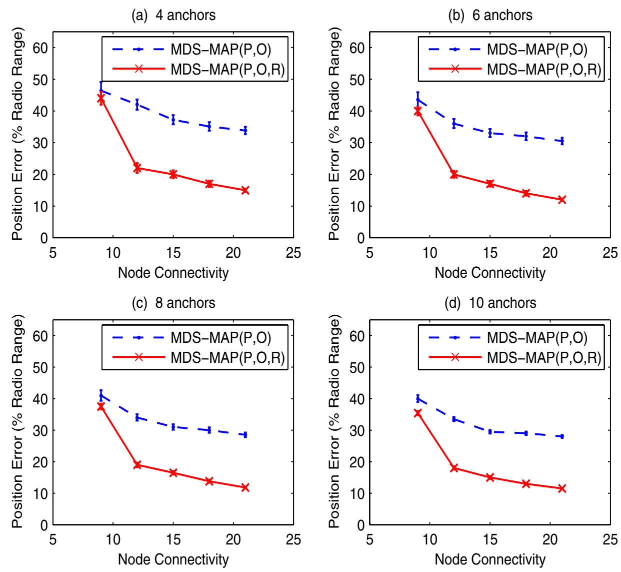

Fig. 2. Hop-based performance between MDS-MAP(P, O) and MDS-MAP(P, O, R) in a $10r \times 10r$ square network topology with 200 nodes.

error. The performance improvement confirms the conjecture that in sensors' localization problem, the use of the monotonic constraints in ordinal MDS is more appropriate than the use of linear constraints in classical MDS.

As the average connectivity level increases, the confidence intervals reduce in size. This shows that dense networks can provide more consistent average error values. This is due to the fact that dense networks have smaller two-hop regions, which in turn lead to more accurate shortest path distances. These distances therefore improve the classical MDS results as well as the ordinal MDS results, since more accurate distances translate into more accurate proximities in the ordinal case.

The accuracy of the MDS-MAP(P, O) localization algorithm can further be improved by using an *optional global relative map refinement* [2]. This optional step is invoked after the patching of the local maps. The least-squares minimization is used for the measured and calculated distances between neighboring nodes. This optional refinement step has a complexity of $O(N^3)$ where N is the total number of sensor nodes. We use the notation MDS-MAP(P, O, R) to denote the original MDS-MAP(P, O) algorithm with global relative map refinement.

Figure 2 shows the performance comparisons between MDS-MAP(P, O) and MDS-MAP(P, O, R) in hop-based scenarios. The number of anchors deployed is varied from 4 to 10. In the hop-based case, there is significant reduction on the position estimation error when the average node connectivity level is above 9. The difference between the results is greater than 30% for high average connectivity levels. Note that the global relative map refinement comes at a cost. A sensor node must process the global map and then propagate the results to all the sensors in the network (e.g., via flooding). This may cause a higher signaling overhead.

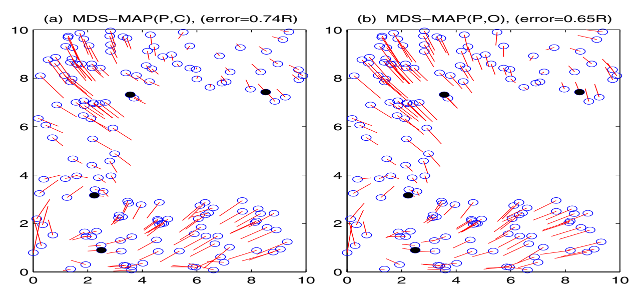

Fig. 3. Nodes' location estimated by hop-based (a) MDS-MAP(P, C) and (b) MDS-MAP(P, O) in a $10r \times 10r$ irregular (C-shaped) network region employing uniform random placement of 160 nodes with connectivity level of 12 and four anchors. Anchors are denoted by shaded circles. Estimation error is represented by lines.

B. Random Irregular Network Topology

Whereas most papers presented have only considered uniform sensor network deployments, the method in which these networks are meant to be deployed may not guarantee uniform coverage. Wireless sensor networks may exhibit regions of sparseness once deployed. Therefore, localization algorithms must be able to perform well under different conditions. In this section, we evaluate the performance of MDS-MAP(P, O) by using the same topology in [2], (i.e., a C-shaped topology). In our simulations, we notice that the position estimation errors are changed when the anchors are placed at different positions. For good performance, we recommend to have at least one anchor on each wing of a C-shaped topology.

Figure 3 shows the topologies estimated by hop-based MDS-MAP(P, C) and MDS-MAP(P, O). The position estimation errors by MDS-MAP(P, C) and MDS-MAP(P, O) are 74% and 65% of the radio range, respectively. The position estimation error of each individual sensor node varies. There

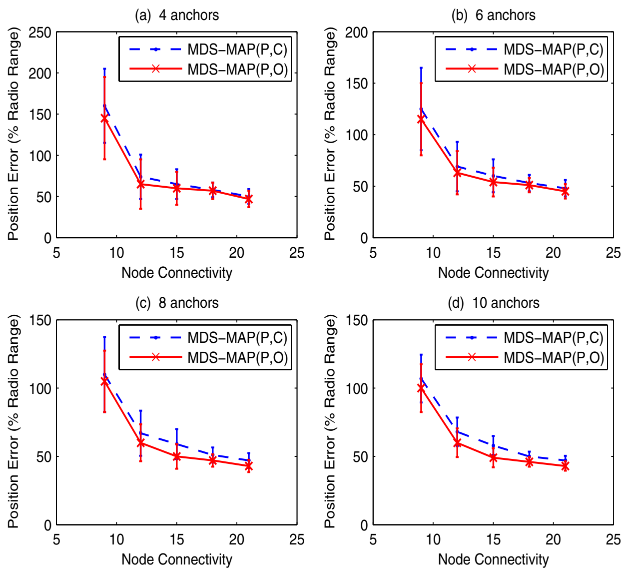

Fig. 4. Hop-based performance between MDS-MAP(P, C) and MDS-MAP(P, O) in a $10r \times 10r$ irregular (C-shaped) network topology with 160 nodes.

is no correlation for sensors that are closer to the anchors to have better position estimation.

Figure 4 shows the position estimation errors as a function of the average connectivity level by hop-based MDS-MAP(P, C) and MDS-MAP(P, O), in a C-shaped network topology. Results show that MDS-MAP(P, O) outperforms MDS-MAP(P, C) by a 9% lower position estimation error when the connectivity is 12. This difference is greater than the square topology case; however, the confidence intervals among the two algorithms show considerable overlap. This is to be expected since the estimated shortest path distances are more prone to errors arising from the geometry of nodes that are within the inside corners of the network.

V. Conclusions

In this paper, we proposed and analyzed the MDS-MAP(P, O) localization algorithm for wireless sensor networks. The MDS-MAP(P, O) algorithm is an extension of the MDS-MAP(P, C) algorithm originally proposed in [1][2]. We extend their work by using the ordinal MDS algorithm instead of the classical MDS algorithm. Our proposed MDS-MAP(P, O) algorithm is essential for future sensor applications which require a high accuracy of nodes' position by using a small number of anchor nodes. The algorithm can be applied not only to the case where nodes are equipped with distance-estimation hardware (range-based), but also to the case where only connectivity information (hop-based) is available. We conducted simulation studies under both regular (square) and irregular (C-shaped) topologies. Simulation results show that MDS-MAP(P, O) provides a lower position estimation error than MDS-MAP(P, C) in both hop-based and range-based scenarios [15]. Further work includes investigating the overhead for control packet exchange and the energy involved in each sensor node for computation.

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