ON ALMOST AUTOMORPHIC OSCILLATIONS

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ABSTRACT. Almost automorphic dynamics have been given a notable amount of attention in recent years with respect to the study of almost periodically forced monotone systems. There are solid evidences that these types of dynamics should also be of fundamental importance in non-monotone especially conservative systems due to the interaction of multi-frequencies. This article will give a preliminary discussion in this regard by reviewing certain known cases and raising some problems for potential future studies.

1. Introduction

Almost automorphic functions, generating the (Bohr) almost periodic ones, were first introduced by Bochner ([12]) in 1955 in a differential geometry context, and, with a minor modification, they were shown in [27, 94] to coincide with the Levitan class of N-almost periodic functions ([56]). Let T be a locally compact, σ -compact, Abelian, first countable topological group and let E be a complete metric space. A function $f \in C(T, E)$ is said to be *almost automorphic* if whenever $\{t_n\}$ is a sequence such that $f(t_n + t) \to g(t) \in C(T, E)$ uniformly on compact sets, then $g(t - t_n) \to f(t)$ uniformly on compact sets, as $n \to \infty$. f becomes *almost periodic* if any sequence $\{t_n\}$ admits a subsequence $\{t_{n'}\}$ such that $f(t_{n'}+t)$ converges uniformly on T, as $n \to \infty$. In the literature, the above notion of almost automorphy is referred to as *sequential* or *continuous* almost automorphy. In fact, almost automorphic functions (and flows) can be defined in an abstract fashion on any topological group with respect to pointwise net convergence (see [27, 94, 98] for details). A function is sequential almost automorphic iff it is net almost automorphic and uniformly continuous ([98]). We choose the sequential notion in this paper because of its convenience in the applications to differential equations.

An almost periodic function is necessarily almost automorphic, but not vice versa. One can define Fourier series for both almost periodic and almost automorphic functions valued in a Banach space but the one for an almost periodic function is unique and converges uniformly in terms of Bochner-Fejer summation, while the one for an almost automorphic function is in general non-unique and its Bochner-Fejer sum only converges pointwise ([98]). Although there can be many Fourier series associated with a given almost automorphic function, one can define the *frequency module* of an almost automorphic function in the usual way as the smallest Abelian group containing a Fourier spectrum - the set of Fourier exponents associated with a Fourier series, and, it has been shown that such a frequency module is uniquely defined ([106]). In the above sense, both almost periodic and almost

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automorphic functions can be viewed as natural generalizations to the periodic ones in the strongest and the weakest sense respectively.

Almost automorphic minimal flows were first introduced and studied by Veech ([97]-[100]). A flow (E, T) is called *almost automorphic* (*almost periodic*) *minimal* if E is the closure of an almost automorphic (almost periodic) orbit. An almost automorphic minimal flow contains residually many almost automorphic points and becomes almost periodic only if every point in E is almost automorphic ([98, 100]). Typical examples of almost automorphic minimal sets include the well known Toeplitz minimal sets in symbolic dynamics ([19, 61]), the Denjoy set ([17]) on the circle, and the Aubry-Mather sets ([5, 62]) on an annulus. Unlike an almost periodic minimal flow, an almost automorphic one can be non-uniquely ergodic and can admit positive topological entropy ([61]), and its general measure theoretical characterization is completely random ([31]). Topologically, while an almost periodic minimal set is always a compact topological group ([22]), a non-almost periodic, almost automorphic one is only an almost 1-cover of a topological group ([98]) and can be topologically complicated ([46]). Hence, on one hand, an almost automorphic flow resembles an almost periodic one harmonically, but on the other hand, it presents certain complicated dynamical, topological and measure theoretical features which are significantly different from an almost periodic one.

Systematic studies of almost automorphic dynamics in differential equations were made in a series of recent works of the author with Shen ([89]-[93]) with respect to almost periodically forced monotone systems which are roughly those admitting no internal frequencies. Consider a skew-product semi-flow π_t over an almost periodic minimal base flow. Loosely speaking, it has been shown that if π_t is fiber-wise totally monotone (e.g., skew-product flows and semi-flows generated by almost periodically forced scalar ODEs and almost periodically forced parabolic PDEs in one space dimension, respectively), then all its minimal sets are almost automorphic with frequency modules responding to that of the base flow harmonically, and, if it is fiber-wise strongly monotone (e.g., skew-product flows and semi-flows generated by cooperative almost periodic systems of ODEs and almost periodically forced parabolic PDEs in higher space dimensions, respectively), then each linearly stable minimal set is almost automorphic with frequency module responding to that of the base flow sub-harmonically, and moreover, a minimal set in the fiber-wise strongly monotone skew-product semi-flow π_t over an almost periodic minimal base flow becomes almost periodic if it is uniformly stable. It is well known that even in the simplest almost periodically forced monotone system such as a quasi-periodically forced scalar ODE with only two frequencies and a linear scalar ODE with limiting periodic coefficients, almost periodic motions need not exist ([26, 46, 56, 75, 93]). Thus, the finding of almost automorphic dynamics in [89]-[93] actually shows a fundamental phenomenon in almost periodically forced monotone systems, i.e., almost automorphic solutions largely exist but almost periodic ones need not. We refer the readers to [85, 93] for references on the study of almost automorphic dynamics in almost periodically forced differential systems and to [3, 41, 42, 74, 76, 86, 88] for recent developments in the subject.

Comparing with periodically forced monotone systems in which periodic solutions are generically expected, the existence of almost automorphic solutions in almost periodically forced monotone systems reflects a general harmonic nature of the systems due to the interactions of several frequencies especially when they are close to the resonance. It is therefore well expected that almost automorphic dynamics should also largely exist in almost periodically forced non-monotone systems (i.e., systems with self-excitations or containing internal frequencies) and even in autonomous conservative systems. Some examples of periodically forced nonlinear oscillators and Hamiltonian systems have already supported this assertion but no systematic investigation is made to the existence of almost automorphic dynamics in general almost periodically forced non-monotone systems yet. In this article, we will review some known cases of the existence of almost automorphic dynamics in non-monotone systems and formulate some general open problems as potential starting points in the subject. We are also attempting to link the study of almost automorphic oscillations with other active areas in dynamical systems such as the almost periodic Floquet theory, Aubry-Mather theory, chaos, Hamiltonian systems, non-chaotic strange attractors, nonlinear oscillations, and toral flows. Problems arising in multi-frequency, non-monotone systems are far more complicated and challenging than those in monotone systems. At this stage, we have more questions than answers. Therefore, instead of an expository article as it was supposed to be, the present article is rather a research note, aiming at making some preliminary discussions in this interesting subject, which is by no means complete or well thought out.

Throughout the paper, we let (Y, R) be an almost periodic minimal flow and denote $y \cdot t$ as the orbit of the flow passing through a point $y \in Y$. We will mainly consider ODEs with almost periodic forcing of the form $f(x, y \cdot t)$, where $x \in R^n$ is a state variable (the global existence of solutions of the ODEs is always assumed). This is without loss of generality. Suppose that an ODE

$$x' = F(x,t),\ x \in R^n, \tag{1.1}$$

is considered with F being uniformly Lipschitz in x and almost periodic in t uniformly with respect to x. Let $Y = H(F) = cl\{F_\tau | \tau \in R\}$ be the *hull* of $y_0 = F$ under the compact open topology (hence Y is compact metric), where $F_\tau(x,t) \equiv F(x, t+\tau)$. Define $f : R^n \times Y \to R^n$: $f(x, y) = y(x, 0)$, $y \in Y$. Then the time translation $y \cdot t = y_t$ defines a natural almost periodic minimal flow (Y, R) and $F(x,t) \equiv f(x, y_0 \cdot t)$. Thus instead of studying the single equation (1.1), we will consider a family of equations

$$x' = f(x, y \cdot t),\ x \in R^n, \tag{1.2}$$

which consist of the translated and limiting equations of (1.1). To study the dynamical behaviors of solutions of (1.1), we note that the equations (1.2) give rise to a skew-product flow $(R^n \times Y, \pi_t)$:

$$\pi_t(x, y) = (X(x, y, t), y \cdot t),$$

where $X(x, y, t)$ denotes the solution of (1.2) with the initial value x. Such a formulation was originated in [67, 84] in studying dynamics of non-autonomous differential equations (see also [48, 105]). We note that if F is quasi-periodic in t with k frequencies, then $Y \simeq T^k$ and (Y, R) is topologically conjugated to the standard quasi-periodic flow on the k-torus.

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2002, the International Conference on Dynamical Methods for Differential Equations, Medina, Spain, 2002, and the International Workshop on Global Analysis of Dynamical Systems, Leiden, The Netherlands, 2001. The author would like to thank the organizers of these conferences for opportunities to present part of the results contained in this article, in particular the organizers for the 7th ICDEA for their invitation to write this article.

2. Almost automorphic functions and flows

In this section, unless specified otherwise, we let T be a locally compact, σ-compact, Abelian, first countable group, and E be a complete metric space.

2.1. **Harmonic properties.**

Fourier theory was first developed in [98] for complex valued almost automorphic functions. Parallel Fourier theory can be also found in [56] for N-almost periodic functions.

Theorem 2.1. *Let f be an almost automorphic function on T valued in a Banach space. Then f admits Fourier series (not necessarily unique) whose Bochner-Fejer sums converge to f pointwise (in fact, uniformly on compact sets).*

The theorem was first proved in [98] for complex valued functions and was generalized in [106] for almost automorphic functions on R valued in a Banach space. The same proof can be carried over for general T.

Let f be an almost automorphic function on T. Then the topological hull $H(f)$ is almost automorphic minimal under the 'time' translation flow which is an almost 1-cover of its maximal almost periodic factor (Y, T) (see Theorem 2.3 below). The almost periodic factor is unique up to flow isomorphism and admits a natural compact Abelian topological group structure inherited from the flow ([22]), e.g., when $T = R$, Y is a solenoidal group ([2]). Hence the dual group Y' is discrete. We define the *frequency module* $\mathcal{M}(f)$ of f as the subgroup of Y' generated by the Fourier spectrum associated with a Fourier series of f. This is well defined because of the following theorem which was originally shown in [106] for $T = R$ but also holds in general.

Theorem 2.2. *With respect to any Fourier series of f, $\mathcal{M}(f) \simeq Y'$.*

The following module containment result can be shown similarly as in [106] for functions defined on R.

Proposition 2.1. *For two almost automorphic functions $f, g \in C(T, E)$, $\mathcal{M}(g) \subset \mathcal{M}(f)$ iff whenever $f(t + t_n) \to f(t)$ for some sequence $\{t_n\} \subset T$ then $g(t + t_n) \to g(t)$, uniformly on compact sets.*

2.2. **Structural properties.**

The following result shown in [98] is known as the Veech *almost automorphic structure theorem*.

Theorem 2.3. *A compact minimal flow (E, T) is almost automorphic iff it is an almost* 1-1 *extension of its maximal almost periodic factor (Y, T), i.e., there is a residual subset $Y_0 \subset Y$ such that each fiber over Y_0 is a singleton.*

Let (E,T), (Y,T) be as in the above. It is shown in [100] that as long as there is a $y_0 \in Y$ corresponding to a singleton fiber, then there are residually many of them. The set of points in E lying in the singleton fibers consists of precisely the almost automorphic points, and, E becomes almost periodic iff all points in E are almost automorphic. Again, as (Y,T) is almost periodic minimal, Y admits a compact Abelian group structure inherited from the flow whose dual group Y' is discrete. We thus define the *frequency module* $\mathcal{M}(E)$ of E as the dual group Y'. This definition is in consistent with that for an almost automorphic function which we introduced above.

Theorem 2.4. 1) *A function $f \in C(T,E)$ is almost automorphic iff it is a pointwise limit of a sequence of jointly almost automorphic, almost periodic functions.*

2) *$f \in C(T,E)$ is almost automorphic iff there exists a dense subgroup G of a compact group G_*, a continuous homomorphism $\phi : T \to G$, and a function $g : G_* \to E$ which is continuous on G such that $f = g \circ \phi$ is uniformly continuous.*

3) *If $\phi : T \to D$, where D is complete metric, is almost automorphic, $g : D \to E$ is continuous on $R(\phi)$, then $f = g \circ \phi \in C(T,E)$ is almost automorphic if it is uniformly continuous.*

Part 1) of the theorem was shown in [98] for complex valued (sequential) almost automorphic functions. Part 2) of the theorem was stated and shown in [96] for complex valued function f as follows: f is (net) almost automorphic iff there is a continuous homomorphism $\phi : T \to G$, where G is a totally bounded group, and a complex valued function g which is continuous on G such that $f = g \circ \phi$. Part 3) of the theorem was stated and shown in [99] for complex valued functions f, g as follows: if $\phi : T \to C$ is (net) almost automorphic and $g : R(\phi) \to C$ is continuous, then $f = g \circ \phi$ is (net) almost automorphic. With proper modifications as the above, all these results can be extended to the ones stated in the theorem by using similar arguments as originally used in [96, 98, 99].

Parts 2) 3) of the above theorem is particularly useful in constructing non-quasi-periodic, almost automorphic functions with finite many frequencies. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_k)$
$\in R^k$ be a given non-resonant vector and $f : T^k \to R^n$ be a measurable function which is continuous on an orbit $\{\theta_0 + \omega t : t \in T\}$ of the quasi-periodic flow on T^k with the frequency ω, here $T = R$ or Z. As $\{\omega t : t \in R\}$ is embedded into T^k as a dense subgroup (or the map $R \to T^k : t \mapsto \theta_0 + \omega t$ is almost periodic hence almost automorphic), an immediate application of part 2) or 3) of Theorem 2.4 and Proposition 2.1 yields the following.

Corollary 2.1. *Let f, ω, θ_0 be as in the above. Then $F(t) = f(\theta_0 + \omega t)$, $t \in R$ or Z, is almost automorphic whose frequency module $\mathcal{M}(F)$ is contained in $\mathcal{M}(\omega)$ - the additive subgroup of R generated by $\omega_1, \omega_2, \cdots, \omega_k$.*

Remark 2.1. *For each $\theta \in T^k$, the function $f(\theta + \omega t)$, where $t \in R$ or Z, may be regarded as a* weak quasi-periodic *function. Weak quasi-periodic orbits are known to existence in monotone twist maps on an annulus as orbits lying in the Aubry-Mather sets* ([62]) *and in scalar ODEs with quasi-periodic time dependence* ([76]).

The existence of such orbits in these dynamical systems were shown under certain monotonicity and properties that $f : T^k \to R^1$ *is semi-continuous and admits bounded directional derivative along* ω*. Indeed, the semi-continuity implies that* f *admits a residual set* Y_0 *of points of continuity, and the directional differentiability together with the continuous dependence on initial conditions imply the continuity of* f *on each orbit* $\{\theta_0 + \omega t\}$ *if* $\theta_0 \in Y_0$*. It is clear that a weak quasi-periodic orbit* $f(\theta + \omega t)$ *becomes almost automorphic precisely when* θ *is a point of continuity of* f*, and becomes quasi-periodic only if* f *itself is a continuous function on the torus.*

It is important to note that the family of weak quasi-periodic orbits such defined always contains almost automorphic ones. This is because of the harmonic similarity between almost automorphic functions with finite many frequencies and quasi-periodic ones. Indeed, if θ *is not a point of continuity of* f*, then the function* $f(\theta + \omega t)$ *needs not even be recurrent and needs not admit any Fourier series whose Bochner-Fejer sum is pointwise convergent, due to the lack of continuity of* f *on the set* $\{\theta + \omega t\}$*.*

2.3. **Inherit properties.** Similar to almost periodic minimal flows ([22]), it was shown in [9] that an almost automorphic minimal flow (X, T) also admits an inheritance property as follows.

Theorem 2.5. *Let* S *be a syndetic subgroup of* T*. Then* (X, T) *is almost automorphic minimal with maximal almost periodic factor* (Y, T) *iff* (X, S) *is almost automorphic minimal with maximal almost periodic factor* (Y, S)*.*

The above inherit property can be easily extended to a Poincaré map associated with a real compact flow (X, R). Recall that a closed subset $Z \subset X$ is a *global cross section of* (X, R) if i) all orbits meet Z; ii) there is a positive continuous function $T : Z \to R$, called *first return time*, such that $z \cdot T(z) \in Z$ and $z \cdot t \notin Z$ for all $z \in Z$ and $0 < t < T(z)$. In case a global cross section Z exists, one can define the *Poincaré map* $P : Z \to Z : \; z \mapsto z \cdot T(z)$, which is a homeomorphism on Z. The following can be shown similarly to a special case considered in [9].

Proposition 2.2. *Suppose that* (X, R) *admits a global cross section* Z *and let* $P : Z \to Z$ *be the associated Poincaré map. Then* (X, R) *is almost automorphic minimal iff* (Z, P) *is.*

Let $P : S^1 \to S^1$ be an orientation preserving homeomorphism with irrational rotation number. Then P has a unique minimal set E which is either topologically conjugated to a pure rotation ([17]) or is an almost 1-1 extension of a pure rotation on S^1 ([60]), both with zero topological entropy. In any case, E is (discrete) almost automorphic.

Now consider a C^1 fixed-point-free flow π_t on the 2-torus T^2. Then π_t admits a Poincaré section and its *rotation vector*

$$\omega = \lim_{t \to \infty} \frac{\tilde{\pi}_t(y)}{t}$$

exists and is independent of $y \in T^2$, where $\tilde{\pi}_t : R^2 \to R^2$ denotes a continuous lift of π_t. We assume further that ω is non-resonant, or equivalently, π_t admits no periodic orbits. An easy application of the Denjoy theory implies that π_t admits a unique minimal set E which is topologically equivalent to a suspension of either a pure rotation or a Denjoy Cantor set on the circle (hence E is uniquely ergodic with zero topological entropy). Applying Proposition 2.2, we further conclude that E is actually almost automorphic.

Similarly, let f be an area preserving, orientation preserving, boundary components preserving, twist homeomorphism of an annulus A. Let $\rho_0 < \rho_1$ denote the rotation numbers of f restricted to the lower and upper boundaries of A, respectively. Then it has been shown in [62] that for any $\rho \in [\rho_0, \rho_1]$ irrational, f admits a minimal set M_ρ which is topologically conjugated to either a pure rotation of the circle or a Denjoy set of the circle, with the rotation number ρ. Hence, M_ρ is an almost automorphic minimal set. In fact, Fourier series and their pointwise convergence for the almost automorphic orbits (referred to as quasi-periodic orbits by Mather) in M_ρ were also discussed in [62] which fit in the harmonic nature of general almost automorphic functions described in Theorem 2.1.

To summarize, we have the following.

Theorem 2.6. *A Denjoy minimal set of a circle homeomorphism, a Denjoy minimal set of a toral flow, and Aubry-Mather sets of an area preserving monotone twist map on an annulus are all almost automorphic minimal sets with zero topological entropy.*

3. Almost periodically forced circle flows

Consider an almost periodically forced circle flow generated by the following almost periodically forced scalar ODE:

$$\phi' = f(\phi, y \cdot t), \tag{3.1}$$

where $\phi \in R^1$, $f : R^1 \times Y \to R^1$ is Lipschitz continuous, periodic in the first variable with period 1. Using the identification $\phi \equiv \phi' \pmod 1$, (3.1) clearly generates a skew-product flow $\Lambda_t : S^1 \times Y \to S^1 \times Y$:

$$\Lambda_t(\phi_0, y_0) = (\phi(\phi_0, y_0, t), y_0 \cdot t) \tag{3.2}$$

when both $\phi(\phi_0, y_0, t)$ and ϕ_0 are identified modulo 1, where $\phi(t) = \phi(\phi_0, y_0, t)$ is the solution of (3.1) with $y = y_0$, $\phi(0) = \phi_0$.

3.1. Rotation number and mean motion.

Using arguments of [49] and the Birkhoff ergodic theorem, it is easily seen that the equation (3.1), or equivalently, the skew-product flow (3.2) admits a well defined *rotation number*

$$\rho = \lim_{t\to\infty} \frac{\phi(\phi_0, y_0, t)}{t}$$

i.e., the limit exists and is independent of initial values $(\phi_0, y_0) \in R^1 \times Y$.

We say that the equation (3.1), or equivalently, the skew-product flow (3.2) admits *mean motion* if

$$\sup_{t\in R^1} |\phi(\phi_0, y_0, t) - \phi_0 - \rho t| < \infty$$

for all $(\phi_0, y_0) \in R^1 \times Y$.

Remark 3.1. 1) *It is well known that if either* (3.1) *is periodically dependent on time* ([38])*, i.e.,* $(Y, R) = (S^1, R)$ *is a pure rotation of the circle, or almost periodically dependent on time but admits an almost periodic solution* ([25])*, then it always admits mean motion. This is however not the case for general almost periodic time dependence. As an extreme example, let* $Y = T^k$*,* $f \equiv f(\omega t)$ *be quasi-periodic in* (3.1)*, where* $\omega \in R^k$ *is a non-resonant frequency vector. Then* $\rho = [f]$ *- the mean value of* f*, and, it is well known that* $\phi(\phi_0, t) - \phi_0 - \rho t = \int_0^t (f(\omega s) - [f]) \mathrm{d}s$ *can be unbounded if* $k > 1$ *unless some additional conditions are assumed on* f *and* ω (*e.g.,* f *is sufficiently smooth and* ω *is Diophantine*).

2) *A natural question here is that under what conditions* (3.1) *can admit mean motion. Another question is that whether the mean motion property is generic, e.g., among the class of real analytic functions* $f : S^1 \times T^k \to R^1$ *endowed with sup-norm, with* $(Y, R) = (T^k, R)$ *being a fixed Diophantine, quasi-periodic flow.*

We do not have answers to these questions but conjecture that the answer to the second question should be affirmative. Below, we give some equivalent conditions for mean motion to hold in (3.1).

Proposition 3.1. *The equation* (3.1) *admits mean motion iff there is a* $(\phi_0, y_0) \in R^1 \times Y$ *such that*

$$|\phi(\phi_0, y_0, t) - \phi_0 - \rho t|$$

is bounded for either $t \geq 0$ *or* $t \leq 0$*.*

Proof. We note that, by the periodicity of f in ϕ, for any $t \in R^1$, $y \in Y$, if $|\phi_1^* - \phi_2^*| < l$ for some positive integer l, then also

$$|\phi(\phi_1^*, y, t) - \phi(\phi_2^*, y, t)| < l. \tag{3.3}$$

Without loss of generality, assume that $(\phi_0, y_0) \in R^1 \times Y$ is such that

$$\sup_{t \geq 0} |\phi(\phi_0, y_0, t) - \phi_0 - \rho t| < \infty.$$

Then it follows from the flow property that

$$\sup_{t \in R^1} |\phi(\phi_*, y_*, t) - \phi_* - \rho t| < \infty$$

for any $(\phi_*, y_*) \in \omega(\phi_0, y_0)$ with respect to the flow (3.2). Hence (3.1) admits mean motion by (3.3). □

Lemma 3.1. *Consider* (3.1)*. Then*

$$|\int_Y \phi(\phi_0, y, t) \mathrm{d}y - \phi_0 - \rho t| \leq 4$$

for any $\phi_0 \in R^1$*.*

Proof. Again, by the periodicity of f in ϕ, for any $t \in R^1$, $y \in Y$, if $\phi_1^* \equiv \phi_2^*$ (mod 1), then

$$\phi(\phi_1^*, y, t) - \phi_1^* = \phi(\phi_2^*, y, t) - \phi_2^*, \tag{3.4}$$

and, if $|\phi_1^* - \phi_2^*| < l$ for some positive integer l, then (3.3) holds.

Now, fix $t, \phi_0 \in R^1$. For any $y \in Y$, $0 \leq |s| \leq |t|$, let $0 \leq \phi_1, \phi_2 < 1$ be such that

$$\phi_1 \equiv \phi_0, \ \phi_2 \equiv \phi(\phi_0, y, s), \ (\mathrm{mod}\, 1).$$

It follows from (3.4) that

$$\phi(\phi_0, y\cdot s, t) - \phi_0 = \phi(\phi_1, y\cdot s, t) - \phi_1, \tag{3.5}$$

$$\phi(\phi_0, y, t+s) - \phi(\phi_0, y, s) = \phi(\phi_2, y\cdot s, t) - \phi_2. \tag{3.6}$$

Hence, by (3.3)-(3.6),

$$\begin{aligned} &|\phi(\phi_0, y, t+s) - \phi(\phi_0, y, s) - \phi(\phi_0, y\cdot s, t) + \phi_0| \\ &= |\phi(\phi_2, y\cdot s, t) - \phi(\phi_1, y\cdot s, t) + \phi_2 - \phi_1| \\ &\le |\phi(\phi_2, y\cdot s, t) - \phi(\phi_1, y\cdot s, t)| + |\phi_2 - \phi_1| \le 4, \end{aligned}$$

i.e.,

$$|\int_s^{t+s} \phi'(\phi_0, y, \lambda)\mathrm{d}\lambda - \phi(\phi_0, y\cdot s, t) + \phi_0| \le 4. \tag{3.7}$$

Let $T \neq 0$. An application of Frobine's theorem yields

$$\begin{aligned} &|\phi_0 - \int_0^t \frac{1}{T}\phi(\phi_0, y, \lambda)\mathrm{d}\lambda + \int_0^t \frac{\phi(\phi_0, y, T+s)}{T}\mathrm{d}s - \frac{1}{T}\int_0^t \phi(\phi_0, y\cdot s, t)\mathrm{d}s| \\ &= |\frac{1}{T}\int_0^T \int_s^{t+s} \phi'(\phi_0, y, \lambda)\mathrm{d}\lambda\mathrm{d}s - \frac{1}{T}\int_0^T \phi(\phi_0, y\cdot s, t)\mathrm{d}s + \phi_0| \le 4. \end{aligned} \tag{3.8}$$

Since

$$\lim_{T\to\infty} \frac{\phi(\phi_0, y, T+s)}{T} = \rho$$

uniformly in $y \in Y$, $s \in [0, t]$, and, by the Birkhoff ergodic theorem,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(\phi_0, y\cdot s, t)\mathrm{d}s = \int_Y \phi(\phi_0, y, t)\mathrm{d}y,$$

the lemma is proved by passing limit $T \to \infty$ in (3.8). □

Theorem 3.1. (3.1) *admits mean motion iff there is a* $(\phi_0, y_0) \in R^1 \times Y$ *such that*

$$|\phi(\phi_0, y_0, t) - \int_Y \phi(\phi_0, y, t)dy|$$

is bounded for $t \ge 0$ *or* $t \le 0$.

Proof. This follows immediately from Proposition 3.1 and Lemma 3.1. □

3.2. **Mean motion and almost automorphic dynamics.** One significance for having mean motion is that it actually implies the existence of almost automorphic dynamics in the skew-product flow (3.2).

First, we consider the following almost periodically forced scalar ODE

$$x' = f(x, y\cdot t), \tag{3.9}$$

where $x \in R^1$, $f : R^1 \times Y \to R^1$ is Lipschitz continuous. Let $\pi_t : R^1 \times Y \to R^1 \times Y$:

$$\pi_t(x_0, y_0) = (x(x_0, y_0, t), y_0 \cdot t) \tag{3.10}$$

be the skew-product flow generated by (3.9), where $x(t) = x(x_0, y_0, t)$ denotes the solution of (3.9) with $y = y_0$ and $x(0) = x_0$. As (3.9) forms a special class of almost periodically forced, 1-dimensional, scalar parabolic PDEs with the Neumann boundary condition, the following lemma follows from the corresponding result in

[91, 93]. As the proofs for the scalar ODE case is much less involved than the parabolic PDE case, we include them here for the readers' convenience.

Lemma 3.2. *Let E be a minimal set of* (3.10)*. Then the following holds.*

1) *(E, R) is almost automorphic and in fact an almost* 1-1 *extension of (Y, R);*
2) *E is uniquely ergodic iff Y_0 admits full Haar measure, where $Y_0 \subset Y$ is the residual set corresponding to the singelton fibers.*
3) *Let $x(x_0, y_0, t)$ be an almost automorphic solution of* (3.9) *for some $y_0 \in Y$. Then $\mathcal{M}(x) \subset \mathcal{M}(f)$.*

Proof. Let $p : R^1 \times Y \to Y$ be the natural projection and define

$$a_M(y) = \max\{x : (x, y) \in E \cap p^{-1}(y)\},\ a_m(y) = \min\{x : (x, y) \in E \cap p^{-1}(y)\}.$$

1) Let 2^E be furnished with Hausdorff metric. Since the map $h : Y \to 2^E$: $y \mapsto E \cap p^{-1}(y)$ is upper semi-continuous, the points of continuity of h form a residual subset $Y_0 \subset Y$. Fix a $y \in Y_0$ and let $t_n \to \infty$ be such that $\pi_{t_n}(a_M(y), y) \to (a_m(y), y)$. By the continuity of h at y, there is a sequence of points $(x_n, y) \in E \cap p^{-1}(y)$ such that $\pi_{t_n}(x_n, y) \to (a_M(y), y)$. Now, by taking limits to both side of the inequalities

$$x(x_n, y, t_n) \le x(a_M(y), y, t_n),$$

we obtain that $a_M(y) \le a_m(y)$. Hence $a_M(y) = a_m(y)$ and $E \cap p^{-1}(y)$ is a singleton.

2) Let μ denote the Haar measure on Y. If $\mu(Y_0) = 1$, then E is clearly uniquely ergodic. Now suppose that $\mu(Y_0) = 0$ (note that Y_0 is invariant). Then the functionals $l_{M,m} : C(E) \to R$,

$$l_{M,m}(f) = \int_Y f(a_{M,m}(y), y) d\mu$$

would define two distinct invariant measures on E.

3) follows immediately from 1) and Proposition 2.1. □

Remark 3.2. 1) *If* (3.9) *is periodically dependent on time, then each minimal set of its associated skew-product flow becomes periodic* (*hence is uniquely ergodic and admits zero topological entropy*)*. We conjecture that in the almost periodic time dependent case, the skew-product flow associated with* (3.9) *can have almost automorphic minimal sets which are non-uniquely ergodic and admit positive topological entropy.*

2) *Assume that* (3.9) *is quasi-periodically dependent on time, i.e., $(Y, R) = (T^k, R) : y \cdot t = y + \omega t$ is a quasi-periodic flow on the k-torus for some k with the toral frequency $\omega \in R^k$. It was shown in* [76] *that if* (3.9) *admits a bounded solution $x(t)$ for some $y = y_0$, then it has two families of weak quasi-periodic solutions $x_-(y + \omega t), x_+(y + \omega t)$, $y \in T^k$, in the sense of* Remark 2.1*, such that $x_-, x_+ : T^k \to R^1$ are lower and upper semi-continuous respectively, and,*

$$\inf x(t) \le x_-(y) \le x_+(y) \le \sup x(t),\quad y \in T^k, \tag{3.11}$$

$$x_-(y_0 + \omega t) \le \phi(t) \le x_+(y_0 + \omega t),\quad t \in R. \tag{3.12}$$

Moreover, if $y_0 \in T^k$ is a point of continuity for x_- (x_+ resp.), then $x_-(y_0 + \omega t)$ ($x_+(y_0 + \omega t)$ resp.) becomes almost automorphic.

Theorem 3.2. *Assume that* (3.1) *admits mean motion and let* $E \subset S^1 \times Y$ *be a minimal set of the skew-product flow* (3.2)*. Then the following holds.*

1) *E is almost automorphic with $\mathcal{M}(E)$ being the additive subgroup of R generated by ρ and $\mathcal{M}(Y)$.*
2) *E is uniquely ergodic iff its set of almost automorphic points admits full measure with respect to an invariant measure on E.*

Proof. 1) Consider the change of variable $x = \phi - \rho t$. This transforms the equation (3.1) into

$$x' = f(x + \rho t, y \cdot t) - \rho \tag{3.13}$$

which generates a skew-product flow π_t on $R^1 \times Y$ or $R^1 \times (S^1 \times Y)$, in the same way as (3.10), depending on whether $\rho \in \mathcal{M}(Y)$.

Since (3.1) admits mean motion,

$$\int_0^t (f(\Lambda_t(\phi_0, y_0)) - \rho)\mathrm{dt}$$

is bounded for all $(\phi_0, y_0) \in E$. It follows from [33] that there is a continuous function $H : E \to R^1$ such that

$$\phi(\phi_0, y_0, t) - \phi_0 - \rho t = H(\Lambda_t(\phi_0, y_0)) - H(\phi_0, y_0) = \int_0^t (f(\Lambda_t(\phi_0, y_0)) - \rho)\mathrm{dt},$$

i.e.,

$$\phi(\phi_0, y_0, t) - H(\Lambda_t(\phi_0, y_0)) = \phi_0 - H(\phi_0, y_0) + \rho t \tag{3.14}$$

for all $t \in R^1$ and all $(\phi_0, y_0) \in E$. We note that either $H(E) \times Y$ or $H(E) \times (S^1 \times Y)$ is a minimal set of π_t, which by Lemma 3.2 is almost automorphic. It follows from (3.14) and the definition of almost automorphic minimal set that E is almost automorphic.

Let $\mathcal{M}$ be the additive subgroup of R generated by ρ and $\mathcal{M}(Y)$. If $(\phi_0, y_0) \in E$ is an almost automorphic point, then $x(t) = H(\Lambda_t(\phi_0, y_0))$ is an almost automorphic solution of (3.13) for $y = y_0$. It follows from Lemma 3.2 that $\mathcal{M}(x) \subset \mathcal{M}$. Since $\Lambda_t(\phi_0, y_0) = (x(t) + \rho t + \phi_0 - x(0), y_0 \cdot t)$, we have by Proposition 2.1 that $\mathcal{M}(E) = \mathcal{M}(\Lambda_t(\phi_0, y_0)) = \mathcal{M}$.

2) follows from 1) and Lemma 3.2 2). □

We now consider a special case of (3.1) in which $Y = T^k$ and (Y, R) is a quasi-periodic minimal flow with non-resonant toral frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_k)$. In this case, more can be said about the structure of a minimal set of (3.2).

Corollary 3.1. *Consider* (3.1) *with* $(Y, R) = (T^k, R)$ *being the quasi-periodic minimal flow on* T^k *with the frequency* ω*. Assume that* (3.1) *admits mean motion and let* E *be a minimal set of* (3.2)*. Then there is an almost periodic minimal flow* (X, R)*, a quasi-periodic minimal flow* $(\hat{Y}, R)$*, and flow epimorphisms* $p : (E, R) \to (X, R)$*,* $q : (X, R) \to (\hat{Y}, R)$ *for which the following holds.*

1) *p is almost 1-1 and q is open.*
2) *$\mathcal{M}(E) = \mathcal{M}(X) = \mathcal{M}(\hat{Y})$ and equals*

$$\{n_0\rho + n_1\omega_1 + \cdots + n_k\omega_k : n_0, n_1, \cdots, n_k \in Z\}.$$

3) *If ρ is rationally independent of ω, then X is a $(k+1)$-solenoid and $\hat{Y}$ is a $(k+1)$-torus. If ρ is rationally dependent of ω, then X is a k-solenoid and $\hat{Y}$ is a k-torus. In fact, if n_0 is the smallest positive number such that $n_0\rho+n_1\omega_1+\cdots+n_k\omega_k=0$, for some relatively prime numbers $n_1,\cdots,n_k$, then $\hat{Y}$ is an n_0-fold cover of $Y=T^k$.*

Proof. 1) Let X be a maximal almost periodic factor of E. The existence of $p:(E,R)\to(X,R)$ follows from Theorem 2.3. In fact, consider the proximal relation

$$P(E)=\{(e_1,e_2)\in E\times E:\inf_{t\in R^1}d(\Lambda_t(e_1),\Lambda_t(e_2))=0\},$$

where d denotes the standard metric on $S^1\times Y$. Then $P(E)$ is a closed (in particular, an equivalence), equivariance relation, and, X can be chosen to be $E/P(E)$ with flow being induced by Λ_t.

Define $q:X\to S^1\times Y$,

$$q(p(\phi_0,y_0))=(\phi_0-H(\phi_0,y_0),y_0)\pmod 1,\ (\phi_0,y_0)\in E.$$

We need to verify that q is well defined. Let $(e_1,e_2)\in P(E)$. Then it is clear that there is a sequence $t_n\to\infty$ and $\phi_1,\phi_2\in S^1$, $y_0\in Y$ such that $e_1=(\phi_1,y_0)$, $e_2=(\phi_2,y_0)$, and, $d(\Lambda_{t_n}(e_1),\Lambda_{t_n}(e_2))\to 0$. It follows from (3.14) that $\phi_1-H(\phi_1,y_0)=\phi_2-H(\phi_2,y_0)\ (mod\,1)$.

Define $\hat{Y}=q(X)$ and the flow on $(\hat{Y},R)$ by

$$(q\circ p(\phi_0,y_0))\cdot t=(\phi(\phi_0,y_0,t)-H(\Lambda_t(\phi_0,y_0),y_0\cdot t)\pmod 1,\ (\phi_0,y_0)\in E.$$

It follows from (3.14) that $(\hat{Y},R)$ is a parallel flow with the frequency vector (ρ,ω), which further shows that $\hat{Y}$ is either a $k+1$ or a k-torus and $(\hat{Y},R)$ is quasi-periodic.

As an epimorphism between two almost periodic flows, q is clearly an open map.

2) By definition, $\mathcal{M}(E)=\mathcal{M}(X)$. Since $(\hat{Y},R)$ is quasi-periodic with the frequency (ρ,ω), we have

$$\mathcal{M}(\hat{Y})=\mathcal{M}\equiv\{n_0\rho+n_1\omega_1+\cdots+n_k\omega_k:n_0,n_1,\cdots,n_k\in Z\}.$$

But by Theorem 3.2 1), $\mathcal{M}(E)=\mathcal{M}$. We thus have $\mathcal{M}(E)=\mathcal{M}(X)=\mathcal{M}(\hat{Y})=\mathcal{M}$.

3) follows from 2) and the fact that

$$\dim X=\mathrm{rank}X'=\mathrm{rank}\mathcal{M}(X)=\mathrm{rank}\mathcal{M}=\dim Y.$$

□

Remark 3.3. 1) *Consider* (3.1) *with periodic time dependence, i.e., $(Y,R)=(S^1,R)$ is a pure rotation of the circle. Then the conclusions of both* Theorem 3.2 *and* Corollary 3.1 *hold without any condition, because the existence of mean motion for* (3.1) *is automatic in the case of periodic time dependence* (Remark 3.1 1)). *In fact, since the Poincaré map in this case is a circle diffeomorphism, this fact also follows from the classical Denjoy theory and* Theorem 2.6*. As each Denjoy minimal set admits zero topological entropy, so does each almost automorphic minimal set of* (3.10) *in the periodic time dependent case.*

2) *We note that the flow Λ_t can well have almost automorphic dynamics without admitting mean motion. An example was given in* [47] *in which an equation of type* (3.1) *with limit periodic time dependence was constructed so that the respective skew-product flow admits an almost automorphic minimal set as an almost* 1-1

extension of the base flow but the corresponding rotation number is not contained in the frequency module of the base (*compare to* Theorem 3.2 1)). *However, this should be a non-generic case as we conjectured earlier.*

3) *In the case of* Corollary 3.1, *the skew-product flow* Λ_t *is a toral flow on* T^{k+1} *whose rotation set is however a single point* (ρ, ω). *Toral flows and maps have been extensively studied in cases that rotation sets have non-empty interiors in which having positive topological entropy was shown to be typical* (*see* [28, 55, 59] *and references therein*) *but little has been known for cases of 'thin' or singleton rotation sets, except for certain cases of nearly quasi-periodic, 'twist' toral flows in which the existence of quasi-periodic orbits was shown in Moser's twist theorem* ([69]). *In this regard,* Corollary 3.1 *seems to suggest that almost automorphic dynamics should be an important subject to look into in toral flows with 'thin' or singleton rotation sets.*

4) *Assume the conditions of* Corollary 3.1. *The result of* [76] (*see* Remark 3.2 2)), *when applying to* (3.13), *implies that any solution* $\phi(t)$ *of* (3.1) *corresponds to two weak quasi-periodic solutions of the form*

$$\phi_{\pm}(t) = \rho t + x_{\pm}(\phi_0 + \rho t, y + \omega t), \;\; \phi_0 \in S^1, \; y \in T^k,$$

where $x_{\pm} : T^{k+1} \to R^1$ *are semi-continuous functions lying in between* $\inf(\phi(t) - \rho t - \phi_0)$ *and* $\sup(\phi(t) - \rho t - \phi_0)$. *Moreover, a such weak quasi-periodic solution becomes almost automorphic if* $(\phi_0, y) \in T^{k+1}$ *is a point of continuity of* $x_{\pm}$.

5) *The study of almost automorphic dynamics for the almost periodically forced skew-product circle flow* (3.2) *is also closely related to the Floquet theory for a two dimensional, almost periodic linear system of ODEs of the form*

$$x' = a(y \cdot t)x, \; x \in R^2, \; tr(a) \equiv 0. \tag{3.15}$$

It is well known that Floquet theory does not hold for such a system in general because the linear skew-product flow (*cocycle*) *generated by* (3.15) *can well have interval Sacker-Sell spectrum* ([82]). *On one hand, there are cases where Floquet theory does hold* (*see* [45, 93] *for certain almost periodic cases and* [21, 39, 51, 53, 73, 104] *for recent develop in quasi-periodic reducibility*), *and, on the other hand, as proposed by Johnson* ([45]) *one can seek for weak Floquet forms concerning finding an almost automorphic* (*not almost periodic*) *strong Perron transformation which transforms* (3.15) *into a canonical* (*upper triangular or diagonal*) *form with almost periodic coefficients. In the later case, an almost automorphic strong Perron transformation is determined by an almost automorphic minimal set* (*if it exists*) *of the skew-product circle flow of the form* (3.2) *which is generated by the angular equation associated with the polar-angle reduction of* (3.15). *We refer the readers to* [45, 85] *for more discussions in this regard.*

4. Almost automorphic oscillations

Multi-frequency oscillations arise naturally in many electrical and mechanical systems. Physical examples include electric networks, power systems, quasi-periodic velocity field of fluid flows, plasma dynamics, mechanical vibrations and coupled biological oscillators, etc., among which almost periodically forced second order oscillators form an important class. In engineering applications, periodic oscillations are referred to as harmonic oscillations and almost periodic ones are interpreted as harmonic oscillations covered with small 'noise'. Accordingly, almost automorphic oscillations can be regarded as harmonic ones covered with big 'noise'.

There have been extensive studies on periodically forced second order oscillators on issues such as strange attractors, period doubling cascades, mode locking, quasi-periodicity, intermittency, etc. (see [15, 16, 34, 36] and references therein). Concerning almost periodically forced second order oscillators, attention has been mainly on the existence of almost periodic oscillations (e.g., [4, 10, 16, 69]) and chaotic dynamics (e.g., [8, 7, 14, 101]). In general, as suggested by recent numerical works, (e.g., [13, 14, 18, 35, 80, 81]), typical non-chaotic dynamics of almost periodically (in particular, quasi-periodically) forced nonlinear, second order oscillators can be far more complicated than almost periodic ones. Here we would like to argue that almost automorphic dynamics should be a main (if not the only) factor largely responsible for such non-chaotic dynamical complexities arising in almost periodically (even periodically) forced second order oscillators.

4.1. **Linear and quasi-linear oscillators.** Consider an almost periodic linear oscillator

$$\ddot{x} + a(y \cdot t)\dot{x} + b(y \cdot t)x = f(y \cdot t),\ y \in Y, \tag{4.1}$$

where $a, b, f \in C(Y)$. In terms of the phase variable $u = (x, \dot{x})$, the equation (4.1) becomes the linear planar system

$$\dot{u} = A(y \cdot t)u + B(y \cdot t), \tag{4.2}$$

where

$$A(y) = \begin{pmatrix} 0 & 1 \\ -b(y) & -a(y) \end{pmatrix},\ \ B(y) = \begin{pmatrix} 0 \\ -f(y) \end{pmatrix}.$$

The system (4.2) clearly generates a skew-product flow π_t on $R^2 \times Y$. The following was stated in [93], Remark 4.3.

Proposition 4.1. *Assume that for some $y_0 \in Y$, $\int_0^t a(y_0 \cdot s)ds$ is bounded and the equation* (4.1) *corresponding to y_0 has a bounded solution with bounded derivative, then there is an almost automorphic minimal set $E \subset R^2 \times Y$ of the flow π_t satisfying $\mathcal{M}(E) \subset \mathcal{M}(Y)$.*

The proposition implies the existence of almost automorphic solutions in (4.1) for residually many $y \in Y$ which harmonically respond to the external frequencies (the frequency module of Y). The existence of almost automorphic dynamics in the case described in the proposition is particularly significant when the homogeneous part of (4.2) admits a nontrivial bounded solution and an unbounded solution for all y (especially when its Sacker-Sell spectrum is a non-degenerate interval) since the existence of almost periodic ones are not generally expected in this case.

Consider the quasi-linear case

$$\ddot{x} + \alpha\dot{x} = f(x, y \cdot t), \tag{4.3}$$

where $\alpha \neq 0$ is a constant. The equations generate a planar skew-product flow, again denoted by π_t, on $R^2 \times Y$, the same way as above. It is shown in [77] that such equations need not admit any almost periodic solutions and the usual upper-lower solutions techniques for periodic time dependence are not applicable to (4.3) to yield the existence of almost periodic solutions. Nevertheless, using the approach in [83], one can obtain almost automorphic solutions of (4.3) for some y (or equivalently, an almost automorphic minimal set of π_t) in between upper and lower solutions.

In the case that f is quasi-periodic time dependent in (4.3), i.e., $(Y, R) = (T^k, R) : y \cdot t = y + \omega t$, for some k, is a quasi-periodic flow on the torus T^k with toral frequency $\omega \in R^k$, the following was shown in [76].

Theorem 4.1. *Assume that for some $y_0 \in T^k$ the equation* (4.3) *has a bounded solution $x(t)$ such that*

$$\frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2} \geq -\frac{\alpha^2}{4}$$

for all $\inf x(t) \leq x_2 < x_1 \leq \sup x(t)$ and $y \in T^k$. Then (4.3) *admits two families of weak quasi-periodic solutions $x_-(y + \omega t)$, $x_+(y + \omega t)$, $y \in T^k$, satisfying* (3.11), (3.12)*, where $x_-, x_+ : T^k \to R^1$ are lower and upper semi-continuous, respectively.*

As remarked in Remark 2.1, each continuity point $y_0 \in T^k$ of $x_\pm$ yields an almost automorphic solution of (4.3) for $y = y_0$, which then gives rise to an almost automorphic minimal set of the skew-product flow $(R^2 \times T^k, R)$ generated by (4.3).

4.2. **Damped nonlinear oscillators.** Below, we let $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ be a small parameter.

There are two types of almost periodically forced nonlinear oscillators for which the results in the previous section can be applied naturally. One type concerns a damped, perturbative, nonlinear oscillator of the form:

$$x'' + a(x, x') = \varepsilon b(x, x', y \cdot t), \ x \in R^1, \tag{4.4}$$

where $a : R^2 \to R^1$ is C^1, $b : R^2 \times Y \to R^1$ are uniformly Lipschitz in $y \in Y$ and C^1 in other variables whose derivatives are also continuous in $y \in Y$.

Proposition 4.2. *Suppose that as $\varepsilon = 0$* (4.4) *admits a hyperbolic periodic solution. Then for ε sufficiently small it has an integral manifold on which the equation reduces to the form*

$$\phi' = 1 + \varepsilon f(\phi, y \cdot t, \varepsilon), \tag{4.5}$$

where $f : S^1 \times Y \times [-\varepsilon_0, \varepsilon_0] \to R^1$ is C^1 in ϕ, ε and Lipschitz in y.

Proof. The condition implies that the unperturbed part of the system

$$\begin{cases} u' &= v, \\ v' &= -a(u, v) + \varepsilon b(u, v, y \cdot t) \end{cases} \tag{4.6}$$

admits an invariant normally hyperbolic cycle. Let $\phi \in S^1$ be an angular coordinate parameterizing the cycle and I be an associated normal coordinate. Then in the vicinity of the cycle there is a change of coordinate of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = h(\phi) + g(I, \phi, y \cdot t),$$

where $h : S^1 \to R^2, g : R^1 \times S^1 \times Y \to R^2$ are C^1 in ϕ, Lipschitz in y, and $g(0, \phi, y) \equiv 0$, under which (4.6) has the form

$$\begin{cases} I' &= \lambda I + \tilde{g}(I, \phi) + \varepsilon \tilde{g}(I, \phi, y \cdot t, \varepsilon), \\ \phi' &= 1 + \tilde{f}(I, \phi) + \varepsilon \tilde{f}(I, \phi, y \cdot t, \varepsilon), \end{cases} \tag{4.7}$$

where $\lambda \neq 0$, $\tilde{g} = O(|I|^2)$, $\tilde{f} = O(|I|)$, and, for fixed I, ε, all terms are multiple periodic with periods 1 ([105]). The proposition now follows from the general integral manifolds theorem ([37, 105]). □

Example (*van der Pol equation*): A special case of (4.4) is the following quasi-periodically forced van der Pol equation

$$x'' - \alpha(1-x^2)x' + x = \varepsilon b(x, x', y\cdot t), \qquad x \in R^1, \tag{4.8}$$

which models an electric circuit with a triode valve, the resistive properties of which change with current, where $\alpha > 0$ is a constant and b is as in (4.4). It is well known that the skew-product system associated with (4.8) admits a normally hyperbolic, asymptotically stable limit cycle for $\varepsilon = 0$, hence admits an integral manifold with flow governed by an equation of form (4.5). Thus, by Theorems 3.2, if for ε sufficiently small (4.5) admits mean motion, then almost automorphic oscillations will occur in the skew-product flow on $R^2 \times Y$ generated by the system associated with (4.8). □

Another type of application of the Theorem 3.2 concerns damped, almost periodically forced non-linear oscillators which admit certain periodic structures. Consider the following system

$$x'' + \beta x' + a(x, y\cdot t) = \varepsilon b(x, x', y\cdot t),\ x \in R^1, \tag{4.9}$$

where $\beta > 0$ is a constant, $a : S^1 \times Y \to R^1, b : S^1 \times R^1 \times Y \to R^1$ are bounded, uniform Lipschitz in y and C^1 in other variables whose derivatives are uniformly bounded. We re-write (4.9) into the following equivalent system

$$\begin{cases} \phi' &= \psi, \\ \psi' &= -\beta\psi - a(\phi, y\cdot t) + \varepsilon b(\phi, \psi, y\cdot t), \end{cases} \tag{4.10}$$

where $\phi \in S^1$.

The following result is a slight generalization of a similar result contained in [87] for the quasi-periodic case.

Proposition 4.3. *Let $\alpha = \sup_{S^1\times Y} |\frac{\partial a}{\partial \phi}|$. If $\beta > 2\sqrt{\alpha}$, then as ε sufficiently small, (4.12) admits an attracting integral manifold on which the equation reduces to the form*

$$\phi' = f(\phi, y\cdot t, \varepsilon), \tag{4.11}$$

where $f : S^1 \times Y \times [-\varepsilon_0, \varepsilon_0] \to R^1$ is C^1 in ϕ, ε and Lipschitz in y.

Proof. We only show the existence and Lipschitz continuity of the integral manifold.

After reversing time $t \to -t$, (4.10) becomes

$$\begin{cases} \phi' &= -\psi, \\ \psi' &= \beta\psi + a(\phi, y\cdot(-t)) - \varepsilon b(\phi, \psi, y\cdot(-t)). \end{cases} \tag{4.12}$$

Fix $\eta > 0, 0 < \alpha' < \alpha$ such that

$$\frac{\beta - \sqrt{\beta^2 - 4(\alpha-\alpha')}}{2} < \eta < \min\{\beta - \sqrt{\alpha}, \frac{\beta + \sqrt{\beta^2 - 4(\alpha-\alpha')}}{2}\}.$$

One can certainly make ε sufficiently small so that

$$\frac{\alpha + \varepsilon\gamma((\eta + \frac{\alpha'}{\beta} + 2) + \varepsilon(\eta + \frac{\alpha'}{\beta}))}{\beta - \eta} \le \eta,$$

$$\frac{\alpha + \varepsilon\gamma}{(\beta-\eta)^2} + \frac{\varepsilon\gamma}{\beta} < 1, \tag{4.13}$$

where $\gamma = \text{Lip } b$.

Let X be the set of Lipschitz continuous functions h on $S^1 \times Y$ with $\max|h| \le K \equiv (\max|a| + \varepsilon \max|b|)/\beta$ and $\mathrm{Lip}_\phi h \le \eta, \mathrm{Lip}_y h \le \eta + \alpha'/\beta$. With the sup-norm $\|\cdot\|$, X is a complete metric space.

Define mapping $T : X \to X$:

(4.14)

$$Th(\phi_0, y_0) = -\int_0^\infty e^{-\beta s}(a(\phi(s), y_0\cdot(-s)) - \varepsilon b(\phi(s), h(\phi(s), y_0\cdot(-s)), y_0\cdot(-s)))\mathrm{d}s,$$

where $\phi(t) = \phi(\phi_0, y_0, h, t)$ is the solution of

$$\left\{\begin{array}{ll}\phi' & = -h(\phi, y_0\cdot(-t)),\\ \phi(0) & = \phi_0.\end{array}\right. \tag{4.15}$$

We first check that T is well defined. Let $h \in X$. It is easy to see that $\|Th\| \le K$ and

$$\begin{aligned}\mathrm{Lip}_{\phi_0}(Th) &\le \int_0^\infty e^{-\beta s}(\alpha + \varepsilon\gamma\eta + \varepsilon\gamma)\mathrm{Lip}_{\phi_0}\phi(s)\mathrm{d}s, & (4.16)\\ \mathrm{Lip}_{y_0}(Th) &\le \int_0^\infty e^{-\beta s}((\alpha + \varepsilon\gamma\eta)\mathrm{Lip}_{\mathrm{y}_0}\phi(s) + \varepsilon(\alpha' + \eta + 2\gamma))\mathrm{d}s. & (4.17)\end{aligned}$$

Using (4.15) and applying Gronwall's inequality ([38]), we have $\mathrm{Lip}_{\phi_0,y_0}\phi(t) \le e^{\eta t}$, which, when substituting into (4.16), (4.17) yields that

$$\mathrm{Lip}_{\phi_0}(Th) \le \eta,\ \ \mathrm{Lip}_{y_0}(Th) \le \eta + \frac{\alpha'}{\beta}.$$

Next, we show that T is a uniform contraction mapping. Let $h_j \in X$ and $\phi_j(t) = \phi(\phi_0, y_0, h_j, t)$ be the solution of (4.15) with $h := h_j$ for $j = 1, 2$ respectively. An application of Gronwall's inequality yields that

$$|\phi_1(t) - \phi_2(t)| \le \|h_1 - h_2\| t e^{\eta t}.$$

It then follows from (4.14) that

$$\begin{aligned}\|Th_1 - Th_2\| &\le \int_0^\infty e^{-\beta s}((\alpha + \varepsilon\gamma)|\phi_1(s) - \phi_2(s)| + \varepsilon\gamma\|h_1 - h_2\|)\mathrm{d}s\\ &\le \int_0^\infty e^{-\beta s}((\alpha + \varepsilon\gamma)s e^{\eta s} + \varepsilon\gamma)\mathrm{d}s\|h_1 - h_2\|\\ &= (\frac{\alpha + \varepsilon\gamma}{(\beta - \eta)^2} + \frac{\varepsilon\gamma}{\beta})\|h_1 - h_2\|.\end{aligned}$$

Thus, by (4.13), T is a uniform contraction mapping.

Now, let h_* be the fixed point of T. It is easy to see from (4.14) that

$$M = \{(h_*(\phi, y), \phi, y) : \phi \in S^1, y \in Y\}$$

is an (topological) invariant manifold of the skew-product flow generated by (4.12), i.e., $h_*(\phi, y\cdot(-t))$ defines an integral manifold of (4.12) with flow governed by the equation

$$\phi' = -h_*(\phi, y\cdot(-t)).$$

Changing t back to $-t$, we then have

$$\phi' = h_*(\phi, y\cdot t)$$

which is invariant to the system (4.10).

To show that M is attracting, we let $x = \psi - h(\phi, y \cdot t)$. By (4.10),

$$x' = -\beta x + \varepsilon b(\phi, h(\phi, y \cdot t), y \cdot t).$$

It follows from Gronwall's inequality that

$$|x(t)| \leq x(0)e^{-(\beta - O(\varepsilon))t} \to 0, \text{ as } t \to \infty.$$

□

The following example was considered in [87] with respect to quasi-periodic forcing, which also holds in the case of almost periodic forcing.

Example (*Josephson junction*): A particular case of (4.7) is the almost periodically forced Josephson junction equation, or, Damped pendulum:

$$x'' + \beta x' + \sin x = F(y \cdot t) \tag{4.18}$$

which arises in many applications ranging from supersensitive detectors to superfast computers, where $F : Y \to R^1$ is a Lipschitz function. By considering the change of variable $x \to 2\pi x$, the equation (4.18) becomes

$$x'' + \beta x' + \frac{\sin 2\pi x}{2\pi} = \frac{F(y \cdot t)}{2\pi}$$

which is in the form of (4.9) with $b \equiv 0$ and

$$a(x, y) \equiv \frac{1}{2\pi}(\sin 2\pi x - F(y)).$$

Applying Proposition 4.3 to (4.18) for $\alpha = 1$, we then obtain a desired integral manifold for $\beta > 2$, with flow governed by an equation in the form of (4.11). Thus, by Theorems 3.2 the skew-product flow generated by the system associated with (4.18) will admit almost automorphic oscillations as long as it admits mean motion.

As observed in [87], mean motion holds on the integral manifold above when $\|F\|$ is sufficiently small. To see this, we note that when $F \equiv 0$, the integral manifold clearly contains a relative equilibrium $\phi = 0$ which is also hyperbolic. It follows that as long as $\|F\|$ is sufficiently small the skew-product flow on the integral manifold admits an almost periodic minimal set which is an 1-cover of the base. Hence by Remark 3.2 1) the skew-product flow on the integral manifold admits mean motion with rotation number $\rho = 0$. □

Remark 4.1. *Equations* (4.8), (4.18) *have been used as models to numerically study the existence of so-called* non-chaotic strange attractors *which typically arise in damped, quasi-periodically forced oscillators such as the Josephson junction and van der Pol* (*e.g.,* [14, 35, 81]). *Roughly speaking, a non-chaotic strange attractor is one which is geometrically strange* (*it is neither a finite set, a closed curve, a smooth surface, nor a volume bounded by a piecewise smooth closed surface*) *but not dynamically chaotic* (*i.e., the Lyapunov exponents on the attractor are non-positive*).

In the case that mean motion exists the above discussions suggest that basic dynamics on a non-chaotic strange attractor of almost periodically forced Josephson junction or van der Pol are almost automorphic (*i.e., such an attractor is formed by almost automorphic minimal sets along with 'connecting orbits'*). *The topological structure of such an attractor will depend on whether the respective rotation number is in resonant with the forcing frequencies. As an almost* 1-1 *extension of a solenoid* *as* Corollary 3.1 *suggests, an almost automorphic minimal set on such a non-chaotic strange attractor can be a fractal* (*geometric complexity*) *and be almost everywhere non-locally connected* (*topological complexity*). *Therefore, the structures of almost automorphic minimal sets should be responsible for part of the complexity of a non-chaotic strange attractor.*

To be able to capture the full complexity of a non-chaotic strange attractor, one should also consider the existence of weak quasi-periodic orbits lying in the attractor (*see* Remark 3.3 4)*, because, as shown in* [76]*, an quasi-periodically forced system can have a complicated attractor with 'simple' almost automorphic orbits.*

4.3. **Chaotic almost automorphic oscillations.** Consider a planar system of the form

$$\dot{u} = f(u) + \varepsilon g(u, y \cdot t), \ u \in R^2. \tag{4.19}$$

Assume that for $\varepsilon = 0$ (4.19) is a Hamiltonian system admitting a hyperbolic equilibrium u_0 along with a homoclinic orbit $u = \gamma(t)$. It follows that the skew-product flow $(R^2 \times Y, \pi_t)$ generated by (4.19) admits a smooth family of almost periodic minimal sets Λ_ε which are 1-covers of Y and skew-hyperbolic (i.e., hyperbolic in the u-direction, see e.g., [105]).

Define the Melnikov functional

$$M(y)(t) = \int_{-\infty}^{\infty} f(\gamma(s)) \wedge g(\gamma(s), y \cdot (t+s)) ds.$$

Suppose that the zero set

$$Z = \{y \in Y : M(y)(0) = 0\}$$

is *simple*, i.e.,

$$\frac{d}{dt} M(y)(t)|_{t=0} \neq 0$$

for all $y \in Z$. Then Z is a global cross section for the flow on Y ([66]). Hence $\Sigma = R^2 \times Z$ is a global cross section for the skew-product flow π_t on $R^2 \times Y$. Let $\Psi : R^2 \times Z \to R^2 \times Z$ be the induced Poincaré map.

Theorem 4.2. ([66]) *Suppose that Z is simple. Then there exists an $\varepsilon_0 > 0$ such that for every $0 < |\varepsilon| < \varepsilon_0$ there is a positive integer n_0 such that for all $n \geq n_0$ there is a compact invariant set $\Omega_n \subset R^2 \times Z$ of Ψ whose dynamics is topologically conjugated to the skew-shift map $A \otimes \eta$ on the Bernoulli bundle $\Sigma \times Z$, where A is the full shift on the space of Σ of bi-infinite sequences on n symbols and η is the Poincaré map on Y based on the global cross section Z.*

Chaotic almost automorphic minimal sets have been known to largely exist in symbolic flows. It was first observed in [30, 60] that almost automorphic symbolic flows can exhibit positive topological entropy and lack unique ergodicity. Recently, Toeplitz sequences as a special class of almost automorphic sequences have received considerable attention (see [1, 19, 20, 32, 43, 44, 60, 78, 103] and references therein). While regular Toeplitz sequences are uniquely ergodic with zero topological entropy, it was shown that irregular Toeplitz sequences are typically not uniquely ergodic and exhibit positive topological entropy, though they may also be uniquely ergodic with positive entropy ([32, 43]).

In [9], chaotic almost automorphic symbolic arrays were also shown to be mainly responsible for the complexity (e.g., spatial chaos) of certain lattice dynamical systems. As a particular case, we have the following result for symbolic sequences.

Theorem 4.3. *Consider the full shift dynamics A on the space Σ of bi-infinite sequences of n-symbols. Let Σ_+ denote the space of half infinite sequences of the same symbols. Then the following holds.*

1) *There is a dense set $\Lambda \subset R$ of irrational numbers such that for any given $\rho \in (0, \log n)$ and $\gamma \in \Lambda$ there exists a residual set $\mathcal{R} \subset \Sigma_+$ such that each $\omega \in \mathcal{R}$ corresponds to an almost automorphic minimal set $M(\omega)$ of (Σ, A) which has the (discrete) circle rotation (S^1, γ) as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h(M(\omega)) > \rho$.*
2) *Given a prime number p there is a residual set $\mathcal{R} \subset \Sigma_+$ such that each $\omega \in \mathcal{R}$ corresponds to an almost automorphic minimal set $M(\omega)$ of (Σ, A) which has the p-adic odometer group $(\Delta, \oplus)$ as its maximal almost periodic factor, is not uniquely ergodic and has topological entropy $h(M(\omega)) = \log n$.*

The theorem says that the response of chaos in symbolic dynamics to almost automorphic sequences can be made optimal: there are almost automorphic sequences with one frequency whose orbit closures admit topological entropy arbitrarily close to the maximal entropy of the system and there are almost automorphic sequences with infinitely many frequencies whose orbit closures attain the maximal entropy. Recall that having infinitely many periodic sequences is a main feature of chaos in symbolic dynamics. A chaotic almost automorphic sequence attaining maximal entropy should be the (non-uniform) limit of these periodic ones. This links the two commonly adopted definitions of chaos for symbolic dynamics, i.e., the one defined by positive topological entropy and the one requires sensitivity dependence on initial conditions, topological transitivity, and a dense set of periodic orbits.

In the periodically forced case, the Melnikov functional above reduces to the classical Melnikov function. It is known that if the Melnikov function admits a simple zero then the Poincaré map associated with (4.19) admits Smale horseshoe which is topologically conjugated to the full shift on two symbols ([36]).

Using Proposition 2.2 and Theorem 4.3, we then have the following.

Theorem 4.4. *Consider the skew-product flow $(R^2 \times Y, \pi_t)$ generated by (4.19) with periodic time dependence, i.e., $(Y, R) = (S^1, R)$ is a pure rotation of the circle. If Z is simple, then there is an $\varepsilon_0 >$ such that the following holds for all $0 < |\varepsilon| \le \varepsilon_0$.*

1) *There is a dense set $\Lambda \subset R$ of irrational numbers such that for any given $\rho \in (0, \log 2)$, $\gamma \in \Lambda$ there is an almost automorphic minimal set E_ε, with $\mathcal{M}(E_\varepsilon)$ being the additive subgroup of R generated by $\{\gamma, 1\}$, which is an almost 1-cover of the 2-torus, is not uniquely ergodic and has the topological entropy $h(E_\varepsilon) > \rho$.*
2) *Given a prime number p, there is an almost automorphic minimal set E_ε, with $\mathcal{M}(E_\varepsilon)$ being the additive subgroup of R generated by $\mathcal{P} = \{1/p^l : l = 0, 1, \cdots\}$, which is an almost 1-cover of the 2-solenoid, is not uniquely ergodic and has the topological entropy $h(E_\varepsilon) = \log 2$.*

Remark 4.2. *The above result shows the existence of rather chaotic almost automorphic oscillations in periodically forced nonlinear oscillators of form* (4.19) *if the Melnikov function admits a simple zero. A natural problem is to extend* Theorem 4.3 *to Bernoulli bundles then use* Theorem 4.2 *to obtain chaotic almost automorphic oscillations in the almost periodically forced case.*

4.4. **Hamiltonian systems.** Hamiltonian systems is an area in which the existence of almost automorphic dynamics is much less understood, especially for higher degrees of freedom. The well known Denjoy and Aubry-Mather theories have already given strong evidence for such existence in the case of lower degrees of freedom.

As shown in [70], given an area preserving, orientation preserving, boundary components preserving, twist homeomorphism ϕ of the annulus $A = \{(x,u) \in S^1 \times R^1, a \le u \le b\}$, there exists a time periodic Hamiltonian $H(x,u,t)$, $(x,u) \in S^1 \times R^1$ of period 1, with $H_x|_{u=a,b} = 0$, $H_{uu} > 0$, whose Poincaré map of the associated motion agrees with ϕ. This, when coupled with Proposition 2.2 and Theorem 2.6, provides an evidence for the existence of almost automorphic solutions of two frequencies in Hamiltonian systems of one and one-half degrees of freedom.

Remark 4.3. *Given a time quasi-periodic Hamiltonian system* $H(x,u,\omega t)$, $(x,u) \in S^1 \times R^1$ *with* k (> 1) *frequencies* $\omega \in R^k$, *a natural problem is to develop a variational technique to show the existence of almost automorphic minimal sets* (*higher dimensional analogue to the Aubry-Mather sets*) *of* $k+1$ *frequencies* (ρ, ω), *where* ρ *is a rotation number, in the associated Hamiltonian skew-product flow under suitable conditions.*

This problem is already significant when $H(x,u,\omega t)$ takes the special form $u^2/2 + V(x,\omega t)$, corresponding to the second order undamped, quasi-periodic time dependent, nonlinear oscillator

$$\ddot{x} + V_x(x,\omega t) = 0,\ x \in S^1,\ V \in C^\infty(T^{k+1}, R^1), \tag{4.20}$$

which can be viewed as a degenerate elliptic equation on the $k+1$-torus. The existence of Aubry-Mather (in fact, almost automorphic) action minimizing solutions for an elliptic equation on $k+1$-torus has been shown in [71, 72] using a variational approach. Similar to [71], a crucial step in tackling the above problem with respect to (4.20) would be to obtain the mean motion property, i.e., for any lifted solution $x(t) \in R^1$ of (4.20) or equivalently any minimizer of the respective Lagrangian there is a unique rotation number ρ such that $|x(t) - \rho t|$ is bounded. This is also closely related to the action-minimizing problems in Hamiltonian systems of higher degrees of freedom (see [63, 64, 65]).

An easier problem is to consider a variational problem for (4.20) without assuming the periodic dependence of V on x. It seems that under reasonable conditions one can obtain almost automorphic dynamics of the respective skew-product flow which respond to ω harmonically. We will discuss this problem in more detail elsewhere.

Almost automorphic dynamics can also exist in Poisson-Hamilton systems defined on a Poisson manifold rather than a symplectic one. In action-angle coordinates, such a system has the form

$$\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = J(y,x)\nabla H(y,x), \quad y \in R^l,\ x \in T^n, \tag{4.21}$$

where J is called a *structure matrix* - a skew-symmetric matrix-valued function satisfying the Jacobi identity (see [57, 58] and references therein). Now let us consider the action-angle-angle case (i.e. $l = 1, n = 2$) with $H_{yy} \neq 0$ for all (x, y) and

$$J = \begin{pmatrix} 0 & -\alpha a(y) & -\beta a(y) \\ \alpha a(y) & 0 & -\gamma \\ \beta a(y) & \gamma & 0 \end{pmatrix},$$

where α, β, γ are constants, $|\alpha|+|\beta|+|\gamma| \neq 0$, and $a(y) \neq 0$ for all y. Then it is clear that each energy surface $M_E = \{H(y,x) = E\}$ is an invariant 2-torus. Moreover, if (α, β) is non-resonant, then the flow on any such invariant 2-torus M_E admits neither fix points nor periodic orbits. By Theorem 2.6, M_E admits a unique 2-frequency almost automorphic minimal set which is not necessarily almost periodic in general. However, for certain nearly integrable case, it was shown in [58] that if (α, β) is Diophantine, then the majority of these 2-tori will be quasi-periodic.

Remark 4.4. 1) *A related problem is to develop a variational framework for Poisson-Hamilton systems and use it to study the existence of almost automorphic dynamics.*

2) *Almost automorphic dynamics may exist even in quasi-periodically forced or autonomous, nearly integrable Hamiltonian systems, as possible intermediate orbits in between KAM tori and stochastic layers. Such intermittency has been investigated numerically as the so-called* Cantori *whose topological natures are quite similar to that of almost automorphic minimal sets. The existence of non-KAM intermediate orbits has been theoretically justified in* [40] *based on an idea of J. C. Yoccoz as the so-called* Lagrangian tori *using a topological approach for fixed small non-integrable Hamiltonian perturbations and in* [11] *using a variational approach when the perturbations increase in sizes and the KAM theory becomes invalid. Indeed, the approach of taking weak limits to KAM tori adopted in* [40] *is in spirit quite similar to* Theorem 2.4 1) *in obtaining almost automorphic functions* (*and minimal sets*). *It would be interesting to know whether some of these immediate orbits can actually be almost automorphic. Linking this problem with* Theorem 2.4 1), *an immediate question is whether some KAM tori form a joint almost automorphic sequence.*

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