

# Higher Dimensional Multi-Fractal Model: Filtering via Simulation

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## **Abstract**

Multi-fractal processes have recently been introduced as a new tool for modeling the stylized facts in financial time series, and it is becoming more and more popular due to its well capturing the long memory property. In this paper, we introduce multivariate multi-fractal model based on our previous proceeding works. We implement its estimation via GMM and bayesian approach, as its high dimensional structure, we employ particle filter to implement simulation based inference. Monte Carlo studies for their estimation performances are conducted.

Keyword: Long memory, Multi-fractal, Multivariate model, Particle filter.

## **1 Introduction**

Financial markets display some similarities to fluid turbulence. For example, both turbulent fluctuations and financial fluctuations display intermittency at all scales. A cascade of energy flux is known to occur from the large scale of injection to the small scale of dissipation and it is typically modeled by a multiplicative cascade, which then leads to multi-fractal field. the Multi-Fractal (MF) Model has been recently introduced as an alternative formalization to financial time series, the model conceives volatility as a hierarchical, multiplicative process with heterogeneous components. The essential new feature of MF models is their ability of generating different degrees of long-term dependence in various powers of returns - a feature pervasively found in empirical financial data, see Ding et al. (1993).

Research on Multi-Fractal models originated from statistical physics, cf. Mandelbrot (1974). The initial MF proposed by Mandelbrot et al. (1997),

named the Multi-Fractal Model of Assets Returns (MMAR), assumes that returns  $x(t)$  follow a compound process:

$$x(t) = B_H[\theta(t)] \quad (1)$$

in which an incremental fractional Brownian motion with index  $H$ ,  $B_H[\cdot]$ , is subordinate to the cumulative distribution function  $\theta(t)$  of a multi-fractal measure, in the earlier work of Mandelbrot (1974), it was employed to model the distribution of energy in turbulent dissipation.

However, the model used in physics is of a combinatorial nature and suffer from non-stationarity due to the limitation to a bounded interval and the non-convergence of moments in the continuous-time limit. This major weakness was overcome by introducing a iterative version of the multi-fractal model, cf. Calvet and Fisher (2001), Calvet and Fisher (2004). In their approach, returns are modeled as:

$$x_t = \sigma \left( \prod_{i=1}^k M_t^{(i)} \right)^{1/2} \cdot u_t \quad (2)$$

with a constant scale parameter  $\sigma$  and increment  $u_t$  drawn from a standard Normal distribution  $N(0, 1)$ . Thus, instantaneous volatility being determined by the product of  $k$  volatility components or multipliers  $M_t^{(1)}$ ,  $M_t^{(2)}$  ...,  $M_t^{(k)}$ , certain constrains can be arbitrary imposed in  $E[M_t^{(i)}]$  or  $E[\sum M_t^{(i)}]$  for the sake of normalizing the time-varying components of volatility. Furthermore, each volatility component is renewed at time  $t$  with probability  $\gamma_i$  depending on its rank within the hierarchy of multipliers or remains unchanged with probability  $1 - \gamma_i$ . The transition probabilities are specified as:

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^{k-i})} \quad (3)$$

with parameters  $\gamma_1 \in [0, 1]$  and  $b \in (1, \infty)$ . Whereas Lux (2003) and Lux (2005) use analogous specification:

$$\gamma_i = 2^{-(k-i)} \quad (4)$$

which leads to an very close approximate to that of eq. 3 with parsimonious parameters (by fixing  $b = 2$  and  $\gamma_1 = 2^{-(k-1)}$ ), and eq. 4 is sufficient to capture the hierarchical structure in the transitional probability, that is, volatility component in a lower frequency state will be renewed as twice as often as its neighbor (a higher one), and it happens with certainty for the component of highest frequency ( $i = k$ ).

The main attraction of Multi-Fractal model is that it shares certain properties of asset returns: fat tails and long memory (asymptotic power-law behavior of the autocovariance function, i.e.  $Cov(|x_t|^q, |x_{t+\tau}|^q) \propto \tau^{2d(q)-1}$ ). Furthermore, multifractality implies that different powers of the measure have different decay rates of their autocovariances, see Calvet and Fisher (2004). In this sense,

other alternatives like FIGARCH or ARFIMA models belong to the catalogue of uni-fractal model, i.e. they have the same decay rate for all moments.

As the increasing requirements in empirical research, multi-variate settings are necessary, in particular it is now well accepted that financial volatilities move together over time across assets and markets. This is particularly important when considering asset allocation, value-at-risk and portfolio hedging strategies for a basket assets. Secondly, since the information on the source of long memory in the volatility process is quite limited, the Multivariate model may provide additional insight into the factors responsible for long memory.

The rest of this paper is organized as follows: Section 2 provides the multi-variate Multi-fractal model of financial returns and implement its estimation via GMM and Maximum likelihood approaches; Section 3 introduces a simulation based inference for multivariate MF model; A preliminary conclusion in Section 4.

## 2 The Higher Dimensional Multi-Fractal Model

By extending the Bivariate MF case, time series for  $N$  assets are assumed to have the same overall number of volatility cascades, which contain the  $k$  joint cascades level, after  $k$ th level each series has separate additional multifractal components.

$$r_{q,t} = \sigma_q \cdot \left[ \left( \prod_{i=1}^k M_t^{(i)} \right) \cdot \left( \prod_{l=k+1}^n M_t^{(l)} \right) \right]^{1/2} \cdot u_{q,t} \quad (5)$$

$q = 1, 2, \dots, N$  is the number of assets.  $\sigma_q$  are the unconditional standard deviation of the return series, it is scale parameter.  $u_{q,t}$  is  $N \times 1$  vector, whose elements are  $N$ -variate normal distribution, by considering three asset portfolio  $N = 3$ , that is, a multivariate normal distribution in three variables (tri-variate Normal) with correlation matrix:

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

In addition, we restrict the specification of the transition probabilities to:

$$\gamma_j = 2^{-(k'-j)} \quad (6)$$

Each component is renewed at time  $t$  with probability  $\gamma_i$  depending on its rank within the hierarchy of multipliers and remains unchanged with probability  $1 - \gamma_i$ .

We specify the multipliers to be random draws from either a Binomial or Lognormal distribution, In the binomial case in which we assume two draws

$m_0 \in (0, 2)$  and alternative  $m_1 = 2 - m_0$ , for the latter, we assume  $-\log_2 M \sim N(\lambda, \sigma^2)$ , and assign constraint  $E[M_t^{(i)}] = 0.5$  which leads to  $\sigma_m^2 = 2(\lambda - 1)/\ln 2$ .

Figure 1 and Figure 2 show simulations of the tri-variate multi-fractal model ( $k = 4, n = 15$ ) with Binomial distribution of its multipliers together with its ACFs. The simulation apparently shares some of the stylized facts of financial time series, namely volatility clustering and hyperbolic decay of the autocorrelation function. One also easily recognizes the correlation in the volatility of both time series.

Historically, the first attempt at estimating the multi-fractal models is the scaling estimator. Since multifractal measures are characterized by a non-linear scaling function of moments (scaling law), through a Legendre transformation, parameter estimation is achieved by matching the empirical and hypothetical spectrum of Hölder exponents. In our proceeding higher dimensional MF model, we will, however, exclude the scaling estimator due to its bias and lack of asymptotic distribution theory, cf. Lux (2003), Lux (2004).

## 2.1 Generalized Method of Moments Estimation

We adopt the GMM (Generalized Method of Moments) approach by Hansen (1982) with analytical solutions of a set of appropriate moment conditions. In the GMM approach, the vector of parameter estimates of the model, say  $\beta$ , can be obtained as:

$$\hat{\beta} = \arg \min_{\beta \in \Theta} \bar{M}(\beta)' W \bar{M}(\beta) \quad (7)$$

with  $\beta$  the parameter vector,  $\bar{M}(\beta)$  the vector of differences between sample moments and analytical moments, and  $W$  a positive definite weighting matrix, which controls the over-identification when applying GMM. Implementing (7), one typically starts with the identity matrix, then the inverse of the covariance matrix obtained from the first round estimation is used as the weighting matrix in the next step, and the procedure will continue until the estimates and weighting matrices converge. Under suitable conditions,  $\hat{\beta}$  is consistent and asymptotically converges to  $T^{1/2}(\hat{\beta} - \beta_0) \sim N(0, \Xi)$  with covariance matrix  $\Xi$ .

The applicability of GMM for multi-fractal models has been discussed by Lux (2003). The approach recommended in this paper is using log differences of absolute returns together with the pertinent analytical moment conditions, i.e. to transform the observed data  $r_t$  into  $T$ th differences of the log observations:

$$R_{t,T} = \ln |r_t| - \ln |r_{t-T}| \quad (8)$$

with  $\varepsilon_t^{(i)} = \ln \left( M_t^{(i)} \right)$ .

GMM provides more convenient and efficient way towards the estimation of higher dimensional MF model, it allows to treat each couple of time series as bivariate case. In order to exploit as much information as possible, the moment conditions that we consider include two categories:

the first set of conditions is obtained by considering moments for cross time series observations:

$$Cov[R_{t,1}, R_{t,1}^-];$$

$R_{t,1}^-$  represents any one other time series rather than  $R_{t,1}$ . In trivariate case, we have 3 moments corresponding to 3 pairs time series. The second set of moment conditions is the moments for autocovariance:

$$Cov[R_{t+T,T}, R_{t,T}];$$

for  $T = 1, 5, 10, 20$ . Furthermore, for either category, we select moment conditions for  $R_{t,T}$  and  $R_{t,T}^2$ , i.e. moments of the raw transferred observations and square transferred observations. The detailed analytical moments for each pair are given in Liu and Lux (2005). Table 1 and Table 2 show the performance of the GMM estimator.

We proceed by conducting several Monte Carlo experiments as Andersen and Sorensen (1996) and Lux (2003) to explore the performance of the GMM estimation. We simulate 100,000 data in each iteration, and randomly choose three different sub-sample with sample sizes  $N_1 = 2000$ ,  $N_2 = 5000$ , and  $N_3 = 10000$ , which is robust to discover the estimation performance. We start with the Binomial Model ( $n = 12$ ) with number of joint cascades  $k = 3$ , we fixed multipliers  $m_0 = 1.3$ , the scale parameters (unconditional variance)  $\sigma_1 = \sigma_2 = \sigma_3 = 1$ , and the three correlation parameters. Table 1 shows the statistical result of the GMM estimator: for the Binomial distribution parameter  $\hat{m}_0$ , not only the bias but also the finite sample standard deviation and root mean squared error show quite encouraging behavior. It is also the case for other parameters, even in the small sample size  $N = 2000$  and  $N = 5000$ , the average bias of the Monte Carlo estimates is moderate, and particularly close to zero for the larger sample sizes  $N = 10000$ .

It is also interesting to note that our estimates are in harmony with  $T^{\frac{1}{2}}$  consistency, and the Hansen's  $J$  test reveals that there is not disappointing concerning the over-identification restrictions (see Figure 3). All these results can be viewed as a positive signal of the log transformation in practice.

One advantage of GMM is that it allows to estimate MF with a continuous distribution, following the work of Mandelbrot et al. (1997) and Calvet and Fisher (2002), we use the same Lognormal distribution  $-\log_2 M \sim N(\lambda, \sigma^2)$  in tri-variate model, and impose a restriction of  $E[M] = 0.5$ . In our Monte Carlo simulations reported in Table 2, we cover parameter values  $\lambda = 1.40$ , and other initial setting as in the Binomial case above. As can be seen, results are not too different from those obtained with the Binomial model: Biases are moderate again, SD and RMSE significantly decrease with increasing the sun-sample size

we also make the Monte Carlo study towards other different choice of parameter values, we skip them for saving some space as no significant divination is observed comparing with the performance above. All in all, the performance from both the Binomial and Lognormal Monte Carlo simulation and estimation shows that GMM seems to work quite well for multi-fractal processes both in the discrete and in the continuous state space.

### 3 Particle filter

The MF dynamics can be interpreted as a special case of a Markov-switching process with a large state space, this makes Maximum Likelihood Estimation feasible. In our tri-variate MF model, the state spaces is relatively finite when the multipliers follow a discrete distribution (i.e. Binomial distribution). The likelihood function can be derived by determining the exact form of each possible component in the transition matrix, and is similar to the likelihood function in Calvet et al. (2006), but differs in so far as the transition matrix of each multifractal component contains two starting cascade levels. We denote  $r_t$  is the set of joint return observations  $\{r_{q,t}\}$ , and have the likelihood function as below:

$$f(r_1, \dots, r_T; \Theta) = \prod_{t=1}^T f(r_t | M_t = m^i) \cdot (\pi_{t-1} A). \quad (9)$$

With the conditional density  $f(r_t | M_t = m^i)$ , and the transition matrix  $A$  which has components  $A_{ij} = P(M_{t+1} = m^j | M_t = m^i)$ , it is a  $8^n \times 8^n$  matrix in our tri-variate case. Both  $M_t$  and  $m^{(\cdot)}$  are vectors,  $M_t = (M_t^1, \dots, M_t^k, \dots, M_t^n)$ ,  $m_k^i$  denotes the  $k$ th component of vector  $m^i$ . The last unknown component within the likelihood function above is  $\pi_t$ , which is the conditional probability of volatility state given observations  $\pi_t^i = P(M_t = m^i | r_1, \dots, r_t)$ . We denote  $\pi_t = (\pi_t^1, \pi_t^2, \dots, \pi_t^R)$  and  $R = 8^n$ , we also know  $\sum_{i=1}^R \pi_t^i = 1$ . The conditional probability can be recursively defined through Bayesian updating, we get<sup>1</sup>

$$\pi_{t+1} = \frac{f(r_{t+1} | M_{t+1} = m^i) \otimes (\pi_t A)}{\sum f(r_{t+1} | M_{t+1} = m^i) \otimes (\pi_t A)} \quad (10)$$

Applicability of the ML approach is constrained by its computational demands: First, it is not applicable for models with an infinite state space, i.e. continuous distributions of the multipliers such as Lognormal distribution we use here. Secondly, even for the discrete distributions, say the Binomial case, we may also realize that Eq (9) requires to evaluate the transition matrix which has the size of  $R \times R$  in the tri-variate case, current computational limitations make choices of  $n$  beyond 3 unfeasible. Table 3 reports the Monte Carlo study (designed as previous section) of the ML estimator. It would be not too surprising that the ML estimators are more efficient compared with the two previous tables, as ML extracts all the information in the data. We also need to realize that Table 3 uses  $n = 3$  which is much smaller  $n = 12$  in Table 1, and more cascade levels generate more fluctuations.

To overcome this up-bound limit, we seek some support from simulation based inference. Recalling the bayesian updating of eq. (10) above, we can

<sup>1</sup> $\otimes$  represents element by element product.

think of

$$\sum_{j=1}^R P(M_{t+1} = m^i | M_t = m^j) \pi_t^j \quad (11)$$

as a prior probability  $P(M_{t+1} = m^i | r_t)$ , then combine it with the conditional density  $f(r_{t+1} | M_{t+1} = m^i)$  to generate a posterior, that is  $(P(M_{t+1} = m^i | r_{t+1}))$ . This procedure can also be expressed by considering that the conditional probabilities of current states ( $\pi_t$ ) are input, passing through a system of dynamic transformation which is transition probability matrix  $A$  here, to produce the conditional probabilities of future states ( $\pi_{t+1}$ ) as output. This procedure is called filtering.

As pointed out before, the likelihood becomes difficult-to-compute due to the dimension of the transition probability matrix increasing with exponential rate. Numerous attempts have been made to provide algorithms that approximate this filtering probabilities (cf. Gordon et al. (1993); Jacquier et al. (1994); Berzuini et al. (1997); Kim et al. (1998)). We use so called particle filter which is the class of simulation based filters that recursively approximate the filtering random variable by certain finite number of particles which are certain discrete points viewed as approximated samples from the prior. In our case, we evaluate eq. (10) by combining the conditional density with eq. (11) up to proportionality:

$$\pi_{t+1}^i \propto f(r_{t+1} | M_{t+1} = m^i) \sum_{j=1}^R P(M_{t+1} = m^i | M_t = m^j) \pi_t^j. \quad (12)$$

As particle filters treat the discrete support generate by the particles as the “true” filtering density, which allows us to produce an approximation to the prediction probability density  $P(M_{t+1} = m^i | r_t)$ , by using the discrete support of the particles, and then one step ahead conditional probability is

$$\pi_{t+1}^i \propto f(r_{t+1} | M_{t+1} = m^i) \sum_{b=1}^B P(M_{t+1} = m^i | M_t = m^{(b)}) \quad (13)$$

This leaves us only one issue - how to make these finite number of draws? We adopt Sampling/Importance Resampling (SIR) introduced by Rubin (1987); Pitt and Shephard (1999).  $B$  draws  $M^{(1)}, \dots, M^{(B)}$  are taken independently from the prior, and then associate each of draws with a weight  $w_j$ , where

$$w_j = \frac{f(r_{t+1} | M_{t+1} = m^{(j)})}{\sum_{i=1}^B f(r_{t+1} | M_{t+1} = m^{(i)})}, \quad j = 1, \dots, B. \quad (14)$$

Instead of evaluating each exact component of  $R$  numbers of  $A$  associated with  $\pi_t^{(\cdot)}$ , SIR produces  $B$  draws (“particles”) from prior  $P(M_{t+1} = m^i | r_t)$  which are used to generate the approximation of the corresponding one step of

head conditional probability as eq. (13), and it will converge as increasing  $B$  (cf. Pitt and Shephard (1999)). This procedure avoids the extreme high dimensional evaluation.

By using particle filter, one step ahead density hence becomes:

$$\begin{aligned} f(r_t|r_1, \dots, r_{t-1}) &= \sum_{i=1}^R f(r_t|M_t = m^i)P(M_t = m^i|I_{t-1}) \\ &\approx \frac{1}{B} \sum_{b=1}^B f(r_t|M_t = m^b), \end{aligned} \tag{15}$$

then the approximate likelihood function is given below:

$$\begin{aligned} g(r_1, \dots, r_T; \Theta) &= \prod_{t=1}^T f(r_t|r_1, \dots, r_{t-1}) \\ &\approx \prod_{t=1}^T \left[ \sum_{b=1}^B f(r_t|M_t = m^{(b)}) \right]. \end{aligned} \tag{16}$$

Table 4 reports estimation results through particle filter. Simulations are based on the tri-variate Binomial Multi-Fractal process with  $n = 6$ ,  $k = 2$ , which is not possible to implement via ML under the computational ability of personal computer. We make Monte Carlo study designed as previous tables, the results also demonstrates the positive performance.

## 4 Conclusion

In this paper we extend our previous work to the higher dimensional Multi-Fractal model, and implement its estimation by both GMM and exact Maximum Likelihood estimation. To overcome the computational restrictions on the choice of the number of cascade levels, we also employ simulation based inference via particle filter.

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Table 1: GMM estimation for the trivariate multifractal Binomial model

$\hat{\theta}$	Sub-sample Size	<i>Bias</i>	<i>SD</i>	<i>RMSE</i>
$\hat{m}_0$	$N_1$	-0.1214	0.1198	0.1705
	$N_2$	-0.0464	0.0724	0.0859
	$N_3$	-0.0150	0.0405	0.0432
$\hat{\sigma}_1$	$N_1$	-0.0088	0.0813	0.0817
	$N_2$	-0.0010	0.0560	0.0559
	$N_3$	-0.0004	0.0397	0.0396
$\hat{\sigma}_2$	$N_1$	0.0001	0.0868	0.0867
	$N_2$	-0.0008	0.0565	0.0564
	$N_3$	-0.0034	0.0393	0.0394
$\hat{\sigma}_3$	$N_1$	-0.0016	0.0894	0.0893
	$N_2$	-0.0026	0.0526	0.0526
	$N_3$	-0.0020	0.0402	0.0402
$\hat{\rho}_{12}$	$N_1$	0.0328	0.1194	0.1237
	$N_2$	0.0150	0.0826	0.0838
	$N_3$	-0.0004	0.0632	0.0631
$\hat{\rho}_{23}$	$N_1$	0.0145	0.0704	0.0718
	$N_2$	0.0075	0.0483	0.0488
	$N_3$	-0.0044	0.0345	0.0347
$\hat{\rho}_{13}$	$N_1$	0.0086	0.0455	0.0462
	$N_2$	-0.0015	0.0379	0.0379
	$N_3$	-0.0033	0.0227	0.0229

Note: Simulations are based on the Trivariate Binomial Multi-Fractal process with  $n = 12$ ,  $k = 4$ , and initial value  $m_0 = 1.3$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $\sigma_3 = 1$ ,  $\rho_{12} = 0.3$ ,  $\rho_{23} = 0.5$ ,  $\rho_{13} = 0.7$ . Sample lengths are  $N_1 = 2,000$ ,  $N_2 = 5,000$  and  $N_3 = 10,000$ . Bias denotes the distance between the given and estimated parameter value, SD and RMSE denote the standard deviation and root mean squared error, respectively. For each scenario, 400 Monte Carlo simulations have been carried out.

Table 2: GMM estimation for the trivariate multifractal Lognormal model

$\hat{\theta}$	Sub-sample Size	<i>Bias</i>	<i>SD</i>	<i>RMSE</i>
$\hat{\lambda}$	$N_1$	-0.0435	0.0418	0.0603
	$N_2$	-0.0122	0.0295	0.0319
	$N_3$	-0.0007	0.0193	0.0193
$\hat{\sigma}_1$	$N_1$	-0.0094	0.1945	0.1945
	$N_2$	-0.0025	0.1225	0.1224
	$N_3$	-0.0031	0.0843	0.0842
$\hat{\sigma}_2$	$N_1$	-0.0294	0.1987	0.2006
	$N_2$	-0.0098	0.1225	0.1227
	$N_3$	0.0010	0.0896	0.0895
$\hat{\sigma}_3$	$N_1$	0.0005	0.2094	0.2091
	$N_2$	0.0017	0.1248	0.1246
	$N_3$	0.0058	0.0864	0.0865
$\hat{\rho}_{12}$	$N_1$	0.0176	0.1335	0.1345
	$N_2$	-0.0013	0.0951	0.0950
	$N_3$	-0.0243	0.0733	0.0772
$\hat{\rho}_{23}$	$N_1$	0.0159	0.0694	0.0711
	$N_2$	-0.0033	0.0490	0.0490
	$N_3$	-0.0151	0.0344	0.0375
$\hat{\rho}_{13}$	$N_1$	0.0003	0.0396	0.0395
	$N_2$	-0.0075	0.0259	0.0269
	$N_3$	-0.0122	0.0179	0.0217

Note: Simulations are based on the Trivariate Binomial Multi-Fractal process with  $n = 12$ ,  $k = 4$ , and initial value  $\lambda = 1.2$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $\sigma_3 = 1$ ,  $\rho_{12} = 0.3$ ,  $\rho_{23} = 0.5$ ,  $\rho_{13} = 0.7$ . Sample lengths are  $N_1 = 2,000$ ,  $N_2 = 5,000$  and  $N_3 = 10,000$ . Bias denotes the distance between the given and estimated parameter value, SD and RMSE denote the standard deviation and root mean squared error, respectively. For each scenario, 400 Monte Carlo simulations have been carried out.

Table 3: ML estimation

$\hat{\theta}$	Sub-sample Size	<i>Bias</i>	<i>SD</i>	<i>RMSE</i>
$\hat{m}_0$	$N_1$	-0.0106	0.0170	0.0200
	$N_2$	-0.0096	0.0118	0.0152
	$N_3$	-0.0101	0.0075	0.0126
$\hat{\sigma}_1$	$N_1$	-0.0016	0.0222	0.0221
	$N_2$	-0.0004	0.0140	0.0140
	$N_3$	0.0002	0.0086	0.0086
$\hat{\sigma}_2$	$N_1$	-0.0033	0.0227	0.0228
	$N_2$	-0.0017	0.0128	0.0128
	$N_3$	-0.0010	0.0087	0.0087
$\hat{\sigma}_3$	$N_1$	-0.0033	0.0227	0.0228
	$N_2$	-0.0017	0.0128	0.0128
	$N_3$	-0.0010	0.0087	0.0087
$\hat{\rho}_{12}$	$N_1$	-0.0033	0.0227	0.0228
	$N_2$	-0.0017	0.0128	0.0128
	$N_3$	-0.0010	0.0087	0.0087
$\hat{\rho}_{23}$	$N_1$	-0.0033	0.0227	0.0228
	$N_2$	-0.0017	0.0128	0.0128
	$N_3$	-0.0010	0.0087	0.0087
$\hat{\rho}_{13}$	$N_1$	0.0099	0.0208	0.0230
	$N_2$	0.0108	0.0123	0.0163
	$N_3$	0.0110	0.0079	0.0135

Note: Simulations are based on the Trivariate Binomial Multi-Fractal process with  $n = 3$ ,  $k = 1$ , which is almost the limit of computational feasibility, and initial value  $m_0 = 1.3$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $\sigma_3 = 1$ ,  $\rho_{12} = 0.3$ ,  $\rho_{23} = 0.5$ ,  $\rho_{13} = 0.7$ . Sample lengths are  $N_1 = 2,000$ ,  $N_2 = 5,000$  and  $N_3 = 10,000$ . Bias denotes the distance between the given and estimated parameter value, SD and RMSE denote the standard deviation and root mean squared error, respectively. For each scenario, 400 Monte Carlo simulations have been carried out.

Table 4: Particle filter

$\hat{\theta}$	Sub-sample Size	<i>Bias</i>	<i>SD</i>	<i>RMSE</i>
$\hat{m}_0$	$N_1$	0.0056	0.0110	0.0113
	$N_2$	0.0146	0.0294	0.0318
	$N_3$	0.0339	0.0228	0.0404
$\hat{\sigma}_1$	$N_1$	0.0329	0.0189	0.0390
	$N_2$	0.0445	0.0108	0.0457
	$N_3$	0.0399	0.0167	0.0430
$\hat{\sigma}_2$	$N_1$	0.0333	0.0165	0.0383
	$N_2$	0.0477	0.0122	0.0421
	$N_3$	0.0359	0.0247	0.0430
$\hat{\sigma}_3$	$N_1$	0.0326	0.0182	0.0384
	$N_2$	0.0468	0.0136	0.0486
	$N_3$	0.0409	0.0176	0.0442
$\hat{\rho}_{12}$	$N_1$	0.0320	0.0210	0.0394
	$N_2$	0.0407	0.0207	0.0423
	$N_3$	0.0383	0.0194	0.0426
$\hat{\rho}_{23}$	$N_1$	0.0360	0.0177	0.0413
	$N_2$	0.0417	0.0190	0.0455
	$N_3$	0.0379	0.0208	0.0429
$\hat{\rho}_{13}$	$N_1$	0.0245	0.0191	0.0318
	$N_2$	0.0429	0.0069	0.0434
	$N_3$	0.0382	0.0167	0.0415

Note: Simulations are based on the Trivariate Binomial Multi-Fractal process with  $n = 4$ ,  $k = 2$ , which is almost the limit of computational feasibility, and initial value  $m_0 = 1.3$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $\sigma_3 = 1$ ,  $\rho_{12} = 0.4$ ,  $\rho_{23} = 0.5$ ,  $\rho_{13} = 0.6$ . Sample lengths are  $N_1 = 2,000$ ,  $N_2 = 5,000$  and  $N_3 = 10,000$ . Bias denotes the distance between the given and estimated parameter value, SD and RMSE denote the standard deviation and root mean squared error, respectively. For each scenario, 400 Monte Carlo simulations have been carried out.

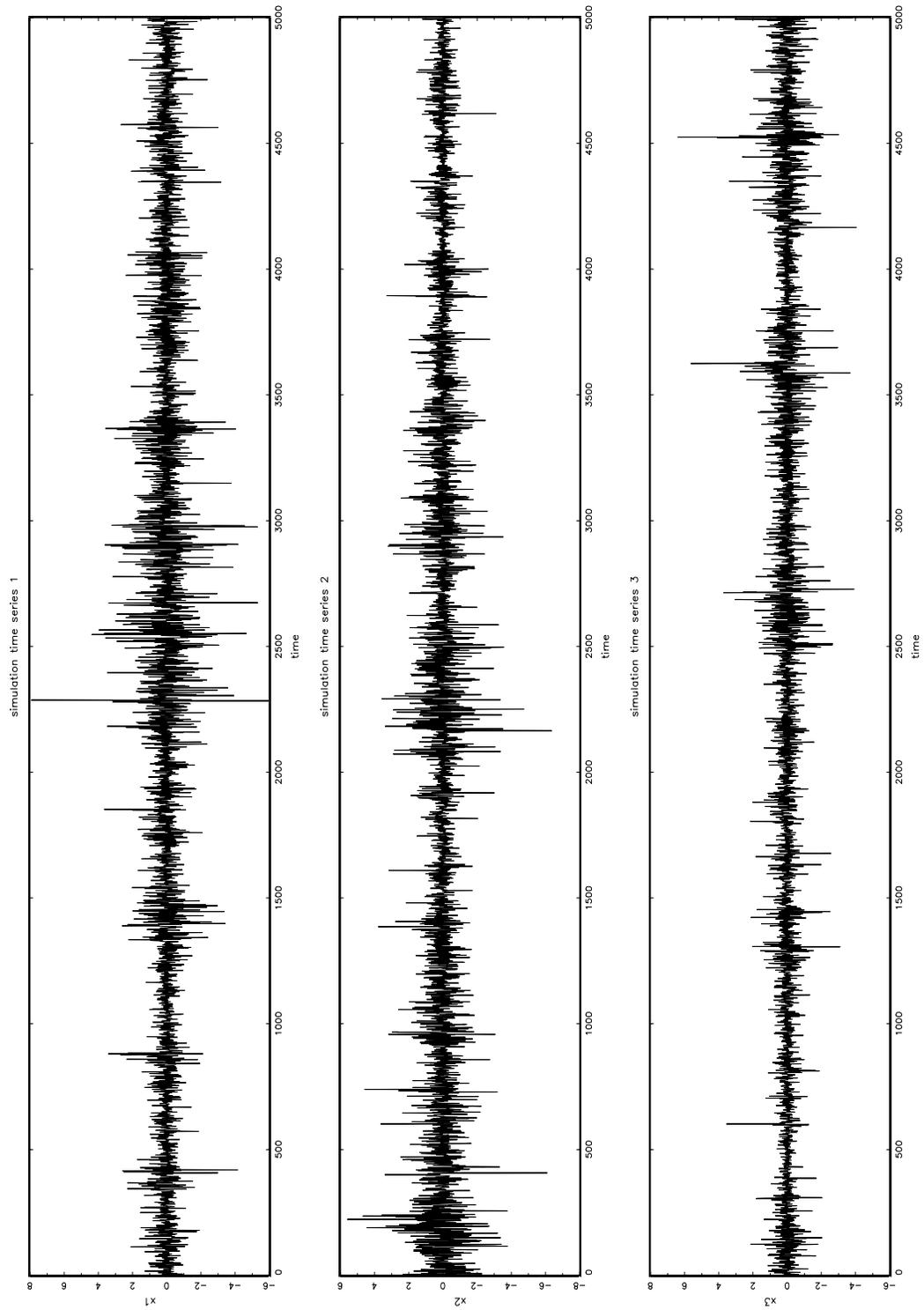


Figure 1: Simulation of the multivariate Binomial Multi-Fractal Model with  $m_0 = 1.3$ ,  $\sigma_1 = \sigma_2 = 1$ ,  $\sigma_3 = 1$ ,  $\rho = 0.4$ ,  $\rho = 0.5$ ,  $\rho = 0.6$ .

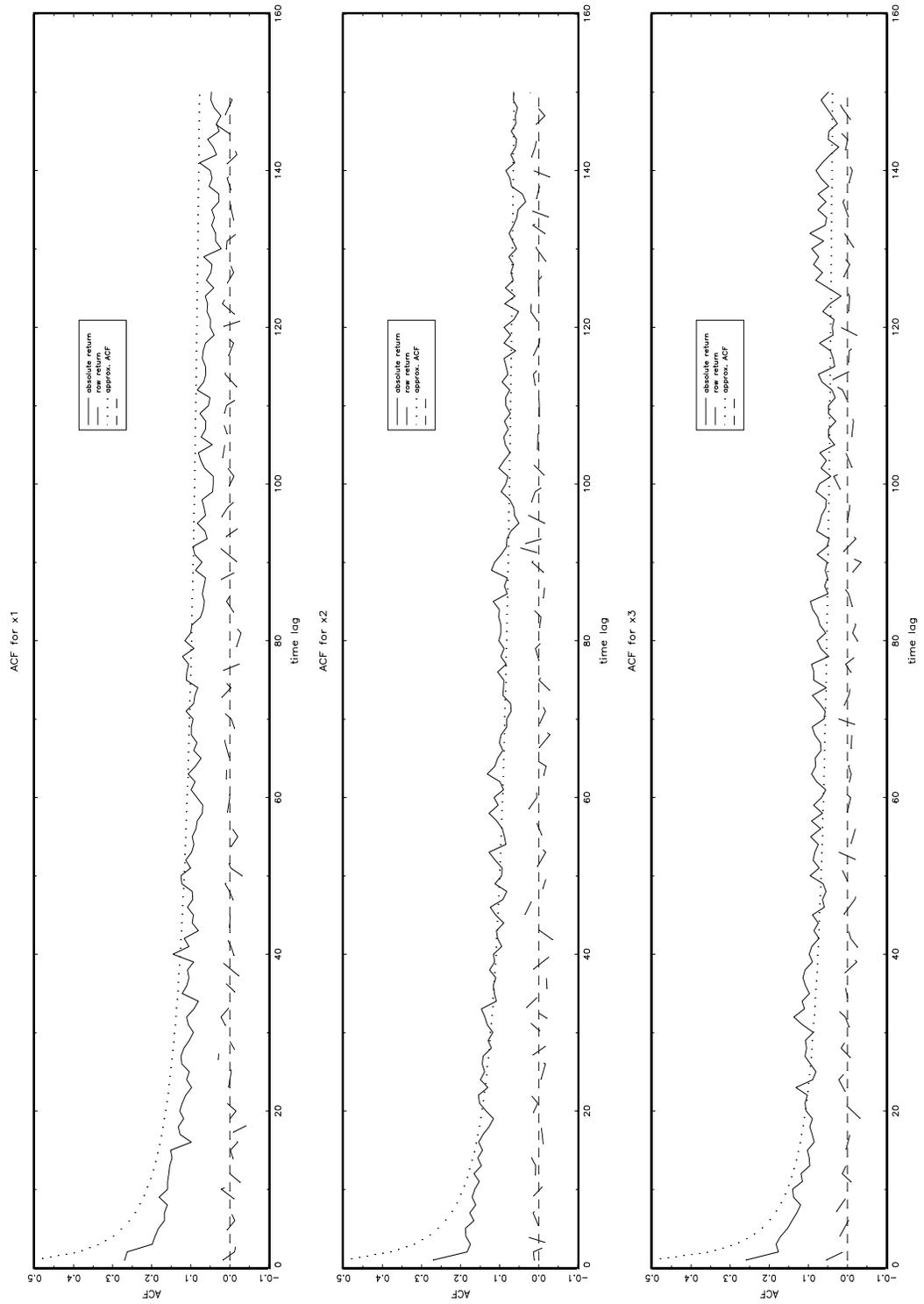


Figure 2: ACF for the Simulation of the multivariate Binomial Multi-Fractal Model above.

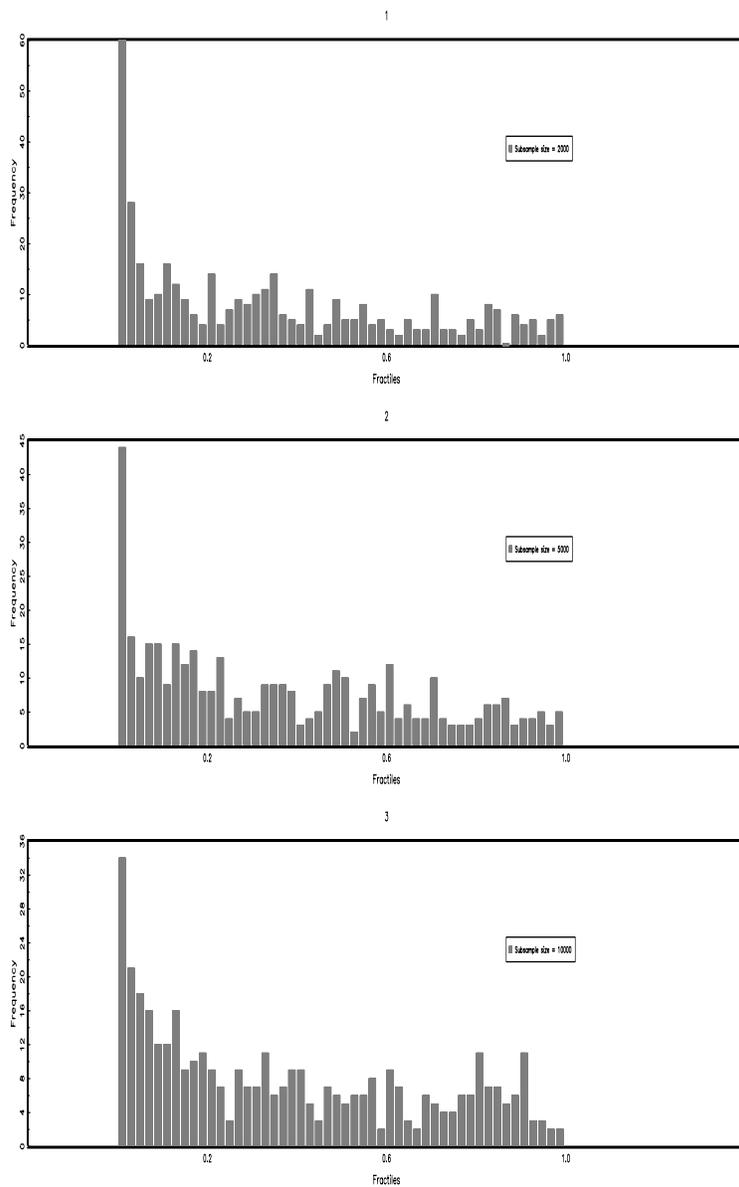


Figure 3: The distribution of  $p$  value for the test of overidentification restrictions for trivariate Binomial MF. Three figures from up to down corresponding to three different sample size:  $N_1 = 2,000$ ,  $N_2 = 5,000$  and  $N_3 = 10,000$ .