

LOCALIZATION THEOREMS IN TOPOLOGICAL HOCHSCHILD HOMOLOGY AND TOPOLOGICAL CYCLIC HOMOLOGY

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ABSTRACT. We construct localization cofiber sequences for the topological Hochschild homology (THH) and topological cyclic homology (TC) of spectral categories. Using a “global” construction of the THH and TC of a scheme in terms of the perfect complexes in a spectrally enriched version of the category of unbounded complexes, the sequences specialize to localization cofiber sequences associated to the inclusion of an open subscheme. These are the targets of the cyclotomic trace from the localization sequence of Thomason-Trobaugh in K -theory. We also deduce versions of Thomason’s blow-up formula and the projective bundle formula for THH and TC .

1. INTRODUCTION

Algebraic K -theory provides a powerful and subtle invariant of schemes. The K -theory of a scheme encodes many of its arithmetic and algebraic properties, captures information about its geometry and singularities, and is closely connected to its étale and motivic cohomology. One of the fundamental underpinnings of the subject is the localization theorem of Thomason and Trobaugh [38, 7.4], which for a quasi-separated quasi-compact scheme X provides a cofiber sequence of (non-connective) K -theory spectra

$$K(X \text{ on } (X - U)) \longrightarrow K(X) \longrightarrow K(U),$$

for U a quasi-compact open subscheme contained in X . Here $K(X \text{ on } (X - U))$ denotes the K -theory of the category of perfect complexes on X which are supported on the complement of U in X . This localization sequence and the closely related Mayer-Vietoris sequence for K -theory allow global assembly of local information.

Keller [25] constructed the analogue of the Thomason-Trobaugh localization sequence for Hochschild homology (HH) and for the variants of cyclic homology, including negative cyclic homology (HC^-). The Dennis trace (or Chern character) connects the localization sequence in K -theory to the localization sequence in HC^- . Using this, together with generalizations to blow-ups along regular sequences and some resolution of singularity results, Cortiñas, Haesemeyer, Schlichting, and Weibel [8, 9, 10] recently resolved Weibel’s conjecture bounding below the negative K -groups and Vorst’s conjecture that K_{d+1} -regularity implies regularity, for finite-type schemes of dimension d over a field of characteristic zero.

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The purpose of this paper is to generalize Keller’s localization sequences to topological Hochschild homology (THH) and topological cyclic homology (TC). Over the course of the last two decades, THH and TC have revolutionized K -theory computations. Roughly, topological Hochschild homology for a ring is obtained by promoting the ring to a ring spectrum and substituting the smash product of spectra for the tensor product of rings in the Hochschild complex [4]. The THH spectrum comes with a “cyclotomic” structure (which involves an S^1 -action and extra structure maps), and for each prime p , topological cyclic homology is then defined as a certain homotopy limit over the fixed point spectra. The Dennis trace map lifts to a “cyclotomic trace” map from K -theory to TC [5], and McCarthy [29] showed that this captures all the relative information at p for surjections with nilpotent kernel, just as HC^- does rationally [18]. Starting from Quillen’s computation of the K -theory of finite fields, Hesselholt and Madsen have used TC to make extensive computations in K -theory [19, 20, 21].

Between TC and THH is an intermediate theory called TR that has the structure of a “Witt complex” (the structure whose universal example is the de Rham-Witt complex of Bloch-Deligne-Illusie). The Hesselholt-Madsen computations proceed by studying this structure on TR . Hesselholt has observed that in all known examples, the de Rham-Witt complex has the same relationship to TR that Milnor K -theory has to algebraic K -theory. This led Hesselholt and Madsen to conjecture an “additive” motivic spectral sequence converging to a modified version of TR with edge homomorphism the universal map from the de Rham-Witt complex. Recent work of Levine [26] axiomatizes the role of localization and Mayer-Vietoris theorems in the construction of the motivic spectral sequence [2, 16], and such theorems for TR should provide key input to the construction of this conjectural “additive” motivic spectral sequence. We prove the following results in this direction.

Theorem 1.1. *Let X be a quasi-compact and semi-separated scheme. For a quasi-compact open subscheme U , there are homotopy cofiber sequences*

$$\begin{aligned} THH(X \text{ on } (X - U)) &\longrightarrow THH(X) \longrightarrow THH(U) \\ TR(X \text{ on } (X - U)) &\longrightarrow TR(X) \longrightarrow TR(U) \\ TC(X \text{ on } (X - U)) &\longrightarrow TC(X) \longrightarrow TC(U), \end{aligned}$$

where $THH(X \text{ on } (X - U))$ denotes the THH of the spectral category of perfect complexes on X which are supported on $X - U$.

For quasi-compact open subschemes U, V with $X = U \cup V$, the squares

$$\begin{array}{ccccc} THH(X) & \longrightarrow & THH(U) & & TR(X) & \longrightarrow & TR(U) & & TC(X) & \longrightarrow & TC(U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ THH(V) & \longrightarrow & THH(U \cap V) & & TR(V) & \longrightarrow & TR(U \cap V) & & TC(V) & \longrightarrow & TC(U \cap V) \end{array}$$

are homotopy cocartesian.

Geisser and Hesselholt [17] proved the second statement for THH of rings and used it to define THH of schemes in terms of Thomason’s hypercohomology construction [36, 1.33]. The relative term $THH(X \text{ on } (X - U))$ does not have an intrinsic description in the context of the Geisser-Hesselholt definition of THH .

Instead, it is most naturally described in terms of a construction of THH for *spectral categories*, i.e., categories enriched over symmetric spectra, the stable homotopy theory refinement of DG-categories.

Dundas and McCarthy [14] generalized Bökstedt's construction of THH to spectral categories. We build on the foundations there and study more general invariance properties; see in particular Theorems 4.9 and 4.12 below. We use these invariance properties to generalize the localization theorem of Keller to the setting of spectral categories. Roughly, we show that the THH of a triangulated quotient is the cofiber on THH ; theorems 6.1 and 6.2 provide precise statements. Although we work in the context of spectral categories, our localization theorem specializes to the setting of DG-categories, as DG-categories may be functorially converted to spectral categories with the same objects and spectral refinements of the Hom complexes; see for example [31, §6], [13, App A], or Appendix A, among others. Just as THH of a ring captures much more torsion information than HH of the ring, THH provides a richer invariant of a DG-category than HH . Using the model theory comparison of spectral categories and DG-categories, we make the following observation at the end of Section 3.

Theorem 1.2. *THH , TR , and TC as defined in Section 3 are functors from the category of small DG-categories and DG-functors to the stable category.*

We define THH of a scheme in terms of a spectral category refinement $\mathcal{D}_{\text{parf}}^S(X)$ of the DG-category quotient $\mathcal{D}_{\text{parf}}^{DG}(X)$ modeling the derived category of perfect complexes. In Section 7, we prove the following consistency theorem that compares this definition to the definition of Geisser-Hesselholt.

Theorem 1.3. *Let X be a quasi-compact and semi-separated scheme, and $\mathcal{D}_{\text{parf}}^S(X)$ a spectral category refinement of $\mathcal{D}_{\text{parf}}^{DG}(X)$. Then $THH(\mathcal{D}_{\text{parf}}^S(X))$ is equivalent to the Thomason hypercohomology of the Zariski presheaf of symmetric spectra $U \mapsto THH(\mathcal{O}_U)$.*

This theorem in particular constructs a trace map from the K -theory of the scheme to $THH(\mathcal{D}_{\text{parf}}^S(X))$ and $TC(\mathcal{D}_{\text{parf}}^S(X))$. In Section 8, we show that the trace map factors through Thomason-Trobaugh's Bass non-connective K -theory spectrum using their spectral version of Bass' fundamental theorem. (Appendix B describes a direct construction of the trace for spectral categories like $\mathcal{D}_{\text{parf}}^S(X)$ that come from certain complicial Waldhausen categories.)

In addition to Theorem 1.1, we also establish THH and TC versions of two classical geometric calculations in algebraic K -theory using our general localization machinery. First, we prove the following formula for blow-ups along regular sequences. Discussions with Haesemeyer, Schlichting, and Weibel have led us to be optimistic that this will allow extension of their work with Cortiñas [9] to interesting cases in characteristic p . We state the theorem using the same notation [9, §1], and prove it in Section 7.

Theorem 1.4. *Let X be a quasi-compact and semi-separated scheme. Let $i: Y \subset X$ be a regularly embedded closed subscheme, $p: X \rightarrow X'$ the blowup along Y , $j: Y' \subset X'$ the exceptional divisor, and write q for the map $Y' \rightarrow Y$. Then the*

squares

$$\begin{array}{ccc}
 THH(X) & \xrightarrow{Lp^*} & THH(X') \\
 \downarrow Li^* & & \downarrow Lj^* \\
 THH(Y) & \xrightarrow{Lq^*} & THH(Y')
 \end{array}
 \qquad
 \begin{array}{ccc}
 TC(X) & \xrightarrow{Lp^*} & TC(X') \\
 \downarrow Li^* & & \downarrow Lj^* \\
 TC(Y) & \xrightarrow{Lq^*} & TC(Y')
 \end{array}$$

are homotopy cocartesian.

We also prove a projective bundle theorem [38, 4.1,7.3] in Section 7.

Theorem 1.5. *Let X be a quasi-compact and semi-separated scheme. Let \mathcal{E} be an algebraic vector bundle of rank r over X , and let $\pi: \mathbb{P}\mathcal{E}_X \rightarrow X$ be the associated projective bundle. Then a spectral lift of the derived functor*

$$\bigoplus_{i=0}^{r-1} \mathcal{O}_{\mathbb{P}\mathcal{E}_X}(-i) \otimes L\pi^*(-)$$

induces a weak equivalence

$$\prod_{i=0}^{r-1} THH(X) \longrightarrow THH(\mathbb{P}\mathcal{E}_X).$$

The paper is organized as follows. In Section 2, we review the basic definitions for spectral categories (categories enriched in symmetric spectra). As indicated above, this is the appropriate setting for studying THH , TR , and TC , and is a stable-homotopy theory generalization of the setting of DG-categories. In Section 3, we review the definition of THH of spectral categories due to Bökstedt [4] and Dundas-McCarthy [14]. Because of the work of Shipley [33], all technical hypotheses may now be omitted. We take the viewpoint, first articulated by Dwyer and Kan, that enriched mapping spaces (or spectra) encode the “higher homotopy theory” of a category, and we view THH , TR , and TC as invariants of the higher homotopy theory of the category, as is K -theory [39, 3]. In Section 4, we list several invariance theorems for THH in this context. Section 5 reviews an elementary tilting argument for THH , Proposition 5.2, originally due to Dennis and Waldhausen [40, p. 391]. We demonstrate how to apply the tilting argument to prove powerful comparison theorems. Using these techniques, in Section 6 we prove the general localization theorems 6.1 and 6.2, which we apply in Section 7 to prove Theorems 1.1, 1.3, 1.4, and 1.5 above. In Section 8, we extend the cyclotomic trace over Bass non-connective K -theory, using Thomason and Trobaugh’s spectral version of Bass’ fundamental theorem.

The paper contains three appendices. The first reviews the relationship of the homotopy theory of small DG-categories to the homotopy theory of small spectral categories. The second discusses the \mathcal{S}_\bullet construction and a canonical definition of a trace map for DG-Waldhausen categories. The third gives a version of Theorem 6.1 that is more useful in the context of spectral model categories.

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2. REVIEW OF SPECTRAL CATEGORIES

Modern constructions of the stable category with point-set level smash products allow easy generalization of the concepts of simplicial category or DG-category to the context of spectra. Symmetric spectra in particular often arise naturally as the refinement of mapping sets. In fact, symmetric ring spectra (the analogue of DG-rings) and categories enriched in symmetric spectra (the analogue of DG-categories) predated Smith's insight that the homotopy theory of symmetric spectra models the stable category. In older K -theory literature, they were called FSPs (or FSPs defined on spheres) and FSPs with many objects, respectively, and treatments generally included hypotheses on connectivity or convergence. A modern approach to THH and TC , taking advantage of [24] and especially [33] obviates the need for any such connectivity or convergence hypotheses. In this section, we review the definition of spectral categories, modules, and bimodules over spectral categories in terms of enriched category theory.

Definition 2.1. A *spectral category* is a category enriched over symmetric spectra. Specifically, a spectral category \mathcal{C} consists of:

- (i) A collection of objects $\text{ob } \mathcal{C}$ (which may form a proper class),
- (ii) A symmetric spectrum $\mathcal{C}(a, b)$ for each pair of objects $a, b \in \text{ob } \mathcal{C}$,
- (iii) A unit map $S \rightarrow \mathcal{C}(a, a)$ for each object $a \in \text{ob } \mathcal{C}$, and
- (iv) A composition map $\mathcal{C}(b, c) \wedge \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ for each triple of objects $a, b, c \in \text{ob } \mathcal{C}$,

satisfying the usual associativity and unit properties. We say that a spectral category is small when the objects $\text{ob } \mathcal{C}$ form a set.

We emphasize that the data in (iii) and (iv) consist of point-set maps (rather than maps in the stable category) and that “ \wedge ” denotes the point-set smash product of symmetric spectra. The definition of spectral functor between spectral categories is the usual definition of an enriched functor:

Definition 2.2. Let \mathcal{C} and \mathcal{D} be spectral categories. A *spectral functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ is an enriched functor. Specifically, a spectral functor consists of:

- (i) A function on objects $F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$, and
- (ii) A map of symmetric spectra $F_{a,b}: \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ for each pair of objects $a, b \in \text{ob } \mathcal{C}$,

which is compatible with the units and the compositions in the obvious sense.

Again, we emphasize that the compatibility condition holds in the point-set category of symmetric spectra rather than in the stable category. Often we use the term *weak equivalence* to mean a spectral functor that is a bijection on objects and a weak equivalence (stable equivalence of symmetric spectra) on all mapping spectra. See Definition 4.1 for a more general kind of equivalence.

We have the evident concepts of module and bimodule over spectral categories:

Definition 2.3. Let \mathcal{C} and \mathcal{D} be spectral categories. A left \mathcal{C} -module is a spectral functor from \mathcal{C} to symmetric spectra. A right \mathcal{D} -module is a spectral functor from \mathcal{D}^{op} to symmetric spectra. A $(\mathcal{D}, \mathcal{C})$ -bimodule is a spectral functor from $\mathcal{D}^{\text{op}} \wedge \mathcal{C}$ to symmetric spectra.

Here \mathcal{D}^{op} denotes the spectral category with the same objects and mapping spectra as \mathcal{D} but the opposite composition map. The spectral category $\mathcal{D}^{\text{op}} \wedge \mathcal{C}$ has as its objects the cartesian product of the objects,

$$\text{ob}(\mathcal{D}^{\text{op}} \wedge \mathcal{C}) = \text{ob} \mathcal{D}^{\text{op}} \times \text{ob} \mathcal{C},$$

and as its mapping spectra the smash product of the mapping spectra

$$(\mathcal{D}^{\text{op}} \wedge \mathcal{C})((d, c), (d', c')) = \mathcal{D}^{\text{op}}(d, d') \wedge \mathcal{C}(c, c'),$$

with unit maps the smash product of the unit maps and composition maps the smash product of the composition maps for \mathcal{D}^{op} and \mathcal{C} . Explicitly, a $(\mathcal{D}, \mathcal{C})$ -bimodule \mathcal{M} consists of a choice of symmetric spectrum $\mathcal{M}(d, c)$ for each d in $\text{ob} \mathcal{D}$ and c in $\text{ob} \mathcal{C}$, together with maps

$$\mathcal{C}(c, c') \wedge \mathcal{M}(d, c) \wedge \mathcal{D}(d', d) \longrightarrow \mathcal{M}(d', c')$$

for each d' in $\text{ob} \mathcal{D}$ and c' in $\text{ob} \mathcal{C}$, making the obvious unit and associativity diagrams commute. In particular, for any spectral category \mathcal{C} , the mapping spectra $\mathcal{C}(-, -)$ define a $(\mathcal{C}, \mathcal{C})$ -bimodule. (This example motivates the convention of listing the right module structure first.)

Older K -theory literature required “convergence” hypotheses on spectral categories and bimodules, asking for the homotopy groups of the constituent spaces in each mapping spectrum to stabilize. These hypotheses appeared necessary at the time to analyze the homotopy colimits arising in Bökstedt’s construction of THH . It was thought that these homotopy colimits could be wrong for a non-convergent symmetric spectrum because the homotopy groups they computed generally differed from the homotopy groups expected from the underlying prespectrum. Because of [24, 33], we now understand that it is the homotopy groups of the underlying prespectrum that may be wrong: The homotopy groups of the prespectrum underlying a symmetric spectrum X do not necessarily agree with the homotopy groups of the object represented by X in the stable category. In general, every symmetric spectrum X admits a (stable) weak equivalence $X \rightarrow \tilde{X}$ to a symmetric Ω -spectrum \tilde{X} , i.e., one whose underlying prespectrum is an Ω -spectrum (level fibrant with adjoint structure maps $\tilde{X}_n \rightarrow \Omega \tilde{X}_{n+1}$ weak equivalences). The correct homotopy groups of X are the homotopy groups of the underlying prespectrum of \tilde{X} ; when these agree under the comparison map with the homotopy groups of the underlying prespectrum of X , then X is said to be *semistable*. In particular, symmetric Ω -spectra and (more generally) convergent symmetric spectra are semistable. Since we do not include convergence or even semistability hypotheses, for brevity and clarity we adhere to the following convention.

Convention. The *homotopy groups* of a symmetric spectrum X will always mean the homotopy groups of X as an object of the stable category, and we will denote these as $\pi_* X$. In the rare cases when we need to refer to the homotopy groups of the underlying prespectrum of X , we will call them the homotopy groups of the underlying prespectrum, and we introduce no notation for these. Thus, a *weak equivalence* of symmetric spectra is precisely a map that induces an isomorphism on homotopy groups; it does not necessarily induce an isomorphism of the homotopy groups of the underlying prespectra.

Although we do not require convergence hypotheses, they tend to hold for examples of interest. In fact, we can replace an arbitrary spectral category with

a weakly equivalent spectral category that has the same objects but has mapping spectra that are symmetric Ω -spectra. One way of doing this is to apply the Quillen model category structures on small enriched categories with a fixed set of objects described in [31, §6]. The maps in the category are the spectral functors that are the identity on object sets, the fibrations are the maps $\mathcal{C} \rightarrow \mathcal{D}$ that restrict to fibrations of symmetric spectra $\mathcal{C}(x, y) \rightarrow \mathcal{D}(x, y)$ for all x, y and the weak equivalences are the maps that restrict to weak equivalences $\mathcal{C}(x, y) \rightarrow \mathcal{D}(x, y)$ for all x, y . Fibrant approximation then gives the following proposition.

Proposition 2.4. ([31, 6.3]) *Given a small spectral category \mathcal{C} , there exists a small spectral category \mathcal{C}^Ω and a spectral functor $R: \mathcal{C} \rightarrow \mathcal{C}^\Omega$ such that:*

- (i) \mathcal{C}^Ω has the same objects as \mathcal{C} and R is the identity map on objects,
- (ii) For every x, y objects of \mathcal{C} , $\mathcal{C}^\Omega(x, y)$ is a symmetric Ω -spectrum (i.e., fibrant in the stable model structure on symmetric spectra), and
- (iii) For every x, y objects of \mathcal{C} , the map $\mathcal{C}(x, y) \rightarrow \mathcal{C}^\Omega(x, y)$ is a weak equivalence of symmetric spectra.

The construction of \mathcal{C}^Ω can be made functorial in spectral categories with the same object sets. Because it is constructed by fibrant replacement, it can be made compatible with arbitrary spectral functors: For a spectral functor $F: \mathcal{C} \rightarrow \mathcal{D}$, choosing $\mathcal{D} \rightarrow \mathcal{D}^\Omega$ as above, we can pull back to a category $F^*\mathcal{D}^\Omega$ with the same objects as \mathcal{C} and with mapping spectra $F^*\mathcal{D}^\Omega(x, y) = \mathcal{D}^\Omega(Fx, Fy)$. Factoring the spectral functor $\mathcal{C} \rightarrow F^*\mathcal{D}$ in the Quillen model category of spectral ob \mathcal{C} -categories, we obtain a spectral functor $\mathcal{C} \rightarrow \mathcal{C}^\Omega$ as above and a strictly commuting diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}^\Omega & \longrightarrow & \mathcal{D}^\Omega. \end{array}$$

This approach extends to certain more complicated kinds of diagrams. Alternatively, functorial lifts of all diagrams follow from the model structure on all small spectral categories described in Appendix A where the maps are arbitrary spectral functors (without restriction on the object set).

Applying cofibrant approximation in a model structure, we obtain the following complementary proposition.

Proposition 2.5. ([31, 6.3]) *Given a small spectral category \mathcal{C} , there exists a small spectral category $\mathcal{C}^{\text{Cell}}$ and a spectral functor $Q: \mathcal{C}^{\text{Cell}} \rightarrow \mathcal{C}$ such that:*

- (i) $\mathcal{C}^{\text{Cell}}$ has the same objects as \mathcal{C} and Q is the identity map on objects,
- (ii) For every x, y objects of \mathcal{C} , $\mathcal{C}^{\text{Cell}}(x, y)$ is a cofibrant symmetric spectrum, and
- (iii) For every x, y objects of \mathcal{C} , the map $\mathcal{C}^{\text{Cell}}(x, y) \rightarrow \mathcal{C}(x, y)$ is a level equivalence of symmetric spectra.

We also use an analogous proposition in the setting of bimodules.

Proposition 2.6. *If \mathcal{M} is a cofibrant $(\mathcal{D}^{\text{Cell}}, \mathcal{C}^{\text{Cell}})$ -bimodule, then $\mathcal{M}(d, c)$ is cofibrant for every c in \mathcal{C} and d in \mathcal{D} .*

In addition to providing the formal technical results above, the model theory of enriched categories also explains the relationship of spectral categories to DG-categories. Sharp statements involve categories enriched over $H\mathbb{Z}$ -modules

(in symmetric spectra of simplicial sets) or Quillen equivalently, categories enriched over symmetric spectra of simplicial abelian groups. For brevity, we will call these $H\mathbb{Z}$ -categories and *Ab-spectral categories*, respectively. Note that the category of $H\mathbb{Z}$ -modules is symmetric monoidal under $\wedge_{H\mathbb{Z}}$ and its derived category is symmetric monoidally equivalent to the derived category of \mathbb{Z} (in particular, $\wedge_{H\mathbb{Z}}$ is more like $\otimes_{\mathbb{Z}}$ than like \wedge). Shipley [34, §2.2] produces a zigzag of “weak monoidal Quillen equivalences” relating $H\mathbb{Z}$ -modules to symmetric spectra of simplicial abelian groups to differential graded modules. For a fixed object set O , applying Proposition 6.4 of [31] (or [13, A.3]) to this zigzag gives a zigzag of Quillen equivalences between the model categories of DG-categories with object set O , Ab-spectral categories with object set O , and $H\mathbb{Z}$ -categories with object set O .

Definition 2.7. Given a small DG-category, an *associated Ab-spectral category model* or *associated $H\mathbb{Z}$ -category model* is an Ab-spectral category or $H\mathbb{Z}$ -category (respectively) obtained from the zigzag of Quillen equivalences above.

By neglect of structure, an Ab-spectral category or $H\mathbb{Z}$ -category is in particular a spectral category. We then get an *associated spectral category model* from any associated Ab-spectral or $H\mathbb{Z}$ -category model. The associated $H\mathbb{Z}$ -category and associated spectral category models are unique up to weak equivalence. Using functorial fibrant and cofibrant approximation, we can obtain models that are functorial in DG-functors that are the identity on object sets. Moreover, we can use the same trick as we did with fibrant approximations above to get a map of associated $H\mathbb{Z}$ -category models and associated spectral category models for an arbitrary DG-functor between DG-categories. A straightforward argument proves the following proposition.

Proposition 2.8. *The zigzags of Quillen equivalences above assemble into a functor from the homotopy category of small DG-categories to the homotopy category of small spectral categories.*

Using the model structure of Appendix A, we can do better. Applying functorial fibrant and cofibrant approximation functors, we obtain a functor from small DG-categories to small spectral categories that preserves object sets and weak equivalences.

3. REVIEW OF THH , TR , AND TC

In this section, we review the definition of THH , TR , and TC of spectral categories. We begin with a review of the cyclic bar construction for spectral categories and the variant defined by Bökstedt [4] and Dundas-McCarthy [14] necessary for the construction of TC . We finish with a brief review of the definition of cyclotomic spectra and the construction of TR and TC .

The following cyclic bar construction gives the “topological” analogue of the Hochschild-Mitchell complex.

Definition 3.1. For a small spectral category \mathcal{C} and $(\mathcal{C}, \mathcal{C})$ -bimodule \mathcal{M} , let

$$N_q^{\text{cy}}(\mathcal{C}; \mathcal{M}) = \bigvee \mathcal{C}(c_{q-1}, c_q) \wedge \cdots \wedge \mathcal{C}(c_0, c_1) \wedge \mathcal{M}(c_q, c_0),$$

where the sum is over the $(q+1)$ -tuples (c_0, \dots, c_q) of objects of \mathcal{C} . This becomes a simplicial object using the usual cyclic bar construction face and degeneracy maps: The unit maps of \mathcal{C} induce the degeneracy maps, and the two action maps on \mathcal{M}

(for d_0 and d_q) and the composition maps in \mathcal{C} (for d_1, \dots, d_{q-1}) induce the face maps. We denote the geometric realization as $N^{\text{cy}}(\mathcal{C}; \mathcal{M})$ and write $N^{\text{cy}}(\mathcal{C})$ for $N^{\text{cy}}(\mathcal{C}; \mathcal{C})$.

The previous construction turns out to be slightly inconvenient to use as the definition of the topological Hochschild homology of a spectral category. This construction typically only has the correct homotopy type when the smash products that comprise the terms of the sum represent the derived smash product. The analogous problem arises in the context of Hochschild homology of DG-categories, where the tensor product may fail to have the right quasi-isomorphism type when the mapping complexes are not DG-flat. Just as in that context, this problem can be overcome using resolutions, such as the ones in Proposition 2.5 and 2.6. There is a further more subtle difficulty with this construction, however. While $N^{\text{cy}}(\mathcal{C})$ obtains an S^1 -action by virtue of being the geometric realization of a cyclic complex, the resulting equivariant spectrum does not have the necessary additional structure to define TC (a well-known problem with this kind of cyclic bar construction definition of THH in modern categories of spectra). The correct definition, due to Bökstedt [4] for symmetric ring spectra and generalized by Dundas-McCarthy [14] to spectral categories, does not suffer from either of these deficiencies.

We give a revisionist explanation of the Bökstedt-Dundas-McCarthy construction, taking advantage of later results of Shipley [33] on the derived smash product of symmetric spectra. Let \mathcal{I} be the category with objects the finite sets $\mathbf{n} = \{1, \dots, n\}$ (including $\mathbf{0} = \{\}$), and with morphisms the injective maps. For a symmetric spectrum T , write T_n for the n -th space. The association $\mathbf{n} \mapsto \Omega^n |T_n|$ extends to a functor from \mathcal{I} to spaces, where $|-|$ denotes geometric realization. More generally, given symmetric spectra T^0, \dots, T^q and a space X , we obtain a functor from \mathcal{I}^{q+1} to spaces that sends $\vec{\mathbf{n}} = (\mathbf{n}_0, \dots, \mathbf{n}_q)$ to

$$\Omega^{n_0 + \dots + n_q} (|T_{n_q}^q \wedge \dots \wedge T_{n_0}^0| \wedge X),$$

which is also natural in X . Defining

$$D_n(T^q, \dots, T^0) = \text{hocolim}_{\vec{\mathbf{n}} \in \mathcal{I}^{q+1}} \Omega^{n_0 + \dots + n_q} (|T_{n_q}^q \wedge \dots \wedge T_{n_0}^0| \wedge S^n),$$

we obtain a symmetric spectrum (of topological spaces) $D(T^q, \dots, T^0)$. The following is the main lemma of [33].

Proposition 3.2. ([33, 4.2.3]) *$D(T^q, \dots, T^0)$ is canonically isomorphic in the stable category to the derived smash product of the T^i .*

This motivates the following definition.

Definition 3.3. Given a small spectral category \mathcal{C} , a $(\mathcal{C}, \mathcal{C})$ -bimodule \mathcal{M} , and a space X , let $V(\mathcal{C}; \mathcal{M}; X)_{\vec{\mathbf{n}}}$ be the functor from \mathcal{I}^{q+1} to spaces defined on $\vec{\mathbf{n}} = (\mathbf{n}_0, \dots, \mathbf{n}_q)$ by

$$\Omega^{n_0 + \dots + n_q} (\bigvee |\mathcal{C}(c_{q-1}, c_q)_{n_q} \wedge \dots \wedge \mathcal{C}(c_0, c_1)_{n_1} \wedge \mathcal{M}(c_q, c_0)_{n_0}| \wedge X),$$

and let

$$THH_q(\mathcal{C}; \mathcal{M})(X) = \text{hocolim}_{\vec{\mathbf{n}} \in \mathcal{I}^{q+1}} V(\mathcal{C}; \mathcal{M}; X)_{\vec{\mathbf{n}}}.$$

This assembles into a simplicial space, functorially in X , as follows. The degeneracy maps are induced by the unit maps $S^0 \rightarrow \mathcal{C}(c_i, c_i)_0$ and the functor

$$(\mathbf{n}_0, \dots, \mathbf{n}_q) \mapsto (\mathbf{n}_0, \dots, \mathbf{0}, \dots, \mathbf{n}_q)$$

from I^{q+1} to I^{q+2} . The face maps are induced by the two action maps on \mathcal{M} (for d_0 and d_q) and the composition maps in \mathcal{C} (for d_1, \dots, d_{q-1}) together with a functor $\mathcal{I}^{q+1} \rightarrow \mathcal{I}^q$ induced by the appropriate disjoint union isomorphism $(\mathbf{n}_i, \mathbf{n}_{i+1}) \mapsto \mathbf{n}$ or $(\mathbf{n}_q, \mathbf{n}_0) \mapsto \mathbf{n}$ for $n = n_i + n_{i+1}$ or $n = n_q + n_0$. We write $THH(\mathcal{C}; \mathcal{M})(X)$ for the geometric realization.

$THH(\mathcal{C}; \mathcal{M})(X)$ is a continuous functor in the variable X , and so by restriction to the spheres S^n specifies a symmetric spectrum which we denote $THH(\mathcal{C}; \mathcal{M})$. In fact, the principal virtue of the construction above for $THH(\mathcal{C}) = THH(\mathcal{C}; \mathcal{C})$ is that it can be regarded as an orthogonal S^1 -spectrum indexed on a complete S^1 -universe (or alternatively, a Lewis-May genuine equivariant prespectrum) by instead restricting the continuous functor to the representation spheres S^V . The fact that the symmetric spectrum THH is the restriction of an orthogonal spectrum implies that it is semistable and so the object that it represents in the stable category agrees with its underlying prespectrum. With additional hypotheses of ‘‘convergence’’ and ‘‘connectivity’’, THH is often an Ω -spectrum [20, 1.4].

The following propositions, which are essentially the ‘‘many objects’’ versions of [33, 4.2.8-9] and an easy consequence of the theory developed in [33], show that THH is simply a homotopically well-behaved model of the Hochschild-Mitchell complex.

Proposition 3.4. *There is a natural map in the stable category from $THH(\mathcal{C})$ to $N^{\text{cy}}(\mathcal{C})$ that is an isomorphism when the mapping spectra in \mathcal{C} are cofibrant.*

Proposition 3.5. *A weak equivalence of spectral categories $\mathcal{C} \rightarrow \mathcal{C}'$ induces a weak equivalence $THH(\mathcal{C}) \rightarrow THH(\mathcal{C}')$.*

Using the results of [31] extracted in Proposition 2.5, we can always replace a given small spectral category \mathcal{C} with the cofibrant replacement $\mathcal{C}^{\text{Cell}} \rightarrow \mathcal{C}$, where $\mathcal{C}^{\text{Cell}}$ has the same objects as \mathcal{C} and each mapping spectrum in $\mathcal{C}^{\text{Cell}}$ is a cofibrant symmetric spectrum, level equivalent to the corresponding mapping spectrum in \mathcal{C} . The induced functor $THH(\mathcal{C}^{\text{Cell}}) \rightarrow THH(\mathcal{C})$ is a weak equivalence of symmetric spectra (and even of genuine S^1 -spectra); as a consequence, $THH(\mathcal{C})$ always has the correct homotopy type even when $N^{\text{cy}}(\mathcal{C})$ does not.

We now list the usual bimodule properties of THH that we require in this paper. Proofs of these properties appear in the literature [14] under more restrictive hypotheses (i.e., connectivity and convergence).

Proposition 3.6. *Let \mathcal{C} be a small spectral category.*

- (i) *A weak equivalence of $(\mathcal{C}, \mathcal{C})$ -bimodules $\mathcal{M} \rightarrow \mathcal{M}'$ induces a weak equivalence $THH(\mathcal{C}; \mathcal{M}) \rightarrow THH(\mathcal{C}; \mathcal{M}')$.*
- (ii) *A cofiber sequence of $(\mathcal{C}, \mathcal{C})$ -bimodules $\mathcal{M} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}'' \rightarrow \Sigma\mathcal{M}$ induces a homotopy cofiber sequence on THH .*
- (iii) *A fiber sequence of level fibrant $(\mathcal{C}, \mathcal{C})$ -bimodules $\Omega\mathcal{M}'' \rightarrow \mathcal{M} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}''$ induces a homotopy fiber sequence on THH .*

Proof. For the first statement, the weak equivalence $\mathcal{M} \rightarrow \mathcal{M}'$ induces a weak equivalence $THH_q(\mathcal{C}; \mathcal{M}) \rightarrow THH_q(\mathcal{C}; \mathcal{M}')$ for all q . Since THH_\bullet is always a proper simplicial object, the geometric realization is then a weak equivalence. For the second statement, we can identify THH_\bullet levelwise as the homotopy colimit (over $\mathcal{I}^{\bullet+1}$) of the orthogonal spectra $V(\mathcal{C}; \mathcal{M}; S^{(-)})_{\bar{\mathbf{n}}}$. The second statement now follows from the observation that V preserves homotopy cofiber sequences in the \mathcal{M}

variable and that homotopy colimits and geometric realization preserve homotopy cofiber sequences. The third statement follows from the second since homotopy fiber sequences and homotopy cofiber sequences agree up to sign. \square

We now give a minimal review of the definition of TR and TC ; we refer the reader interested in more details to the excellent discussions of TR and TC in [20, 21]. The constructions start with the fundamental fixed point homeomorphism

$$(THH(\mathcal{C})(X))^H \cong THH(\mathcal{C})(X^H)$$

for S^1 -spaces X and subgroups H of S^1 . This induces weak equivalences (in fact isomorphisms) of Lewis-May genuine equivariant spectra

$$r_H : \rho_H^* \Phi^H THH(\mathcal{C}) \longrightarrow THH(\mathcal{C})$$

where ρ_H is the isomorphism $S^1 \cong S^1/H$ and Φ^H denotes the geometric fixed points. The maps r fit together compatibly for $H' \leq H$. This kind of structure is called a cyclotomic spectrum; see [20, 1.2] for precise details. For simplicity, in this paper a map of cyclotomic spectra will mean a map of Lewis-May genuine equivariant S^1 -spectra that preserves the above structure maps r_H on the point-set level, since this is what we obtain on THH from a spectral functor; a proper treatment of cyclotomic spectra should allow more general maps.

For a fixed prime p and each n , let $C_{p^n} \subset S^1$ denote the cyclic subgroup of order p^n . We have maps

$$F, R : THH(\mathcal{C})^{C_{p^n}} \longrightarrow THH(\mathcal{C})^{C_{p^{n-1}}}$$

where F is the inclusion of the fixed points and R is the map induced by the composite of the map from the fixed point spectrum to the geometric fixed point spectrum $THH(\mathcal{C})^{C_p} \rightarrow \Phi^{C_p} THH(\mathcal{C})$ and the cyclotomic structure map $r_{C_p} : \Phi^{C_p} THH(\mathcal{C}) \rightarrow THH(\mathcal{C})$. Technically, we must spectrify the THH prespectrum before taking fixed point spectra; again, precise details can be found in [20, §1.5].

Definition 3.7. $TR^\bullet(\mathcal{C})$ is the pro-spectrum $\{THH(\mathcal{C})^{C_{p^n}}\}$ under the maps R , and $TR(\mathcal{C})$ is the homotopy limit. $TC(\mathcal{C})$ is the spectrum (or prospectrum) obtained from $TR(\mathcal{C})$ as the homotopy equalizer of the maps F and R .

A map of genuine equivariant S^1 -spectra induces a weak equivalence on fixed point spectra for all finite subgroups of S^1 if and only if it induces a (non-equivariant) weak equivalence on geometric fixed point spectra for all finite subgroups [28, XVI.6.4]. It follows that a cyclotomic map of cyclotomic spectra induces a weak equivalence of fixed point spectra for all finite subgroups of S^1 if and only if it is a non-equivariant weak equivalence. In particular, we obtain the following proposition.

Proposition 3.8. *A spectral functor of spectral categories $\mathcal{C} \rightarrow \mathcal{D}$ that induces a weak equivalence on THH induces a weak equivalence on TR and TC .*

Likewise, using the same principle on the cofiber of a map of cyclotomic spectra, we obtain the following proposition. Applying this proposition in examples when $THH(\mathcal{C})$ is contractible, localization cofiber sequences on TR and TC follow from ones on THH .

Proposition 3.9. *For a strictly commuting square of spectral categories*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D}, \end{array}$$

if the induced square on THH is homotopy cocartesian, then so are the induced squares on TR and TC .

Finally, we turn to DG-categories. For a DG-category \mathcal{C}^{DG} , we can consider THH of the associated spectral category \mathcal{C}^S . Proposition 3.5 shows that up to weak equivalence of cyclotomic spectra, $THH(\mathcal{C}^S)$ does not depend on the particular model chosen. Moreover, Propositions 2.8 and 3.8 show that defining THH , TR , and TC of \mathcal{C}^{DG} in terms of $THH(\mathcal{C}^S)$ constructs THH , TR , and TC as functors from the category of DG-categories and DG-functors to the stable category; this is Theorem 1.2.

4. SPECTRAL CATEGORIES, HOMOTOPY CATEGORIES, AND INVARIANCE OF THH

In this section, we continue the discussion of the basic properties of spectral categories and study the natural conditions under which spectral functors induce equivalences on THH . We review the concept of “Dwyer-Kan equivalence” (Definition 4.1) of spectral categories, which provides a more sophisticated notion of weak equivalence of spectral categories; Theorem 4.9 below indicates that Dwyer-Kan equivalences induce equivalences of THH . The mapping spectra of a spectral category \mathcal{C} give rise to an associated “homotopy category” that is an invariant of the Dwyer-Kan equivalences. Under rather general conditions (q.v. Definition 4.4), the homotopy category has a triangulated structure and this allows us to formulate useful “cofinality” and “thick subcategory” criteria for spectral functors to induce equivalences of THH in Theorems 4.11 and 4.12. Proofs of Theorems 4.9, 4.11, and 4.12 require the technical tools developed in the next section and are given there.

Definition 4.1. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a spectral functor. We say that F is a *Dwyer-Kan embedding* or *DK-embedding* when for every $a, b \in \text{ob } \mathcal{C}$, the map $\mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is a weak equivalence.

We say that F is a *Dwyer-Kan equivalence* or *DK-equivalence* when F is a DK-embedding and for every $d \in \text{ob } \mathcal{D}$, there exists a $c \in \text{ob } \mathcal{C}$ such that $\mathcal{D}(-, d)$ and $\mathcal{D}(-, Fc)$ represent naturally isomorphic enriched functors from \mathcal{D}^{op} to the stable category.

We can rephrase this definition in terms of “homotopy categories”: Associated to a spectral category \mathcal{C} , we have the following notion of homotopy category.

Definition 4.2. For a spectral category \mathcal{C} , the *homotopy category* $\pi_0\mathcal{C}$ is the Ab-category with the same objects, with morphism abelian groups $\pi_0\mathcal{C}(a, b)$, and with units and composition induced by the unit and composition maps of \mathcal{C} . The *graded homotopy category* is the Ab_* -category with objects $\text{ob } \mathcal{C}$ and morphisms $\pi_*\mathcal{C}(a, b)$.

We remind the reader that by convention, $\pi_0\mathcal{C}(a, b)$ and $\pi_*\mathcal{C}(a, b)$ denote the homotopy groups of $\mathcal{C}(a, b)$ viewed as an object of the stable category.

Without any further hypotheses on the spectral categories in question, the following proposition is a straightforward consequence of the definitions and the Yoneda lemma for enriched functors.

Proposition 4.3. *A spectral functor $\mathcal{C} \rightarrow \mathcal{D}$ is a Dwyer-Kan equivalence if and only if it induces an equivalence of graded homotopy categories $\pi_*\mathcal{C} \rightarrow \pi_*\mathcal{D}$.*

As we will see in Theorems 4.5 and 4.6 below, the homotopy category in practice often has a triangulated structure compatible with the mapping spectra. We formalize this in the following definition.

Definition 4.4. A spectral category \mathcal{C} is *pretriangulated* means:

- (i) There is an object 0 in \mathcal{C} such that the right \mathcal{C} -module $\mathcal{C}(-, 0)$ is trivial (weakly equivalent to the constant functor on the one-point symmetric spectrum $*$).
- (ii) Whenever a right \mathcal{C} -module \mathcal{M} has the property that $\Sigma\mathcal{M}$ is weakly equivalent to a representable \mathcal{C} -module $\mathcal{C}(-, c)$ (for some object c in \mathcal{C}), then \mathcal{M} is weakly equivalent to a representable \mathcal{C} -module $\mathcal{C}(-, d)$ for some object d in \mathcal{C} .
- (iii) Whenever the right \mathcal{C} -modules \mathcal{M} and \mathcal{N} are weakly equivalent to representable \mathcal{C} -modules $\mathcal{C}(-, a)$ and $\mathcal{C}(-, b)$ respectively, then the homotopy cofiber of any map of right \mathcal{C} -modules $\mathcal{M} \rightarrow \mathcal{N}$ is weakly equivalent to a representable \mathcal{C} -module.

The first condition guarantees the existence of a zero object in the homotopy category; the usual argument shows that the left module $\mathcal{C}(0, -)$ is also trivial. The second condition gives a desuspension functor on the homotopy category. For the third condition, note that maps of right modules from $\mathcal{C}(-, a)$ to $\mathcal{C}(-, b)$ are in one to one correspondence with the vertices of the zeroth simplicial set of $\mathcal{C}(a, b)$; using weakly equivalent \mathcal{M} and \mathcal{N} , the maps then represent arbitrary elements of $\pi_0\mathcal{C}(a, b)$. In the case when all of the mapping spectra of \mathcal{C} are fibrant symmetric spectra (e.g., after replacing \mathcal{C} by \mathcal{C}^Ω), condition (iii) can be simplified to considering just the homotopy cofibers of maps $\mathcal{C}(-, a) \rightarrow \mathcal{C}(-, b)$. We explain this interpretation of condition (iii) in more detail at the end of the section in the proof of the following theorem.

Theorem 4.5. *Any small spectral category \mathcal{C} DK-embeds in a small pretriangulated spectral category $\tilde{\mathcal{C}}$.*

The category $\tilde{\mathcal{C}}$ is closely related to the category of right \mathcal{C} -modules, essentially the closure of (the Yoneda embedding of) \mathcal{C} under desuspensions and cofiber sequences. The third condition in the definition of pretriangulated spectral category then indicates the sequences in \mathcal{C} that are equivalent to cofiber sequences in $\tilde{\mathcal{C}}$. We therefore use the third condition to define the analogue of Puppe (cofibration) sequences. We prove the following theorem at the end of the section.

Theorem 4.6. *If the spectral category \mathcal{C} is pretriangulated, then its homotopy category is triangulated with distinguished triangles the four term Puppe sequences. A spectral functor between pretriangulated spectral categories induces a triangulated functor on homotopy categories.*

Corollary 4.7. *A spectral functor $\mathcal{C} \rightarrow \mathcal{D}$ between pretriangulated spectral categories is a Dwyer-Kan equivalence if and only if it induces an equivalence of homotopy categories $\pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$.*

In the context of DG-categories, various analogous conditions have been given to ensure that the homotopy category of the DG-category is triangulated [25, 6, 11]. Following [6] and [11], we refer to such DG-categories as pretriangulated. We have the following consistency result that is clear from the model category theory.

Proposition 4.8. *If \mathcal{D} is a pretriangulated DG-category, then its associated spectral category is pretriangulated.*

As indicated by Proposition 4.3 and Corollary 4.7, we take the perspective that the mapping spectra encode the homotopy theory of the spectral category. From this viewpoint, DK-equivalences clearly represent the correct general notion of weak equivalence of spectral categories. An alternative perspective would not require the mapping spectra of a spectral category \mathcal{C} to encode all of the homotopy theory, but rather also include an additional notion of weak equivalence of objects of \mathcal{C} . For example, this is appropriate in the context of enriched model categories. For model categories enriched over symmetric spectra, the homotopy theory is a localization of the intrinsic homotopy theory of the associated spectral category. The full spectral subcategory of the cofibrant-fibrant objects is the spectral category whose mapping spectra encode the homotopy theory of the enriched model category. This subcategory tends not to be preserved under most interesting functors. Under properness hypotheses, a “cofiber” version of THH works somewhat better; see Appendix C for more details.

We prove the following invariance theorem for DK-equivalences in the next section.

Theorem 4.9. *A DK-equivalence $\mathcal{C} \rightarrow \mathcal{D}$ induces a weak equivalence $THH(\mathcal{C}) \rightarrow THH(\mathcal{D})$.*

We also prove the following more general theorem for bimodule coefficients. In the statement, the $(\mathcal{C}, \mathcal{C})$ -bimodule $F^*\mathcal{N}$ is the functor from $\mathcal{C}^{\text{op}} \wedge \mathcal{C}$ to symmetric spectra defined by first applying F to each variable and then applying \mathcal{N} .

Theorem 4.10. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a DK-equivalence, \mathcal{M} a $(\mathcal{C}, \mathcal{C})$ -bimodule and \mathcal{N} a $(\mathcal{D}, \mathcal{D})$ -bimodule. A weak equivalence $\mathcal{M} \rightarrow F^*\mathcal{N}$ induces a weak equivalence $THH(\mathcal{C}; \mathcal{M}) \rightarrow THH(\mathcal{D}; \mathcal{N})$.*

We now move on from weak equivalences to Morita equivalences. For objects a and c of \mathcal{D} , say that c is a *homotopy factor* of a if it is a factor in the graded homotopy category $\pi_*\mathcal{D}$, i.e., if there exists an object b in \mathcal{D} and a natural isomorphism $\pi_*\mathcal{D}(-, c) \cong \pi_*\mathcal{D}(-, a) \times \pi_*\mathcal{D}(-, b)$ of contravariant functors from $\pi_*\mathcal{D}$ to the category of graded abelian groups. We say that a spectral functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *homotopy cofinal* if it induces weak equivalences on mapping spaces and each object of \mathcal{D} is a homotopy factor of the image of some object in \mathcal{C} . We prove the following theorem in the next section.

Theorem 4.11. *A homotopy cofinal spectral functor $\mathcal{C} \rightarrow \mathcal{D}$ induces a weak equivalence $THH(\mathcal{C}) \rightarrow THH(\mathcal{D})$.*

The previous theorem admits a more sophisticated variant. Given a collection C of objects in a pretriangulated spectral category \mathcal{D} , the *thick closure* of C is the collection of objects in the thick subcategory of $\pi_0\mathcal{D}$ generated by C . In terms of the spectral category \mathcal{D} , the thick closure of C is the smallest collection \bar{C} of objects of \mathcal{D} containing C and satisfying:

- (i) If a is a homotopy factor of an object of \bar{C} , then a is in \bar{C} .
- (ii) If the right \mathcal{D} -module $\Sigma\mathcal{D}(-, a)$ is weakly equivalent to $\mathcal{D}(-, c)$ for some c in \bar{C} , then a is in \bar{C} .
- (iii) If the right \mathcal{D} -module $\mathcal{D}(-, a)$ is weakly equivalent to the cofiber of a map of right \mathcal{D} -modules $\mathcal{M} \rightarrow \mathcal{M}'$ with $\mathcal{M}, \mathcal{M}'$ weakly equivalent to $\mathcal{D}(-, c), \mathcal{D}(-, c')$ for c, c' in \bar{C} , then a is in \bar{C} .

A collection is *thick* if it is its own thick closure. Since any small spectral category \mathcal{C} embeds as a full spectral subcategory of a pretriangulated spectral category \mathcal{D} , the following theorem, proved in the next section, in particular allows us to always reduce questions in THH to the case of pretriangulated spectral categories.

Theorem 4.12. *Let \mathcal{D} be a pretriangulated spectral category. Let C be a set of objects of \mathcal{D} , \bar{C} its thick closure, and C' a set containing C and contained in \bar{C} . Let \mathcal{C} and \mathcal{C}' be the full spectral subcategories of \mathcal{D} on the objects in C and C' respectively. Then the inclusion $\mathcal{C} \rightarrow \mathcal{C}'$ induces a weak equivalence $THH(\mathcal{C}) \rightarrow THH(\mathcal{C}')$.*

We close the section with the proof of Theorems 4.5 and 4.6. The argument involves the well-known properties of categories of enriched functors into a Quillen closed model category. For any small spectral category \mathcal{C} , the category $\mathfrak{Mod}_{\mathcal{C}}$ of right \mathcal{C} -modules has a *standard model structure* (or projective model structure) that is proper and compactly generated, where the generating cofibrations and generating acyclic cofibrations are the maps $\mathcal{C}(-, c) \wedge f$ for c in \mathcal{C} and f varying through the generating cofibrations and generating acyclic cofibrations (respectively) of the stable model structure on symmetric spectra described in [24, 3.3.2, 3.4.9] (see also [24, 3.4.2.1, 3.4.16]). Consequently, the weak equivalences and fibrations are the maps that are objectwise weak equivalences and objectwise fibrations (respectively) in the stable model structure on symmetric spectra. The representable right \mathcal{C} -modules $\mathcal{C}(-, c)$ are cofibrant and *compact*, meaning that maps out of $\mathcal{C}(-, c)$ preserve sequential colimits. In fact, the set of maps, simplicial set of maps, and symmetric spectrum of maps out of $\mathcal{C}(-, c)$ preserves arbitrary colimits, by the enriched Yoneda lemma.

In the case when \mathcal{C} is small, we can use this theory to prove Theorem 4.6 as follows. Using Quillen's theory of cofibration sequences, we obtain a triangulated structure on the Quillen homotopy category $\mathrm{Ho}\mathfrak{Mod}_{\mathcal{C}}$ of $\mathfrak{Mod}_{\mathcal{C}}$. The homotopy category $\pi_0\mathcal{C}$ embeds as a full subcategory of $\mathrm{Ho}\mathfrak{Mod}_{\mathcal{C}}$, and the conditions in the definition of pretriangulated spectral category implies that $\pi_0\mathcal{C}$ is closed under desuspensions, suspensions, and triangles in $\mathrm{Ho}\mathfrak{Mod}_{\mathcal{C}}$. In the case when \mathcal{C} is not small, $\mathfrak{Mod}_{\mathcal{C}}$ does not typically have small Hom-sets. Of course, we can upgrade the Grothendieck universe, and apply the same arguments as above; a direct argument is also possible by restricting to the cell modules and using small object arguments to construct replacements whose values on a given small set of objects are fibrant symmetric spectra.

Now given a spectral functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between small pretriangulated spectral categories, left Kan extension produces a functor $\mathrm{Lan}_F: \mathfrak{Mod}_{\mathcal{C}} \rightarrow \mathfrak{Mod}_{\mathcal{D}}$ left adjoint to the functor $F^*: \mathfrak{Mod}_{\mathcal{D}} \rightarrow \mathfrak{Mod}_{\mathcal{C}}$. Since F^* preserves fibrations and weak equivalences in the model structure above, Lan_F and F^* form a Quillen adjoint pair. In particular Lan_F preserves Quillen cofiber sequences and Quillen suspensions. It follows that the left derived functor of Lan_F on Quillen homotopy categories is triangulated; on the representable functors, the left derived functor Lan_F is just

$\pi_0 F: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$. Again, the case when \mathcal{C} or \mathcal{D} is not small may be handled by upgrading the Grothendieck universe or by a straightforward direct argument in terms of the cell modules.

We also use this model theory to prove Theorem 4.5. By Proposition 2.4, we can assume without loss of generality that all of the mapping spectra $\mathcal{C}(x, y)$ are fibrant in the stable model structure on symmetric spectra, and so the representable right \mathcal{C} -modules $\mathcal{C}(-, c)$ are both cofibrant and fibrant in the model structure on $\mathfrak{Mod}_{\mathcal{C}}$. In order to remain in the setting of small categories, we do the following cardinality trick: Let U be the full subcategory of the category of sets consisting of the sets that are canonical countable colimits of the union of the underlying simplicial sets of the mapping spectra in \mathcal{C} crossed with the underlying sets of countable products of standard simplices. Because \mathcal{C} is small, U is a small set. Writing $U\mathfrak{Mod}_{\mathcal{C}}$ for the full subcategory of $\mathfrak{Mod}_{\mathcal{C}}$ consisting of the functors that take values in symmetric spectra whose underlying sets are in U , then $U\mathfrak{Mod}_{\mathcal{C}}$ is small and closed under all the typical (countable) constructions of homotopy theory. In particular, $U\mathfrak{Mod}_{\mathcal{C}}$ is a Quillen model category with cofibrations, fibrations, and weak equivalences the maps that are such in $\mathfrak{Mod}_{\mathcal{C}}$. We then get a (closed model) category $\mathfrak{Mod}_{U\mathfrak{Mod}_{\mathcal{C}}}$.

Let $\tilde{\mathcal{C}}$ be the full spectral subcategory of $U\mathfrak{Mod}_{\mathcal{C}}$ consisting of the cofibrant-fibrant objects. The enriched Yoneda lemma embeds \mathcal{C} as a full spectral subcategory of $\tilde{\mathcal{C}}$. Properties (i) and (ii) for $\tilde{\mathcal{C}}$ in the definition of pretriangulated spectral category are clear. For property (iii), consider a map of right $\tilde{\mathcal{C}}$ -modules $\mathcal{M} \rightarrow \mathcal{N}$. Since the model structure on $\mathfrak{Mod}_{\tilde{\mathcal{C}}}$ is left proper, after replacing \mathcal{M} and \mathcal{N} with fibrant approximations, we obtain an equivalent homotopy cofiber, and so we can assume without loss of generality that \mathcal{M} and \mathcal{N} are fibrant. We assume that \mathcal{M} is weakly equivalent to $\tilde{\mathcal{C}}(-, a)$ and \mathcal{N} is weakly equivalent to $\tilde{\mathcal{C}}(-, b)$ for objects a, b in $\tilde{\mathcal{C}}$; since $\tilde{\mathcal{C}}(-, a)$ and $\tilde{\mathcal{C}}(-, b)$ are cofibrant and \mathcal{M} and \mathcal{N} are fibrant, we can choose weak equivalences $\tilde{\mathcal{C}}(-, a) \rightarrow \mathcal{M}$ and $\tilde{\mathcal{C}}(-, b) \rightarrow \mathcal{N}$. Furthermore, as $\tilde{\mathcal{C}}(a, b)$ and $\mathcal{N}(a)$ are both fibrant, we can lift the composite map $\tilde{\mathcal{C}}(-, a) \rightarrow \mathcal{N}$ to a homotopic map $\tilde{\mathcal{C}}(-, a) \rightarrow \tilde{\mathcal{C}}(-, b)$. We get a weak equivalence on the homotopy cofibers. The map $\tilde{\mathcal{C}}(-, a) \rightarrow \tilde{\mathcal{C}}(-, b)$ comes from a map $a \rightarrow b$ by the Yoneda lemma. A fibrant approximation of the homotopy cofiber in $U\mathfrak{Mod}_{\mathcal{C}}$ is in $\tilde{\mathcal{C}}$ and represents the homotopy cofiber of $\mathcal{M} \rightarrow \mathcal{N}$ in $\mathfrak{Mod}_{\tilde{\mathcal{C}}}$. This completes the proof of Theorem 4.5.

5. THE DENNIS-WALDHAUSEN MORITA ARGUMENT

In this section, we consider the invariance properties of THH from the perspective of generalized Morita theory. Dennis and Waldhausen gave a very concrete argument for the Morita invariance of the Hochschild homology of rings using an explicit bisimplicial construction [40, p. 391]. We give a broad generalization of this argument to the setting of spectral categories that provides the technical foundations for the proofs of the theorems of the previous section as well as the arguments in the remainder of the paper.

Before explaining the argument, we need notation for a version of Bökstedt's construction that provides a flexible model for the two-sided bar construction.

Definition 5.1. Let \mathcal{C} be a spectral category, \mathcal{M} a right \mathcal{C} -module, and \mathcal{N} a left \mathcal{C} -module. The *Bökstedt two-sided bar construction* $TB(\mathcal{M}; \mathcal{C}; \mathcal{N})$ is the geometric

realization of the simplicial (orthogonal) spectrum $TB_\bullet(\mathcal{M}; \mathcal{C}; \mathcal{N})$, where

$$TB_q(\mathcal{M}; \mathcal{C}; \mathcal{N})(V) = \text{hocolim}_{I^{q+2}} \Omega^{n_0 + \dots + n_{q+1}} (W_{n_0, \dots, n_{q+1}} \wedge S^V)$$

and

$$W_{n_0, \dots, n_{q+1}} = \bigvee_{c_0, \dots, c_q} |\mathcal{M}(c_q)_{n_{q+1}} \wedge \mathcal{C}(c_{q-1}, c_q)_{n_q} \wedge \dots \wedge \mathcal{C}(c_0, c_1)_{n_1} \wedge \mathcal{N}(c_0)_{n_0}|.$$

Here we have written $\mathcal{M}(c)_n$ for the n -th simplicial set in the symmetric spectrum $\mathcal{M}(c)$ and similarly for $\mathcal{N}(c)_n$ and $\mathcal{C}(c, d)_n$.

The following is the main technical proposition of this section. In it and elsewhere when necessary for clarity, we write

$$TB(\mathcal{M}(x); x, y \in \mathcal{C}; \mathcal{N}(y)) \quad \text{and} \quad THH(x, y \in \mathcal{C}; \mathcal{P}(x, y))$$

for $TB(\mathcal{M}; \mathcal{C}; \mathcal{N})$ and $THH(\mathcal{C}; \mathcal{P})$, especially when \mathcal{M} , \mathcal{N} , and/or \mathcal{P} depend on other variables.

Proposition 5.2 (Dennis-Waldhausen Morita Argument). *Let \mathcal{C} and \mathcal{D} be spectral categories. Let \mathcal{P} be a $(\mathcal{D}, \mathcal{C})$ -bimodule and \mathcal{Q} a $(\mathcal{C}, \mathcal{D})$ -bimodule. Then there is a natural isomorphism in the stable category*

$$\begin{aligned} THH(x, y \in \mathcal{C}; TB(\mathcal{P}(w, y); w, z \in \mathcal{D}; \mathcal{Q}(x, z))) \\ \simeq THH(w, z \in \mathcal{D}; TB(\mathcal{Q}(x, z); x, y \in \mathcal{C}; \mathcal{P}(w, y))). \end{aligned}$$

The basic idea behind this proposition is easiest to explain using the Hochschild-Mitchell complex and the usual two-sided bar construction in place of the Bökstedt-Dundas-McCarthy THH and Bökstedt two-sided bar construction. With these substitutions, we can identify both

$$N^{\text{cy}}(\mathcal{C}; B(\mathcal{P}; \mathcal{D}; \mathcal{Q})) \quad \text{and} \quad N^{\text{cy}}(\mathcal{D}; B(\mathcal{Q}; \mathcal{C}; \mathcal{P}))$$

as the geometric realization of the bisimplicial spectrum with (q, r) -simplices as pictured.

$$\begin{array}{ccc} & \mathcal{C}(c_{q-1}, x) \wedge \dots \wedge \mathcal{C}(y, c_1) & \\ & \wedge & \wedge \\ \mathcal{Q}(x, z) & & \mathcal{P}(w, y) \\ & \wedge & \wedge \\ & \mathcal{D}(z, d_1) \wedge \dots \wedge \mathcal{D}(d_{r-1}, w) & \end{array}$$

These two constructions are therefore isomorphic in the point-set category of symmetric spectra (of topological spaces).

In the case of the Bökstedt construction, we have maps from both

$$THH(\mathcal{C}; TB(\mathcal{P}; \mathcal{D}; \mathcal{Q})) \quad \text{and} \quad THH(\mathcal{D}; TB(\mathcal{Q}; \mathcal{C}; \mathcal{P}))$$

to a third construction

$$THH(\mathcal{C}; \mathcal{P}; \mathcal{D}; \mathcal{Q})$$

defined as the geometric realization of the bisimplicial spectrum that in bidegree (q, r) is the homotopy colimit over $\mathcal{I}^{r+1+q+1}$ of

$$\begin{aligned} \Omega^n \bigvee & |\mathcal{C}(c_{q-1}, x)_{n_{r+1+q}} \wedge \dots \wedge \mathcal{C}(y, c_1)_{n_{r+1+1}} \wedge \\ & \mathcal{P}(w, y)_{n_{r+1}} \wedge \mathcal{D}(d_{r-1}, w)_{n_r} \wedge \dots \wedge \mathcal{D}(z, d_1)_{n_1} \wedge \mathcal{Q}(x, z)_{n_0}| \wedge S^V, \end{aligned}$$

where $n = n_0 + \cdots + n_{r+1+q}$ and the wedge is over

$$y, c_1, \dots, c_{q-1}, x \in \text{ob } \mathcal{C}, \quad z, d_1, \dots, d_{r-1}, w \in \text{ob } \mathcal{D}.$$

Schematically, akin to the picture above, this is:

$$\Omega^n \bigvee \left| \begin{array}{ccc} & \mathcal{C}(c_{q-1}, x)_{n_{r+1+q}} \wedge \cdots \wedge \mathcal{C}(y, c_1)_{n_{r+1+1}} & \\ & \wedge & \wedge \\ \mathcal{Q}(x, z)_{n_0} & & \mathcal{P}(w, y)_{n_{r+1}} \\ & \wedge & \wedge \\ & \mathcal{D}(z, d_1)_{n_1} \wedge \cdots \wedge \mathcal{D}(d_{r-1}, w)_{n_r} & \end{array} \right| \wedge S^V.$$

The maps to $THH(\mathcal{C}; \mathcal{P}; \mathcal{D}; \mathcal{Q})$ are induced by pulling outside the inside homotopy colimit and loops. These maps are not isomorphisms, but they are weak equivalences: Using the cofibrant approximations from Proposition 2.5 and 2.6, Proposition 5.2 follows from Proposition 3.4 (and its generalization to the two-sided bar construction).

The following lemma complements Proposition 5.2. Its proof is the usual simplicial contraction (see for example [27, 9.8]).

Lemma 5.3 (Two-Sided Bar Lemma). *Let \mathcal{C} be a spectral category, let \mathcal{M} be a right \mathcal{C} -module, and let \mathcal{N} be a left \mathcal{C} -module. For any object c in \mathcal{C} , the composition maps*

$$TB_\bullet(\mathcal{M}; \mathcal{C}; \mathcal{C}(c, -)) \longrightarrow \mathcal{M}(c) \quad \text{and} \quad TB_\bullet(\mathcal{C}(-, c); \mathcal{C}; \mathcal{N}) \longrightarrow \mathcal{N}(c)$$

are simplicial homotopy equivalences.

Proposition 5.2 provides a tool for converting objectwise equivalence conditions into equivalences on THH . We illustrate this with the proof of the theorems of the previous section.

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a spectral functor. Let \mathcal{M} be a $(\mathcal{C}, \mathcal{C})$ -bimodule, \mathcal{N} a $(\mathcal{D}, \mathcal{D})$ -bimodule and $\mathcal{M} \rightarrow F^*\mathcal{N}$ a weak equivalence. We describe a criterion, Proposition 5.6 below, for the map

$$THH(\mathcal{C}; \mathcal{M}) \longrightarrow THH(\mathcal{D}; \mathcal{N})$$

to be a weak equivalence. For Theorem 4.10, we apply this in the notation above. For Theorem 4.9, we apply this with $\mathcal{M} = \mathcal{C}$ and $\mathcal{N} = \mathcal{D}$, and for Theorem 4.12, \mathcal{D} will be the spectral category \mathcal{C}' in the statement.

Consider the commutative diagram

$$(5.4) \quad \begin{array}{ccc} THH(\mathcal{C}; TB(\mathcal{M}(-, -); \mathcal{C}; \mathcal{C}(-, -))) & \xrightarrow{\simeq} & THH(\mathcal{C}; \mathcal{M}) \\ \downarrow & & \downarrow \simeq \\ THH(\mathcal{C}; TB(\mathcal{N}(-, F-); \mathcal{D}; \mathcal{D}(F-, -))) & \xrightarrow{\simeq} & THH(\mathcal{C}; F^*\mathcal{N}) \\ \downarrow & & \downarrow \\ THH(\mathcal{D}; TB(\mathcal{N}(-, -); \mathcal{D}; \mathcal{D}(-, -))) & \xrightarrow{\simeq} & THH(\mathcal{D}; \mathcal{N}). \end{array}$$

The arrows marked “ \simeq ” are weak equivalences by the Two-Sided Bar Lemma and Proposition 3.6 above. The map on right is the map we are interested in, and so

our goal is to describe a tool for showing that the two maps on the left are weak equivalences. The first map is induced by a map of \mathcal{C} -bimodules

$$TB(\mathcal{M}(-, -); \mathcal{C}; \mathcal{C}(-, -)) \longrightarrow TB(\mathcal{N}(-, F-); \mathcal{D}; \mathcal{D}(F-, -)),$$

which is easily seen to be a weak equivalence by the Two-Sided Bar Lemma and the hypothesis that $\mathcal{M} \rightarrow F^*\mathcal{N}$ is a weak equivalence. Understanding the second map is where we apply Proposition 5.2: We obtain a commutative diagram with the maps labeled “ \simeq ” weak equivalences

$$\begin{array}{ccc} THH(\mathcal{C}; TB(\mathcal{N}(-, F-); \mathcal{D}; \mathcal{D}(F-, -))) & \xrightarrow{\simeq} & THH(\mathcal{D}; TB(\mathcal{D}(F-, -); \mathcal{C}; \mathcal{N}(-, F-))) \\ \downarrow & & \downarrow \\ THH(\mathcal{D}; TB(\mathcal{N}(-, -); \mathcal{D}; \mathcal{D}(-, -))) & \xrightarrow{\simeq} & THH(\mathcal{D}; TB(\mathcal{D}(-, -); \mathcal{D}; \mathcal{N}(-, -))) \end{array}$$

by applying Proposition 5.2 with $\mathcal{P} = \mathcal{N}(-, F-)$ and $\mathcal{Q} = \mathcal{D}(F-, -)$ on the top and $\mathcal{P} = \mathcal{N}(-, -)$ and $\mathcal{Q} = \mathcal{D}(-, -)$ (for $\mathcal{C} = \mathcal{D}$) on the bottom. This reduces the question of the map $THH(\mathcal{C}; \mathcal{M}) \rightarrow THH(\mathcal{C}; \mathcal{N})$ being a weak equivalence to showing that the map

$$(5.5) \quad TB(\mathcal{D}(F-, z); \mathcal{C}; \mathcal{N}(w, F-)) \longrightarrow TB(\mathcal{D}(-, z); \mathcal{D}; \mathcal{N}(w, -))$$

is a weak equivalence for every (fixed) pair of objects w, z in \mathcal{D} . We summarize this in the following proposition.

Proposition 5.6. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a spectral functor. Let \mathcal{M} be a $(\mathcal{C}, \mathcal{C})$ -bimodule, \mathcal{N} a $(\mathcal{D}, \mathcal{D})$ -bimodule and $\mathcal{M} \rightarrow F^*\mathcal{N}$ a weak equivalence. If the map (5.5) is a weak equivalence for all w, z in \mathcal{D} , then the map*

$$THH(\mathcal{C}; \mathcal{M}) \longrightarrow THH(\mathcal{D}; \mathcal{N})$$

is a weak equivalence.

We apply this criterion in the proof of Theorem 4.9, Theorem 4.10, Theorem 4.11, and Theorem 4.12.

Proof of Theorem 4.9. Using the Two-Sided Bar Lemma, it suffices to show that the composition map

$$TB(\mathcal{D}(F-, z); \mathcal{C}; \mathcal{D}(w, F-)) \longrightarrow \mathcal{D}(w, z)$$

is a weak equivalence. Viewing $TB(\mathcal{D}(F-, z); \mathcal{C}; \mathcal{D}(w, F-))$ as a simplicial object in the stable category, up to simplicial isomorphism, it only depends on $\mathcal{D}(F-, z)$ as a functor from \mathcal{C} to the stable category. By hypothesis, there exists an object c in \mathcal{C} such that $\mathcal{D}(-, z)$ and $\mathcal{D}(-, Fc)$ are isomorphic as functors from \mathcal{D}^{op} to the stable category. Since this is just a comparison of simplicial objects in the stable category, we do not get a direct comparison on geometric realizations (but see also the proof of Theorem 4.10 below). Nonetheless, since TB is a proper simplicial orthogonal spectrum, the homotopy groups of TB_\bullet are the E_1 -term of a spectral sequence that computes the homotopy groups of TB . The E_1 differential comes from the simplicial face maps, and applying the Two-Sided Bar Lemma, we see that this spectral sequence degenerates at E_2 and that (5.5) is a weak equivalence. \square

For the proof of Theorem 4.10, we note that the map (5.5) admits a right \mathcal{D} -module generalization, replacing $\mathcal{D}(-, z)$ with an arbitrary right \mathcal{D} -module ϕ :

$$(5.7) \quad TB(F^*\phi; \mathcal{C}; \mathcal{N}(w, F-)) \longrightarrow TB(\phi; \mathcal{D}; \mathcal{N}(w, -)).$$

We have used ϕ to denote the right \mathcal{D} -module to avoid possible confusion between the different roles played by the right module ϕ and the bimodule \mathcal{N} .

Proof of Theorem 4.10. The generalization (5.7) of (5.5) is natural in the right \mathcal{D} -module ϕ . We take advantage of this as follows. Let ϕ be a right \mathcal{D} -module fibrant approximation of $\mathcal{D}(-, z)$. By hypothesis, viewing ϕ as an enriched functor from \mathcal{D} to the stable category, we have a natural isomorphism $\tilde{f}: \mathcal{D}(-, Fz') \rightarrow \phi$ for some z' in \mathcal{C} ; by the Yoneda lemma for enriched functors, this corresponds to an element $\tilde{f} \in \pi_0(\phi(Fz'))$. Since $\phi(Fz')$ is fibrant, we can choose a vertex f in $\phi(Fz')_0$ representing \tilde{f} . Again by the Yoneda lemma, f represents a map of right \mathcal{D} -modules $\mathcal{D}(-, Fz') \rightarrow \phi$ that induces the natural isomorphism \tilde{f} of enriched functors to the stable category. In particular, f is a weak equivalence. The map (5.7) is a weak equivalence for $\mathcal{D}(-, Fz')$, and so is a weak equivalence for ϕ and for $\mathcal{D}(-, z)$. \square

Theorem 4.11 can be proved using essentially the same argument as the proof of Theorem 4.9 above, using the fact that a direct sum of maps in the stable category is an isomorphism if and only if it is an isomorphism on each factor. On the other hand, given Theorem 4.5, Theorem 4.11 follows from Theorem 4.9 and Theorem 4.12, which we now prove.

Proof of 4.12. By the discussion preceding the proof of Theorem 4.10 above, we can take advantage of the right module generalization (5.7) to show that the map (5.5) is a weak equivalence. We have that (5.7) is a weak equivalence when ϕ is weakly equivalent to $\mathcal{C}'(-, z)$ for z in \mathcal{C} . Using the fact that both sides preserve homotopy cofiber sequences in the ϕ variable, it follows that (5.7) is a weak equivalence for $\tilde{\mathcal{C}}(-, z)$ for any z in the thick subcategory of $\pi_0\tilde{\mathcal{C}}$ generated by $\pi_0\mathcal{C}$. This completes the proof of Theorem 4.12. \square

6. GENERAL LOCALIZATION METHOD

In this section, we discuss a general method for producing localization cofiber sequences in THH . The basic strategy takes advantage of the fact that THH preserves (co)fiber sequences in the bimodule variable: We apply the Dennis-Waldhausen Morita argument to identify THH of a spectral category with THH of another spectral category with coefficients in a bimodule. We obtain all of our localization fiber sequences by reinterpreting sequences of THH of spectral categories

$$THH(\mathcal{A}) \longrightarrow THH(\mathcal{B}) \longrightarrow THH(\mathcal{C})$$

as the THH of a single spectral category with coefficients in a sequence of judiciously chosen bimodules

$$THH(\mathcal{B}; \mathcal{L}_{\mathcal{A}}^{\mathcal{B}}) \longrightarrow THH(\mathcal{B}; \mathcal{B}) \longrightarrow THH(\mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}).$$

where $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ is the cofiber of a map of $(\mathcal{B}, \mathcal{B})$ -bimodules $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}} \rightarrow \mathcal{B}$. Although we can make more general statements, the situation we are most interested in is when the sequence of spectral categories models a triangulated quotient. We prove the following theorem.

Theorem 6.1. *Let $F: \mathcal{B} \rightarrow \mathcal{C}$ be a spectral functor between small pretriangulated spectral categories, and let \mathcal{A} be the full spectral subcategory of \mathcal{B} consisting of the objects a such that $F(a)$ is isomorphic to zero in the homotopy category $\pi_0\mathcal{C}$. If the induced map from the triangulated quotient $\pi_0\mathcal{B}/\pi_0\mathcal{A}$ to $\pi_0\mathcal{C}$ is cofinal, then*

$THH(\mathcal{C})$ is weakly equivalent through cyclotomic maps to the homotopy cofiber of $THH(\mathcal{A}) \rightarrow THH(\mathcal{B})$.

In general, we call $(\mathcal{B}, \mathcal{A})$ a *localization pair* when \mathcal{B} is a pretriangulated spectral category \mathcal{B} and \mathcal{A} is a full spectral subcategory such that $\pi_0\mathcal{A}$ is thick in $\pi_0\mathcal{B}$; we say that the localization pair is small when the spectral category \mathcal{B} is small. This definition of localization pair differs slightly from that of Keller [25, 2.4] in that we do not require a well-behaved ambient category (our additional requirement that \mathcal{A} be thick is for convenience rather than necessity by Theorem 4.12).

In Theorem 6.1, letting \mathcal{Z} the full subcategory of objects of \mathcal{C} in the thick closure of the image of \mathcal{A} , then $(\mathcal{C}, \mathcal{Z})$ is a localization pair and $(\mathcal{B}, \mathcal{A}) \rightarrow (\mathcal{C}, \mathcal{Z})$ is a map of localization pairs: It is a spectral functor $\mathcal{B} \rightarrow \mathcal{C}$ that takes \mathcal{A} into \mathcal{Z} . Note that for any objects x, y in \mathcal{Z} , the spectrum $\mathcal{Z}(x, y)$ is trivial, and so $THH(\mathcal{Z})$ is trivial. The inclusion of $THH(\mathcal{C})$ in the homotopy cofiber of $THH(\mathcal{Z}) \rightarrow THH(\mathcal{C})$ is a weak equivalence and a cyclotomic map. We have the cyclotomic map of homotopy cofibers

$$C(THH(\mathcal{A}) \rightarrow THH(\mathcal{B})) \longrightarrow C(THH(\mathcal{Z}) \rightarrow THH(\mathcal{C})),$$

and we prove Theorem 6.1 by showing that this map is a weak equivalence. Theorem 6.1 then naturally appears as a special case of the following theorem, which essentially says that the cofiber of THH is an invariant of the localization pair.

Theorem 6.2. *Let $F: (\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$ be a map of small localization pairs. If the induced map of triangulated quotients*

$$\pi_0\mathcal{B}_1/\pi_0\mathcal{A}_1 \longrightarrow \pi_0\mathcal{B}_2/\pi_0\mathcal{A}_2$$

is cofinal, then the induced cyclotomic map

$$C(THH(\mathcal{A}_1) \rightarrow THH(\mathcal{B}_1)) \longrightarrow C(THH(\mathcal{A}_2) \rightarrow THH(\mathcal{B}_2))$$

is a weak equivalence.

Following Keller [25], we define THH of a localization pair as the cofiber on THH . The previous theorem provides a perspective and justification for the following definition.

Definition 6.3. Let $(\mathcal{B}, \mathcal{A})$ be a small localization pair. We write $CTHH(\mathcal{B}/\mathcal{A})$ for the cyclotomic spectrum obtained as the cofiber of the map $THH(\mathcal{A}) \rightarrow THH(\mathcal{B})$.

The rest of the section is devoted to the proof of Theorem 6.2. As indicated above, we use the Dennis-Waldhausen Morita argument, Proposition 5.2, to rewrite $THH(\mathcal{A})$ as $THH(\mathcal{B}; \mathcal{L}_{\mathcal{A}}^{\mathcal{B}})$ for an appropriate $(\mathcal{B}, \mathcal{B})$ -bimodule $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}$.

Lemma 6.4. *Let $(\mathcal{B}, \mathcal{A})$ be a small localization pair, and let $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}$ be the $(\mathcal{B}, \mathcal{B})$ -bimodule defined by*

$$\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}(x, y) = TB(\mathcal{B}(-, y); \mathcal{A}; \mathcal{B}(x, -)).$$

Then $THH(\mathcal{A})$ is naturally weakly equivalent to $THH(\mathcal{B}; \mathcal{L}_{\mathcal{A}}^{\mathcal{B}})$.

Proof. We apply Proposition 5.2 with $\mathcal{C} = \mathcal{A}$, $\mathcal{D} = \mathcal{B}$, $\mathcal{P} = \mathcal{B}$, and $\mathcal{Q} = \mathcal{B}$ to obtain natural weak equivalences

$$THH(\mathcal{B}; \mathcal{L}_{\mathcal{A}}^{\mathcal{B}}) \xrightarrow{\cong} THH(\mathcal{B}; \mathcal{L}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{B}) \xleftarrow{\cong} THH(\mathcal{A}; TB(\mathcal{B}; \mathcal{B}; \mathcal{B})).$$

The natural map

$$THH(\mathcal{A}; TB(\mathcal{B}; \mathcal{B}; \mathcal{B})) \longrightarrow THH(\mathcal{A}; \mathcal{B}) = THH(\mathcal{A}).$$

is a weak equivalence since the composition map of $(\mathcal{A}, \mathcal{A})$ -bimodules $TB(\mathcal{B}; \mathcal{B}; \mathcal{B}) \rightarrow \mathcal{B}$ is a weak equivalence by the Two-Sided Bar Lemma 5.3. \square

For a small localization pair $(\mathcal{B}, \mathcal{A})$, write $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ for the $(\mathcal{B}, \mathcal{B})$ -bimodule obtained as the cofiber of the composition map $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}} \rightarrow \mathcal{B}$. Then by the previous lemma, we have a natural weak equivalence

$$THH(\mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}) \simeq CTHH(\mathcal{B}/\mathcal{A}).$$

Naturality here refers to the fact that a map of small localization pairs F induces a map of $(\mathcal{B}_1, \mathcal{B}_1)$ -bimodules $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1} \rightarrow F^* \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}$ and therefore a map

$$(6.5) \quad THH(\mathcal{B}_1; \mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}) \longrightarrow THH(\mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}).$$

Looking at the proof of Lemma 6.4, we see that this map is compatible under the weak equivalences above with the map on cofibers in the statement of Theorem 6.2. Thus, to prove Theorem 6.2, we just need to show that the map (6.5) is a weak equivalence.

For a small localization pair $(\mathcal{B}, \mathcal{A})$ and fixed object b in \mathcal{B} , the right \mathcal{B} -module $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}(-, b)$ is the enriched homotopy left Kan extension along $\mathcal{A} \rightarrow \mathcal{B}$ of the enriched functor $\mathcal{B}(-, b)$ from \mathcal{A} to symmetric spectra. Philosophically, the cofiber of the map $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}(-, b) \rightarrow \mathcal{B}(-, b)$ should then represent the right \mathcal{C} -module of maps into the image of b in any spectral category \mathcal{C} representing the triangulated quotient, cf. [11, (1.3)]. From this perspective and viewed through the principles of the Dennis-Waldhausen Morita argument, $THH(\mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$ should be equivalent to $THH(\mathcal{C})$. This is the idea behind the following lemma proved at the end of the section.

Lemma 6.6. *Under the hypotheses of Theorem 6.2, the map of $(\mathcal{B}_1, \mathcal{B}_1)$ -bimodules $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1} \rightarrow F^* \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}$ is a weak equivalence.*

A fundamental property of $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ is that $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(a, -)$ and $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, a)$ are trivial for any object a in \mathcal{A} : The Two-Sided Bar Lemma 5.3 implies that the composition maps $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}(a, -) \rightarrow \mathcal{B}(a, -)$ and $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}(-, a) \rightarrow \mathcal{B}(-, a)$ are weak equivalences. This leads to the following technical observation needed below to analyze the map (6.5).

Lemma 6.7. *For a small localization pair $(\mathcal{B}, \mathcal{A})$, the maps of bimodules*

$$TB(\mathcal{B}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}) \longrightarrow TB(\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}) \quad \text{and} \quad TB(\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{B}) \longrightarrow TB(\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$$

induced by $\mathcal{B} \rightarrow \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ are weak equivalences.

Proof. We prove the first equivalence; the argument for the second is similar. Expanding $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ in terms of its definition, we see that $TB(\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$ is the cofiber of the bimodule map $TB(\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}) \rightarrow TB(\mathcal{B}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$, and so by the Two-Sided Bar Lemma 5.3, it suffices to see that $TB(\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$ is trivial. Since $\mathcal{L}_{\mathcal{A}}^{\mathcal{B}} = TB(\mathcal{B}; \mathcal{A}; \mathcal{B})$, the bimodule $TB(\mathcal{L}_{\mathcal{A}}^{\mathcal{B}}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$ is weakly equivalent to

$$TB(\mathcal{B}; \mathcal{A}; TB(\mathcal{B}; \mathcal{B}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})),$$

and applying the Two-Sided Bar Lemma 5.3 again, we see that it is weakly equivalent to $TB(\mathcal{B}; \mathcal{A}; \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}})$. This is trivial because the restriction of $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(x, -)$ to \mathcal{A} is trivial for any x . \square

We can extend $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ to be a $(\mathcal{B}, \mathfrak{Mod}_{\mathcal{B}})$ -bimodule, where $\mathfrak{Mod}_{\mathcal{B}}$ denotes the category of right \mathcal{B} -modules. For x an object of \mathcal{B} and ϕ a right \mathcal{B} -module, let $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(x, \phi)$ be the cofiber of the composition map

$$TB(\phi(-); \mathcal{A}; \mathcal{B}(x, -)) \longrightarrow \phi(x).$$

Clearly, $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(x, \phi)$ is isomorphic to $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(x, y)$ when $\phi = \mathcal{B}(-, y)$, and $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(x, -)$ sends cofiber sequences of right \mathcal{B} -modules to cofiber sequences of symmetric spectra and sends weak equivalences of right \mathcal{B} -modules to weak equivalences of symmetric spectra. The usual category of fractions description of the triangulated quotient $\pi_0\mathcal{B}/\pi_0\mathcal{A}$ and the fact that $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}$ is trivial when either variable is in \mathcal{A} then implies that $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, y)$ and $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, y')$ are weakly equivalent right \mathcal{B} -modules when y and y' are isomorphic in $\pi_0\mathcal{B}/\pi_0\mathcal{A}$. Moreover, when z is isomorphic to $w \vee y$ in $\pi_0\mathcal{B}/\pi_0\mathcal{A}$, $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, z)$ is weakly equivalent as a right \mathcal{B} -module to $\mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, w) \vee \mathcal{Q}_{\mathcal{A}}^{\mathcal{B}}(-, y)$. Using these observations and the lemmas above, we can now prove Theorem 6.2.

Proof of Theorem 6.2. We need to show that the map (6.5) is a weak equivalence. Consider the following commutative diagram

$$\begin{array}{ccccc} THH(\mathcal{B}_1; F^*TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}; \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2})) & \xleftarrow{\simeq} & THH(\mathcal{B}_1; F^*TB(\mathcal{B}_2; \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2})) & \xrightarrow{\simeq} & THH(\mathcal{B}_1; F^*\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}) \\ \downarrow & & \downarrow & & \downarrow \\ THH(\mathcal{B}_2; TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}; \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2})) & \xleftarrow{\simeq} & THH(\mathcal{B}_2; TB(\mathcal{B}_2; \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2})) & \xrightarrow{\simeq} & THH(\mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}). \end{array}$$

The lefthand horizontal maps are weak equivalences by Lemma 6.7 and the righthand horizontal maps are weak equivalences by the Two-Sided Bar Lemma 5.3. The map (6.5) is the composite of the righthand vertical map and the induced map on THH of the map of bimodules $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1} \rightarrow F^*\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}$, which is a weak equivalence by Lemma 6.6. Thus, to see that (6.5) is a weak equivalence, it suffices to show that one of the vertical maps is a weak equivalence.

Focusing on the lefthand vertical map and applying the Dennis-Waldhausen Morita argument 5.2, it suffices to show that the map

$$(6.8) \quad TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(F-, y); \mathcal{B}_1; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, F-)) \longrightarrow TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, y); \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, -))$$

is a weak equivalence for every pair of objects x, y in \mathcal{B}_2 . It is clear from Lemmas 6.6 and 6.7 that (6.8) is an equivalence when either x or y is in the image of \mathcal{B}_1 . By the remarks above, if an object y in \mathcal{B}_2 is isomorphic in $\pi_0\mathcal{B}_2/\pi_0\mathcal{A}_2$ to Fy' for some object y' in \mathcal{B}_1 , then $\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, y)$ is weakly equivalent as a right \mathcal{B}_2 -module to $\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, y')$ and the map (6.8) is a weak equivalence for all x . Since $\pi_0\mathcal{B}_1/\pi_0\mathcal{A}_1$ is cofinal in $\pi_0\mathcal{B}_2/\pi_0\mathcal{A}_2$, for any y in \mathcal{B}_2 , there exists w in \mathcal{B}_2 such that the sum $w \vee y$ in $\pi_0\mathcal{B}_2/\pi_0\mathcal{A}_2$ is isomorphic to Fz for some z in \mathcal{B}_1 ; then as noted above, the right \mathcal{B}_2 -module $\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, w) \vee \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, y)$ is weakly equivalent to $\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, Fz)$. We get compatible weak equivalences

$$\begin{aligned} & TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(F-, w); \mathcal{B}_1; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, F-)) \vee TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(F-, y); \mathcal{B}_1; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, F-)) \\ & \simeq \\ & TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(F-, Fz); \mathcal{B}_1; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, F-)) \end{aligned}$$

and

$$\begin{aligned} TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, w); \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, -)) \vee TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, y); \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, -)) \\ \simeq \\ TB(\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(-, Fz); \mathcal{B}_2; \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(x, -)), \end{aligned}$$

and we see that (6.8) is a weak equivalence for x and y . \square

It still remains to prove Lemma 6.6. The proof makes use of Bousfield localization [7], [22, §3.3] in the category of right \mathcal{B} -modules for a small pretriangulated spectral category \mathcal{B} . As discussed in Section 4, the category $\mathfrak{Mod}_{\mathcal{B}}$ of right \mathcal{B} -modules has a standard compactly generated stable model structure where the weak equivalences and fibrations are the maps that are objectwise weak equivalences and fibrations in the stable model structure on symmetric spectra. The generating cofibrations and generating acyclic cofibrations are the maps $\mathcal{B}(-, b) \wedge f$ for b in \mathcal{B} and f varying through the generating cofibrations and generating acyclic cofibrations, respectively, of the stable model structure on symmetric spectra. The representable right \mathcal{B} -modules $\mathcal{B}(-, b)$ are both cofibrant and compact.

Now let $(\mathcal{B}, \mathcal{A})$ be a localization pair. We say that a right \mathcal{B} -module ψ is \mathcal{A} -local if it is fibrant and $\psi(a)$ is trivial for every object a of \mathcal{A} . In this context, we say that a map of right \mathcal{B} -modules $f: \phi \rightarrow \phi'$ is an \mathcal{A} -local equivalence if it induces a bijection of morphism sets in the Quillen homotopy category, $[\phi', \psi] \rightarrow [\phi, \psi]$, for every \mathcal{A} -local right \mathcal{B} -module ψ . The \mathcal{A} -local model structure on right \mathcal{B} -modules has the same cofibrations as the standard stable model structure but has weak equivalences the \mathcal{A} -local equivalences [22, 4.1.2]. This is a compactly generated model structure with the acyclic cofibrations generated by the acyclic cofibrations in the standard stable model structure together with the maps of the form $\mathcal{B}(-, a) \wedge f$ for a in \mathcal{A} and f a generating cofibration in the stable model structure on symmetric spectra. The fibrant objects in the \mathcal{A} -local model structure are the \mathcal{A} -local right \mathcal{B} -modules.

More specifically, every acyclic cofibration in the \mathcal{A} -local model structure is a retract of a sequential colimit of pushouts along arbitrary coproducts of the generating acyclic cofibrations indicated above. The cofiber of such a pushout is weakly equivalent (in the standard stable model structure) to a wedge of objects of \mathcal{A} . Since objects of \mathcal{B} are compact, a standard argument [23, 2.3.17], [30, 2.1] shows that if a representable right \mathcal{B} -module $\mathcal{B}(-, b)$ is \mathcal{A} -acyclic (\mathcal{A} -locally equivalent to $*$), then in the Quillen homotopy category of the standard stable model structure, $\mathcal{B}(-, b)$ is in the thick subcategory generated by the representables from \mathcal{A} , and so b is in \mathcal{A} . This implies the following proposition.

Proposition 6.9. *Let $(\mathcal{B}, \mathcal{A})$ be a localization pair. The Yoneda functor that includes \mathcal{B} in $\mathfrak{Mod}_{\mathcal{B}}$ as the representable functors induces a triangulated embedding of $\pi_0\mathcal{B}/\pi_0\mathcal{A}$ in the Quillen homotopy category of the \mathcal{A} -local model structure on $\mathfrak{Mod}_{\mathcal{B}}$.*

We now turn to the proof of Lemma 6.6.

Proof. Fixing objects x, y in \mathcal{B}_1 , it suffices to show that the map $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, Fy)$ is a weak equivalence.

We take advantage of the functoriality of $\mathcal{Q}_{\mathcal{A}_i}^{\mathcal{B}_i}$ generalized to modules and the previous proposition. Choose an \mathcal{A}_1 -local \mathcal{B}_1 -module ψ_1 and an \mathcal{A}_1 -local acyclic

cofibration $q: \mathcal{B}_1(-, y) \rightarrow \psi_1$. It is clear from the characterization of the generating \mathcal{A}_1 -local acyclic cofibrations that q induces a weak equivalence $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, \psi_1)$. Moreover, since $\psi_1(a)$ is trivial for every object a in \mathcal{A}_1 , we have that the map $\psi_1(x) \rightarrow \mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, \psi_1)$ is a weak equivalence.

The functor F^* from right \mathcal{B}_2 -modules to right \mathcal{B}_1 -modules has a left adjoint Lan_F defined by left Kan extension. Since Lan_F takes $\mathcal{B}_1(-, b)$ to $\mathcal{B}_2(-, Fb)$ for any object b in \mathcal{B}_1 , Lan_F takes the generating cofibrations and generating acyclic cofibrations of the \mathcal{A}_1 -local model structure into the generating cofibrations and generating acyclic cofibrations of the \mathcal{A}_2 -local model structure, i.e., Lan_F, F^* is a Quillen adjunction on the local model structures. In particular, Lan_F takes q to an \mathcal{A}_2 -local acyclic cofibration $\mathcal{B}_2(-, Fy) \rightarrow \text{Lan}_F \psi_1$. Choose an \mathcal{A}_2 -local object ψ_2 and an \mathcal{A}_2 -local acyclic cofibration $\text{Lan}_F \psi_1 \rightarrow \psi_2$. Now we have weak equivalences

$$\mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, Fy) \longrightarrow \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, \text{Lan}_F \psi_1) \longrightarrow \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, \psi_2).$$

Moreover, since $\psi_2(a)$ is trivial for every object a in \mathcal{A}_2 , we have that the map $\psi_2(Fx) \rightarrow \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, \psi_2)$ is a weak equivalence.

Applying Proposition 6.9 and the hypothesis that $\mathcal{B}_1 \rightarrow \mathcal{B}_2$ induces an embedding of $\pi_0 \mathcal{B}_1 / \pi_0 \mathcal{A}_1$ into $\pi_0 \mathcal{B}_2 / \pi_0 \mathcal{A}_2$, we see that $\psi_1(b) \rightarrow \psi_2(Fb)$ is a weak equivalence for every object b in \mathcal{B}_1 . Thus, we have shown that in the commutative diagram

$$\begin{array}{ccccc} \psi_1(x) & \xrightarrow{\simeq} & \mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, \psi_1) & \xleftarrow{\simeq} & \mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, y) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ \psi_2(Fx) & \xrightarrow{\simeq} & \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, \psi_2) & \xleftarrow{\simeq} & \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, Fy), \end{array}$$

the arrows marked “ \simeq ” are weak equivalences. It follows that the map $\mathcal{Q}_{\mathcal{A}_1}^{\mathcal{B}_1}(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_2}^{\mathcal{B}_2}(Fx, Fy)$ is a weak equivalence. \square

7. APPLICATIONS

We now turn to the applications of the general theory of the preceding sections to *THH* and *TC* of schemes. We begin with a discussion of the spectral enrichment of the derived category of a scheme. Recent work shows that any stable category can be regarded as enriched in symmetric spectra [12, 13, 32] and one approach would be to apply this theory in the setting of categories of unbounded complexes to construct a spectral derived category from first principles. On the other hand, such an approach would demand a comparison with the DG-category structures that arise in nature on categories of complexes. For this reason, we take the simpler approach of lifting DG-categories to associated spectral categories.

For a scheme X , let $\mathcal{K}^{DG}(X)$ denote the pretriangulated DG-category of unbounded complexes of sheaves of \mathcal{O}_X -modules; its homotopy category $\pi_0 \mathcal{K}^{DG}(X)$ is the triangulated category typically denoted $\mathcal{K}(X)$ of unbounded complexes and chain homotopy classes of maps. The derived category $\mathcal{D}(X)$ is the localization of $\mathcal{K}(X)$ obtained by inverting the quasi-isomorphisms, or equivalently, the triangulated quotient of $\mathcal{K}(X)$ by the full triangulated subcategory of acyclic complexes. The derived category of perfect complexes $\mathcal{D}_{\text{parf}}(X)$ is the full triangulated subcategory of unbounded complexes locally quasi-isomorphic to strictly bounded complexes of vector bundles. By choosing a large enough cardinal \aleph and restricting to

the perfect complexes whose underlying sets are in \aleph , we can find a small full pretriangulated subcategory $\mathcal{K}_{\text{parf}}^{DG}(X)$ of $\mathcal{K}^{DG}(X)$ consisting of perfect complexes and having the property that the triangulated quotient of the homotopy category by the full subcategory of acyclics is equivalent to $\mathcal{D}_{\text{parf}}(X)$ via the canonical map. Moreover, when X is quasi-compact and quasi-separated, the full subcategory $\mathcal{K}_{\text{parf}}^{DG}(X)_{\flat}$ consisting of those complexes in $\mathcal{K}_{\text{parf}}^{DG}(X)$ that are strictly bounded above and degreewise flat \mathcal{O}_X -modules also has the property that the triangulated quotient of the homotopy category by the full subcategory of acyclics is equivalent to $\mathcal{D}_{\text{parf}}(X)$ via the canonical map.

Keller [25] and Drinfeld [11] described “quotient” DG-categories whose homotopy categories model the quotients of triangulated categories. We obtain small DG-categories $\mathcal{D}_{\text{parf}}^{DG}(X)$ and (for X quasi-compact and quasi-separated) $\mathcal{D}_{\text{parf}}^{DG}(X)_{\flat}$ whose homotopy categories are equivalent to the derived category $\mathcal{D}_{\text{parf}}$. We obtain small spectral categories $\mathcal{D}_{\text{parf}}^S(X)$ and $\mathcal{D}_{\text{parf}}^S(X)_{\flat}$ associated to $\mathcal{D}_{\text{parf}}^{DG}(X)$ and $\mathcal{D}_{\text{parf}}^{DG}(X)_{\flat}$. We now prove Theorems 1.1, 1.4, and 1.3 from the introduction; in all cases, the results for TR and TC follow from the corresponding results for THH by Proposition 3.9.

Theorem 1.1 follows from Theorem 6.1: For the first statement, the Thomason-Trobaugh localization sequence, we apply Theorem 6.1 with $\mathcal{B} = \mathcal{D}_{\text{parf}}^S(X)$, \mathcal{A} the full spectral subcategory of $\mathcal{D}_{\text{parf}}^S(X)$ consisting of those complexes that are supported on $X - U$, and $\mathcal{C} = \mathcal{D}_{\text{parf}}^S(U)$, using a lift $\mathcal{B} \rightarrow \mathcal{C}$ of the DG-functor $j^*: \mathcal{D}_{\text{parf}}^{DG}(X) \rightarrow \mathcal{D}_{\text{parf}}^{DG}(U)$. The Mayer-Vietoris statement follows from the localization statement and Corollary 4.7 since the inclusion in X of any open set V containing $X - U$ induces an equivalence on the derived categories of perfect complexes supported on $X - U = V - U \cap V$.

For Theorem 1.3, we choose an affine open cover $\{U_1, \dots, U_r\}$ of X . For each i_1, \dots, i_n let $U_{i_1, \dots, i_n} = U_{i_1} \cap \dots \cap U_{i_n}$ and let $A_{i_1, \dots, i_n} = \mathcal{O}_{U_{i_1, \dots, i_n}}$. Since X is semi-separated, $U_{i_1, \dots, i_n} = \text{Spec } A_{i_1, \dots, i_n}$. We now construct a Čech complex on THH associated to this cover as follows.

Let \mathcal{A} denote the full subcategory of $\mathcal{K}_{\text{parf}}^{DG}(X)$ consisting of the acyclic complexes, and for each (non-empty) U_{i_1, \dots, i_n} , let $\mathcal{A}_{i_1, \dots, i_n}$ denote the full subcategory of $\mathcal{K}_{\text{parf}}^{DG}(X)$ of objects acyclic on U_{i_1, \dots, i_n} . We have defined $\mathcal{D}_{\text{parf}}^{DG}(X)$ by Drinfeld’s quotient category construction $\mathcal{K}_{\text{parf}}^{DG}(X)/\mathcal{A}$. For all i_1, \dots, i_n and $U = U_{i_1} \cap \dots \cap U_{i_n}$, the DG-functor $j^*: \mathcal{K}_{\text{parf}}^{DG}(X) \rightarrow \mathcal{K}_{\text{parf}}^{DG}(U)$ (associated to $j: U \subset X$) induces a DG-functor $\mathcal{K}_{\text{parf}}^{DG}(X)/\mathcal{A}_{i_1, \dots, i_n} \rightarrow \mathcal{D}_{\text{parf}}^{DG}(U)$ that is a DG-equivalence onto its image. Moreover, this functor is cofinal in that every perfect complex on U is a direct summand of j^* of a perfect complex on X . We apply functorial factorization to construct the associated (fibrant) spectral categories $\mathcal{D}_{i_1, \dots, i_n}$. This constructs a strictly commuting diagram of spectral functors associated to intersections of the open sets in the cover. Moreover, comparing this construction with the argument above for Theorem 1.1, we see that the cyclotomic map

$$THH(\mathcal{D}_{\text{parf}}^S(X)) \longrightarrow \text{holim}_{\mathcal{S}} THH(\mathcal{D}_{i_1, \dots, i_n})$$

is a weak equivalence, where \mathcal{S} is the partially-ordered set of non-empty subsets of $1, \dots, r$.

Each of the categories $\mathcal{D}_{i_1, \dots, i_n}$ has an object called \mathcal{O}_X whose endomorphism spectrum is an Eilenberg-Mac Lane ring spectrum of A_{i_1, \dots, i_n} . Write HA_{i_1, \dots, i_n} for

$\mathcal{D}_{i_1, \dots, i_n}(\mathcal{O}_X, \mathcal{O}_X)$. Since the objects called \mathcal{O}_X in $\mathcal{D}_{i_1, \dots, i_n}$ are compatible under inclusion of intersections, we obtain a map

$$\mathrm{holim}_{\mathcal{S}_r} THH(HA_{i_1, \dots, i_n}) \longrightarrow \mathrm{holim}_{\mathcal{S}_r} THH(\mathcal{D}_{i_1, \dots, i_n}).$$

The lefthand spectrum is easily seen to be equivalent to the Čech cohomology of the Zariski presheaf of symmetric spectra $THH(\mathcal{O}_{(-)})$ associated to the cover $\{U_1, \dots, U_r\}$. Geisser and Hesselholt [17, 3.2.1] showed that the homotopy groups of THH of a commutative ring form a quasi-coherent sheaf, and so the lefthand homotopy limit computes the hypercohomology spectrum of $THH(\mathcal{O}_{(-)})$, i.e., $THH(X)$ as defined by [17, 3.2.3].

Thus, Theorem 1.3 for quasi-compact semi-separated schemes reduces to showing that each map $THH(HA_{i_1, \dots, i_n}) \rightarrow THH(\mathcal{D}_{i_1, \dots, i_n})$ is a weak equivalence. This follows from Theorem 4.12.

Next we prove Theorem 1.4. We have functors

$$\begin{aligned} Li^* : \mathcal{D}_{\mathrm{parf}}^{DG}(X)_b &\longrightarrow \mathcal{D}_{\mathrm{parf}}^{DG}(X')_b \\ Lj^* : \mathcal{D}_{\mathrm{parf}}^{DG}(Y)_b &\longrightarrow \mathcal{D}_{\mathrm{parf}}^{DG}(Y')_b. \end{aligned}$$

Each of these is a DG-equivalence to its image. Let $\mathcal{B}_1 = \mathcal{D}_{\mathrm{parf}}^S(X')_b$ and $\mathcal{B}_2 = \mathcal{D}_{\mathrm{parf}}^S(Y')_b$, and let \mathcal{A}_1 denote the full spectral subcategory of \mathcal{B}_1 consisting of objects equivalent to those in the image of Li^* and \mathcal{A}_2 denote the full spectral subcategory of \mathcal{B}_2 of objects in the image of Lj^* . The map Lp^* lifts to a map $\mathcal{D}_{\mathrm{parf}}^S(X)_b \rightarrow \mathcal{D}_{\mathrm{parf}}^S(X')_b$ that lands in \mathcal{A}_1 and is a DK-equivalence to \mathcal{A}_1 . Likewise, Lq^* induces a DK-equivalence of $\mathcal{D}_{\mathrm{parf}}^S(Y)$ with \mathcal{A}_2 . In this way, we obtain a strictly commuting spectral model for the DG-functors Lp^* , Lq^* , Li^* , and Lj^* as a map of localization pairs $(\mathcal{B}_1, \mathcal{A}_1) \rightarrow (\mathcal{B}_2, \mathcal{A}_2)$. By [37, §2.7] or [9, 1.5], this map induces an equivalence on quotient triangulated categories, and therefore a weak equivalence $CTHH(\mathcal{B}_1/\mathcal{A}_1) \rightarrow CTHH(\mathcal{B}_2/\mathcal{A}_2)$. Theorem 1.4 now follows.

We close the section with the proof of Theorem 1.5. Let $\pi : \mathbb{P}\mathcal{E}_X \rightarrow X$ be the projective bundle of an algebraic vector bundle \mathcal{E} of rank r . Thomason [37, §2.7] constructed a triangulated filtration

$$0 \simeq \mathcal{A}_r \subset \mathcal{A}_{r-1} \subset \dots \subset \mathcal{A}_0 = \mathcal{D}_{\mathrm{parf}}(\mathbb{P}\mathcal{E}_X)$$

of the derived category as follows. Let \mathcal{A}_k denote the full subcategory of $\mathcal{D}_{\mathrm{parf}}(\mathbb{P}\mathcal{E}_X)$ consisting of complexes Z such that $R\pi_*(Z \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}_X}(i)) = 0$ for $0 \leq i < k$. By definition, $\mathcal{A}_0 = \mathcal{D}_{\mathrm{parf}}^{DG}(X)(\mathbb{P}\mathcal{E}_X)$, and since \mathcal{E} is rank r , \mathcal{A}_r is trivial [37, 2.5]. Furthermore, \mathcal{A}_k admits the alternate description as the thick subcategory of $\mathcal{D}_{\mathrm{parf}}(\mathbb{P}\mathcal{E}_X)$ generated by $L\pi^*(-) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}_X}(-j)$ for $k \leq j < r$ (see also [9, 1.2]). Let $\mathcal{A}'_{r-1}, \dots, \mathcal{A}'_0$ denote the corresponding filtration on $\mathcal{D}_{\mathrm{parf}}^S(\mathbb{P}\mathcal{E}_X)$.

The functor $L\pi^*(-) \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}_X}(-k)$ from $\mathcal{D}_{\mathrm{parf}}(X)$ to \mathcal{A}_k admits a refinement to a DG-functor $\mathcal{D}_{\mathrm{parf}}^{DG}(X) \rightarrow \mathcal{D}_{\mathrm{parf}}^{DG}(\mathbb{P}\mathcal{E}_X)$, which we can lift to a spectral functor $\mathcal{D}_{\mathrm{parf}}^S(X) \rightarrow \mathcal{A}'_k$. Viewed as map of localization pairs $(\mathcal{D}_{\mathrm{parf}}^S(X), 0) \rightarrow (\mathcal{A}'_k, \mathcal{A}'_{k+1})$, the induced map of triangulated quotients $\mathcal{D}_{\mathrm{parf}}(X) \rightarrow \mathcal{A}_k/\mathcal{A}_{k+1}$ is an equivalence [37, §2.7]. Theorem 6.2 then shows that the induced map $THH(X) \rightarrow CTHH(\mathcal{A}'_k, \mathcal{A}'_{k+1})$ is a weak equivalence. In particular, we obtain split cofiber sequences of cyclotomic spectra

$$THH(\mathcal{A}'_{k+1}) \longrightarrow THH(\mathcal{A}'_k) \longrightarrow THH(X),$$

and hence weak equivalence of cyclotomic spectra

$$THH(\mathcal{A}'_k) \simeq THH(\mathcal{A}'_{k+1}) \times THH(X).$$

for $k = 0, \dots, r-1$. This completes the proof of Theorem 1.5.

8. THE CYCLOTOMIC TRACE FROM K^B

In this section we show that the cyclotomic trace from K -theory to THH and TC factors through Thomason-Trobaugh's construction of Bass non-connective K -theory K^B [38, §6]. Using a version of the Bass fundamental theorem for THH , we factor the cyclotomic trace map from connective K -theory on affine schemes (commutative rings) through K^B . This factorization holds more generally for maps from K -theory to any theory satisfying the appropriate analogue of Bass' fundamental theorem, and is natural for such functors to a (strict point-set) category of spectra. Since the trace map admits a model in which it is a map of presheaves restricted to affine covers, we obtain the factorization $K^B \rightarrow THH$ on the level of presheaves, which we show lifts to a map of presheaves $K^B \rightarrow TC$. For quasi-compact semi-separated schemes, K^B is equivalent to the Čech hypercohomology spectrum of its presheaf by [38, 8.4]. The work of the previous section (and [17]) shows that for such schemes THH and TC are each equivalent to both the hypercohomology spectrum and Čech hypercohomology spectrum of their respective presheaves. This then constructs trace maps $K^B \rightarrow TC \rightarrow THH$ for all quasi-compact, semi-separated schemes.

We begin by discussing the analogue of Bass' fundamental theorem that we need. For the purposes of this section, we say that a covariant functor F from commutative rings to some (point-set) category of spectra is a *Bass functor* when it comes with a natural transformation $\tau: \Sigma F(R) \rightarrow F(R[t, t^{-1}])$ and satisfies:

- (i) For any R , $F(R)$ is connective ($\pi_n F(R) = 0$ for $n < 0$),
- (ii) For any R and any $n \geq 0$, the sequence

$$0 \longrightarrow \pi_n F(R) \longrightarrow \pi_n F(R[t]) \oplus \pi_n F(R[t^{-1}]) \longrightarrow \pi_n F(R[t, t^{-1}])$$

induced by the inclusion maps is exact, and

- (iii) For any R and any $n > 0$, the composite map

$$\begin{aligned} \pi_{n-1} F(R) = \pi_n \Sigma F(R) &\longrightarrow \pi_n F(R[t, t^{-1}]) \\ &\longrightarrow \text{Coker}(\pi_n F(R[t]) \oplus \pi_n F(R[t^{-1}]) \longrightarrow \pi_n F(R[t, t^{-1}])) \end{aligned}$$

induced by τ is an isomorphism.

A map of Bass functors is a natural transformation $F \rightarrow G$ that commutes with the maps τ . The key fact we need to apply this theory is the following (well-known) theorem, whose proof we review at the end of this section.

Theorem 8.1. *The K -theory functor and the THH functor admit models that are Bass functors with the trace map a map of Bass functors.*

As an immediate consequence of the definition, a Bass functor in particular comes with a natural 4-term exact sequences

$$0 \longrightarrow \pi_n F(R) \longrightarrow \pi_n F(R[t]) \oplus \pi_n F(R[t^{-1}]) \longrightarrow \pi_n F(R[t, t^{-1}]) \longrightarrow \pi_{n-1} F(R) \longrightarrow 0$$

for $n > 0$ with the map $\pi_n F(R[t, t^{-1}]) \rightarrow \pi_{n-1} F(R)$ naturally split. This exact sequence and splitting are functorial in maps of Bass functors. Bass' construction extends these sequences to all n :

Definition 8.2 (Bass' Construction). For a Bass functor F , let $\beta_n F = \pi_n F$ and let

$$\tau_n: \beta_n F(R) \longrightarrow \beta_{n+1} F(R[t, t^{-1}])$$

be the map induced by τ for $n \geq 0$. Inductively, for $n \leq 0$, define

$$\beta_{n-1} F(R) = \text{Coker}(\beta_n F(R[x]) \oplus \beta_n F(R[x^{-1}]) \longrightarrow \beta_n F(R[x, x^{-1}]))$$

and $\tau_n: \beta_{n-1} F(R) \rightarrow \beta_n F(R[t, t^{-1}])$ to be the induced map on cokernels

$$\text{Coker} \left(\begin{array}{c} \beta_n F(R[x]) \\ \oplus \\ \beta_n F(R[x^{-1}]) \\ \downarrow \\ \beta_n F(R[x, x^{-1}]) \end{array} \right) \longrightarrow \text{Coker} \left(\begin{array}{c} \beta_{n+1} F(R[x, t, t^{-1}]) \\ \oplus \\ \beta_{n+1} F(R[x^{-1}, t, t^{-1}]) \\ \downarrow \\ \beta_{n+1} F(R[x, x^{-1}, t, t^{-1}]) \end{array} \right)$$

Applied to the K -theory functor, Bass' construction defines Bass' negative K -groups. Applied to the THH functor, $\beta_n THH = 0$ for $n < 0$ since the map

$$\pi_0 THH(R[x]) \oplus \pi_0 THH(R[x^{-1}]) \longrightarrow \pi_0 THH(R[x, x^{-1}])$$

is surjective. (It is the map $R[x] \oplus R[x^{-1}] \rightarrow R[x, x^{-1}]$.) Thomason and Trobaugh extended Bass' construction to a construction on spectra suitable for application to general Bass functors as defined above. The following is essentially a simplification of [38, 6.3].

Lemma 8.3. *Let F be a Bass functor. There exists a functor F^B from commutative rings to spectra and a natural transformation $F \rightarrow F^B$ that is an isomorphism on π_n for $n \geq 0$ and induces (as indicated in [38, 6.3]) a natural isomorphism $\beta_n F \rightarrow \pi_n F^B$ for $n < 0$. The functor and natural transformation are functorial in maps of Bass functors.*

We need a few of the details of the construction. Thomason and Trobaugh construct F^B as the homotopy colimit of a sequence of functors

$$F = F_0 = F'_0 \longrightarrow F'_{-1} \longrightarrow F'_{-2} \longrightarrow \cdots F'_{-k} \longrightarrow \cdots$$

The functor F'_{-k-1} is formed inductively as a homotopy pushout

$$\begin{array}{ccc} \Omega^k F_{-k} & \longrightarrow & F'_{-k} \\ \downarrow & & \\ \Omega^{k+1} F_{-k-1} & & \end{array}$$

for functors F_{-k} which come with natural transformations $\Sigma F_{-k} \rightarrow F_{-k-1}$. The functor F_{-k-1} is defined inductively as the homotopy cofiber of the natural map

$$F_{-k}(R[x]) \overset{h}{\cup}_{F_{-k}(R)} F_{-k}(R[x^{-1}]) \longrightarrow F_{-k}(R[x, x^{-1}]).$$

The map $\Sigma F_{-k} \rightarrow F_{-k-1}$ comes from the canonical map $\Sigma F_{-k}(R) \rightarrow F_{-k}(R[t, t^{-1}])$, constructed just as in Bass' construction in algebra, as the induced map on cofibers

coming from the natural commutative diagram

$$\begin{array}{ccc} \Sigma F_{-k}(R[x]) \overset{h}{\cup}_{\Sigma F_{-k}(R)} \Sigma F_{-k}(R[x^{-1}]) & \longrightarrow & \Sigma F_{-k}(R[x, x^{-1}]) \\ \downarrow & & \downarrow \\ F_{-k}(R[t, t^{-1}, x]) \overset{h}{\cup}_{F_{-k}(R[t, t^{-1}])} F_{-k}(R[t, t^{-1}, x^{-1}]) & \longrightarrow & F_{-k}(R[t, t^{-1}, x, x^{-1}]). \end{array}$$

Our notation differs slightly from that of [38, 6.3]; our F_{-k} is their $\Sigma^k F^{-k}$. As a consequence, we get the following observation.

Proposition 8.4. *If F is a Bass functor and factors through cyclotomic spectra (with τ a natural map of cyclotomic spectra), then the functor F^B factors through cyclotomic spectra and the natural transformation $F \rightarrow F^B$ is a natural transformation of cyclotomic spectra.*

Combining the Thomason-Trobaugh lemma with Theorem 8.1, we get a natural transformation of functors

$$K^B \longrightarrow THH^B \simeq THH.$$

As THH^B is a cyclotomic spectrum by Proposition 8.4, we can form a functor TC as the appropriate limit (or pro-object), to obtain the following commutative diagram of functors.

$$\begin{array}{ccc} K^B & \longrightarrow & TC^B \xleftarrow{\simeq} TC \\ & & \downarrow \quad \downarrow \\ & & THH^B \xleftarrow{\simeq} THH \end{array}$$

As explained in the introduction to this section, this extends the trace to non-connective K -theory.

The remainder of the section proves Theorem 8.1. We begin by observing that THH algebraically satisfies the analogue of Bass' fundamental theorem. For a commutative ring R , the Eilenberg Mac Lane spectrum HR is a commutative ring spectrum in any of the modern categories of spectra. We have a weak equivalence of associative ring spectra $HR \wedge TS \rightarrow HR[t]$, where TS is the free associative ring spectrum on the sphere spectrum (or a cofibrant model of it). We then get a weak equivalences of $THH(R)$ -modules

$$\begin{aligned} THH(R[t]) &\simeq THH(R \wedge TS) \simeq THH(R) \wedge THH(TS) \\ &\cong THH(R) \wedge_{HR} (HR \wedge THH(TS)). \end{aligned}$$

We also have the identifications

$$HR \wedge THH(TS) \cong THH^{HR}(HR \wedge TS) \simeq THH^{HR}(R[t]),$$

where $THH^{HR}(R[t])$ is as defined in [15, ???] and is essentially the spectrum whose homotopy groups are $HH_*^R(R[t])$. Since this Hochschild homology is a free module over R , we obtain the computation

$$\pi_* THH(R[t]) \cong \pi_* THH(R) \otimes_R HH_*^R(R[t]) \cong \pi_* THH(R) \otimes_R R[t] \langle 1, \sigma_t \rangle,$$

where 1 is in degree zero and σ_t is in degree one. This is an isomorphism of $\pi_* THH(R)$ -modules, and is natural in R and TS , though not obviously in $R[t]$.

Writing $TS[t^{-1}]$ for the localization of TS under multiplication by the generator of $\pi_0 S$ (which we are thinking of as t), we have a weak equivalence of associative ring spectra $HR \wedge TS[t^{-1}] \rightarrow HR[t, t^{-1}]$, and as above, we get the weak equivalences

$$THH(R[t, t^{-1}]) \simeq THH(R \wedge TS[t^{-1}]) \simeq THH(R) \wedge_{HR} THH^{HR}(HR[t, t^{-1}])$$

and the computation

$$\begin{aligned} \pi_* THH(R[t, t^{-1}]) &\cong \pi_* THH(R) \otimes_R HH_*^R(R[t, t^{-1}]) \\ &\cong \pi_* THH(R) \otimes_R R[t, t^{-1}] \langle 1, \sigma_t \rangle, \end{aligned}$$

Again, this is an isomorphism of $\pi_* THH(R)$ -modules, and is natural in R and $TS[t^{-1}]$, though not obviously in $R[t, t^{-1}]$.

The map $R[x] \rightarrow R[t, t^{-1}]$ sending x to t^{-1} is induced by a map of associative ring spectra $TS \rightarrow TS[t^{-1}]$, namely, the map induced by the map $S \rightarrow TS[t^{-1}]$ sending the generator of $\pi_0 S$ to t^{-1} in $\pi_0 TS[t^{-1}] \cong \mathbb{Z}[t, t^{-1}]$. Thus, we can compute the maps in Bass' sequence for $\pi_* THH$ in terms of Hochschild homology. This then becomes an easy computation with resolutions: the inclusion $R[t^{-1}] \rightarrow R[t, t^{-1}]$ induces the map of $\pi_* THH(R) \otimes_R R[t^{-1}]$ -modules that sends 1 to 1 and $\sigma_{t^{-1}}$ to $-t^{-2}\sigma_t$. It follows that the sequence of graded abelian groups

$$0 \longrightarrow \pi_* THH(R) \longrightarrow \pi_* THH(R[t]) \oplus \pi_* THH(R[t^{-1}]) \longrightarrow \pi_* THH(R[t, t^{-1}])$$

is exact and the map $\pi_{*-1} THH(R) \rightarrow \pi_* THH(R[t, t^{-1}])$ induced by the inclusion and multiplication by $t^{-1}\sigma_t$ induces an isomorphism from $\pi_{*-1} THH(R)$ onto the cokernel of the last map above.

Thomason and Trobaugh [38, §6] proves an analogous formulation of Bass' fundamental theorem for K -theory: The three term sequence is exact and the map $K_{n-1}R \rightarrow K_n R[t, t^{-1}]$ induced by the inclusion and multiplication by t (in $K_1 R[t, t^{-1}]$) induces an isomorphism onto the cokernel for $n > 1$. Since the Dennis trace map takes the element t in $K_1(\mathbb{Z}[t, t^{-1}])$ to the element $t\sigma_{t^{-1}} = -t^{-1}\sigma_t$ in $HH_1(\mathbb{Z}[t, t^{-1}])$, multiplication by the image of t under the trace to THH also provides an isomorphism from $\pi_{*-1} THH(R)$ to the cokernel for $\pi_* THH$. We now have what we need to prove Theorem 8.1.

Proof of Theorem 8.1. We use the model of the trace map described in [14, §2.1.6], as modified by [17, §6.3] which is a functor from exact categories (and exact functors) to symmetric spectra of orthogonal spectra. In order to construct this as a functor on the category of commutative rings, we need a model of the exact category of finitely generated projective modules that is strictly functorial in maps of commutative rings. For this, consider the category $\mathcal{P}(R)$ whose objects are pairs (P, m) where P is a projective submodule of R^m , and whose maps $(P, m) \rightarrow (Q, n)$ are the R -module maps $P \rightarrow Q$. This is an exact category in the evident way. A map of rings $R \rightarrow R'$ induces a map $\mathcal{P}(R) \rightarrow \mathcal{P}(R')$ by extension of scalars and the canonical identification $R' \otimes_R R^m \cong R'^m$; this makes \mathcal{P} a functor from commutative rings to exact categories. Defining $K(R) = K(\mathcal{P}(R))$ and $THH(R) = TH(\mathcal{P}(R))$ (in the notation of [17]), we obtain functors from commutative rings to symmetric spectra of orthogonal spectra and a natural transformation of such functors $K(R) \rightarrow THH(R)$.

We have a bi-exact strictly associative tensor product on $\mathcal{P}(R)$ defined by the usual tensor product and the (lexicographical order) identification $R^m \otimes_R R^n \cong$

R^{mn} . As observed in [17, §6.3], it follows that $K(R)$ and $THH(R)$ are naturally associative symmetric ring spectra of orthogonal spectra and the natural map $K(R) \rightarrow THH(R)$ is a ring map. After applying the functor D of [33] in the symmetric spectrum direction (for each object V in the orthogonal spectrum direction) or a fibrant approximation functor in an appropriate category, we can assume without loss of generality that the element t in $K_1\mathbb{Z}$ is represented by a point-set map $T: S^1 \rightarrow K(\mathbb{Z}[t, t^{-1}])$. We fix a choice of T .

We define $\tau: \Sigma K(R) \rightarrow K(R[t, t^{-1}])$ to be the natural transformation induced by multiplication with the point-set representative T of t . Likewise, we define $\tau: \Sigma THH(R) \rightarrow THH(R[t, t^{-1}])$ to be the natural transformation induced by multiplication by the image of t . This constructs K and THH as Bass functors and the cyclotomic trace as a map of Bass functors. Looking closely at the construction of THH in [17, §6.3], we see that the map τ is a natural transformation of cyclotomic spectra. \square

APPENDIX A. A MODEL STRUCTURE ON THE CATEGORY OF SMALL SPECTRAL CATEGORIES

In this section we describe a model structure on the category of small spectral categories (without requiring a fixed set of objects) where the weak equivalences are the weak equivalences defined in Section 2. Specifically, the weak equivalences are the spectral functors that are bijections on the objects and weak equivalences on each mapping spectrum. As we have indicated before, from our perspective, the DK-equivalences rather than these weak equivalences define the correct homotopy theory on small spectral categories. However, this model structure is significantly easier to construct and suffices for the application we need: constructing a functor from the category of DG-categories to the category of spectral categories. (For a model structure based on DK-equivalences, see the recent preprint of Tabuada [35] based on the work of Bergner [1] in the simplicial context).

Our model structure on small spectral categories glues together the model structures on spectral categories on fixed sets of objects obtained by viewing them as ring spectra with many objects [31, §6], [13, App A]. For a set O , a map of spectral O -categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is a spectral functor that is the identity on the object set (which is O for both \mathcal{C} and \mathcal{D}). In the model structure on spectral O -categories, F is a fibration or weak equivalence if it induces a fibration or weak equivalence on each mapping spectrum $\mathcal{C}(x, y) \rightarrow \mathcal{D}(x, y)$ for all x, y in O . Cofibrations admit a description in terms of freely attaching generating cofibrations to mapping spectra, but are defined by the left lifting property.

For an arbitrary spectral functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between arbitrary small spectral categories, we have two ways of factoring F through a map of spectral O -categories. The easier way, already discussed in Section 2, is the factorization

$$\mathcal{C} \longrightarrow F^*\mathcal{D} \longrightarrow \mathcal{D}$$

through the map of spectral $\text{ob}\mathcal{C}$ -categories $\mathcal{C} \rightarrow F^*\mathcal{D}$. We recall that $F^*\mathcal{D}$ has the same objects as \mathcal{C} and mapping spectra $F^*\mathcal{D}(x, y) = \mathcal{D}(Fx, Fy)$ for each x, y in $\text{ob}\mathcal{C}$. We can regard F^* as a functor from spectral $\text{ob}\mathcal{D}$ -categories to spectral $\text{ob}\mathcal{C}$ -categories, which only depends on $F: \text{ob}\mathcal{C} \rightarrow \text{ob}\mathcal{D}$. As such, the functor F^* has a left adjoint F_* from spectral $\text{ob}\mathcal{C}$ -categories to spectral $\text{ob}\mathcal{D}$ -categories. This

gives the factorization

$$\mathcal{C} \longrightarrow F_*\mathcal{C} \longrightarrow \mathcal{D}$$

through the map of spectral ob \mathcal{D} -categories $F_*\mathcal{C} \rightarrow \mathcal{D}$.

Definition A.1. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a spectral functor between small spectral categories. We define F to be:

- (i) A *cofibration* if $F_*\mathcal{C} \rightarrow \mathcal{D}$ is a cofibration of spectral ob \mathcal{D} -categories,
- (ii) A *fibration* if $\mathcal{C} \rightarrow F^*\mathcal{D}$ is a fibration of spectral ob \mathcal{C} -categories, and
- (iii) A *weak equivalence* if F is a bijection on object sets and induces a weak equivalence of mapping spectra $\mathcal{C}(x, y) \rightarrow \mathcal{D}(Fx, Fy)$ for all x, y in $\text{ob}\mathcal{C}$.

With these definitions, it is straightforward to use the model structure on spectral \mathcal{O} -categories to prove the following theorem.

Theorem A.2. *With the definitions above, the category of small spectral categories becomes a closed model category.*

Proof. It is clear that the category of small spectral categories has all limits and colimits, and that the weak equivalences satisfy two-out-of-three and are closed under retracts. Thus, it suffices to prove the factorization and lifting properties.

The factorization properties are clear: Given $F: \mathcal{C} \rightarrow \mathcal{D}$, factoring $\mathcal{C} \rightarrow F^*\mathcal{D}$ in the model category of spectral ob \mathcal{C} -categories as an acyclic cofibration followed by a fibration induces a factorization of F as an acyclic cofibration followed by a fibration as defined above; similarly, factoring $F_*\mathcal{C} \rightarrow \mathcal{D}$ in the model category of spectral ob \mathcal{D} -categories as a cofibration followed by an acyclic fibration induces a factorization of F as a cofibration followed by an acyclic fibration, as defined above.

For the lifting properties consider the commutative square on the left

$$\begin{array}{ccc} \begin{array}{ccc} \mathcal{A} & \xrightarrow{A} & \mathcal{X} \\ \downarrow C & & \downarrow F \\ \mathcal{B} & \xrightarrow{B} & \mathcal{Y} \end{array} & \begin{array}{ccc} \mathcal{A} & \longrightarrow & A^*\mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & B^*\mathcal{Y} \end{array} & \begin{array}{ccc} C_*\mathcal{A} & \longrightarrow & B^*\mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & B^*\mathcal{Y} \end{array} \end{array}$$

where C is a cofibration and F is a fibration as defined above. In the case when C is a weak equivalence, by replacing \mathcal{B} by an isomorphic spectral category, we can assume without loss of generality that $\text{ob}\mathcal{B} = \text{ob}\mathcal{A}$ and C is the identity on objects. Then we can make sense of the middle diagram, which is a commutative square of spectral ob \mathcal{A} -categories. In it, the left-hand vertical map is an acyclic cofibration and the right-hand vertical map is a fibration (of spectral ob \mathcal{A} -categories). We therefore obtain a lift $\mathcal{B} \rightarrow B^*\mathcal{X}$ from the model structure on spectral ob \mathcal{A} -categories, and this gives a lift $\mathcal{B} \rightarrow \mathcal{X}$ in the original square. On the other hand, when F is a weak equivalence, likewise we can assume without loss of generality that it is the identity on object sets, and we can then make sense of the diagram on the right above. This is a commutative square in the category of spectral ob \mathcal{B} -categories, with the left-hand vertical map a cofibration and the right-hand vertical map an acyclic fibration. We obtain a lift $\mathcal{B} \rightarrow B^*\mathcal{X}$ from the model structure on spectral ob \mathcal{B} -categories, and this gives a lift $\mathcal{B} \rightarrow \mathcal{X}$ in the original square. \square

For the application, we also need the analogous variants of this model structure for small categories enriched in the intermediate categories that arise in the comparison between DG-modules, symmetric Ab -spectra, and $H\mathbb{Z}$ -modules. The

argument above is purely formal given the model structures on enriched categories with fixed object sets. This formal argument then establishes all of the required model structures for small DG-categories, small spectral Ab -categories, small $H\mathbb{Z}$ -categories, and the intermediate categories of small enriched categories. Using functorial cofibrant and fibrant approximation, we obtain the following corollary.

Corollary A.3. *There is a functor from the category of small DG-categories to the category of small spectral categories that takes each DG-category to an associated spectral category.*

We note that the functor in this corollary lifts strictly commuting diagrams of DG-categories to strictly commuting diagrams of associated spectral categories. The corollary does not make any direct implication about diagrams that commute up to natural isomorphism; however, one can use the usual trick of regarding a natural isomorphism of functors $\mathcal{C} \rightarrow \mathcal{D}$ as a functor from an expanded category $\mathcal{C}' \rightarrow \mathcal{D}$ to apply the corollary.

APPENDIX B. THE CYCLOTOMIC TRACE FOR DG-WALDHAUSEN CATEGORIES

In Section 7, we implicitly constructed the cyclotomic trace connecting the K -theory of a scheme to the TC and THH of the associated spectral derived category via the comparison to the Geisser-Hesselholt definition of these spectra in terms of hypercohomology. This streamlined approach allowed us to avoid the lengthy technical development necessary for a more intrinsic construction of the cyclotomic trace, and was sufficient for our applications. In this section, we complete the theory of TC and THH of spectral derived categories by describing an intrinsic construction of the cyclotomic trace.

Our construction of the cyclotomic trace follows the perspective of [14, §2.1.6] that the trace should be regarded as “the inclusion of the objects” from a Waldhausen category to a model of THH which mixes the cyclic bar construction and Waldhausen’s S_\bullet construction. In order to enable this mixing, we work with a class of Waldhausen categories equipped with a DG-enrichment that is compatible with the Waldhausen structure. We call these DG-Waldhausen categories; they are in particular complicial biWaldhausen categories [38, 1.2.11].

Definition B.1. A *DG-Waldhausen category* consists of a small full subcategory \mathcal{C} of the category of complexes of an abelian category $Ab_{\mathcal{C}}$ (which is part of the structure), and a subcategory $w\mathcal{C}$ of \mathcal{C} called the weak equivalences, satisfying the following properties.

- (i) \mathcal{C} contains zero.
- (ii) \mathcal{C} is closed under pushouts along degreewise split monomorphisms and pullbacks along degreewise split epimorphisms.
- (iii) \mathcal{C} is closed under cones and cocones.
- (iv) The weak equivalences contain the quasi-isomorphisms of complexes, are preserved by pushout along degreewise-split monomorphisms and pullback along degreewise-split epimorphisms, and satisfy Waldhausen’s saturation and extension properties.

A *DG-exact functor* from $(\mathcal{C}, Ab_{\mathcal{C}}, w\mathcal{C})$ to $(\mathcal{C}', Ab_{\mathcal{C}'}, w\mathcal{C}')$ is an additive functor $Ab_{\mathcal{C}} \rightarrow Ab_{\mathcal{C}'}$ that takes \mathcal{C} into \mathcal{C}' and $w\mathcal{C}$ into $w\mathcal{C}'$.

By abuse of language, we usually call \mathcal{C} the DG-Waldhausen category. In the definition, the cone CX and cocone $C'X$ of a complex X are the usual contractible complexes that fit into the short exact sequences

$$\begin{aligned} 0 &\longrightarrow X \longrightarrow CX \longrightarrow X[1] \longrightarrow 0 \\ 0 &\longrightarrow X[-1] \longrightarrow C'X \longrightarrow X \longrightarrow 0. \end{aligned}$$

Waldhausen's saturation property on the weak equivalences means that $w\mathcal{C}$ satisfies "two-out-of-three": for composable maps f, g in \mathcal{C} , if any two of the maps f, g , and $g \circ f$ are in $w\mathcal{C}$ then so is the third. Waldhausen's extension property means that when

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z \longrightarrow 0 \end{array}$$

is a commutative diagram of degree-wise split short exact sequences with the maps $X \rightarrow X'$ and $Z \rightarrow Z'$ in $w\mathcal{C}$, then the map $Y \rightarrow Y'$ is in $w\mathcal{C}$. As a consequence, the subcategory \mathcal{C}^w of $w\mathcal{C}$ -acyclic objects (those objects weakly equivalent to 0) is closed under extensions; the extension property is equivalent to this closure condition.

A DG-Waldhausen category obtains the structure of a pretriangulated DG-category with the usual mapping complexes and also the structure of a Waldhausen category (in fact a complicial biWaldhausen category) with the cofibrations the degreewise-split monomorphisms. Therefore we can construct both its algebraic K -theory (using the Waldhausen category structure), as well as its THH and TC (lifting the DG-category structure to an associated spectral category structure). The weak equivalences of the Waldhausen category structure specify additional homotopical data beyond that in the mapping spectra: The natural homotopy category of the Waldhausen category structure is the localization of the homotopy category associated to the DG-category with respect to the weak equivalences. In the terminology of Section 6, this homotopy category is the triangulated quotient of \mathcal{C} by the subcategory \mathcal{C}^w of $w\mathcal{C}$ -acyclics. Thus, the proper notion of THH and TC are the THH and TC of the localization pair, $CTHH(\mathcal{C}, \mathcal{C}^w)$ and $CTC(\mathcal{C}, \mathcal{C}^w)$.

We now review the construction of algebraic K -theory in preparation for constructing the trace map. Recall Waldhausen's S_\bullet construction produces a simplicial Waldhausen category from a Waldhausen category. In the case of a DG-Waldhausen category \mathcal{C} , the S_\bullet construction produces a simplicial DG-Waldhausen category. Let $\text{Ar}[n]$ denote the category with objects (i, j) for $0 \leq i \leq j \leq n$ and a unique map $(i, j) \rightarrow (i', j')$ for $i \leq i'$ and $j \leq j'$. $S_n\mathcal{C}$ is defined to be the full subcategory of the category of functors $A: \text{Ar}[n] \rightarrow \mathcal{C}$ such that:

- $A_{i,i} = 0$ for all i ,
- The map $A_{i,j} \rightarrow A_{i,k}$ is a cofibration (degreewise-split monomorphism) for all $i \leq j \leq k$, and
- The diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout square for all $i \leq j \leq k$,

where we write $A_{i,j}$ for $A(i,j)$. The last two conditions can be simplified to the hypothesis that each map $A_{0,j} \rightarrow A_{0,j+1}$ is a cofibration and the induced maps $A_{0,j}/A_{0,i} \rightarrow A_{i,j}$ are isomorphisms. This becomes a DG-Waldhausen category by taking the abelian category to be the category of functors $\text{Ar}[n] \rightarrow \text{Ab}_{\mathcal{C}}$ and defining a map $A \rightarrow B$ to be a weak equivalence when each $A_{i,j} \rightarrow B_{i,j}$ is a weak equivalence in \mathcal{C} . Note that $A \rightarrow B$ is a degreewise-split monomorphism when each $A_{i,j} \rightarrow B_{i,j}$ and each induced map $A_{i,k} \cup_{A_{i,j}} B_{i,j} \rightarrow B_{i,k}$ is a degreewise-split monomorphism. An ordered map $\{1, \dots, m\} \rightarrow \{1, \dots, n\}$ induces a functor $\text{Ar}[m] \rightarrow \text{Ar}[n]$ and hence a DG-exact functor $S_n \mathcal{C} \rightarrow S_m \mathcal{C}$, making $S_{\bullet} \mathcal{C}$ a simplicial DG-Waldhausen category. Because each $S_p \mathcal{C}$ is itself a DG-Waldhausen category, the S_{\bullet} construction can be iterated to form multisimplicial DG-Waldhausen categories.

For any DG-Waldhausen category \mathcal{D} , let $w_q \mathcal{D}$ denote the DG-category whose objects consist of a sequence of q composable weak equivalences in \mathcal{D} (with $w_0 \mathcal{D} = \mathcal{D}$). Using this construction and iterating the S_{\bullet} construction, we obtain multisimplicial DG-categories $w_{\bullet} S_{\bullet}^{(n)} \mathcal{C}$. The inclusion of \mathcal{D} as $S_1 \mathcal{D}$ induces an $(n+2)$ -simplicial map

$$\text{ob}(w_{\bullet} S_{\bullet}^{(n)} \mathcal{C}) \wedge S_{\bullet}^1 \longrightarrow \text{ob}(w_{\bullet} S_{\bullet}^{(n+1)} \mathcal{C}),$$

where S_{\bullet}^1 denotes the standard simplicial model of the circle (with one non-degenerate vertex and one non-degenerate 1-simplex). These structure maps together with the natural Σ_n action on the categories $w_q S_p^{(n)} \mathcal{C}$ give the collection of simplicial sets

$$\{\text{diag ob}(w_{\bullet} S_{\bullet}^{(n)} \mathcal{C}) \mid n \geq 0\}$$

the structure of a symmetric spectrum. Waldhausen showed that the adjoint attaching maps

$$|\text{diag ob}(w_{\bullet} S_{\bullet}^{(n)} \mathcal{C})| \longrightarrow \Omega |\text{diag ob}(w_{\bullet} S_{\bullet}^{(n+1)} \mathcal{C})|$$

are weak equivalences for $n > 0$; i.e., the geometric realization is a positive fibrant symmetric spectrum of topological spaces.

Definition B.2 (Waldhausen). $K\mathcal{C}$ is the (symmetric) spectrum

$$K\mathcal{C}(n) = \text{diag ob}(w_{\bullet} S_{\bullet}^{(n)} \mathcal{C}).$$

To mix the S_{\bullet} construction with the cyclic bar construction, we use the more convenient DG-categories $\bar{w}_q \mathcal{D}$ in place of the DG-categories $w_q \mathcal{D}$. For a DG-Waldhausen category \mathcal{D} , let $\bar{w}_q \mathcal{D}$ be the DG-category that is the full subcategory of $w_q \mathcal{D}$ consisting of those objects where each weak equivalence in the the sequence is also a cofibration (degreewise-split monomorphism). The advantage of $\bar{w}_q \mathcal{D}$ over $w_q \mathcal{D}$ is that the limit defining its mapping complexes is a homotopy limit. Waldhausen also used this construction; the following is a special case of Lemma 1.6.3 of [41].

Proposition B.3. *For a DG-Waldhausen category \mathcal{D} , the inclusion of $\text{ob}(\bar{w}_{\bullet} \mathcal{D})$ in $\text{ob}(w_{\bullet} \mathcal{D})$ is a weak equivalence.*

For the construction of the cyclotomic trace, we use the associated spectral category functor of the previous section to lift the multi-simplicial DG-categories $\bar{w}_{\bullet} S_{\bullet}^{(n)} \mathcal{C}$ to multi-simplicial spectral categories, which by abuse, we denote with the same notation. For any space X , the spaces

$$|THH(\bar{w}_{\bullet} S_{\bullet}^{(n)} \mathcal{C})(X)|$$

then fit together into a symmetric spectrum (indexed on n) of topological spaces. Let \mathcal{U} be a complete S^1 -universe (a countable dimensional S^1 -representation with equivariant inner product containing infinitely many copies of each irreducible representation). For each finite dimensional subspace V of \mathcal{U} , let $WTHH_V(\mathcal{C})$ be the symmetric spectrum defined by

$$WTHH_V(\mathcal{C})(n) = \operatorname{colim}_{V \subset W \subset \mathcal{U}} \Omega^{W-V} |THH(\bar{w}_\bullet S_\bullet^{(n)} \mathcal{C})(S^W)|,$$

where $W - V$ denotes the orthogonal complement. Then for each n , $WTHH(\mathcal{C})(n)$ is a genuine S^1 -equivariant Lewis-May spectrum, namely, the spectrification of the prespectrum $|THH(\bar{w}_\bullet S_\bullet^{(n)} \mathcal{C})(S^V)|$. This defines $WTHH(\mathcal{C})$ as a symmetric spectrum in Lewis-May spectra.

The spectrum $WTHH(\mathcal{C})$ lies between $K\mathcal{C}$ and $CTHH(\mathcal{C}, \mathcal{C}^w)$. Writing $\bar{K}\mathcal{C}$ for the symmetric spectrum of topological spaces $\bar{K}\mathcal{C}(n) = |\operatorname{ob} \bar{w}_\bullet S_\bullet^{(n)} \mathcal{C}|$, the suspension spectra $\Sigma^\infty \bar{K}\mathcal{C}(n)$ assemble to an equivalent symmetric spectrum of Lewis-May spectra. The inclusion of objects (via the identity) induces a map of symmetric spectra of Lewis-May spectra

$$K\mathcal{C} \simeq \Sigma^\infty \bar{K}\mathcal{C} \longrightarrow WTHH(\mathcal{C}),$$

natural in DG-exact functors. Likewise, using the free functor from spaces to symmetric spectra (and Lewis-May spectrifying), we obtain a map of symmetric spectra of Lewis-May spectra

$$THH(\mathcal{C}) \simeq F_0 THH(\mathcal{C}) \longrightarrow WTHH(\mathcal{C}),$$

natural in DG-exact functors, induced by the identification of $w_0 S_\bullet^{(0)} \mathcal{C}$ as \mathcal{C} . Finally, we obtain the comparison map

$$CTHH(\mathcal{C}, \mathcal{C}^w) \simeq F_0 THH(\mathcal{C}, \mathcal{C}^w) \longrightarrow WTHH(\mathcal{C}),$$

using the fact that $CTHH(\mathcal{C}, \mathcal{C}^w)$ is defined as the cofiber of the map $THH(\mathcal{C}^w) \rightarrow THH(\mathcal{C})$: The functor $\mathcal{C}^w \rightarrow \bar{w}_1 \mathcal{C}$ that sends a $w\mathcal{C}$ -acyclic object a to the weak equivalence $0 \rightarrow a$ induces a map from the cone on $THH(\mathcal{C}^w)$ to $|THH(w_\bullet(\mathcal{C}))|$ that restricts on the face $THH(\mathcal{C}^w)$ to the inclusion of $THH(\mathcal{C}^w)$ in $THH(\mathcal{C}) = THH(\bar{w}_0 \mathcal{C})$. This then extends to the map from $CTHH(\mathcal{C}, \mathcal{C}^w)$ above. Similar observations construct the symmetric spectrum of Lewis-May spectra

$$WTC(\mathcal{C}) = |TC(w_\bullet S_\bullet^{(-)} \mathcal{C})|$$

and maps

$$K\mathcal{C} \simeq \Sigma^\infty \bar{K}\mathcal{C} \longrightarrow WTC(\mathcal{C}) \longleftarrow F_0 CTC(\mathcal{C}, \mathcal{C}^w) \simeq CTC(\mathcal{C}, \mathcal{C}^w).$$

The following is the main theorem of this appendix and is proved below.

Theorem B.4. *For a DG-Waldhausen category \mathcal{C} , the maps*

$$F_0 CTHH(\mathcal{C}, \mathcal{C}^w) \longrightarrow WTHH(\mathcal{C}) \quad \text{and} \quad F_0 CTC(\mathcal{C}, \mathcal{C}^w) \longrightarrow WTC(\mathcal{C})$$

are level equivalences of symmetric spectra of Lewis-May spectra.

We can now define the trace.

Definition B.5. The cyclotomic trace map from K -theory to TC and from K -theory to THH are the zigzags

$$\begin{array}{ccccc}
& & WTC(\mathcal{C}) & \xleftarrow{\cong} & F_0CTC(\mathcal{C}, \mathcal{C}^w) \simeq CTC(\mathcal{C}, \mathcal{C}^w) \\
& \nearrow & \downarrow & & \downarrow \\
KC \simeq \Sigma^\infty \bar{K}\mathcal{C} & & & & \\
& \searrow & WTHH(\mathcal{C}) & \xleftarrow{\cong} & F_0CTHH(\mathcal{C}, \mathcal{C}^w) \simeq CTHH(\mathcal{C}, \mathcal{C}^w)
\end{array}$$

Every map in the diagram is natural in DG-exact functors.

When we restrict to appropriate categories of schemes or pairs of schemes as in [38, §6], we can factor the trace above through non-connective K -theory. Essentially, we take \bar{K} and $WTHH$ as our model functors to spectra (which here would be the point-set category of symmetric spectra of Lewis-May spectra) applied to the appropriate DG-Waldhausen category model for perfect complexes (as in [38, §3]), depending on the kind of naturality required for the maps of schemes. For any of these models, we get natural and suitably associative pairings

$$\bar{K}(X \text{ on } (X - U)) \wedge K_f(\mathbb{Z}[t, t^{-1}]) \longrightarrow \bar{K}(X[t, t^{-1}] \text{ on } (X[t, t^{-1}] - U[t, t^{-1}]))$$

and

$$\begin{aligned}
WTHH(X \text{ on } (X - U)) \wedge K_f(\mathbb{Z}[t, t^{-1}]) \\
\longrightarrow WTHH(X[t, t^{-1}] \text{ on } (X[t, t^{-1}] - U[t, t^{-1}])),
\end{aligned}$$

where $K_f(\mathbb{Z}[t, t^{-1}])$ denotes the Waldhausen K -theory symmetric ring spectrum of the exact category with objects the canonical free modules

$$0, \mathbb{Z}[t, t^{-1}], (\mathbb{Z}[t, t^{-1}])^2, (\mathbb{Z}[t, t^{-1}])^3, \dots$$

The arguments presented in Section 8 extend to this context to construct the non-connective cyclotomic trace.

The remainder of the section proves Theorem B.4. A version of the Additivity Theorem, as always, provides the key lemma. Given DG-Waldhausen categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and DG-exact functors $\phi: \mathcal{A} \rightarrow \mathcal{B}$, $\psi: \mathcal{C} \rightarrow \mathcal{B}$, let $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be the DG-Waldhausen category where an object consists of:

- (i) A tuple (a, b, c) of objects $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$, and
- (ii) A degreewise-split short exact sequence in \mathcal{B} ,

$$0 \longrightarrow \phi a \longrightarrow b \longrightarrow \psi c \longrightarrow 0.$$

The mapping complex in $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ from (a, b, c) to (a', b', c') is

$$(B.6) \quad \mathcal{A}(a, a') \times_{\mathcal{B}(\phi a, b')} \mathcal{B}(b, b') \times_{\mathcal{B}(\psi c, \psi c')} \mathcal{C}(c, c'),$$

which is isomorphic to

$$(B.7) \quad \mathcal{A}(a, a') \times_{\mathcal{B}(\phi a, \phi a')} \mathcal{B}(b, b') \times_{\mathcal{B}(b, \psi c')} \mathcal{C}(c, c').$$

Note that each of the maps

$$\begin{aligned} \mathcal{B}(b, b') &\longrightarrow \mathcal{B}(\phi a, b'), \\ \mathcal{B}(b, b') &\longrightarrow \mathcal{B}(b, \psi c'), \\ \mathcal{A}(a, a') \times_{\mathcal{B}(\phi a, b')} \mathcal{B}(b, b') &\longrightarrow \mathcal{B}(\psi c, \psi c'), \\ \mathcal{B}(b, b') \times_{\mathcal{B}(b, \psi c')} \mathcal{C}(c, c') &\longrightarrow \mathcal{B}(\phi a, \phi a') \end{aligned}$$

is a degreewise-split epimorphisms, and so the limits in (B.6) and (B.7) are homotopy limits.

We have DG-exact functors

$$\begin{aligned} \alpha: \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) &\longrightarrow \mathcal{A} \\ \beta: \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) &\longrightarrow \mathcal{B} \\ \gamma: \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) &\longrightarrow \mathcal{C} \end{aligned}$$

induced by the forgetful functor and a DG-exact functor

$$\sigma: \mathcal{A} \times \mathcal{C} \longrightarrow \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$$

induced by $\sigma(a, c) = (a, \phi a \oplus \psi c, c)$ (and the split short exact sequence). The version of the additivity theorem we prove compares the maps induced on THH by σ and $\alpha \vee \gamma$.

Theorem B.8 (Additivity Theorem). *The functors*

$$THH(\mathcal{A}) \vee THH(\mathcal{C}) \longrightarrow THH(\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})) \longrightarrow THH(\mathcal{A}) \vee THH(\mathcal{C})$$

induced by σ and $\alpha \vee \gamma$ are inverse weak equivalences.

Proof. Consider the DG-exact functor $\phi': \mathcal{A} \rightarrow \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ that takes a in \mathcal{A} to $(a, \phi a, 0)$. By (B.6), we see that this is a DK-embedding. Now by Theorem 6.1, it suffices to show that the functor $\psi': \mathcal{B} \rightarrow \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ (sending c to $(0, \psi c, c)$) induces an equivalence from homotopy category $\pi_0 \mathcal{B}$ to the triangulated quotient $\pi_0 \mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) / \pi_0 \mathcal{A}$. This is a straightforward calculation from (B.7). \square

We can apply this to understand the effect both of \bar{w}_q and S_p on THH . An element of \bar{w}_q of \mathcal{C} is a sequence of degreewise-split maps

$$c_0 \longrightarrow \cdots \longrightarrow c_q$$

such that each quotient c_i/c_{i-1} is in \mathcal{C}^w . Choosing quotients, we get a DK-equivalent DG-category \bar{W}_q that is a DG-Waldhausen category. Furthermore, we can identify $\bar{W}_{q+1}\mathcal{C}$ as $\mathcal{E}(\bar{W}_q\mathcal{C}, \mathcal{C}, \mathcal{C}^w)$, for the functor $\phi: \bar{W}_q\mathcal{C} \rightarrow \mathcal{C}$ that sends the sequence pictured above to c_q . As a consequence we get the following corollary.

Corollary B.9. *For all q , the map*

$$\underbrace{THH(\mathcal{C}^w) \vee \cdots \vee THH(\mathcal{C}^w)}_{q \text{ factors}} \vee THH(\mathcal{C}) \longrightarrow THH(\bar{w}_q\mathcal{C})$$

induced by the map that sends (a_1, \dots, a_q, c) to

$$c \longrightarrow c \oplus a_1 \longrightarrow \cdots \longrightarrow c \oplus a_1 \oplus \cdots \oplus a_q$$

is a weak equivalence.

Similarly, for any DG-Waldhausen category \mathcal{D} , the DG-Waldhausen category $S_r\mathcal{D}$ is DK-equivalent (via a DG-exact functor) to the DG-category $\mathcal{E}(S_{r-1}\mathcal{D}, \mathcal{D}, \mathcal{D})$ for the functor $\phi: S_r\mathcal{D} \rightarrow \mathcal{D}$ that takes $\{A_{i,j}\}$ to $A_{0,r-1}$. We use this observation to prove the following corollary.

Corollary B.10. *For each n and q , the map*

$$\Sigma|THH(\bar{w}_q S_\bullet^{(n)}\mathcal{C})| \longrightarrow |THH(\bar{w}_q S_\bullet^{(n+1)}\mathcal{C})|$$

is a weak equivalence.

Proof. We can write $\Sigma|THH(\bar{w}_q S_\bullet^{(n)}\mathcal{C})|$ as the geometric realization of a multi-simplicial object with one more simplicial direction, $THH(\bar{w}_q S_\bullet^{(n)}\mathcal{C}) \wedge S_\bullet^1$, where S_\bullet^1 denotes the standard simplicial model of the circle. The map in the statement is induced by the map on geometric realizations of the map of multi-simplicial objects

$$THH(\bar{w}_q S_\bullet^{(n)}\mathcal{C}) \wedge S_\bullet^1 \longrightarrow THH(\bar{w}_q S_\bullet^{(n+1)}\mathcal{C}).$$

Using the standard isomorphisms

$$S_r\bar{w}_q \cong \bar{w}_q S_r, \quad S_r S_p \cong S_p S_r,$$

and writing $\mathcal{D} = \bar{w}_q S_{p_1} \cdots S_{p_n} \mathcal{C}$, we are looking at maps of the form

$$\bigvee_r THH(\mathcal{D}) \longrightarrow THH(S_r\mathcal{D}).$$

Using the relationship of $S_r\mathcal{D}$ and $\mathcal{E}(S_{r-1}\mathcal{D}, \mathcal{D}, \mathcal{D})$ as above, we see by induction that this map is a weak equivalence. \square

Combining these two corollaries, we prove Theorem B.4.

Proof of Theorem B.4. We can identify the map $CTHH(\mathcal{C}, \mathcal{C}^w) \rightarrow |THH(\bar{w}_\bullet \mathcal{C}^w)|$ described above as the induced map on geometric realization of the map of simplicial objects

$$\underbrace{THH(\mathcal{C}^w) \vee \cdots \vee THH(\mathcal{C}^w)}_{\bullet \text{ factors}} \vee THH(\mathcal{C}) \longrightarrow THH(\bar{w}_\bullet \mathcal{C}),$$

and is a weak equivalence by Corollary B.9. For $n > 0$, the n -th level of the symmetric spectrum of spectra $F_0CTHH(\mathcal{C}, \mathcal{C}^w)$ is $\Sigma^n CTHH(\mathcal{C}, \mathcal{C}^w)$. It now follows from Corollary B.10, that the map

$$F_0CTHH(\mathcal{C}, \mathcal{C}^w) \longrightarrow WTHH(\mathcal{C})$$

is a level equivalence of symmetric spectra of spectra. \square

APPENDIX C. THH AND TC OF SMALL SPECTRAL MODEL CATEGORIES

Our treatment of THH and TC of spectral categories in the body of the paper took the perspective of having all the homotopy information encoded in the mapping spectra. In the context of closed model categories enriched over symmetric spectra, the weak equivalence encode an additional localization. We can extract a spectral category satisfying the hypotheses of the main discussion of the paper from such a model category by restricting to the full spectral subcategory of cofibrant-fibrant objects. However, this subcategory is not usually preserved by naturally-occurring functors between model categories, which tend to preserve only cofibrant or only fibrant objects.

In this appendix, we present a construction of THH of a spectral model category in terms of either the full subcategory of cofibrant or fibrant objects. The construction is in terms of a “cofiber THH ” description, exactly as in the THH of localization pairs constructed in Section 6. Since the quotient of the subcategory of cofibrants by the acyclic cofibrants is the homotopy category of the model category, we can regard this pair as analogous to a localization pair, although it may not satisfy the hypotheses of the definition. Nevertheless, a similar (but easier) proof applies to compare the THH of this pair to the THH of the cofibrant-fibrants. The main theorem of this section is the following.

Theorem C.1. *Let \mathcal{M} be a small closed model category enriched over symmetric spectra (satisfying the symmetric spectrum version of SM7). Write \mathcal{A} for the subcategory of acyclic objects (objects weakly equivalent to the zero object), and subscripts c and f for the subcategories of cofibrant and fibrant objects, respectively, of \mathcal{M} and \mathcal{A} . In the following diagram of cyclotomic spectra, the vertical map is always a weak equivalence, the left-hand map is a weak equivalence if \mathcal{M} is left proper, and the right-hand map is a weak equivalence if \mathcal{M} is right proper.*

$$\begin{array}{ccc} & THH(\mathcal{M}_{cf}) & \\ & \downarrow & \\ CTHH(\mathcal{M}_f, \mathcal{A}_f) & \longleftarrow CTHH(\mathcal{M}_{cf}, \mathcal{A}_{cf}) & \longrightarrow CTHH(\mathcal{M}_c, \mathcal{A}_c) \end{array}$$

Since for any pair of objects in \mathcal{A}_{cf} , the symmetric spectrum of maps is trivial, $THH(\mathcal{A}_{cf})$ is trivial, and it then follows that the vertical map is a weak equivalence.

Of the remaining statements in the theorem, we treat the case of the right horizontal map in detail; the case of the left horizontal map is similar (and in fact follows by considering the opposite category). As in Section 6, we define the $(\mathcal{M}_c, \mathcal{M}_c)$ -bimodule $\mathcal{L}_{\mathcal{A}_c}^{\mathcal{M}_c}$ by

$$\mathcal{L}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y) = TB(\mathcal{M}_c(-, y); \mathcal{A}_c; \mathcal{M}_c(x, -))$$

and $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}$ as the cofiber of the composition map $\mathcal{L}_{\mathcal{A}_c}^{\mathcal{M}_c} \rightarrow \mathcal{M}_c$. The following lemma lists the properties of $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}$ we need in the proof of the theorem.

Lemma C.2. *Let x be an object of \mathcal{M}_c .*

- (i) *For y in \mathcal{M}_{cf} , the map $\mathcal{M}_c(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y)$ is a weak equivalence.*
- (ii) *If \mathcal{M} is right proper, then $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, -)$ preserves weak equivalences.*

Proof. Since the mapping spectrum from a cofibrant acyclic object to a fibrant object is trivial, for any object y in \mathcal{M}_{cf} , $\mathcal{L}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y)$ is trivial, and the map $\mathcal{M}_c(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y)$ is a weak equivalence. This proves (i). To prove (ii), it suffices to show that for any object y and any fibrant approximation $y \rightarrow y'$, the map $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y')$ is a weak equivalence. Factor the initial map $* \rightarrow y'$ as an acyclic cofibration followed by a fibration $a' \rightarrow y'$, and let a be a cofibrant approximation of the pullback $y \times_{y'} a'$.

$$\begin{array}{ccccc} a & \xrightarrow{\cong} & y \times_{y'} a' & \longrightarrow & a' \\ & & \downarrow & & \downarrow \\ & & y & \xrightarrow{\cong} & y' \end{array}$$

We obtain from this fibration pullback square (and the symmetric spectrum version of SM7) the homotopy (co)cartesian square of \mathcal{M}_c -modules on the left below, and from this, the homotopy cocartesian square of symmetric spectra on the right below.

$$\begin{array}{ccc} \mathcal{M}_c(-, a) & \longrightarrow & \mathcal{M}_c(-, a') \\ \downarrow & & \downarrow \\ \mathcal{M}_c(-, y) & \longrightarrow & \mathcal{M}_c(-, y') \end{array} \quad \begin{array}{ccc} \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, a) & \longrightarrow & \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, a') \\ \downarrow & & \downarrow \\ \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y) & \longrightarrow & \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y') \end{array}$$

The hypothesis that \mathcal{M} is right proper implies that the map $a \rightarrow a'$ is a weak equivalence and therefore that a is in \mathcal{A}_c . It follows that $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, a)$ and $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, a')$ are trivial, and that $\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y) \rightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y')$ is a weak equivalence. \square

As in Section 6, we have a natural weak equivalence relating $CTHH(\mathcal{M}_c, \mathcal{A}_c)$ with $THH(\mathcal{M}_c; \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c})$, compatible with the map from $THH(\mathcal{M}_c)$. Applying part (i) of the lemma, to complete the proof of the theorem, it suffices to show that the map

$$THH(\mathcal{M}_{cf}; \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}) \longrightarrow THH(\mathcal{M}_c; \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c})$$

is a weak equivalence.

Our strategy as in Section 6 is to apply the Dennis-Waldhausen Morita argument 5.2 to reduce to proving an objectwise statement. For this, we use the weak equivalence of $(\mathcal{M}_c, \mathcal{M}_c)$ -bimodules $TB(\mathcal{M}_c; \mathcal{M}_c; \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}) \rightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}$. Then the Dennis-Waldhausen Morita argument gives us a weak equivalence

$$THH(\mathcal{M}_{cf}; TB(\mathcal{M}_c; \mathcal{M}_c; \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c})) \simeq THH(\mathcal{M}_c; TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}; \mathcal{M}_{cf}; \mathcal{M}_c))$$

and likewise the analogous weak equivalence with \mathcal{M}_{cf} replaced by \mathcal{M}_c . Now it suffices to show that the map

$$THH(\mathcal{M}_c; TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}; \mathcal{M}_{cf}; \mathcal{M}_c)) \longrightarrow THH(\mathcal{M}_c; TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}; \mathcal{M}_c; \mathcal{M}_c))$$

is a weak equivalence. This map is induced by the map of $(\mathcal{M}_c, \mathcal{M}_c)$ -bimodules

$$TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}; \mathcal{M}_{cf}; \mathcal{M}_c) \longrightarrow TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}; \mathcal{M}_c; \mathcal{M}_c)$$

and so it suffices to show that the map

$$TB(\mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(-, y); \mathcal{M}_{cf}; \mathcal{M}_c(x, -)) \longrightarrow \mathcal{Q}_{\mathcal{A}_c}^{\mathcal{M}_c}(x, y)$$

is a weak equivalence for each pair of objects x, y in \mathcal{M}_c . This is clear from the Two-Sided Bar Construction Lemma and part (i) of the lemma when y is in \mathcal{M}_{cf} . It then follows for arbitrary y by part (ii) of the lemma.

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