Asymptotic Behavior of Internet Congestion Controllers in a Many-Flows Regime

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Abstract

Congestion controllers for the Internet are typically designed based on deterministic delay differential equation models. In this paper, we consider the case of a single link accessed by many TCP-like congestion-controlled flows, and uncontrolled flows which are modeled as stochastic disturbances. We show that if the number of flows is large and the link capacity is scaled in proportion to the number of users, then, under appropriate conditions, the trajectory of the stochastic system is eventually well approximated by the trajectory of a delay-differential equation. Our analysis also throws light on the choice of various parameters which ensures *global asymptotic stability* of the limiting deterministic system in the presence of feedback delay. Numerical examples with some popular congestion feedback mechanisms validate the parameter choices from the analysis. The results indicate that a system with multiple TCP-like flows is globally stable (and thus, a deterministic model is reasonable if the number of flows is large) as long as *the product of the throughput and feedback delay per flow is not very small*.

1 Introduction

The design philosophy of the current Internet is based on the end-to-end paradigm, wherein most of the intelligence resides at the end hosts. The network's task is to simply notify the end systems whenever it detects congestion in the network. Congestion detection is based on the *aggregate flow behavior* at the router, and the end-hosts are notified by simply dropping or by marking packets using the explicit congestion notification (ECN) bit (Floyd 1994). The end-host reacts to this information by decreasing its transmission rate, thus adapting to network congestion. In this manner, end-to-end control is maintained using only minimal network resources.

This end-to-end design philosophy has motivated a lot of work (Kelly, Maulloo & Tan 1998, Kunniyur & Srikant 2000, Paganini, Doyle & Low 2001) using a utility function maximization framework, leading a class of end-to-end rate control mechanisms for Internet congestion control. Based on the choice of the utility function, various types of fairness amongst users can be achieved. Further, it has been shown in (Kunniyur & Srikant 2000) that the congestion-avoidance phase of TCP flow-control can be considered a special case of the above framework for appropriately chosen utility functions. Such deterministic rate-based models have led to a better understanding of TCP and also allows us to improve existing congestion control and congestion feedback mechanisms used in the Internet.

Deterministic rate update models which explicitly account for round-trip delay have been the focus of much study in the recent past. An important question that has been addressed deals with stability of network controllers based on these deterministic rate adaptation mechanisms. In (Kelly 2000), a stability condition for a single proportionally fair congestion controller with delayed feedback was provided. Since then, this result has been extended to networks in (Johari & Tan 2001, Massoulie 2000, Vinnicombe 2001), and in (Paganini et al. 2001), similar results were shown for a different class of controllers. We also refer the reader to (Hollot, Misra, Towsley & Gong 2001, Kunniyur & Srikant 2001) for other related analysis of congestion controllers with delay. All of the above work dealt with *local stability* of the linearized controllers in the presence of round-trip delay. More recently, sufficient conditions were derived in (Deb & Srikant 2002) for *global exponential stability* for the case of a single flow accessing a link.

However, one may ask why we should consider *deterministic* models for Internet congestion control. In realistic systems, there are two sources of randomness. First, there can be flows that do not react to congestion control. For instance, these could be *web-mice*, which are short flows which terminate before they can react to congestion control. Such uncontrolled flows can be modeled as stochastic disturbances in the router.

Second, the marking decisions at the router could be probabilistic. To see this, consider a particular time-instance where the router decides to mark 20% of the packets. Due to the constraint that the router can toggle the state of only a single bit in the packet header, a possible strategy is to *mark each packet independently, and with probability* 0.2.

In this paper we concern ourselves with randomness generated due to uncontrolled short flows, in a many-flows regime. We refer the reader to (Baccelli & Hong 2002) for an analysis of TCP behavior in the context of probabilistic marking at the router.

In (Shakkottai & Srikant 2002), the authors justified the use of deterministic delay-differential equation models for studying *proportionally fair* congestion controllers. They showed that, in the many-flows regime, the trajectory of the average rate at the router converges to that of the deterministic model with the noise replaced by its mean.

1.1 Main Contribution

In this paper, we consider a system consisting of a single link accessed by a large number of *TCP-like* flows, each with identical feedback delay, but with (possibly) different initial conditions and also accessed by a large number of uncontrolled flows. By a *TCP-like* mechanism, we refer to the rate control model of the congestion avoidance phase of TCP proposed in (Kunniyur & Srikant 2000), which is also closely related to the model in (Hollot et al. 2001). We are interested in relating this stochastic model to a deterministic model where the noise process is replaced by its mean.

We show that, in the presence of uncontrolled flows modeled as stochastic noise, the deterministic delay-differential equation model with the noise replaced by its mean value is accurate in the following sense: The *average rate* of the flows behaves like a single flow *asymptotically in the number of flows and time*. Thus, unlike in the proportionally-fair case studied in (Shakkottai & Srikant 2002) where convergence was shown for each time (as opposed to asymptotically in time), here, the trajectory of the stochastic system *does not* converge to that of the deterministic system in the many-flows regime.

However, if the number of flows is large enough, the *global stability criterion* for a single flow (with minor modifications) is also a global stability condition, in an appropriate sense, for the stochastic system with multiple flows. Thus, the implication is that *parameter design can be carried out using deterministic analysis based on the single flow model.*

Further, for some standard marking functions used in literature, we show that TCP-like flows with standard TCP parameters satisfy the stability criterion when the bandwidth-delay product (i.e., the product of the throughput and the feedback delay) per flow is sufficiently large.

1.2 Organization of this Paper

We begin with a description of the model in Section 2. We consider multiple *TCP-like* flows along with uncontrolled flows in the model. In Section 3, we state the main results of our paper. In the two subsequent sections we go on to prove these results. Towards this end, in Section 4, we study a deterministic system by simply considering the mean of the uncontrolled flow rate through the link. We present conditions on the congestion control gain for the global stability of such a system. In Section 5, we prove the results stated in Section 3. In Section 6 we study the conditions derived in the context of standard TCP parameters and give examples to illustrate the results. We conclude in Section 7.

2 Model, Assumptions and Preliminaries

2.1 System Model

Our model is that of a single bottleneck link being accessed by many *TCP-like* flows. The delay in the forward and the reverse path is $d/2$ so that the round-trip delay of each flow is d. Such a model can be applicable in a scenario when multiple users behind an ISP access a server through a common bottleneck link as in Figure 1. The number of flows in the system is N, which is also the scaling parameter. We consider a sequence of such systems indexed by N. In the N-th system, there are N flows accessing the link and the capacity of the link is scaled as Nc so that capacity per flow is maintained at c. Further, in the N-th system there are N uncontrolled flows accessing the link. Before we describe the rate update mechanism for the flows, we first comment on the marking function of the link.

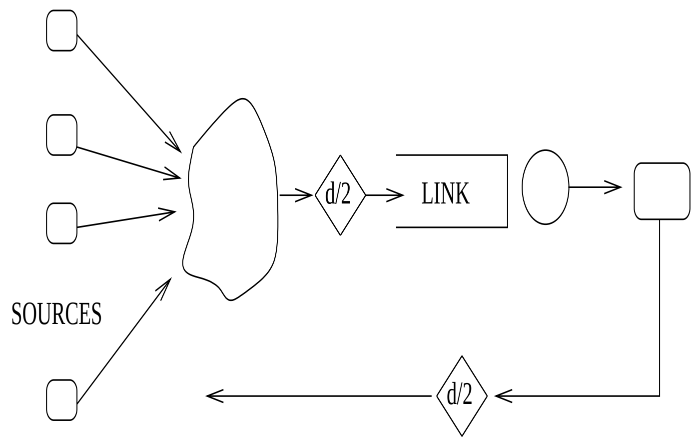

Figure 1: The system model

The link has a marking function $p(\lambda, C)$ which denotes the fraction of packets marked when the total arrival rate into the link is λ and the link capacity is C (where $C = Nc$). The marking function is assumed to satisfy the following conditions.

Assumption 1. *(Properties of Marking Function)*

1. *The function $p(\lambda, C)$ is increasing in λ and is Lipschitz continuous in λ.*
2. *We further assume that $p(\lambda, C) = p(\lambda/C, 1)$.*

$\square$

The first assumption is obvious since $p(\lambda, C)$ is the fraction of packets marked. The second assumption says that the fraction of packets marked simply depends on the ratio of the total arrival rate and the link capacity. To understand this property in the context of our scaling, suppose in the N-th system, the rate of the i-th controlled flow is x_i and the rate of the i-th uncontrolled flow is e_i for $1 \leq i \leq N$. Then the marking function for the N-th system is,

$$p(\lambda, Nc) = p\left(\frac{\sum_i (x_i + e_i)}{Nc}, 1\right) = p\left(\frac{x+e}{c}, 1\right) ,$$

where x and e are the average rate of the controlled and the uncontrolled flows respectively. Thus, under this assumption, the marking function in the N-th system simply depends on the *average flow rate* through the link for some fixed capacity per flow. Two examples of marking functions which have this property are:

$$1.\ \ p(\lambda, C) = \left(\frac{\lambda}{C}\right)^B \qquad\qquad 2.\ \ p(\lambda, C) = \frac{a\lambda}{C - (1-a)\lambda}\ .$$

The first marking function has the interpretation of the queue size being B or larger in an $M/M/1$ queue with arrival rate x. The second marking function can be used as a rate based model for a marking scheme called Random Early Marking (REM) (Kelly 2000). We remark that any reasonable marking function should satisfy the second assumption. This ensures the scalability of the marking function in the number of flows. Thus, from now on, we will interpret the arguments x and c of the marking function $p(x, c)$, as the average arrival rate and the capacity per flow respectively. Further, in the systems we consider from now on, the capacity per flow c will remain constant, and the only time-varying parameter is the average rate x. Thus, to avoid unnecessary notation, we will hide the dependence of $p(\cdot, c)$ on c, and let

$$p(x) \equiv p(x, c) = p(x/c, 1)$$

In addition to the controlled flows, we assume that the system is accessed by uncontrolled flows. These are flows that do not react to congestion signals and are modeled as stochastic processes with mean a. In the N-th system, there are N uncontrolled flows $\{e_i^{(N)}[k] + a\}_{i=1}^N$, where $(e_i^{(N)}[k] + a)$ is the rate of the i-th uncontrolled flow at k-th time epoch in the N-th system. We model $\{e_i^{(N)}[k]\}_{i=1}^N$ as i.i.d and bounded stochastic process with mean 0.

We now describe our model for the controlled flows. We consider a fluid model for the rate update of the controlled flows. Denote by $y_i^{(N)}[k]$ the flow rate of the i-th flow at time slot k when there are N such flows present in the system. Further, denote by $x^{(N)}[k]$ the average flow rate of the controlled flows through the link at time k and so

$$x^{(N)}[k] = \frac{1}{N} \sum_{i=1}^{N} y_i^{(N)}[k] \ .$$

Similarly, denote by $(e^{(N)}[k] + a)$ the average flow rate of the N uncontrolled flows through the link. The fraction of packets marked by the link is $p(x^{(N)}[k] + a + e^{(N)}[k])$, where the average flow rate at the link consists of the average flow rate of the controlled flows $x^{(N)}(t)$ and the average flow rate due to N uncontrolled flows $a + e^{(N)}(t)$.

We now describe the time interval over which the flows update their rates. The flows update their rates at discrete time slots. We can view each time slot as a measurement interval over which rates are measured in the system and control actions by the routers and flows are updated. Typically, this measurement interval is measured in terms of the number of packets that can be processed by a typical router. For example, the time-step could be "100 packets long." Thus, by scaling both the time-step and the capacity, we can maintain a constant time-step, as measured in packets (Shakkottai & Srikant 2002). To this end, let each time time-step in the N-th system be $1/N$. Thus, the update of the i-th system at the $(k+1)$-th time-step can be described by the following.

$$\begin{aligned} y_i^{(N)}[k+1] &= y_i^{(N)}[k] + \frac{\kappa}{N}\left[w - y_i^{(N)}[k] y_i^{(N)}[k-Nd] p(x^{(N)}[k-Nd] + e^{(N)}[k-Nd] + a)\right] , \\ & \quad i \in \{1, 2, \ldots, N\} \end{aligned} \tag{1}$$

Note that, since the delay d as measured in seconds is fixed, the delay in the N-th system corresponds to Nd time slots. Further note that, we assume that the feedback delay is constant. This is reasonable if we employ early congestion notification schemes using virtual queues which lead to negligible queueing delay at the router.

A continuous time model can now be embedded as

$$y_i^{(N)}(t) = y_i^{(N)}[Nt] \text{ ,for } Nt \in \mathbb{N}$$

with a straight line approximation used between integers. Similarly, the average rate process $x^{(N)}(t)$ and the noise process $e^{(N)}(t)$ can be defined. Thus, we have the following rate update model in continuous time.

$$\begin{aligned} \dot{y}_i^{(N)}(t) &= \kappa\left[w - y_i^{(N)}\left(\frac{\lfloor Nt \rfloor}{N}\right) y_i^{(N)}\left(\frac{\lfloor N(t-d) \rfloor}{N}\right) p\left(x^{(N)}\left(\frac{\lfloor N(t-d) \rfloor}{N}\right) + a + e^{(N)}\left(\frac{\lfloor N(t-d) \rfloor}{N}\right)\right)\right] , \\ & \quad i \in \{1, 2, \ldots, N\}, \end{aligned} \tag{2}$$

We remark that the above update equation has to be interpreted as a representation for the unique trajectory satisfying the integral equation

$$\begin{aligned} & y_i^{(N)}(t) = y_i^{(N)}(0) + \\ & \quad \kappa \int_{s=0}^{t} \left[w - y_i^{(N)}\left(\frac{\lfloor Ns \rfloor}{N}\right) y_i^{(N)}\left(\frac{\lfloor N(s-d) \rfloor}{N}\right) p\left(x^{(N)}\left(\frac{\lfloor N(s-d) \rfloor}{N}\right) + a + e^{(N)}\left(\frac{\lfloor N(s-d) \rfloor}{N}\right)\right)\right] ds \quad . \end{aligned}$$

It is in this sense that all the differential equations in the rest of the paper are to be interpreted. We comment that choosing $\kappa = 2/3$ and $\kappa w = 1/d^2$ in (2) result in the rate control model of standard TCP (Kunniyur & Srikant 2000).

In the rest of the paper, we find conditions under which the system converges to the unique equilibrium point given by the solution of

$$y_i^* \sqrt{p(y_i^* + a)} = \sqrt{w} \ ,$$

in the presence of feedback delay, with and without stochastic disturbance introduced by the uncontrolled flows. We also discuss the implication of our results in the context of TCP.

2.2 Some Preliminaries

In this subsection we discuss two key results which will be useful in some of our derivations.

The first result is on the boundedness of the average trajectory of congestion controlled flows in the presence of delay and without the stochastic noise. Consider a continuous-time deterministic model of congestion-controlled flows where the noise is modeled by a constant process of rate a. There are N flows and the the rate update of the i-th flow is given by

$$\dot{y}_i(t) \quad = \quad \kappa[w - y_i(t)y_i(t-d)p(x(t-d)+a)] \ , \quad i = 1, 2, \ldots, N \ ,$$

The trajectory of the average flow rate through the link $x(t)$ can be described by

$$\dot{x}(t) = \kappa[w - \left[\frac{1}{N}\sum_{1}^{N} y_i(t)y_i(t-d)\right] p(x(t-d)+a)] \ .$$

We are interested in conditions on κ so that the average flow rate $x(t)$ is bounded. The following result which is derived in (Shakkottai, Srikant & Meyn 2001) provides such a condition.

Lemma 2.1. *(Shakkottai et al. 2001) Suppose $\kappa d < \beta$. Fix $\delta > 0$, where δ can be arbitrarily small. Then there exists $t_0(\delta) < \infty$ such that for all $t \geq t_0$*

$$x(t) \leq M_\beta,$$

where M_β is the smallest positive number satisfying

$$M_\beta^2 p(M_\beta + a - 2w)\left(1 - \frac{2w\beta}{M_\beta}\right) \geq w + \delta$$

□

For the rest of this paper, we will assume that δ in the preceding lemma is a very small number, fixed at, say, $\delta = 0.0001$.

We next state a useful result on the global exponential stability of linear time varying delay differential equations (Deb & Srikant 2002). We will use this in our subsequent analysis of the model.

Lemma 2.2. *(Deb & Srikant 2002) Consider the delay differential equation given by*

$$\dot{x}(t) = a(t)x(t) + b(t)x(t-d)$$

with some initial condition, $x(t) = \phi(t), \ t \in [-d, 0]$. If there exists $q > 1$ such that $a(t)$ and $b(t)$ satisfy

$$d\sqrt{q} \max_{t-d \leq s \leq t}(|a(s)| + |b(s)|) < -sgn(b(t)) - \frac{a(t)}{|b(t)|} \ ,$$

for all $t \leq t_1$, then

$$V(t) < qV(0)e^{-\alpha t} \ , \quad \forall \ t \leq t_1,$$

where $V(x(t)) = \sup_{t-2d \leq s \leq t} x^2(s)$ and $\alpha > 0, \ q > 1$ are constants. □

3 Main Result: Multiple TCP-like flows with Identical Round-trip Time and Stochastic Noise

Before we state our main result, we state a result on the upper bound of the average flow rate, which is easy to show using the upper bound on the average given in Lemma 2.1. Consider the system in (2) and let $x^{(N)}(t)$ denote the average rate of the flows, i.e.,

$$x^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} y_i^{(N)}(t) \ .$$

Lemma 3.1. *Suppose $\kappa d < \beta$. Then given $\epsilon' > 0$, there exists $\overline{N}$ and $\overline{t}(\epsilon')$ such that $\forall (N \geq \overline{N})$ and $\forall (t \geq \overline{t})$,*

$$x^{(N)}(t) \leq M_\beta + \epsilon' \ ,$$

where M_β is as given in Lemma 2.1 □

Thus, we see that the average rate eventually gets arbitrarily close to M_β for large enough N. Without loss of generality, we can study the system evolution from time $\overline{t}$ onwards. In the following, we thus assume that $x^{(N)}(t) \leq M_\beta$ for all $t \in [-d, 0]$ and for all $N \geq 1$. It is more precise to assume that $x^{(N)}(t) \leq M_\beta + \epsilon'$ for all $t \in [-d, 0]$ and N large enough. However, our assumption is less notationally cumbersome and does not lead to any loss of generality. Suppose the initial value of the average rate lies in some compact set $[0, K]$ in which the equilibrium point of the system is included. Then, clearly, there exists a β for which $M_\beta \geq K$. We now assume that the initial condition for each flow satisfies the following.

Assumption 2. *(Initial Condition) The initial trajectory for any user $i \in \{1, 2 \ldots N\}$ satisfies*

$$M_\beta(1-\epsilon) \leq y_i^{(N)}(s) < M_\beta(1+\epsilon), \ \forall \ s \in [-d, 0]$$

for some $\epsilon < 1$. □

Essentially, this says that the initial values of the individual user rates are not too far away from each other. Since the value of M_β is larger than the equilibrium rate, not allowing the initial user rates to be more than twice the value of M_β is a reasonable assumption. We remark that for all N, the initial conditions are assumed to satisfy the conditions given in Assumption 2. We also state below our assumption on the noise process.

Assumption 3. *(Noise Process) The disturbances due to the uncontrolled flows are bounded, i.e., $\exists K < \infty$ such that $|e_i^{(N)}[k]| < K$ for all $i = 1, 2, \ldots N$. Further, we assume that these uncontrolled flows are i.i.d., stationary and ergodic. Under these assumptions it can be shown (Shakkottai & Srikant 2002) that the average noise process $e^{(N)}(t)$ satisfies*

$$\lim_{N\to\infty} \sup_{t\in[-d,NT]} |e^{(N)}(t)| = 0 \quad a.s.$$

□

Finally, consider the following system given by,

$$\dot{u} = \kappa[w - u(t)u(t-d)p(u(t-d)+a)] \ . \tag{3}$$

This is the rate update model of single flow accessing a single link with marking function $p(.)$, and, when the noise is modeled by a process of constant rate a. Let R be any value such that the above system (3) is semi-globally, exponentially stable for $\kappa d < R$. Techniques for finding such an R are given in (Deb & Srikant 2002).

We have the following main result of the paper.

Theorem 3.1. *Suppose*

$$\kappa d < \min\left[\beta,\ R,\ \frac{1}{6M_\beta p(M_\beta + a)}\right] \ ,$$

Then, under Assumption 1, 2 and 3, given $\epsilon' > 0$, $\exists\ N'(\epsilon')$ such that $\forall N \geq N'$,

$$|y_k^{(N)}(NT) - y^*| \leq \epsilon' \ , \ a.s. \quad k \in \{1, 2, \ldots N\} \ ,$$

where y^* is the solution of $w = y^2 p(y+a)$. □

In the next few sections we prove the above result. We first establish some intermediate results before we prove Theorem 3.1. One of our results in the next section also provides conditions for a deterministic version of the congestion control model to be globally asymptotically stable.

4 Multiple TCP-like Flows with Constant Noise

In this section, we study a deterministic, continuous-time model of congestion control when the noise process due to the uncontrolled flows is simply modeled by a constant process of rate a. For the purposes of this section we keep N, the number of flows, fixed. We denote the rate of the i-th flow by $y_i(t)$ and the average flow rate by $x(t)$.

The rate update of the i-th flow is,

$$\dot{y}_i(t) \;=\; \kappa[w - y_i(t)y_i(t-d)p(x(t-d)+a)] \;, \quad i = 1, 2, \ldots, N \;, \tag{4}$$

and the trajectory of the average flow rate through the link $x(t)$ can be described by

$$\dot{x}(t) = \kappa[w - \left[\frac{1}{N}\sum_{1}^{N} y_i(t)y_i(t-d)\right] p(x(t-d)+a)] \;.$$

Note from Lemma 2.1 that, $x(t) \le M_\beta$ for all $t \ge t_0$. Thus, by shifting the time axis appropriately, we assume that $x(t) \le M_\beta$ for all $t \ge 0$. Further, the initial trajectory of the individual flows satisfy the conditions given by Assumption 2.

Our goal is to find suitable conditions on κ for the system given by (4) to be globally asymptotically stable.

We now introduce the following notation for every pair of flows.

$$r_{ij}(t) = y_i(t) - y_j(t) \;, \quad (i,j) \in \{1, 2, \ldots, N\}^2 \tag{5}$$

Our goal is to show that $r_{ij}(t)$ converges to zero for appropriately chosen κ. This will enable us to show that the system indeed converges to the unique equilibrium point under suitable conditions.

First, note that the dynamics of $r_{ij}(t)$ can be described by the following.

$$\begin{aligned}
&\dot{r}_{ij}(t) \\
&\quad = \; -\kappa p(x(t-d)+a)[y_i(t)y_i(t-d) - y_j(t)y_j(t-d)] \\
&\quad = \; -\kappa p(x(t-d)+a)[y_i(t)y_i(t-d) - y_i(t)y_j(t-d) + y_i(t)y_j(t-d) - y_j(t)y_j(t-d)] \\
&\quad = \; -\kappa p(x(t-d)+a)[y_i(t)r_{ij}(t-d) + y_j(t-d)r_{ij}(t)]
\end{aligned}$$

We are now in a position to state and prove the following result on the convergence of $r_{ij}(t)$.

Theorem 4.1. *If κd satisfies*

$$\kappa d < \min\left[\beta, \; \frac{1}{6M_\beta p(M_\beta + a)}\right] \;,$$

then

$$\lim_{t\to\infty} \sup_{(i,j)\in\{1,2,\ldots,N\}^2} r_{ij}^2(t) \le A\exp(-\alpha t) \ ,$$

where $A > 0$ and $\alpha > 0$ are suitable constants.

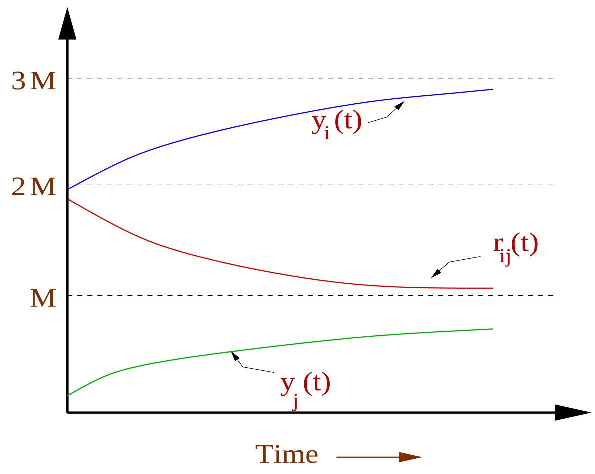

Figure 2: Main proof idea in Theorem 4.1

Remark: Before we go into the details of the proof we will illustrate the key idea in the proof informally. First, we can show that as long as all the flow rates $\{y_i(t)\}$ are less than $3M$, the difference between their rates $\{r_{ij}(t)\}$ will decrease. This will follow from the single flow global stability condition from Lemma 2.2. Second, recall from Lemma 2.1 that the average rate is upper-bounded as well, i.e., $x(t) < M$ (see Figure 2).

Suppose at some time, say t_1, it happens that for a particular flow l, $y_l(t_1) = 3M$, and up to time t_1, we have all the individual flow rates strictly less than $3M$. As the average rate at this time $x(t_1) < M$, *there will be some flow k whose rate $y_k(t_1) < M$.*

On the other hand, by assumption, the initial value of $\{r_{ij}(.)\}$ is less than $2M$. From the decreasing property of $\{r_{ij}(.)\}$, it follows that $r_{lk}(t_1) < 2M$. This, along with the fact that $y_k(t_1) < M$ implies that $y_l(t_1) < 3M$ leading to a contradiction. Therefore, we must have that all the flow rates are strictly less than $3M$ for all time, and the required result will follow. We now formally prove this result.

Proof of Theorem 4.1: Since β is assumed to be fixed throughout the proof, we drop the subscript in M_β, and simply use M in this proof. Define

$$t_1 = \inf_{t>0}\{t : \max_{i\in\{1,2,\ldots,N\}} [y_i(t)] \ge 3M\} \ , \tag{6}$$

where we include the possibility of t_1 being infinity (which would mean $\max_i[y_i(t)] < 3M]$ for all $t > 0$). Further, define the function $V_{ij}(t)$ as

$$V_{ij}(t) = \sup_{t-2d \le s \le t} r_{ij}^2(s) \ .$$

We divide the proof into two steps. In the first step we show that under the condition on κd given by the statement of the theorem, $V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t)$ for all $t < t_1$, for every (i,j) and for some constants q and α. In the second step we show that $t_1 = \infty$ and use this to conclude that $r_{ij}(t)$ converges to zero.

Step 1: Consider the delay-differential equation,

$$\dot{r}_{ij}(t) = -\kappa p(x(t-d)+a)[y_i(t) r_{ij}(t-d) + y_j(t-d) r_{ij}(t)]$$

for every pair (i,j). Suppose that

$$\kappa d \max_{t-2d \le s \le t} [p(x(t-d)+a)(y_i(t) + y_j(t-d))] \quad < \quad 1 + \frac{y_j(t-d)}{y_i(t)} \ .$$

Since we are considering t such that $t < t_1$, we have $0 < y_i(t) < 3M$ for all $i \in \{1, 2, \ldots, N\}$. Further the average rate $x(t) < M$ by our assumption. Thus a sufficient condition for the preceding inequality to be satisfied is

$$\kappa d[6Mp(M+a)] < 1 \ .$$

Since we also have $\kappa d < \beta$ for M to be an eventual upper bound on $x(t)$, thus if κd satisfies the condition given by the statement of the theorem, and q is chosen to satisfy

$$1 < \sqrt{q} < \frac{1}{6Mp(M+a)\kappa d} \ ,$$

then by Lemma 2.2,

$$V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t) \quad \forall t < t_1 \ ,$$

where α is a function of q. In this case choose q such that,

$$1 < \sqrt{q} < \min\left(\frac{1}{\epsilon}, \frac{1}{6Mp(M+a)\kappa d}\right) \ ,$$

where ϵ is such that the initial rates of the individual flows lie in $[M(1-\epsilon), M(1+\epsilon)]$. We will consider q in the above range for the rest of the proof.

Step 2: We now show that t_1 as defined by (6) is not finite. We will show it by contradiction. Suppose $t_1 < \infty$. Since the trajectories of $y_i(t)$'s are continuous in t, we have $\max_i[y_i(t_1)] = 3M$. Suppose

$$k = \arg \max_{i \in \{1,2,\ldots,N\}} y_i(t_1) \ .$$

If κd satisfies the conditions given in the statement of the theorem, we further have from *Step 1* that, $V_{kj}(t) < qV_{kj}(0)\exp(-\alpha t)$ for all $t < t_1$. Since, $V_{kj}(0) < (2M\epsilon)^2$ for all j from our assumption on the initial condition, we have $V_{kj}(t) < q(2M\epsilon)^2 \exp(-\alpha t)$ for all $t < t_1$. From the continuity of $V_{kj}(t)$ in t,

$$V_{kj}(t_1) < q(2M\epsilon)^2$$

which in turn implies

$$\sup_{t_1-2d\leq s\leq t_1} |r_{kj}(t_1)| < 2M\epsilon\sqrt{q}$$

which further implies

$$|y_k(t_1) - y_j(t_1)| < 2M\epsilon\sqrt{q} \quad \forall j \ .$$

We also have

$$\frac{1}{N}\sum_{i=1}^{N} y_i(t_1) < M$$

since $x(t) < M$ for all $t > 0$. If the average of N quantities is less than M, there must be at least one of them less than M. Let that element be indexed by l so that $y_l(t_1) < M$. Note that $l \neq k$ since $y_k(t_1) = 3M$. We thus have,

$$y_k(t_1) \leq |y_k(t_1) - y_l(t_1)| + y_l(t_1) < 2M\sqrt{q}\epsilon + M < 3M \ , \tag{7}$$

where we have used the fact that $\sqrt{q}\epsilon < 1$. But, $y_k(t_1) = 3M$. Thus we have arrived at a contradiction and so for all $t > 0$, $y_i(t) < 3M$ for all $i \in \{1, 2, \ldots, N\}$. This along with *Step 1* implies that

$$V_{ij}(t) < qV_{ij}(0)\exp(-\alpha t) \quad \forall\ t > 0 \ .$$

This proves the convergence of $r_{ij}(t)$. To show that the convergence is uniform in all pairs (i, j) we simply note that the exponent in the exponential convergence only depends on the choice of κ and not on any specific flow. □

Thus, we have shown that the trajectories of all the flows get coupled if their round-trip delays are the same. Using this we can now study the stability with multiple flows by using stability results from the single flow case. First note that the average flow rate through the link $x(t)$ can be written as

$$\begin{aligned} x(t) &= \frac{1}{N}\sum_{i=1}^{N} y_i(t) \\ &= y_k(t) + \frac{1}{N}\sum_{i=1}^{N}[y_i(t) - y_k(t)] \\ &= y_k(t) + \delta(t) \ , \end{aligned}$$

where $y_k(t)$ is any particular flow and $\delta(t)$ is a term which goes to zero exponentially. Next we rewrite the update equation for flow k as follows.

$$
\begin{aligned}
\dot{y}_k(t) &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d) + a + \delta(t-d))] \\
&= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d) + a)] + \\
&\quad \kappa[y_k(t)y_k(t-d)p(y_k(t-d) + a) - y_k(t)y_k(t-d)p(y_k(t-d) + a + \delta(t-d))] \\
&= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d) + a)] - y_k(t)y_k(t-d)p'(\beta_k(t) + a)\delta(t-d) \\
\Rightarrow \quad \dot{y}_k &= \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d) + a)] + \eta_k(t) \qquad (8)
\end{aligned}
$$

The second last step follows from the mean-value theorem and $\beta_k(t) = y_k(t-d) + f\delta(t-d))$ for some f such that $0 < f < 1$. Note that since $|\delta(t)| \to 0$ exponentially and all the other terms are bounded, $|\eta_k(t)| \to 0$ exponentially. Thus we can view the trajectory of the k-th flow as a single flow accessing the link except for an additional term which is negligible for large t. It is thus natural to believe that the stability criterion for

$$
\dot{u} = \kappa[w - u(t)u(t-d)p(u(t-d) + a)] \qquad (9)
$$

is sufficient to guarantee the stability of the system with multiple flows. We show this in our next theorem, which is a simple extension of the global stability result with single flow in (Deb & Srikant 2002).

Theorem 4.2. *Suppose*

$$
\kappa d < \min\left[\beta,\ R,\ \frac{1}{6M_\beta p(M_\beta + a)}\right] ,
$$

where $R > 0$ is such that $\kappa d < R$ is a sufficient condition for (9) to be globally stable. Then the system described by (8) is globally exponentially stable.

Before we prove the above result we state a useful result on functional differential equation (Kolmanovskii & Nosov 1986).

Lemma 4.1 (page 79, (Kolmanovskii & Nosov 1986)). *Consider the retarded functional differential equation*

$$
\dot{x}(t) = f(x_t),\ x_0 = \phi \qquad (10)
$$

where $x_t = \{x(t+\theta) : -d \le \theta \le 0\} \in CB[-d, 0]$ and $\phi \in CB[-d, 0]$. Assume $f : CB[-d, 0] \to R^n$ is continuous, Lipschitz, and, $f(0) = 0$. Then (10) is exponentially stable if and only if there exists *a functional* $V(t,\phi)$ *such that*

$$\begin{array}{ll} c_1\|\phi\| \quad \leq V(t,\phi) \leq \quad c_2\|\phi\| & \\ \dot{V} \leq -c_3\|x_t\| \leq -\frac{c_3}{c_2}V, & |V(t,\phi)-V(t,\xi)| \leq c_4\|\phi-\xi\|, \end{array}$$

where the norms of the functions are defined as $\|\phi\| = \sup_{-d\leq\theta\leq 0}|\phi(\theta)-\xi(\theta)|$, *and,* c_i *are some positive constants.* □

Proof of Theorem 4.2: Since,

$$\dot{y}_k = \kappa[w - y_k(t)y_k(t-d)p(y_k(t-d)+a)]$$

is globally exponentially stable (Deb & Srikant 2002), we have from Lemma 4.1, the existence of a Lyapunov function $V_k(t, y_{kt})$ such that

$$\dot{V}_k \leq -\gamma V_k$$

and satisfying the properties given in Lemma 4.1. Now, apply the Lyapunov functional

$$V(t) \equiv V(t,y_t) = \frac{1}{N}\sum_{k=1}^{N} V_k(t,y_{kt}) \tag{11}$$

to the system given by(8). Since

$$\sup_k |\eta_k(t)| \leq K_1 \exp(-\alpha t) \ ,$$

it is easy to see that

$$\dot{V} \leq -\gamma V + K_1 c_4 \exp(-\alpha t) \ ,$$

from which it follows that

$$V(t) \leq V(0)\exp(-\gamma t) - \frac{K_1 c_4}{\gamma-\alpha}[\exp(-\gamma t) - \exp(-\alpha t)].$$

The result thus follows since all the initial conditions are assumed to lie in a compact set. We note that the exponent in the exponential stability can be chosen as $\min(\gamma, \alpha)$. □

We now use the results derived in this section to prove the main results of this paper as stated in Section 3.

5 Proof of the Main Result in Section 3

In this section, we show that the global stability criterion of the system given by (4) is sufficient to ensure the stability of the system with stochastic noise in an appropriate sense.

Before we prove our main result, we will prove Theorem 3.1 under the following slightly relaxed assumption.

$$\lim_{N\to\infty} \sup_{t\in[0,\infty)} |e^{(N)}(t)| = 0 \ \ a.s. \ , \tag{12}$$

We introduce some notations next. Define

$$r_{ij}^{(N)}(t) = y_i^{(N)}(t) - y_j^{(N)}(t) \ , \quad (i,j) \in \{1,2,\ldots,N\}^2 \ . \tag{13}$$

Further, let

$$y_{it}{}^{(N)} = \{y_i^{(N)}(s) : t-d \le s \le t\}$$

and similarly define $y_t{}^{(N)}$ for the vector $y^{(N)}(t) = [y_i^{(N)}(t)]_{i=1}^N$. Let

$$\|y_{it}{}^{(N)}\| = \sup_{t-d\le s\le t} y_i^{(N)}(s) \ .$$

Now note that the update of the i-th flow can be written as follows.

$$\begin{aligned}\dot{y}_i^{(N)}(t) &= \kappa\left[w - y_i^{(N)}(\frac{\lfloor Nt\rfloor}{N})y_i^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})p\left(x^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N}) + a + e^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})\right)\right] \\ &= \kappa[w - y_i^{(N)}(t)y_i^{(N)}(t-d)p\left(x^{(N)}(\frac{\lfloor N(t-d)\rfloor)}{N} + a + e^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})\right)] + g_i(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)}) \ ,\end{aligned}$$

where

$$\begin{aligned} g_i(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)}) \ &= \ \kappa y_i^{(N)}(t)y_i^{(N)}(t-d)p\left(x^{(N)}(\frac{\lfloor N(t-d)\rfloor)}{N} + a + e^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})\right) - \\ &\quad \kappa y_i^{(N)}(\frac{\lfloor Nt\rfloor}{N})y_i^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})p\left(x^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N}) + a + e^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N})\right)\end{aligned}$$

Now we note that the argument of $p(.)$ is bounded. Further, since $(t - \lfloor Nt\rfloor/N) \le 1/N$, by applying mean value theorem to the expression for $g_i(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)})$, it can be shown after some straightforward algebraic manipulations that,

$$|g_i(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)})| \le \frac{1}{N} h_i(\|y_{it}{}^{(N)}\|, \|y_{i\frac{\lfloor Nt\rfloor}{N}}{}^{(N)}\|, \|y_{i\frac{\lfloor N(t-d)\rfloor}{N}}{}^{(N)}\|),$$

where $h_i(.,.,.)$ is a polynomial in $\|y_{it}{}^{(N)}\|$, $\|y_{i\frac{\lfloor Nt\rfloor}{N}}{}^{(N)}\|$ and $\|y_{i\frac{\lfloor N(t-d)\rfloor}{N}}{}^{(N)}\|$.

As before, it can be shown for (2) that,

$$\begin{aligned}\dot{r}_{ij}^{(N)}(t) \ &= \ -\kappa p\left(x^{(N)}(\frac{\lfloor N(t-d)\rfloor)}{N} + a + e^{(N)}(\frac{\lfloor (t-d)\rfloor}{N})\right)[y_i^{(N)}(t)r_{ij}^{(N)}(t-d) + y_j^{(N)}(t-d)r_{ij}^{(N)}(t)] \\ &\quad + g_i(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)}) - g_j(y_t^{(N)}, y_{\frac{\lfloor Nt\rfloor}{N}}^{(N)})\end{aligned} \tag{14}$$

We also remind the reader that for all N, the initial conditions are assumed to satisfy the conditions given in Assumption 2. We have the following result.

Theorem 5.1. *Suppose the noise process satisfies (12). If*

$$\kappa d < \min\left[\beta,\ \frac{1}{6M_\beta p(M_\beta + a)}\right] ,$$

then, given $\epsilon' > 0$, $\exists(\bar{t}(\epsilon'),\ \overline{N})$ *such that* $\forall(N \geq \overline{N})$,

$$\sup_{t\in[\bar{t},\infty)} (r_{ij}^{(N)}(t))^2 < \epsilon' \quad a.s. \quad .$$

Proof. Without loss of generality, we assume that $\epsilon' < M_\beta \epsilon$. (Recall that the initial condition is such that $|r_{ij}^{(N)}| < 2M_\beta\epsilon$.) The proof of this result is very much similar to the proof of Theorem 4.1. First, we outline the key differences from *Step 1* of the proof of Theorem 4.1.

As in *Step 1* of the proof of Theorem 4.1, first let us consider $t \leq t_1$ where

$$t_1 = \inf_{t>0}\{t : \max_{i\in\{1,2,\ldots,N\}} [y_i^{(N)}(t)] \geq 3M\} .$$

Clearly, there exists $\delta > 0$ such that,

$$\kappa d < \min\left[\beta,\ \frac{1}{6(M_\beta+\delta)p(M_\beta+\delta+a)}\right] . \tag{15}$$

Since $x^{(N)}(s) \leq M_\beta$ for $s \in [-d, 0]$, we have from Lemma 3.1 that, $\exists\ N_1$ such that $\forall\ N \geq N_1$,

$$\sup_{t\in[0,\infty)} x^{(N)}(t) \leq M_\beta + \frac{\delta}{2} .$$

Further, $\exists\ N_2$ such that $\forall\ N \geq N_2$,

$$\sup_{t\in[0,\infty)} |e^{(N)}(t)| < \delta/2 .$$

Let $\overline{N} = \max\{N_1,\ N_2\}$. Note that for $t \in [0, \infty)$, the following is true

$$\kappa d < \frac{1}{6(M_\beta + \frac{\delta}{2})p(M_\beta + \frac{\delta}{2} + a + e^{(N)}(\frac{\lfloor N(t-d)\rfloor}{N}))} . \tag{16}$$

whenever (15) holds. We first look at the following slightly different system given by,

$$\dot{\hat{r}}_{ij}^{(N)}(t) \quad = \quad -\kappa p\left(x(\frac{\lfloor(t-d)\rfloor}{N}) + a + e^{(N)}(\frac{\lfloor(t-d)\rfloor}{N})\right)[y_i^{(N)}(t)\hat{r}_{ij}^{(N)}(t-d) + y_j^{(N)}(t-d)\hat{r}_{ij}^{(N)}(t)],$$

for which, it can be seen from Theorem 4.1 that $\sup_{t-d\leq s\leq t}|\hat{r}_{ij}(t)|^2 < q|\hat{r}_{ij}(0)|^2$ for $t < t_1$ provided (16) holds. This can be proved using Razumikhin's theorem by showing that, under (16), if

$$q(\hat{r}_{ij}^{(N)}(t))^2 > \max_{s\in[t-2d,t]} (\hat{r}_{ij}^{(N)}(t))^2 ,$$

then,

$$\frac{d}{dt}(\hat{r}_{ij}^{(N)}(t))^2 \leq -K_1(\hat{r}_{ij}^{(N)}(t))^2 \ ,$$

where $q > 1, \ K_1 > 0$.

Redoing a similar calculation for the system given by (14) and for $t < t_1$, we can show that, whenever,

$$q(r_{ij}^{(N)}(t))^2 > \max_{s\in[t-2d,t]}(r_{ij}^{(N)}(t))^2 \ ,$$

then,

$$\frac{d}{dt}(r_{ij}^{(N)}(t))^2 \leq -K_1(r_{ij}^{(N)}(t))^2 + \frac{12M}{N}|h_i(3M,3M,3M) + h_j(3M,3M,3M)| \ .$$

The factor $12M$ accounts for the fact that, for $t < t_1$, $|r_{ij}^{(N)}(t)| < 6M$. Choose N large enough so that $12M|h_i(3M,3M,3M) + h_j(3M,3M,3M)|/N < K_1\epsilon'^2/2$. It follows that

$$\frac{d}{dt}(r_{ij}^{(N)}(t))^2 < -\frac{K_1\epsilon'^2}{2}.$$

whenever,

$$q(r_{ij}^{(N)}(t))^2 > \max_{s\in[t-2d,t]}(r_{ij}^{(N)}(t))^2 \ , \quad (r_{ij}^{(N)}(t))^2 > \epsilon'^2, \quad t < t_1 \ .$$

Using an argument similar to *Step 2* of Theorem 4.1, we can show that t_1 is unbounded. The result follows from Razumikhin's Theorem for ultimate boundedness. $\square$

Note that we immediately have from the preceding result that $y_i^{(N)}(t) < M_\beta + \epsilon'$ for all $t > \bar{t}$ and N large enough. By doing calculations similar to that in (8), the update for the i-th flow can be written as,

$$\dot{y}_i^{(N)} \quad = \quad \kappa[w - y_i^{(N)}(t)y_i^{(N)}(t-d)p(y_i^{(N)}(t-d)+a)] + \delta_{kN}(t), \qquad k \in \{1,2,\ldots,N\} \qquad (17)$$

where, given $\epsilon' > 0$, $\exists\overline{N}(\epsilon')$, and $\bar{t}(\epsilon')$ such that

$$\sup_{t\in[\bar{t},\infty)} |\delta_{kN}(t)| < \epsilon', \quad \forall N \geq \overline{N} \ .$$

We now have a result similar to Theorem 3.1.

Lemma 5.1. *Suppose*

$$\kappa d < \min\left[\beta, \ R, \ \frac{1}{6M_\beta p(M_\beta + a)}\right] \ ,$$

If the noise process satisfies (12), then, under Assumption 1 and Assumption 2, given $\epsilon' > 0$, $\exists(\bar{t}(\epsilon'), \ \overline{N})$ *such that* $\forall(N \geq N')$,

$$\sup_{t\in[\bar{t},\infty)} |y_k^{(N)}(t) - y^*| \leq \epsilon' \ , \quad a.s. \quad k \in \{1,2,\ldots N\} \ ,$$

where y^* *is the solution of* $w = y^2p(y+a)$.

Proof. As in the proof of Theorem 4.2, consider the Lyapunov functional given by (11) for the system given by (17). It can be seen that

$$\dot{V} \leq -\gamma V + c_4 \delta_N(t) \ ,$$

where, $\lim_{N\to\infty} \lim_{t\to\infty} |\delta_N(t)| = 0$. The result is easy to show from this by considering all times $t > \bar{t}(\epsilon')$ and sufficiently large N. □

We now prove the main result of the paper, which follows from the preceding lemma along with minor additional arguments.

Proof of Theorem 3.1:

Fix a typical sample path $e^{(N)}(t)$ also fix ϵ'. Consider a process ${e'}^{(N)}(t)$ satisfying

$$\sup_{t\in[0,\infty)} {e'}^{(N)}(t) \to 0$$

and

$${e'}^{(N)}(t) = e^{(N)}(t), \quad \forall (t \leq NT) \ .$$

Note that the update of the i-th flow can be written as,

$$\begin{aligned} \dot{y}_i^{(N)}(t) &= \kappa \left[w - y_i^{(N)}(\frac{\lfloor Nt \rfloor}{N}) y_i^{(N)}(\frac{\lfloor N(t-d) \rfloor}{N}) p \left(x^{(N)}(\frac{\lfloor N(t-d) \rfloor}{N}) + a + {e'}^{(N)}(\frac{\lfloor N(t-d) \rfloor}{N}) \right) \right] \\ &\quad + K(t) |{e'}^{(N)}(\frac{\lfloor N(t-d) \rfloor}{N}) - e^{(N)}(\frac{\lfloor N(t-d) \rfloor}{N})| \ , \end{aligned}$$

where $K(t)$ is a bounded process. The statement of the theorem thus follows from Lemma 5.1, since the $y_i^{(N)}(t)$ process behaves as if the noise process is ${e'}^{(N)}(t)$ for $t \leq NT$. □

6 Stability Conditions with Standard TCP

In previous sections, we have seen that that stability condition with a single TCP connection accessing a link along with some additional condition is enough to ensure the stability and the convergence of multiple TCP-like flows in a many flows regime.

In this section, we use the earlier results to derive stability conditions with the standard TCP parameters. Since the global stability criterion of a single TCP-flow accessing a bottleneck link plays an important role in the many flows regime, we first state the stability condition with a single TCP flow with parameters implied by standard TCP. Recall that the congestion avoidance phase of TCP can be modeled as (Kunniyur & Srikant 2000),

$$\dot{x}(t) = \frac{1}{d^2} - \frac{2}{3} x(t) x(t-d) p(x(t-d) + a, c) \ , \tag{18}$$

where x is in segments (each segment may correspond to 512 bytes) per time unit and a is the mean flow rate due to the uncontrolled flows. We define $w(t)$ as

$$w(t) = dx(t)$$

The evolution of the congestion-avoidance phase of TCP can be re-written as,

$$\dot{w}(t) = \frac{2}{3d}\left[\frac{3}{2} - w(t)w(t-d)p\left(\frac{w(t-d)+ad}{cd}, 1\right)\right] . \tag{19}$$

Since $x(t)$ is the rate in segments per unit time and d is the round trip time, $w(t)$ can be interpreted as the congestion window size in segments. Further, the quantity cd determines the desired bandwidth-delay product per flow at the equilibrium. Thus, (19) is a continuous time version of the window update algorithm of TCP algorithm in the congestion avoidance phase.

The following lemma provides sufficient conditions under which (19) is globally exponentially stable. The condition follows from the results derived in (Deb & Srikant 2002). We state the stability condition of (19) below, which follows from Theorem 2.2 in (Deb & Srikant 2002). For the the rest of this section, we let

$$q(w) = p\left(\frac{w}{cd}, 1\right).$$

Lemma 6.1. *The controller given by (19) is globally exponentially stable if,*

$$\frac{2}{3} < \frac{l^2(\frac{3}{2} + M^2R)}{M^3R(\frac{3}{2} + l^2R)} ,$$

where, $R = \sup_{l \le w \le M}(q(w) + wq'(w))$, M *is the smallest positive number satisfying*

$$M\,(M-1)\,p\,(M-1+ad) \ge \frac{3}{2} ,$$

and l *is the largest positive number satisfying*

$$l\left(l + \frac{2}{3}M^2q(M+ad) - 1\right) q\left(l + \frac{2}{3}M^2q(M+ad) - 1 + ad\right) \;\; \le \;\; \frac{3}{2} \qquad \square$$

Now we consider multiple TCP-flows. We further consider only the mean flow rate of the uncontrolled flows, since the stability of such a system is sufficient to ensure the stability of the system with stochastic disturbances when the number of flows is large. Let there be N flows, with the update equation of the i-th flow described by

$$\dot{w}_i(t) \;\; = \;\; \frac{2}{3d}\left[\frac{3}{2} - w_i(t)w_i(t-d)q(w(t-d)+ad)\right] , \quad i \in \{1, 2, \ldots N\} , \tag{20}$$

where $w(t)$ is the average window size ($w(t) = dx(t)$ where $x(t)$ is the average flow rate) of the N flows. Using Theorem 4.2, we can easily derive stability conditions for the system described by (20). The following result provides such conditions.

Corollary 6.1. *The system given by (20) is globally stable if the following conditions hold:*

1. *We have* $\frac{2}{3}M_1 q(M_1+a) < \frac{1}{6}$*, where* M_1 *is the smallest positive number satisfying*

$$(M_1-1)^2\, q\,(M_1+ad-2)\left(1-\frac{1}{M_1-1}\right) \geq \frac{3}{2}\ .$$

2. *The condition given by Lemma 6.1 is satisfied.* □

We next present examples for two different marking functions to demonstrate the usefulness of the previous result. We are interested in finding the range of the equilibrium bandwidth-delay product to ensure stability and convergence of multiple TCP flows. We now demonstrate that, for the examples considered, we can ensure global stability if the bandwidth-delay product is large enough.

Example 1 ($M/M/1$ type marking function):

We consider the marking function

$$p(x,c) = \left(\frac{x}{c}\right)^B = \left(\frac{w}{cd}\right)^B\ . \tag{21}$$

Here x is the average flow rate of the flows through the link and c is a parameter which can be adjusted for a desired bandwidth per flow at the equilibrium. Such a marking function has the interpretation of probability of the buffer size being larger than B in an $M/M/1$ queue with arrival rate x. The equilibrium rate per flow x^* in this case is given by

$$x^*d = \left(\frac{3}{2}\right)^{\frac{1}{B+2}}(cd)^{\frac{B}{B+2}}\ .$$

We are interested in finding values of equilibrium bandwidth-delay product x^*d to guarantee global stability of multiple TCP flows.

We show that the conditions given by Corollary (6.1) are satisfied as $x^*d \to \infty$. To make our calculations easy, we assume that $(cd)^{B/(B+2)} > 5$ for reasons which will become obvious soon. In other words, we seek values of x^*d in the range $[5(3/2)^{1/(B+2)}, \infty)$ to ensure global stability of multiple identical TCP-like flows. For the given marking function, first note that we have for M given in Lemma 6.1

$$\begin{aligned} M &< 1+(cd)^{\frac{B}{B+2}}(3/2)^{\frac{1}{B+2}} \\ &< (cd)^{\frac{B}{B+2}}\left[\frac{1}{5}+\left(\frac{3}{2}\right)^{\frac{1}{B+2}}\right] = K(cd)^{\frac{B}{B+2}}\ , \end{aligned}$$

where

$$K = \frac{1}{5} + \left(\frac{3}{2}\right)^{\frac{1}{B+2}} \quad .$$

It can also be easily seen that

$$M^2 q(M) < K^{B+2} \ .$$

Also, for l given in Lemma 6.1, we have

$$\begin{aligned} l &> 1 + (cd)^{\frac{B}{B+2}} (3/2)^{\frac{1}{B+2}} - \frac{2}{3} M^2 p(M) \\ &> (cd)^{\frac{B}{B+2}} \left[\left(\frac{3}{2}\right)^{\frac{1}{B+2}} - \frac{1}{5}\left(\frac{2}{3}K^{B+2} - 1\right) \right] \ . \end{aligned}$$

It follows from some algebraic manipulations that Condition 2 in Corollary 6.1 is satisfied when

$$M\left(\frac{3}{2l^2} + R\right) < \frac{3}{2} \ ,$$

which is satisfied if

$$(cd)^{\frac{B}{B+2}} \geq \frac{2}{3} K^{B+1}(1+B) + \frac{K}{\left(\left(\frac{3}{2}\right)^{\frac{1}{B+2}} - \frac{1}{5}\left(1 - \frac{2}{3}K^{B+2}\right)\right)^2} \ .$$

Next note that M_1, the upper bound on the average rate in condition 1 of Corollary 6.1, is such that

$$M_1 < \left(\frac{3}{2}\right)^{\frac{1}{B+2}} (cd)^{\frac{B}{B+2}} + 2 < K_1 (cd)^{\frac{B}{B+2}} \ ,$$

where

$$K_1 = \frac{2}{5} + \left(\frac{3}{2}\right)^{\frac{1}{B+2}} \quad .$$

One can check that a sufficient condition for Condition 1 in Corollary 6.1 to be satisfied is

$$(cd)^{\frac{B}{B+2}} \geq 4K_1^{B+1} \ .$$

Since $x^* d = (3/2)^{1/(B+2)}(cd)^{b/(B+2)}$, a sufficient global stability criterion for multiple TCP-like flows is

$$x^* d \ \geq \ \left(\frac{3}{2}\right)^{\frac{1}{B+2}} \max \left[\frac{2}{3} K^{B+1}(1+B) + \frac{K}{\left(\left(\frac{3}{2}\right)^{\frac{1}{B+2}} - \frac{1}{5}\left(1 - \frac{2}{3}K^{B+2}\right)\right)^2}, \ 4K_1^{B+1}, \ 5 \right] \ .$$

It follows that the system is globally stable as $x^* d \to \infty$.

With a more accurate numerical calculation based on Corollary 6.1, it can be verified that for $B = 8$, the system is globally stable for

$$x^* d \geq 16.66 \ ,$$

which corresponds to at least 8 packets per flow at the equilibrium since each packet approximately consists of 2 segments. For $B = 5$ a sufficient condition is,

$$x^*d \geq 13.43$$

corresponding to at least 7 packets per flow at the equilibrium.

One can obtain similar results for the marking function

$$p(x) = \frac{\left(\frac{x}{c}\right)^B \left(1 - \left(\frac{x}{c}\right)\right)}{1 - \left(\frac{x}{c}\right)^{B+1}} \ ,$$

which can viewed as the blocking probability in an $M/M/1/B$ queue with arrival rate x and service rate c. A sufficient condition with $B = 8$ is

$$x^*d \geq 14.97 \ ,$$

and, with $B = 5$,

$$x^*d \geq 12.44$$

provides a sufficient condition for global stability.

Example 2 (Random Early Marking or REM):

Next, we consider the following marking function:

$$p(x,c) = \frac{\theta\sigma^2 x}{\theta\sigma^2 x + 2(c-x)}, \tag{22}$$

where σ^2 denotes the variability of the traffic and c can be tuned to obtain a desired rate allocation at equilibrium. For our purposes, we can simply assume that marking takes place at a rate given by (22). However, according to (Kelly 2000), under appropriate conditions, (22) can be viewed as an approximation to a well-known marking mechanism called REM (Athuraliya, Lapsley & Low 1999).

In this exampley, we take $\theta\sigma^2 = 0.5$. The equilibrium rate allocation can be obtained by solving

$$\frac{0.5(x^*d)^3}{0.5x^*d + 2(cd - x^*d)} = \frac{3}{2} \ .$$

It can be verified that

$$x^*d \leq (6cd)^{\frac{1}{3}} \ .$$

First we argue that the global stability condition is satisfied as $x^*d \to \infty$. Suppose we consider cd in the range $cd \geq 5$, which corresponds to $x^*d \geq 2.63$. We are interested in finding x^*d in the range $[2.37, \infty)$ to guarantee stability. Note that the parameter M in Lemma 6.1 satisfies

$$K_1(cd)^{\frac{1}{3}} \leq M \leq K_2(cd)^{\frac{1}{3}} ,$$

for suitable constants K_1 and K_2. Further, it can be shown that

$$l \geq K_3(cd)^{\frac{1}{3}}$$

and

$$R = \sup_{l \leq w \leq M} q(w) + wq'(w)) \leq K_4(cd)^{-\frac{2}{3}}$$

for appropriate positive constants K_3 and K_4. It follows that Condition 2 in Corollary 6.1 is satisfied when

$$M\left(\frac{3}{2l^2} + R\right) < \frac{3}{2} ,$$

which is satisfied if

$$(cd)^{\frac{1}{3}} \geq \frac{2}{3}K_2\left(\frac{1}{K_3} + K_4\right) .$$

Similarly, condition 1 is Corollary 6.1 can be expressed as

$$(cd)^{\frac{1}{3}} \geq K_5 .$$

Since $x^*d \leq (6cd)^{\frac{1}{3}}$, a sufficient condition for global stability in this case is

$$x^*d \geq K$$

for a suitable constant K.

One can use numerical calculations to obtain the a sufficient condition for global stability with multiple TCP flows as

$$x^*d \geq 8.67 .$$

This corresponds to at least 5 packets per flow at the equilibrium with packet sizes of 2 segments.

The above examples clearly indicate that:

For reasonable marking functions and large enough target bandwidth-delay product per flow, multiple TCP flows eventually behave like a single flow and the system is globally asymptotically stable.

7 Conclusions

We have studied a system consisting of a single link accessed by a large number of *TCP-like* flows, each with identical delay access, but with (possibly) different initial condition and also accessed by a large number of uncontrolled flows. The contributions of this paper are:

(i) Our main result is that, in the presence of uncontrolled flows (stochastic noise), if the number of flows is large enough, the global exponential stability criterion for a single flow (with minor modifications) is also a *global stability* condition for the stochastic system with multiple flows. Thus, the implication is that *parameter design can be carried out using deterministic analysis based on the single flow model.*

(ii) For the rate adaptation model of TCP (Kunniyur & Srikant 2000), we have shown that the stability is ensured if the target equilibrium delay-bandwidth product (window size) per flow is large enough, and we have derived bounds on this quantity. Thus, we have derived sufficient conditions for global stability. Numerical examples with two popular marking functions indicate that the target window size per flow required to ensure stability is not very large.

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