STABILITY OF GORENSTEIN OBJECTS IN TRIANGULATED CATEGORIES

ZHANPING WANG   CHUNLI LIANG

Abstract Let $\mathcal{C}$ be a triangulated category with a proper class ξ of triangles. Asadollahi and Salarian introduced and studied ξ-Gorenstein projective and ξ-Gorenstein injective objects, and developed Gorenstein homological algebra in $\mathcal{C}$. In this paper, we further study Gorenstein homological properties for a triangulated category. First, we discuss the stability of ξ-Gorenstein projective objects, and show that the subcategory $\mathcal{GP}(\xi)$ of all ξ-Gorenstein projective objects has a strong stability. That is, an iteration of the procedure used to define the ξ-Gorenstein projective objects yields exactly the ξ-Gorenstein projective objects. Second, we give some equivalent characterizations for ξ-Gorenstein projective dimension of object in $\mathcal{C}$.

2010 **Mathematics Subject Classification**: 16E05, 16E10, 18G35
Key words: stability; ξ-Gorenstein projective object; triangulated category

1. Introduction

Triangulated categories were introduced independently in algebraic geometry by Verdier in his thése [21], and in algebraic topology by Puppe [19] in the early sixies, which have by now become indispensable in many areas of mathematics such as algebraic geometry, stable homotopy theory and representation theory [5, 11, 17]. The basic properties of triangulated categories can be found in Neeman's book [18].

Let $\mathcal{C}$ be a triangulated category with triangles Δ. Beligiannis [6] developed homological algebra in $\mathcal{C}$ which parallels the homological algebra in an exact category in the sense of Quillen. By specifying a class of triangles $\xi \subseteq \Delta$, which is called a proper class of triangles, he introduced ξ-projective objects, ξ-projective dimensions and their duals.

Auslander and Bridger generalized in [1] finitely generated projective modules to finitely generated modules of Gorenstein dimension zero over commutative noetherian rings. Furthermore, Enochs and Jenda introduced in [9] Gorenstein projective modules for arbitrary modules over a general ring, which is a generalization of finitely generated modules of Gorenstein dimension zero, and dually they defined Gorenstein injective modules. Gorenstein homological algebra has been extensively studied by many authors, see for example [2, 8, 10, 13].

As a natural generalization of modules of Gorenstein dimension zero, Beligiannis [7] defined the concept of an $\mathcal{X}$-Gorenstein object in an additive category $\mathcal{C}$ for a contravariantly finite subcategory $\mathcal{X}$ of $\mathcal{C}$ such that any $\mathcal{X}$-epic has kernal in $\mathcal{C}$. In an attempt to extend the theory, Asadollahi and Salarian [3] introduced and studied ξ-Gorenstein projective and ξ-Gorenstein injective objects, and then ξ-Gorenstein projective and ξ-Gorenstein injective dimensions of objects in a triangulated

Address correspondence to Zhanping Wang, Department of Mathematics, Northwest Normal University, Lanzhou 730070, P.R. China.

E-mail: wangzp@nwnu.edu.cn (Z.P. Wang).

category which are defined by modifying what Enochs and Jenda have done in an abelian category [10].

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ an additive full subcategory of $\mathcal{A}$. Sather-Wagstaff, Sharif and White introduced in [20] the Gorenstein category $\mathcal{G}(\mathcal{X})$, which is defined as $\mathcal{G}(\mathcal{X}) = \{A$ is an object in $\mathcal{A}$ | there exists an exact sequence $\cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$ in $\mathcal{X}$, which is both $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$ -exact, such that $A \cong \mathrm{Im}(X_0 \to X^0)\}$. Set $\mathcal{G}^0(\mathcal{X}) = \mathcal{X}$, $\mathcal{G}^1(\mathcal{X}) = \mathcal{G}(\mathcal{X})$, and inductively set $\mathcal{G}^{n+1}(\mathcal{X}) = \mathcal{G}^n(\mathcal{G}(\mathcal{X}))$ for any $n \geq 1$. They proved that when $\mathcal{X}$ is self-orthogonal, $\mathcal{G}^n(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ for any $n \geq 1$; and they proposed the question whether $\mathcal{G}^2(\mathcal{X}) = \mathcal{G}(\mathcal{X})$ holds for an arbitrary subcategory $\mathcal{X}$. See [20, 4.10, 5.8]. Recently, Huang [15] proved that the answer to this question is affirmative. This shows that $\mathcal{G}(\mathcal{X})$, in particular the subcategory $\mathcal{GP}(\mathcal{A})$ of all Gorenstein projective objects, has a strong stability. Kong and Zhang give a slight generalization of this stability by a different method [16].

Inspired by the above results, we consider the stability of the subcategory $\mathcal{GP}(\xi)$ of all ξ-Gorenstein projective objects, which is introduced by Asadollahi and Salarian [3]. Set $\mathcal{G}^0\mathcal{P}(\xi) = \mathcal{P}(\xi)$, $\mathcal{G}^1\mathcal{P}(\xi) = \mathcal{GP}(\xi)$, and inductively set $\mathcal{G}^{n+1}\mathcal{P}(\xi) = \mathcal{G}^n(\mathcal{GP}(\xi))$ for any $n \geq 1$. A natural question is whether $\mathcal{G}^n\mathcal{P}(\xi) = \mathcal{GP}(\xi)$.

In Section 2, we give some terminologies and some preliminary results. Section 3 is devoted to answer the above question. We will prove the following theorem.

Main theorem Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, where ξ is a proper class of triangles. Then $\mathcal{G}^n\mathcal{P}(\xi) = \mathcal{GP}(\xi)$ for any $n \geq 1$.

The above theorem shows that the subcategory $\mathcal{GP}(\xi)$ of all ξ-Gorenstein projective objects has a strong stability. That is, an iteration of the procedure used to define the ξ-Gorenstein projective objects yields exactly the ξ-Gorenstein projective objects. Finally, we give some equivalent characterizations for ξ-Gorenstein projective dimension of an object A in $\mathcal{C}$.

2. Some basic facts in triangulated categories

This section is devoted to recall the definitions and elementary properties of triangulated categories used throughout the paper. For the terminology we shall follow [3, 4, 6].

Let $\mathcal{C}$ be an additive category and $\Sigma : \mathcal{C} \to \mathcal{C}$ be an additive functor. Let $\mathrm{Diag}(\mathcal{C}, \Sigma)$ denote the category whose objects are diagrams in $\mathcal{C}$ of the form $A \to B \to C \to \Sigma A$, and morphisms between two objects $A_i \to B_i \to C_i \to \Sigma A_i$, $i = 1, 2$, are a triple of morphisms $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ and $\gamma : C_1 \to C_2$, such that the following diagram commutes:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \xrightarrow{h_1} & \Sigma A_1 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\ A_2 & \xrightarrow{f_2} & B_2 & \xrightarrow{g_2} & C_2 & \xrightarrow{h_2} & \Sigma A_2. \end{array}$$

A triangulated category is a triple $(\mathcal{C}, \Sigma, \Delta)$, where $\mathcal{C}$ is an additive category, Σ is an autoequivalence of $\mathcal{C}$ and Δ is a full subcategory of $\mathrm{Diag}(\mathcal{C}, \Sigma)$ which satisfies the following axioms. The elements of Δ are then called triangles.

(Tr1) Every diagram isomorphic to a triangle is a triangle. Every morphism $f : A \to B$ in $\mathcal{C}$ can be embedded into a triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A$. For any object $A \in \mathcal{C}$, the diagram $A \overset{1_A}{\to} A \to 0 \to \Sigma A$ is a triangle, where 1_A denotes the identity morphism from A to A.

(Tr2) $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A$ is a triangle if and only if $B \overset{g}{\to} C \overset{h}{\to} \Sigma A \overset{-\Sigma f}{\to} \Sigma B$.

(Tr3) Given triangles $A_i \overset{f_i}{\to} B_i \overset{g_i}{\to} C_i \overset{h_i}{\to} \Sigma A_i$, $i = 1, 2$, and morphism $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ such that $f_2\alpha = \beta f_1$, there exists a morphism $\gamma : C_1 \to C_2$ such that (α, β, γ) is a morphism from the first triangle to the second.

(Tr4) (The Octahedral Axiom) Given triangles

$$A \overset{f}{\to} B \overset{i}{\to} C' \overset{i'}{\to} \Sigma A,\ \ B \overset{g}{\to} C \overset{j}{\to} A' \overset{j'}{\to} \Sigma B,\ \ A \overset{gf}{\to} C \overset{k}{\to} B' \overset{k'}{\to} \Sigma A,$$

there exist morphisms $f' : C' \to B'$ and $g' : B' \to A'$ such that the following diagram commutes and the third column is a triangle:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{i} & C' & \xrightarrow{i'} & \Sigma A \\
\| & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle f'} & & \| \\
A & \xrightarrow{gf} & C & \xrightarrow{k} & B' & \xrightarrow{k'} & \Sigma A \\
 & & \downarrow{\scriptstyle j} & & \vdots{\scriptstyle g'} & & \downarrow{\scriptstyle \Sigma f} \\
 & & A' & = & A' & \xrightarrow{j'} & \Sigma B \\
 & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j'\Sigma i} & & \\
 & & \Sigma B & \xrightarrow{\Sigma i} & \Sigma C' & &
\end{array}$$

Some equivalent formulations for the Octahedral Axiom (Tr4) are given in [6, 2.1], which are more convient to use. If $\mathcal{C} = (\mathcal{C}, \Sigma, \Delta)$ satisfies all the axioms of a triangulated category except possible of (Tr4), then (Tr4) is equivalent to each of the following:

Base Change: For any triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \Delta$ and morphism $\varepsilon : E \to C$, there exists a commutative diagram:

$$\begin{array}{ccccccc}
 & & M & = & M & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \\
A & \xrightarrow{f'} & G & \xrightarrow{g'} & E & \xrightarrow{h'} & \Sigma A \\
\| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \varepsilon} & & \| \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \Sigma f'} \\
 & & \Sigma M & = & \Sigma M & \xrightarrow{-\Sigma\alpha} & \Sigma G
\end{array}$$

in which all horizontal and vertical diagrams are triangles in Δ.

Cobase Change: For any triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \Delta$ and morphism $\alpha : A \to D$, there exists a commutative diagram:

$$\begin{array}{ccccccc}
 & & N & = & N & & \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \delta} & & \\
\Sigma^{-1}C & \xrightarrow{-\Sigma^{-1}h} & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\| & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \| \\
\Sigma^{-1}C & \xrightarrow[-\Sigma^{-1}h']{} & D & \xrightarrow[f']{} & F & \xrightarrow[g']{} & C \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \upsilon} & & \downarrow{\scriptstyle -h} \\
 & & \Sigma N & = & \Sigma N & \xrightarrow[-\Sigma\eta]{} & \Sigma A
\end{array}$$

in which all horizontal and vertical diagrams are triangles in Δ.

The following definitions are quoted verbatim from [6, Section 2]. A class of triangles ξ is closed under base change if for any triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \xi$ and any morphism $\varepsilon : E \to C$ as in the base change diagram above, the triangle $A \overset{f'}{\to} G \overset{g'}{\to} E \overset{h'}{\to} \Sigma A$ belongs to ξ. Dually, a class of triangles ξ is closed under cobase change if for any triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \xi$ and any morphism $\alpha : A \to D$ as in the cobase change diagram above, the triangle $D \overset{f'}{\to} F \overset{g'}{\to} C \overset{h'}{\to} \Sigma D$ belongs to ξ. A class of triangles ξ is closed under suspension if for any triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \xi$ and $i \in \mathbb{Z}$, the triangle $\Sigma^i A \xrightarrow{(-1)^i \Sigma^i f} \Sigma^i B \xrightarrow{(-1)^i \Sigma^i g} \Sigma^i C \xrightarrow{(-1)^i \Sigma^i h} \Sigma^{i+1} A$ is in ξ. A class of triangles ξ is called saturated if in the situation of base change, whenever the third vertical and the second horizontal triangles are in ξ, then the triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A$ is in ξ. An easy consequence of the octahedral axiom is that ξ is saturated if and only if in the situation of cobase change, whenever the second vertical and the third horizontal triangles are in ξ, then the triangle $A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A$ is in ξ.

A triangle $(T) : A \overset{f}{\to} B \overset{g}{\to} C \overset{h}{\to} \Sigma A \in \Delta$ is called split if it is isomorphic to the triangle $A \xrightarrow{(1\ 0)} A \oplus C \xrightarrow{\binom{0}{1}} C \overset{0}{\to} \Sigma A$. It is easy to see that (T) is split if and only if f is a section or g is a retraction or $h = 0$. The full subcategory of Δ consisting of the split triangles will be denoted by Δ_0.

Definition 2.1. ([6, 2.2]) Let $\mathcal{C} = (\mathcal{C}, \Sigma, \Delta)$ be a triangulated category. A class $\xi \subseteq \Delta$ is called a proper class of triangles if the following conditions hold.

(1) ξ is closed under isomorphisms, finite coproducts and $\Delta_0 \subseteq \xi \subseteq \Delta$.

(2) ξ is closed under suspensions and is saturated.

(3) ξ is closed under base and cobase change.

It is known that Δ_0 and the class of all triangles Δ in $\mathcal{C}$ are proper classes of triangles. There are more interesting examples of proper classes of triangles enumerated in [6, Example 2.3].

Throughout we fix a proper class ξ of triangles in the triangulated category $\mathcal{C}$.

Definition 2.2. ([6, 4.1]) An object $P \in \mathcal{C}$ (resp., $I \in \mathcal{C}$) is called ξ-projective (resp., ξ-injective) if for any triangle $A \to B \to C \to \Sigma A$ in ξ, the induced sequence of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(P, A) \to \mathrm{Hom}_{\mathcal{C}}(P, B) \to \mathrm{Hom}_{\mathcal{C}}(P, C) \to 0$ (resp., $0 \to \mathrm{Hom}_{\mathcal{C}}(C, I) \to \mathrm{Hom}_{\mathcal{C}}(B, I) \to \mathrm{Hom}_{\mathcal{C}}(A, I) \to 0$) is exact.

It follows easily from the definition that the subcategory $\mathcal{P}(\xi)$ of ξ-projective objects and the subcategory $\mathcal{I}(\xi)$ of ξ-injective objects are full, additive, Σ-stable, and closed under isomorphisms and direct summands. The category $\mathcal{C}$ is said to have enough ξ-projectives (resp., ξ-injectives) if for any object $A \in \mathcal{C}$, there exists a triangle $K \to P \to A \to \Sigma K$ (resp., $A \to I \to L \to \Sigma A$) in ξ with $P \in \mathcal{P}(\xi)$ (resp., $I \in \mathcal{I}(\xi)$). However, it is not so easy to find a proper class ξ of triangles in a triangulated category having enough ξ-projectives or ξ-injectives in general. Here we present a nontrivial example due to Beligiannis, and also, see more nontrivial examples in [6, Sections 12.4 and 12.5], which are of great interest. Take $\mathcal{C}$ to be the unbounded homotopy category of complexes of objects from a Grothendieck category which has enough projectives. Then the so-called Cartan-Eilenberg projective and injective complexes form the relative projective and injective objects for a proper class of triangles in $\mathcal{C}$.

The following lemma is quoted from [6, 4.2].

Lemma 2.3. *Let $\mathcal{C}$ have enough ξ-projectives. Then a triangle $A \to B \to C \to \Sigma A$ is in ξ if and only if for all $P \in \mathcal{P}(\xi)$ the induced sequence $0 \to \mathrm{Hom}_{\mathcal{C}}(P, A) \to \mathrm{Hom}_{\mathcal{C}}(P, B) \to \mathrm{Hom}_{\mathcal{C}}(P, C) \to 0$ is exact.*

Recall that an ξ-exact sequence X is a diagram

$$\cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \longrightarrow \cdots$$

in $\mathcal{C}$, such that for each $n \in \mathbb{Z}$, there exists triangle $K_{n+1} \xrightarrow{f_n} X_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$ in ξ and the differential is defined as $d_n = f_{n-1} g_n$ for any n.

Let $\mathcal{X}$ be a full and additive subcategory of $\mathcal{C}$. A triangle $A \to B \to C \to \Sigma A$ in ξ is called $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{X})$-exact (resp., $\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}, -)$-exact), if for any $X \in \mathcal{X}$, the induced sequence of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X) \to 0$ (resp., $0 \to \mathrm{Hom}_{\mathcal{C}}(X, A) \to \mathrm{Hom}_{\mathcal{C}}(X, B) \to \mathrm{Hom}_{\mathcal{C}}(X, C) \to 0$) is exact.

A complete ξ-projective resolution is a diagram

$$\mathbf{P} : \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \to \cdots$$

in $\mathcal{C}$ such that for any integer n, $P_n \in \mathcal{P}(\xi)$ and there exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact triangles

$$K_{n+1} \xrightarrow{f_n} P_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in ξ and the differential is defined as $d_n = f_{n-1} g_n$ for any n.

Definition 2.4. (see [3]) Let $\mathbf{P}$ be a complete ξ-projective resolution in $\mathcal{C}$. So for any integer n, there exist triangles

$$K_{n+1} \xrightarrow{f_n} P_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in ξ. The objects K_n for any integer n, are called ξ-Gorenstein projective (ξ-$\mathcal{G}$projective for short).

Dually, one can define complete ξ-injective resolution and ξ-Gorenstein injective (ξ-$\mathcal{G}$injective for short) objects.

We denote by $\mathcal{GP}(\xi)$ and $\mathcal{GI}(\xi)$ the subcategory of ξ-$\mathcal{G}$projective and ξ-$\mathcal{G}$injective objects of $\mathcal{C}$ respectively. It is obvious that $\mathcal{P}(\xi) \subseteq \mathcal{GP}(\xi)$ and $\mathcal{I}(\xi) \subseteq \mathcal{GI}(\xi)$; $\mathcal{GP}(\xi)$ and $\mathcal{GI}(\xi)$ are full, additive, Σ-stable, and closed under isomorphisms and direct summands.

3. Main results

Throughout the paper, $\mathcal{C}$ is a triangulated category with enough ξ-projectives and enough ξ-injectives, where ξ is a fixed proper class of triangles.

Recall that in [3] a diagram

$$\mathbf{P} : \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \to \cdots$$

in $\mathcal{C}$ is a complete ξ-projective resolution, if for any n, $P_n \in \mathcal{P}(\xi)$, and there exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact triangles

$$K_{n+1} \xrightarrow{f_n} P_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in ξ and the differential is defined as $d_n = f_{n-1}g_n$ for any n. That is, $\mathbf{P}$ is an ξ-exact sequence of ξ-projective objects, and is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. The objects K_n for any integer n, are called ξ-Gorenstein projective (ξ-$\mathcal{G}$projective for short). Dually, one can define complete ξ-injective resolution and ξ-Gorenstein injective (ξ-$\mathcal{G}$injective for short) objects.

We study only the case of ξ-$\mathcal{G}$projective objects since the study of the ξ-$\mathcal{G}$injective objects is dual.

An ξ-$\mathcal{G}$projective resolution of $A \in \mathcal{C}$ is an ξ-exact sequence $\cdots \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to A \to 0$ in $\mathcal{C}$, such that $G_n \in \mathcal{GP}(\xi)$ for all $n \geq 0$. The definition is different from [3, Definition 4.2].

Lemma 3.1. *Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, $A \in \mathcal{C}$. Then A has an ξ-projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact if and only if A has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact.*

Proof Since $\mathcal{P}(\xi) \subseteq \mathcal{GP}(\xi)$, it is enough to show the "if" part. Assume that A has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. Then there exists a triangle $\delta : B \to G_0 \to A \to \Sigma B \in \xi$ which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact, where $G_0 \in \mathcal{GP}(\xi)$ and B has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. Since $G_0 \in \mathcal{GP}(\xi)$, there exists a triangle $\eta : G_0' \to P_0 \to G_0 \to \Sigma G_0' \in \xi$ such that $G_0' \in \mathcal{GP}(\xi)$, $P_0 \in \mathcal{P}(\xi)$ and it is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. By base change, we have the following commutative diagram:

$$\begin{array}{cccccccc}
 & & & \Sigma^{-1}A & = & \Sigma^{-1}A & & \\
 & & & \downarrow & & \downarrow & & \\
\eta' : & G_0' & \longrightarrow & L & \longrightarrow & B & \longrightarrow & \Sigma G_0' \\
 & \| & & \downarrow & & \downarrow & & \| \\
\eta : & G_0' & \longrightarrow & P_0 & \longrightarrow & G_0 & \longrightarrow & \Sigma G_0' \\
 & & & \downarrow & & \downarrow & & \\
 & & & \Sigma A & = & A & &
\end{array}$$

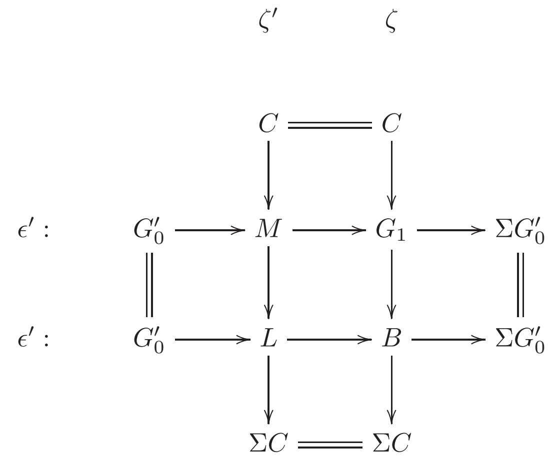
Since ξ is closed under base change, we have $\eta' \in \xi$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi), -)$ to the above diagram, we obtain the following commutative diagram

$$
\begin{array}{lccccccccc}
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),\eta'): & 0 & \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),G_0') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),L) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),B) & \dashrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),\eta): & 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),G_0') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),P_0) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),G_0) & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),A) & = & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi),A) & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}
$$

By snake lemma, we get that the second vertical sequence is exact, and so $L \to P_0 \to A \to \Sigma L \in \xi$ by Lemma 2.3. For any $Q \in \mathcal{P}(\xi)$, applying $\mathrm{Hom}_{\mathcal{C}}(-, Q)$ to the above base change diagram, we obtain the following commutative diagram:

$$
\begin{array}{lccccccccc}
 & & & & & \mathrm{Hom}_{\mathcal{C}}(\eta,Q) & & \mathrm{Hom}_{\mathcal{C}}(\eta',Q) & & \\
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{C}}(\delta,Q): & 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(G_0,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(B,Q) & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{\mathcal{C}}(\delta',Q): & 0 & \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(A,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(P_0,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(L,Q) & \dashrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & \mathrm{Hom}_{\mathcal{C}}(G_0',Q) & = & \mathrm{Hom}_{\mathcal{C}}(G_0',Q) & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}
$$

By snake lemma, one can get that $\mathrm{Hom}_{\mathcal{C}}(\delta', Q)$ and $\mathrm{Hom}_{\mathcal{C}}(\eta', Q)$ are exact. Since B has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact, there exists a triangle $C \to G_1 \to B \to \Sigma C \in \xi$ which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact, where $G_1 \in \mathcal{GP}(\xi)$ and C has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. By base change and the similar method above, we have the following commutative diagram:



such that ϵ' and ζ' are in ξ. Since $G_0' \in \mathcal{GP}(\xi)$ and $G_1 \in \mathcal{GP}(\xi)$, by [3, Theorem 3.11], we have $M \in \mathcal{GP}(\xi)$. Thus L has an ξ-$\mathcal{G}$projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. Note that the triangle $L \to P_0 \to A \to \Sigma L$ is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact. By repeating the preceding process, we have that A has an ξ-projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact, as required. □

An ξ-projective (ξ-$\mathcal{G}$projective) coresolution of $A \in \mathcal{C}$ is an ξ-exact sequence $0 \to A \to X^0 \to X^1 \to \cdots$ in $\mathcal{C}$, such that $X^n \in \mathcal{P}(\xi)$ ($X^n \in \mathcal{GP}(\xi)$) for all $n \geq 0$.

Lemma 3.2. *Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, $A \in \mathcal{C}$. Then A has an ξ-projective coresolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact if and only if A has an ξ-$\mathcal{G}$projective coresolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact.*

Proof It is completely dual to the proof of Lemma 3.1. So we omit it. □

Lemma 3.3. *Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, $A \in \mathcal{C}$. Then the following statements are equivalent:*

(1) A is an ξ-$\mathcal{G}$projective object.

(2) A has an ξ-projective resolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact and has an ξ-projective coresolution which is $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact.

(3) There exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact triangles $K_{n+1} \to P_n \to K_n \to \Sigma K_{n+1} \in \xi$ such that $P_n \in \mathcal{P}(\xi)$ and $K_0 = A$.

Proof It follows from the definition of ξ-$\mathcal{G}$projective object. □

Theorem 3.4. *Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, $A \in \mathcal{C}$. The following are equivalent:*

(1) A is ξ-$\mathcal{G}$projective.

(2) There exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{GP}(\xi))$-exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}(\xi), -)$-exact triangles $K_{n+1} \to G_n \to K_n \to \Sigma K_{n+1} \in \xi$ such that $G_n \in \mathcal{GP}(\xi)$ and $K_0 = A$.

(3) There exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{GP}(\xi))$-exact triangles $K_{n+1} \to G_n \to K_n \to \Sigma K_{n+1} \in \xi$ such that $G_n \in \mathcal{GP}(\xi)$ and $K_0 = A$.

(4) There exist $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{P}(\xi))$-exact triangles $K_{n+1} \to G_n \to K_n \to \Sigma K_{n+1} \in \xi$ such that $G_n \in \mathcal{GP}(\xi)$ and $K_0 = A$.

(5) There exists a $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}(\xi), -)$-exact and $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{GP}(\xi))$-exact triangle $A \to G \to A \to \Sigma A \in \xi$ such that $G \in \mathcal{GP}(\xi)$.

(6) There exists a $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{GP}(\xi))$*-exact triangle* $A\to G\to A\to\Sigma A\in\xi$ *such that* $G\in\mathcal{GP}(\xi)$.
(7) There exists a $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{P}(\xi))$*-exact triangle* $A\to G\to A\to\Sigma A\in\xi$ *such that* $G\in\mathcal{GP}(\xi)$.

Proof
(1)⇒(2) Let A be an ξ-$\mathcal{G}$projective object of $\mathcal{C}$. Consider the triangles

$$0\to A\xrightarrow{1}A\to 0 \text{ and } A\xrightarrow{1}A\to 0\to\Sigma A$$

Since ξ is proper, it contains Δ_0. So the above two triangles are in ξ. It is easy to see that they are $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{GP}(\xi))$-exact and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{GP}(\xi),-)$-exact triangles.
(2)⇒(3) and (3)⇒(4) are clear.
(4)⇒(1) It follows from Lemma 3.1 and Lemma 3.2.
(1)⇒(5) Let A be an ξ-$\mathcal{G}$projective object of $\mathcal{C}$. Consider the split triangle

$$\delta:\ A\xrightarrow{\binom{1}{0}}A\oplus A\xrightarrow{(0\ 1)}A\xrightarrow{0}\Sigma A$$

Since ξ is proper, it contains Δ_0. So δ is in ξ. For any $Q\in\mathcal{GP}(\xi)$, applying the functors $\mathrm{Hom}_{\mathcal{C}}(Q,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,Q)$ to the above triangle, we get the following exact sequences

$$0\to\mathrm{Hom}_{\mathcal{C}}(A,Q)\to\mathrm{Hom}_{\mathcal{C}}(A\oplus A,Q)\to\mathrm{Hom}_{\mathcal{C}}(A,Q)\to 0,$$

$$0\to\mathrm{Hom}_{\mathcal{C}}(Q,A)\to\mathrm{Hom}_{\mathcal{C}}(Q,A\oplus A)\to\mathrm{Hom}_{\mathcal{C}}(Q,A)\to 0.$$

By [3, Theorem 3.11], we can obtain $A\oplus A\in\mathcal{GP}(\xi)$. So we are done.
(5)⇒(6)⇒(7)⇒(1) are clear. □

Denote by $\mathcal{GP}(\xi)$ the subcategory of all ξ-$\mathcal{G}$projective objects. Set $\mathcal{G}^0\mathcal{P}(\xi)=\mathcal{P}(\xi)$, $\mathcal{G}^1\mathcal{P}(\xi)=\mathcal{GP}(\xi)$, and inductively set $\mathcal{G}^{n+1}\mathcal{P}(\xi)=\mathcal{G}^n(\mathcal{GP}(\xi))$ for any $n\geq 1$. Now we can obtain our main theorem.

Theorem 3.5. *Let* $\mathcal{C}$ *be a triangulated category with enough* ξ*-projectives. Then* $\mathcal{G}^n\mathcal{P}(\xi)=\mathcal{GP}(\xi)$ *for any* $n\geq 1$.

Proof It is easy to see that $\mathcal{P}(\xi)\subseteq\mathcal{GP}(\xi)\subseteq\mathcal{G}^2\mathcal{P}(\xi)\subseteq\mathcal{G}^3\mathcal{P}(\xi)\subseteq\cdots$ is an ascending chain of subcategories of $\mathcal{C}$. By (1) ⇔ (3) of the above theorem, we have that $\mathcal{G}^2\mathcal{P}(\xi)=\mathcal{GP}(\xi)$. By using induction on n we get easily the assertion. □

In order to give some equivalent characterizations for ξ-Gorenstein projective dimension of an object A in $\mathcal{C}$, one needs the following lemma.

Lemma 3.6. *Let* $0\to B\to G_1\to G_0\to A\to 0$ *be an* ξ*-exact sequence with* $G_1,G_0\in\mathcal{GP}(\xi)$. *Then there exist the following* ξ*-exact sequences:*

$$0\to B\to P\to G_0'\to A\to 0$$

and

$$0\to B\to G_1'\to Q\to A\to 0$$

where $P,Q\in\mathcal{P}(\xi)$ *and* $G_0',G_1'\in\mathcal{GP}(\xi)$.

Proof Since G_1 is in $\mathcal{GP}(\xi)$, there exists a triangle $G_1\to P\to G_2\to\Sigma G_1\in\xi$ with P ξ-projective and G_2 ξ-$\mathcal{G}$projective. Since $0\to B\to G_1\to G_0\to A\to 0$ is an ξ-exact sequence, there exist triangles $B\to G_1\to K\to\Sigma B\in\xi$ and $K\to G_0\to A\to\Sigma K\in\xi$. Then we have the following commutative diagrams by cobase change.

$$\begin{array}{ccccccc}
 & & B & = & B & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^{-1}G_2 & \longrightarrow & G_1 & \longrightarrow & P & \longrightarrow & G_2 \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}G_2 & \longrightarrow & K & \longrightarrow & C & \longrightarrow & G_2 \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma B & = & \Sigma B & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & \Sigma^{-1}G_2 & = & \Sigma^{-1}G_2 & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^{-1}A & \longrightarrow & K & \longrightarrow & G_0 & \longrightarrow & A \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}A & \longrightarrow & C & \longrightarrow & G'_0 & \longrightarrow & A \\
 & & \downarrow & & \downarrow & & \\
 & & G_2 & = & G_2 & &
\end{array}$$

Since ξ is closed under cobase change, we get $K \to C \to G_2 \to \Sigma K \in \xi$ and $C \to G'_0 \to A \to \Sigma C \in \xi$.

For any $Q \in \mathcal{P}(\xi)$, applying the functor $\mathrm{Hom}_{\mathcal{C}}(Q, -)$ to the above diagrams, we have the following commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \vdots & & & & \\
 & & \mathrm{Hom}_{\mathcal{C}}(Q,B) & = & \mathrm{Hom}_{\mathcal{C}}(Q,B) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,G_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,G_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \dashrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,G_2) & \dashrightarrow & 0 \\
 & & \downarrow & & \vdots & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \vdots & & & & \\
0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,K) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,G_0) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \cdots\to & \mathrm{Hom}_{\mathcal{C}}(Q,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,G_0') & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Q,A) & \cdots\to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Hom}_{\mathcal{C}}(Q,G_2) & = & \mathrm{Hom}_{\mathcal{C}}(Q,G_2) & & & & \\
 & & \downarrow & & \vdots & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By snake lemma and Lemma 2.3, we have $B \to P \to C \to \Sigma B \in \xi$ and $G_0 \to G_0' \to G_2 \to \Sigma G_0 \in \xi$. Because both G_0 and G_2 are in $\mathcal{GP}(\xi)$, G_0' is also in $\mathcal{GP}(\xi)$ by [3, Theorem 3.11]. Connecting the triangles $B \to P \to C \to \Sigma B$ and $C \to G_0' \to A \to \Sigma C$, we get the first desired ξ-exact sequence.

Since G_0 is ξ-$\mathcal{G}$projective, there is a triangle $G_3 \to Q \to G_0 \to \Sigma G_3 \in \xi$ with $Q \in \mathcal{P}(\xi)$ and $G_3 \in \mathcal{GP}(\xi)$. Then we have the following two commutative diagrams by base change:

$$\begin{array}{ccccccc}
 & & G_3 & = & G_3 & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^{-1}A & \longrightarrow & W & \longrightarrow & Q & \longrightarrow & A \\
\| & & \downarrow & & \downarrow & & \| \\
\Sigma^{-1}A & \longrightarrow & K & \longrightarrow & G_0 & \longrightarrow & A \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma G_3 & = & \Sigma G_3 & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & G_3 & = & G_3 & & \\
 & & \downarrow & & \downarrow & & \\
B & \longrightarrow & G_1' & \longrightarrow & W & \longrightarrow & \Sigma B \\
\| & & \downarrow & & \downarrow & & \| \\
B & \longrightarrow & G_1 & \longrightarrow & K & \longrightarrow & \Sigma B \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma G_3 & = & \Sigma G_3 & &
\end{array}$$

Since ξ is closed under base change, we get that the triangles $G_3 \to W \to K \to \Sigma G_3$ and $B \to G_1' \to W \to \Sigma B$ are in ξ. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}(\xi), -)$ to the above two diagrams, by snake lemma and Lemma 2.3 we have that the triangles $W \to Q \to A \to \Sigma W$ and $G_3 \to G_1' \to G_1 \to \Sigma G_3$ are in ξ. Because both G_1 and G_3 are in $\mathcal{GP}(\xi)$, G_1' is also in $\mathcal{GP}(\xi)$ by [3, Theorem 3.11]. Connecting the triangles $B \to G_1' \to W \to \Sigma B$ and $W \to Q \to A \to \Sigma W$, we get the second desired ξ-exact sequence. □

In particular, we have the following corollary.

Corollary 3.7. *Let $G_1 \to G_0 \to A \to \Sigma G_1$ be in ξ with $G_1, G_0 \in \mathcal{GP}(\xi)$. Then there exist the following triangles:*

$$P \to G_0' \to A \to \Sigma P$$

and

$$G_1' \to Q \to A \to \Sigma G_1'$$

in ξ where $P, Q \in \mathcal{P}(\xi)$ and $G_0', G_1' \in \mathcal{GP}(\xi)$.

Proposition 3.8. *Let $\mathcal{C}$ be a triangulated category with enough ξ-projectives, $A \in \mathcal{C}$, and n be a non-negative integer. Then the following statements are equivalent:*

(1) ξ-Gpd$(A) \leq n$.

(2) For every $0 \leq i \leq n$, there is an ξ-exact sequence

$$0 \to P_n \to \cdots \to P_{i+1} \to G \to P_{i-1} \to \cdots \to P_0 \to A \to 0$$

with $P_j \in \mathcal{P}(\xi)$ for all $0 \leq j \leq n$, $j \neq i$, and $G \in \mathcal{GP}(\xi)$.

(3) For every $0 \leq i \leq n$, there is an ξ-exact sequence

$$0 \to G_n \to \cdots \to G_{i+1} \to P \to G_{i-1} \to \cdots \to G_0 \to A \to 0$$

with $G_j \in \mathcal{GP}(\xi)$ for all $0 \leq j \leq n$, $j \neq i$, and $P \in \mathcal{P}(\xi)$.

Proof The case $n = 0$ is trivial. We may assume $n \geq 1$.

(1)⇒(2) we proceed by induction on n. Suppose ξ-Gpd$(A) \leq 1$. Then there exists a triangle $G_1 \to G_0 \to A \to \Sigma G_1$ in ξ with $G_0, G_1 \in \mathcal{GP}(\xi)$. By Corollary 3.7, we get the triangles $P \to G_0' \to A \to \Sigma P$ and $G_1' \to Q \to A \to \Sigma G_1'$ in ξ with $P, Q \in \mathcal{GP}(\xi)$ and $G_0', G_1' \in \mathcal{GP}(\xi)$.

Now suppose $n \geq 2$. Then there exists an ξ-exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to A \to 0$$

with $G_i \in \mathcal{GP}(\xi)$ for all $0 \leq i \leq n$. Applying Proposition 3.6 to the relevant ξ-exact sequence $0 \to K \to G_1 \to G_0 \to A \to 0$, we get an ξ-exact sequence $0 \to K \to G_1' \to P_0 \to A \to 0$ with $G_1' \in \mathcal{GP}(\xi)$ and $P_0 \in \mathcal{P}(\xi)$, which yields an ξ-exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_2 \to G_1' \to P_0 \to A \to 0.$$

Taking into account the relevant ξ-exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_2 \to G_1' \to L \to 0,$$

it follows that ξ-Gpd$(L) \leq n-1$. By the induction hypothesis, there exists an ξ-exact sequence

$$0 \to P_n \to \cdots \to P_{i+1} \to G \to P_{i-1} \to \cdots \to P_1 \to L \to 0$$

with $P_j \in \mathcal{P}(\xi)$ for all $1 \leq j \leq n$, $j \neq i$, and $G \in \mathcal{GP}(\xi)$. Now one can paste the above ξ-exact sequence and the triangle $L \to P_0 \to A \to \Sigma L$ together to obtain the desired ξ-exact sequence.

(2)⇒(1) and (3)⇒(1) are clear.

(1)⇒(3) Suppose ξ-Gpd$(A) \leq n$. Then there exists an ξ-exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to A \to 0$$

with $G_i \in \mathcal{GP}(\xi)$ for all $0 \leq i \leq n$. For every $0 \leq i < n$, considering the relevant ξ-exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_{i+1} \to G_i \to M \to 0,$$

it follows that ξ-Gpd$(M) \leq n-i$. By the proof of (1)⇒(2), we get an ξ-exact sequence

$$0 \to G'_n \to G'_{n-1} \to \cdots \to G'_{i+1} \to P \to M \to 0$$

with $G'_i \in \mathcal{GP}(\xi)$ and $P \in \mathcal{P}(\xi)$. So we obtain the ξ-exact sequence

$$0 \to G'_n \to \cdots \to G'_{i+1} \to P \to G_{i-1} \to \cdots \to G_0 \to A \to 0.$$

Now we only need to prove the result for $i = n$. Applying Corollary 3.7 to the relevant triangle $G_n \to G_{n-1} \to L \to \Sigma G_n$, we get the triangle $P \to G'_{n-1} \to L \to \Sigma P$ with $G'_{n-1} \in \mathcal{GP}(\xi)$ and $P \in \mathcal{P}(\xi)$. Thus we obtain the desired ξ-exact sequence

$$0 \to P \to G'_{n-1} \to G_{n-2} \to \cdots \to G_0 \to A \to 0.$$

□

References

[1] M. Auslander, M. Bridger, Stable Module Theory, Mem. Amer. Math. Soc., vol.94, 1969.

[2] L.L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimensions, Proc. Lond. Math. Soc. 85(2002)393–440.

[3] J. Asadollahi, Sh. Salarian, Gorenstein objects in triangulated categories, J. Algebra 281(2004)264–286.

[4] J. Asadollahi, Sh. Salarian, Tate cohomology and Gorensteinness for triangulated categories, J. Algebra 299(2006)480–502.

[5] A.A. Beilinson,J. Bernstein,P. Deligne, Perverse sheaves, in: Analysis and Topology on Singular Space, I, Luminy, 1981, in:Asterisque,vol.100, 1982, 5–171.

[6] A. Beligiannis, Relative homological algebra and purity in triangulated categories, J. Algebra 227(2000), 268–361.

[7] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization, Comm.Algebra 28(2000)4547–4596.

[8] L.W. Christensen, Gorenstein Dimensions, Lecture Notes in Math., vol.1747, Springer-Verlag, Berlin, 2000.

[9] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, Math.Z. 220(1995)611–633.

[10] E.E. Enochs, O.M.G. Jenda, Relative Homological Algebra, de Gruyter Exp. Math., vol.30, Walter De Gruyter, NewYork, 2000.

[11] D. Happel, Triangulated Categories in the Representation Theory of Finite-dimensional Algebras, London Math. Soc. Lecture Note Ser., vol. 119, Cambridge University Press, Cambridge, UK, 1988.

[12] R. Hartshorne, Residues and Duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, 1966.

[13] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189(2004)167–193.

[14] Z.Y. Huang, Proper resolutions and Gorenstein categories, J. Algebra 393(2013), 142–169.

[15] A. Hubery, Notes on the octahedral axiom. Available at http://www.maths.leeds.ac.uk/

[16] F. Kong, P. Zhang, From CM-finite to CM-free, Preprint. Available at http://arxiv.org/abs/1212.6184

[17] H.R. Margolis, Spectra and the Steenrod Algebra, North-Holland Math. Library, vol.29, North-Holland, Amsterdam, 1983.

[18] A. Neeman, Triangulated Categories, Ann. Math. Stud., vol. 148, Princeton Univ. Press, Princeton, NJ, 2001.

[19] D. Puppe, On the structure of stable homotopy theory, in: Colloquim on Algebraic Topology, Aarhus Universitet Matematisk Institut, 1962, 65–71.

[20] S. Sather-Wagstaff, T. Sharif, D. White, Stability of Gorenstein categories, J. Lond. Math. Soc. 77(2008)481–502.

[21] J.L. Verdier, Catégories dérivées: état 0, in: SGA 412, in: Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin, 1977, 262–311.