AN AUSLANDER-TYPE RESULT FOR GORENSTEIN-PROJECTIVE MODULES

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Abstract. An artin algebra A is said to be CM-finite if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective A-modules. We prove that for a Gorenstein artin algebra, it is CM-finite if and only if every its Gorenstein-projective module is a direct sum of finitely generated Gorenstein-projective modules. This is an analogue of Auslander's theorem on algebras of finite representation type ([3, 4]).

1. Introduction

Let A be an artin R-algebra, where R is a commutative artinian ring. Denote by A-Mod (resp. A-mod) the category of (resp. finitely generated) left A-modules. Denote by A-Proj (resp. A-proj) the category of (resp. finitely generated) projective A-modules. Following [21], a chain complex $P^\bullet$ of projective A-modules is defined to be *totally-acyclic*, if for every projective module $Q \in A$-Proj the Hom-complexes $\mathrm{Hom}_A(Q, P^\bullet)$ and $\mathrm{Hom}_A(P^\bullet, Q)$ are exact. A module M is said to be *Gorenstein-projective* if there exists a totally-acyclic complex $P^\bullet$ such that the 0-th cocycle $Z^0(P^\bullet) = M$. Denote by A-GProj the full subcategory of Gorenstein-projective modules. Similarly, we define finitely generated Gorenstein-projective modules by replacing all modules above by finitely generated ones, and we also get the category A-Gproj of finitely generated Gorenstein-projective modules [17]. It is known that $A\text{-Gproj} = A\text{-GProj} \cap A\text{-mod}$ ([14], Lemma 3.4). Finitely generated Gorenstein-projective modules are also referred as maximal Cohen-Macaulay modules. These modules play a central role in the theory of singularity [11, 12, 10, 14] and of relative homological algebra [9, 17].

An artin algebra A is said to be *CM-finite* if there are only finitely many, up to isomorphisms, indecomposable finitely generated Gorenstein-projective modules. Recall that an artin algebra A is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable finitely generated modules. Clearly, finite representation type implies CM-finite. The converse is not true, in general.

Let us recall the following famous result of Auslander [3, 4] (see also Ringel-Tachikawa [26], Corollary 4.4) :

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Auslander's Theorem An artin algebra A is of finite representation type if and only if every A-module is a direct sum of finitely generated modules, that is, A is left pure semisimple, see [31].

Inspired by the theorem above, one may conjecture the following Auslander-type result for Gorenstein-projective modules: an artin algebra A is CM-finite if and only if every Gorenstein-projective A-module is a direct sum of finitely generated ones. However we can only prove this conjecture in a nice case.

Recall that an artin algebra A is said to be Gorenstein [19] if the regular module A has finite injective dimension both at the left and right sides. Our main result is

Main Theorem Let A be a Gorenstein artin algebra. Then A is CM-finite if and only if every Gorenstien-projective A-module is a direct sum of finitely generated Gorenstein-projective modules.

Note that our main result has a similar character to a result by Beligiannis ([9], Proposition 11.23), and also note that similar concepts were introduced and then similar results and ideas were developed by Rump in a series of papers [28, 29, 30].

2. Proof of Main Theorem

Before giving the proof, we recall some notions and known results.

2.1. Let A be an artin R-algebra. By a subcategory $\mathcal{X}$ of A-mod, we mean a full additive subcategory which is closed under taking direct summands. Let $M \in A$-mod. We recall from [8, 6] that a *right $\mathcal{X}$-approximation* of M is a morphism $f : X \longrightarrow M$ such that $X \in \mathcal{X}$ and every morphism from an object in $\mathcal{X}$ to M factors through f. The subcategory $\mathcal{X}$ is said to be *contravariantly-finite in* A-mod if each finitely generated modules has a right $\mathcal{X}$-approximation. Dually, one defines the notions of *left $\mathcal{X}$-approximations* and *covariantly-finite* subcategories. The subcategory $\mathcal{X}$ is said to be *functorially-finite in* A-mod if it is contravariantly-finite and covariantly-finite. Recall that a morphism $f : X \longrightarrow M$ is said to be *right minimal*, if for each endomorphism $h : X \longrightarrow X$ such that $f = f \circ h$, then h is an isomorphism. A right $\mathcal{X}$-approximation $f : X \longrightarrow M$ is said to be a *right minimal $\mathcal{X}$-approximation* if it is right minimal. Note that if a right approximation exists, so does right minimal ones; a right minimal approximation, if in existence, is unique up to isomorphisms. For details, see [8, 6, 7].

The following fact is known.

Lemma 2.1. *Let A be an artin algebra. Then*
(1). The subcategory A-Gproj *of* A-mod *is closed under taking direct summands, kernels of epimorphisms and extensions, and contains* A-proj.
(2). The category A-Gproj *is a Frobenius exact category* [22]*, whose relative projective-injective objects are precisely contained in* A-proj*. Thus the stable category* $\underline{A\text{-Gproj}}$ *modulo projectives is a triangulated category.*
(3). Let A be Gorenstein. Then the subcategory A-Gproj *of* A-mod *is functorially-finite.*
(4). Let A be Gorenstein. Denote by $\{S_i\}_{i=1}^n$ a complete list of pairwise nonisomorphic simple A-modules. Denote by $f_i : X_i \longrightarrow S_i$ the right minimal A-Gproj*-approximations. Then every finitely generated Gorenstein-projective module M is a*

direct summand of some module M', such that there exists a finite chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M'$ with each subquotient M_j/M_{j-1} lying in $\{X_i\}_{i=1}^n$.

Proof. Note that A-Gproj is nothing but $\mathcal{X}_\omega$ with $\omega = A$-proj in [6], section 5. Thus (1) follows from [6], Proposition 5.1, and (3) follows from [6], Corollary 5.10(1) (just note that in this case, ${}_AA$ is a cotilting module).

Since A-Gproj is closed under extensions, thus it becomes an exact category in the sense of [22]. The property of being Frobenius and the characterization of projective-injective objects follow directly from the definition, also see [14], Proposition 3.1(1). Thus by [18], chapter 1, section 2, the stable category $\underline{A\text{-Gproj}}$ is triangulated.

By (1) and (3), we see that (4) is a special case of [6], Proposition 3.8. ■

Let R be a commutative artinian ring as above. An additive category $\mathcal{C}$ is said be to R-*linear* if all its Hom-spaces are R-modules, and the composition maps are R-bilinear. An R-linear category is said to be *hom-finite*, if all its Hom-spaces are finitely generated R-modules. Recall that an R-*variety* $\mathcal{C}$ means a hom-finite R-linear category which is skeletally-small and idempotent-split (that is, for each idempotent morphism $e : X \longrightarrow X$ in $\mathcal{C}$, there exists $u : X \longrightarrow Y$ and $v : Y \longrightarrow X$ such that $e = v \circ u$ and $\mathrm{Id}_Y = u \circ v$). It is well-known that a skeletally-small R-linear category is an R-variety if and only if it is hom-finite and Krull-Schmidt (i.e., every object is a finite sum of indecomposable objects with local endomorphism rings). See [27], p.52 or [15], Appendix A. Then it follows that any factor category ([7], p.101) of an R-variety is still an R-variety.

Let $\mathcal{C}$ be an R-variety. We will abbreviate the Hom-space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ as (X, Y). Denote by $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ (resp. $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$) the category of contravariant R-linear functors from $\mathcal{C}$ to R-Mod (resp. R-mod). Then $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ is an abelian category and $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ is its abelian subcategory. Denote by $(-, X)$ the representable functor for each $X \in \mathcal{C}$. A functor F is said to be *finitely generated* if there exists an epimorphism $(-, C) \longrightarrow F$ for some object $C \in \mathcal{C}$; F is said to be *finitely presented (= coherent)* [2, 3], if there exists an exact sequence of functors $(-, C_1) \longrightarrow (-, C_0) \longrightarrow F \longrightarrow 0$. Denote by $\mathbf{fp}(\mathcal{C})$ the subcategory of $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ consisting of finitely presented functors. Clearly, $\mathbf{fp}(\mathcal{C}) \subseteq (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. Recall the duality

$$D = \mathrm{Hom}_R(-, E) : R\text{-mod} \longrightarrow R\text{-mod},$$

where E is injective hull of $R/\mathrm{rad}(R)$ as an R-module. Therefore, it induces duality $D : (\mathcal{C}^{\mathrm{op}}, R\text{-mod}) \longrightarrow (\mathcal{C}, R\text{-mod})$ and $D : (\mathcal{C}, R\text{-mod}) \longrightarrow (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. The R-variety $\mathcal{C}$ is called a *dualizing R-variety* [5], if this duality preserves finitely presented functors.

The following observation is important.

Lemma 2.2. *Let A be a Gorenstein artin R-algebra. Then the stable category $\underline{A\text{-Gproj}}$ is a dualizing R-variety.*

Proof. Since A-Gproj $\subseteq A$-mod is closed under taking direct summands, thus idempotents-split. Therefore, we infer that A-Gproj is an R-variety, and its stable category $\underline{A\text{-Gproj}}$ is also an R-variety. By Lemma 2.1(3), the subcategory A-Gproj is functorially-finite in A-mod, then by a result of Auslander-Smalø ([8], Theorem 2.4(b)) A-Gproj has almost-split sequences, and thus theses sequences induce

Auslander-Reiten triangles in $A\text{-}\underline{\mathrm{Gproj}}$ (Let us remark that it is Happel ([19], 4.7) who realized this fact for the first time). Hence the triangulated category $A\text{-}\underline{\mathrm{Gproj}}$ has Auslander-Reiten triangles, and by a theorem of Reiten-Van den Bergh ([25], Theorem I.2.4) we infer that $A\text{-}\underline{\mathrm{Gproj}}$ has Serre duality. Now by [20], Proposition 2.11 (or [13], Corollary 2.6), we deduce that $A\text{-}\underline{\mathrm{Gproj}}$ is a dualizing R-variety. Let us remark that the last two cited results are given in the case where R is a field, however one just notes that the results can be extended to the case where R is a commutative artinian ring without any difficulty. ■

For the next result, we recall more notions on functors over varieties. Let $\mathcal{C}$ be an R-variety and let $F \in (\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ be a functor. Denote by $\mathrm{ind}(\mathcal{C})$ the complete set of pairwise nonisomorphic indecomposable objects in $\mathcal{C}$. The *support* of F is defined to $\mathrm{supp}(F) = \{C \in \mathrm{ind}(\mathcal{C}) \mid F(C) \neq 0\}$. The functor F is *simple* if it has no nonzero proper subfunctors, and F *has finite length* if it is a finite iterated extension of simple functors. Observe that F has finite length if and only if F lies in $(\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ and $\mathrm{supp}(F)$ is a finite set. The functor F is said to be *noetherian*, if its every subfunctor is finitely generated. It is a good exercise to show that a functor is noetherian if and only if every ascending chain of subfunctors in F becomes stable after finite steps (one may use the fact: for a finitely generated functor F with epimorphism $(-, C) \longrightarrow F$, then for any subfunctor F' of F, $F' = F$ provided that $F'(C) = F(C)$). Observe that a functor having finite length is necessarily noetherian by an argument on its total length (i.e., $l(F) = \sum_{C\in \mathrm{ind}(\mathcal{C})} l_R(F(C))$, where l_R denotes the length function on finitely generated R-modules).

The following result is essentially due to Auslander (compare [3], Proposition 3.10).

Lemma 2.3. *Let $\mathcal{C}$ be a dualizing R-variety, $F \in (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$. Then F has finite length if and only if F is finitely presented and noetherian.*

Proof. Recall from [5], Corollary 3.3 that for a dualizing R-variety, functors having finite length are finitely presented. So the "only if" follows.

For the "if" part, assume that F is finitely presented and noetherian. Since F is finitely presented, by [5], p.324, we have the filtration of subfunctors

$$0 = \mathrm{soc}_0(F) \subseteq \mathrm{soc}_1(F) \subseteq \cdots \subseteq \mathrm{soc}_{i+1}(F) \subseteq \cdots$$

where $\mathrm{soc}_1(F)$ is the socle of F, and in general soc_{i+1} is the preimage of the socle of $F/\mathrm{soc}_i(F)$ under the canonical morphism $F \longrightarrow F/\mathrm{soc}_i(F)$. Since F is noetherian, we get $\mathrm{soc}_{i_0}F = \mathrm{soc}_{i_0+1}(F)$ for some i_0, and that is, the socle of $F/\mathrm{soc}_{i_0}(F)$ is zero. However, by the dual of [5], Proposition 3.5, we know that for each nonzero finitely presented functor F, the socle $\mathrm{soc}(F)$ is necessarily nonzero and finitely generated semisimple. In particular, $\mathrm{soc}(F)$ has finite length, and thus it is finitely presented. Note that $\mathbf{fp}(C) \subseteq (\mathcal{C}^{\mathrm{op}}, R\text{-mod})$ is an abelian subcategory, closed under extensions. Thus $F/\mathrm{soc}_1(F)$ is finitely presented. Applying the above argument to $F/\mathrm{soc}_1(F)$, we obtain that $\mathrm{soc}_2(F)$, as the extension between the socles of two finitely presented functors, has finite length. In general, one proves that $F/\mathrm{soc}_i(F)$ is finitely presented and $\mathrm{soc}_{i+1}(F)$ has finite length for all i. Hence $\mathrm{soc}(F/\mathrm{soc}_{i_0}(F)) = 0$ will imply that $F/\mathrm{soc}_{i_0}(F) = 0$, i.e., $F = \mathrm{soc}_{i_0}(F)$, which has finite length. ■

Let us consider the category A-GProj. Similar to Lemma 2.1(1),(2), we recall that $A\text{-GProj} \subseteq A\text{-Mod}$ is closed under taking direct summands, kernels of epimorphisms and extensions, and it is a Frobenius exact category with (relative) projective-injective objects precisely contained in A-Proj. Consider the stable category $A\text{-}\underline{\mathrm{GProj}}$, which is also triangulated and has arbitrary coproducts. Recall that in an additive category $\mathcal{T}$ with arbitrary coproducts, an object T is said to be *compact*, it the functor $\mathrm{Hom}_{\mathcal{T}}(T,-)$ commutes with coproducts. Denote the full subcategory of compact objects by $\mathcal{T}^c$. If we assume further that $\mathcal{T}$ is triangulated, then $\mathcal{T}^c$ is a thick triangulated subcategory. We say that $\mathcal{T}$ is a *compactly generated* [23, 24], if the subcategory $\mathcal{T}^c$ is skeletally-small and for each object X, $X \simeq 0$ provided that $\mathrm{Hom}_{\mathcal{T}}(T,X) = 0$ for every compact object T.

Note that in our situation, we always have an inclusion $A\text{-}\underline{\mathrm{Gproj}} \hookrightarrow A\text{-}\underline{\mathrm{GProj}}$, and in fact, we view it as $A\text{-}\underline{\mathrm{Gproj}} \subseteq (A\text{-}\underline{\mathrm{GProj}})^c$. Next lemma, probably known to experts, states the converse in Gorenstein case. It is a special case of [14], Theorem 4.1 (compare [10], Theorem 6.6). One may note that in the artin case, the category $A\text{-}\underline{\mathrm{Gproj}}$ is idempotent-split.

Lemma 2.4. *Let A be an Gorenstein artin algebra. Then the triangulated category $A\text{-}\underline{\mathrm{GProj}}$ is compactly generated and $A\text{-}\underline{\mathrm{Gproj}} \subseteq (A\text{-}\underline{\mathrm{GProj}})^c$ is dense (i.e., surjective up to isomorphisms).*

2.2. **Proof of Main Theorem:** Assume that A is an artin R-algebra. Set $\mathcal{C} = A\text{-}\underline{\mathrm{Gproj}}$, by Lemma 2.2, $\mathcal{C}$ is a dualizing R-variety. For a finitely generated Gorenstein-projective module M, we will denote by $(-,M)$ the functor $\mathrm{Hom}_{\mathcal{C}}(-,M)$; for an arbitrary module X, we denote by $(-,X)|_{\mathcal{C}}$ the restriction of the functor $\underline{\mathrm{Hom}}_A(-,X)$ to $\mathcal{C}$.

For the "if" part, we assume that each Gorenstein-projective module is a direct sum of finitely generated ones. It suffices to show that the set $\mathrm{ind}(\mathcal{C})$ is finite. For this end, assume that M is a finitely generated Gorenstein-projective module. We claim that the functor $(-,M)$ is noetherian. In fact, given a subfunctor $F \subseteq (-,M)$, first of all, we may find an epimorphism

$$\oplus_{i\in I}(-,M_i) \longrightarrow F,$$

where each $M_i \in \mathcal{C}$ and I is an index set. Compose this epimorphism with the inclusion of F into $(-,M)$, we get a morphism from $\oplus_{i\in I}(-,M_i)$ to $(-,M)$. By the universal property of coproducts and then by Yoneda's Lemma, we have, for each i, a morphism $\theta_i : M_i \longrightarrow M$, such that F is the image of the morphism

$$\sum_{i\in I}(-,\theta_i) : \oplus_{i\in I}(-,M_i) \longrightarrow (-,M).$$

Note that $\oplus_{i\in I}(-,M_i) \simeq (-,\oplus_{i\in I}M_i)|_{\mathcal{C}}$, and the morphism above is also induced by the morphism $\sum_{i\in I}\theta_i : \oplus_{i\in I}M_i \longrightarrow M$. Form a triangle in $A\text{-}\underline{\mathrm{GProj}}$

$$K[-1] \longrightarrow \oplus_{i\in I}M_i \xrightarrow{\sum_{i\in I}\theta_i} M \xrightarrow{\phi} K.$$

By assumption, we have a decomposition $K = \oplus_{j\in J}K_j$ where each K_j is finitely generated Gorenstein-projective. Since the module M is finitely generated, we infer that ϕ factors through a finite sum $\oplus_{j\in J'}K_j$, where $J' \subseteq J$ is a finite subset. In other words, ϕ is a direct sum of

$$M \xrightarrow{\phi'} \oplus_{j\in J'}K_j \quad \text{and} \quad 0 \longrightarrow \oplus_{j\in J\setminus J'}K_j.$$

By the additivity of triangles, we deduce that there exists a commutative diagram

$$\begin{array}{ccc} \oplus_{i\in I} M_i & \xrightarrow{\sum_{i\in I}\theta_i} & M \\ \downarrow & & \Vert \\ M' \oplus (\oplus_{j\in J\backslash J'} K_j)[-1] & \xrightarrow{(\theta',0)} & M \end{array}$$

where the left side vertical map is an isomorphism, and M' and θ' are given by the triangle $(\oplus_{j\in J'} K_j)[-1] \longrightarrow M' \xrightarrow{\theta'} M \xrightarrow{\phi'} \oplus_{j\in J} K_j$. Note that $M' \in \mathcal{C}$, and by the above diagram we infer that F is the image of the morphism $(-,\theta') : (-, M') \longrightarrow (-, M)$, and thus F is finitely-generated. This proves the claim.

By the claim, and by Lemma 2.3, we deduce that for each $M \in \mathcal{C}$, the functor $(-, M)$ has finite length, in particular, $\mathrm{supp}((-, M))$ is finite. Assume that $\{S_i\}_{i=1}^n$ is a complete list of pairwise nonisomorphic simple A-modules. Denote by $f_i : X_i \longrightarrow S_i$ the right minimal A-Gproj-approximations. By Lemma 2.1(4), the module M is a direct summand of M' and we have a finite chain of submodules of M' with factors being among X_i's. Then it is not hard to see that $\mathrm{supp}((-, M)) \subseteq \mathrm{supp}((-, M')) \subseteq \bigcup_{i=1}^n \mathrm{supp}((-, X_i))$ for every $M \in \mathcal{C}$. Therefore we deduce that $\mathrm{ind}(\mathcal{C}) = \bigcup_{i=1}^n \mathrm{supp}((-, X_i))$, which is finite.

For the "only if" part, assume that A is a CM-finite Gorenstein algebra. Then the set $\mathrm{ind}(\mathcal{C})$ is finite, say $\mathrm{ind}(\mathcal{C}) = \{G_1, G_2, \cdots, G_m\}$. Let $B = \mathrm{End}_{\mathcal{C}}(\oplus_{i=1}^m G_i)^{\mathrm{op}}$. Then B is also an artin R-algebra. Note that for each $C \in \mathcal{C}$, the Hom-space $\mathrm{Hom}_{\mathcal{C}}(\oplus_{i=1}^m G_i, C)$ has a natural left B-module structure, moreover, it is a finitely generated projective B-module. In fact, we get an equivalence of categories

$$\Phi = \mathrm{Hom}_{\mathcal{C}}(\oplus_{i=1}^m G_i, -) :\ \mathcal{C} \longrightarrow B\text{-proj}.$$

Then the equivalence above naturally induces the following equivalences, still denoted by Φ

$$\Phi :\ \mathbf{fp}(\mathcal{C}) \longrightarrow B\text{-mod}, \quad \Phi :\ (\mathcal{C}^{\mathrm{op}}, R\text{-Mod}) \longrightarrow B\text{-Mod}.$$

In what follows, we will use these equivalences. By [24], p.169 (or [13], Proposition 2.4), we know that the category $\mathbf{fp}(\mathcal{C})$ is a Frobenius category. Therefore, via Φ, we get that B is a self-injective algebra. Therefore by [1], Theorem 31.9, we get that B-Mod is also a Frobenius category, and by [1], p.319, every projective-injective B-module is of form $\oplus_{i=1}^m Q_m^{(I_i)}$, where $\{Q_1, Q_2, \cdots, Q_m\}$ is a complete set of indecomposable projective B-modules such that $Q_i = \Phi(G_i)$, and each I_i is some index set, and $Q_i^{(I_i)}$ is the corresponding coproduct.

Take $\{P_1, P_2, \cdots, P_n\}$ to be a complete set of pairwise nonisomorphic indecomposable projective A-modules. Let $G \in A$-GProj. We will show that G is a direct sum of some copies of G_i's and P_j's. Then we are done. Consider the functor $(-, G)|_{\mathcal{C}}$, which is cohomological, and thus by [13], Lemma 2.3 (or [24], p.258), we get $\mathrm{Ext}^1(F, (-, G)|_{\mathcal{C}}) = 0$ for each $F \in \mathbf{fp}(\mathcal{C})$, where the Ext group is taken in $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$. Via Φ and applying the Baer's criterion, we get that $(-, G)|_{\mathcal{C}}$ is an injective object, and thus by the above, we get an isomorphism of functors

$$\oplus_{i=1}^m (-, T_i)^{(I_i)} \longrightarrow (-, G)|_{\mathcal{C}},$$

where I_i are some index sets. As in the first part of the proof, we get a morphism $\theta : \oplus_{i=1}^m T_i^{(I_i)} \longrightarrow T$ such that it induces the isomorphism above. Form the triangle in $\underline{A\text{-GProj}}$

$$\oplus_{i=1}^{m} G_i^{(I_i)} \xrightarrow{\theta} T \longrightarrow X \longrightarrow (\oplus_{i=1}^{m} G_i^{(I_i)})[1].$$

For each $C \in \mathcal{C}$, applying the cohomological functor $\mathrm{Hom}_{\underline{A\text{-GProj}}}(C, -)$ and by the property of θ, we obtain that

$$\mathrm{Hom}_{\underline{A\text{-GProj}}}(C, X) = 0, \quad \forall\, C \in \mathcal{C}.$$

By Lemma 2.4, the category $\underline{A\text{-GProj}}$ is generated by $\mathcal{C}$, and thus $X \simeq 0$, and hence θ is an isomorphism in the stable category $\underline{A\text{-GProj}}$. Thus it is well-known (say, by [16], Lemma 1.1) that this will force an isomorphism in the module category

$$\oplus_{i=1}^{m} G_i^{(I_i)} \oplus P \simeq G \oplus Q,$$

where P and Q are projective A-modules. Now by [1], p.319, again, P is a direct sum of copies of P_j's. Hence the combination of Azumaya's Theorem and Crawlay-Jønsson-Warfield's Theorem ([1], Corollary 26.6) applies in our situation, and thus we infer that G is isomorphic to a direct sum of copies of G_i's and P_j's. This completes the proof. ■

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