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Intuitionistic L-Fuzzy Subrings

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Abstract. The aim of this paper is to study the concept of Intuitionistic L-fuzzy subrings and Intuitionistic L-fuzzy ideals of a ring R. In this direction definitions and properties relating to Intuitionistic L-fuzzy subrings of R and Intuitionistic L-fuzzy ideals of R are introduced and discussed. We introduce a special type of Quotient ring.

Keywords: L-fuzzy set, Intuitionistic L-fuzzy set, Intuitionistic L-fuzzy subrings, Intuitionistic L-fuzzy ideals, quotient rings

1. Introduction

The theory of Intuitionistic fuzzy set plays an important role in modern mathematics. The idea of Intuitionistic L-fuzzy set (ILFS) was introduced by Atanassov (1986) [1–3] as a generalization of Zadeh's (1965) [7] fuzzy sets. Many researchers applied the notion of Intuitionistic fuzzy concepts to relation, group theory, topological space, knowledge engineering, natural language, neural network etc. Biswas [4] have applied the concept of Intuitionistic fuzzy sets to the theory of groups. In this paper we study the algebraic nature of Intuitionistic fuzzy subrings and Intuitionistic fuzzy ideals on a lattice $(L, \leq, \wedge, \vee)$. Infact we emphasize the truth of the results relating to the non-membership function of an Intuitionistic fuzzy subring and Intuitionistic fuzzy ideal on a lattice. The proof of the results on the membership function of Intuitionistic fuzzy subring and Intuitionistic fuzzy ideals are omitted to avoid repetitions which are already done by researchers Malik D. S. and Mordeson J. N. [5, 6]. We also introduce Quotient rings of the Intuitionistic L-fuzzy set. This work is a generalization of fuzzy algebraic structures introduced by many researchers.

2. Preliminaries

In this section we list some basic concepts and well known results of Intuitionistic L-fuzzy sets. Throughout this paper $(L, \leq, \wedge, \vee)$ denotes a complete

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distributive lattice with maximal element 1 and minimal element 0 respectively. Let $(R, +, \cdot)$ be a commutative ring.

Definition 2.1. *Let X be a non-empty set. A L-fuzzy set μ of X is a function $\mu : X \to L$.*

Definition 2.2. *Let $(L, \leq)$ be the lattice with an involutive order reversing operation $N : L \to L$. Let X be a non-empty set. An Intuitionistic L-fuzzy set (ILFS) A in X is defined as an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$$

where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

Definition 2.3. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in X\}$ be two Intuitionistic L-fuzzy sets of X. Then we define*

(i) $A \subseteq B$ *iff for all* $x \in X$, $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$

(ii) $A = B$ *iff for all* $x \in X$, $\mu_A(x) = \mu_B(x)$ *and* $\nu_A(x) = \nu_B(x)$

(iii) $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle / x \in X\}$

(iv) $A \cup B = \{\langle x, (\mu_A \vee \mu_B)(x), (\nu_A \wedge \nu_B)(x)\rangle / x \in X\}$

(v) $A \cap B = \{\langle x, (\mu_A \wedge \mu_B)(x), (\nu_A \vee \nu_B)(x)\rangle / x \in X\}$

(vi) For ILFS, A_j of R, $j \in I$, I-index set,

$$\cup_{j\in I} A_j = \{\langle x, \vee_{j\in I}\mu_{A_j}(x), \wedge_{j\in I}\nu_{A_j}(x), \rangle / x \in R\}$$

and

$$\cap_{j\in I} A_j = \{\langle x, \wedge_{j\in I}\mu_{A_j}(x), \vee_{j\in I}\nu_{A_j}(x), \rangle / x \in R\}$$

(vii)

$$A + B = \{\langle x, (\mu_A + \mu_B)(x), (\nu_A + \nu_B)(x)\rangle / x \in R\}$$

where
$(\mu_A + \mu_B)(x) = \vee\{\mu_A(y) \wedge \mu_B(z) / y, z \in R, y + z = x\}$
and
$(\nu_A + \nu_B)(x) = \wedge\{\nu_A(y) \vee \nu_B(z) / y, z \in R, y + z = x\}$

(viii)

$$A - B = \{\langle x, (\mu_A - \mu_B)(x), (\nu_A - \nu_B)(x)\rangle / x \in R\}$$

where
$(\mu_A - \mu_B)(x) = \vee\{\mu_A(y) \wedge \mu_B(z) / y, z \in R, y - z = x\}$
and
$(\nu_A - \nu_B)(x) = \wedge\{\nu_A(y) \vee \nu_B(z) / y, z \in R, y - z = x\}$

(ix)

$$A \circ B = \{\langle x, (\mu_A \circ \mu_B)(x), (\nu_A \circ \nu_B)(x)\rangle / x \in R\}$$

where $(\mu_A \circ \mu_B)(x) = \vee\{\mu_A(y) \wedge \mu_B(z)/y, z \in R, yz = x\}$
and
$(\nu_A \circ \nu_B)(x) = \wedge\{\nu_A(y) \vee \nu_B(z)/y, z \in R, yz = x\}$

(x)

$$AB = \{\langle x, (\mu_A\mu_B)(x), (\nu_A\nu_B)(x)\rangle / x \in R\}$$

where $(\mu_A\mu_B)(x) = \vee\{\wedge_{i=1}^{n}(\mu_A(y_i) \wedge \mu_B(z_i))/y_i, z_i \in R, 1 \le i \le n, n \in \mathbb{N}, \sum_{i=1}^{n} y_i z_i = x\}$
and
$(\nu_A\nu_B)(x) = \wedge\{\vee_{i=1}^{n}(\nu_A(y_i) \vee \nu_B(z_i))/y_i, z_i \in R, 1 \le i \le n, n \in \mathbb{N}, \sum_{i=1}^{n} y_i z_i = x\}$

Definition 2.4. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in X\}$ *be an ILFS of* X. *Let* $Y \subseteq X$. *Then*

$$(\mu_A)_Y(x) = \begin{cases} \mu_A, & x \in Y \\ 0, & x \in X\backslash Y \end{cases}$$

and

$$(\nu_A)_Y(x) = \begin{cases} \nu_A, & x \in Y \\ 0, & x \in X\backslash Y \end{cases}$$

If Y *is a singleton then the above functions are the Intuitionistic L-fuzzy point.*

The following results are immediate from the definition and hence their proof is omitted.

Proposition 2.5. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$,
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be two ILFS. Then* $A \circ B \subseteq AB$.

Proposition 2.6. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$,
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFS. Then for all* $x, y \in R$
(i) $(\mu_A\mu_B)(x + y) \ge (\mu_A\mu_B)(x) \wedge (\mu_A\mu_B)(y)$
(ii) $(\nu_A\nu_B)(x + y) \le (\nu_A\nu_B)(x) \vee (\nu_A\nu_B)(y)$.

Proposition 2.7. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and*
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFS. Then for all* $x, y \in R$
(i) $(\mu_A\mu_B)(x) = (\mu_A\mu_B)(-x)$
(ii) $(\nu_A\nu_B)(x) = (\nu_A\nu_B)(-x)$

3. Intuitionistic L-fuzzy subring

Here we introduce and study Intuitionistic L-fuzzy subrings and Intuitionistic L-fuzzy ideals over a ring R.

Definition 3.1. *A L-fuzzy set* μ *of* R *is a L-fuzzy subring of* R *if for all* $x, y \in R$,
(i) $\mu(x - y) \ge \mu(x) \wedge \mu(y)$
(ii) $\mu(xy) \ge \mu(x) \wedge \mu(y)$

Definition 3.2. *An Intuitionistic L-fuzzy subset* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *of* R *is said to be an Intuitionistic L-fuzzy subring of* R *(ILFSR) if for all* $x, y \in R$,
(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$
(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$
(iv) $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$.

Proposition 3.3. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *is an ILFSR. Then*
(i) $\mu_A(0) \geq \mu_A(x)$ *and* $\nu_A(0) \leq \nu_A(x)$ *for all* $x \in R$
(ii) if R *is a ring with identity 1 then* $\mu_A(1) \leq \mu_A(x)$ *and* $\nu_A(1) \geq \nu_A(x)$*, for all* $x \in R$

Proposition 3.4. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFSR then* $\mu_A(x) = \mu_A(-x)$ *and* $\nu_A(x) = \nu_A(-x)$.

Theorem 3.5. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be two ILFSR. Then* $A \cap B$ *is an ILFSR.*

Theorem 3.6. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be two ILFSR. Then* $A \cup B$ *is an ILFSR iff* $A \subseteq B$ *or* $B \subseteq A$.

Theorem 3.7. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFSR. Then* AB *is an ILFSR.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}(\nu_A\nu_B)(x - y) &= (\nu_A\nu_B)(x + (-y)) \\ &\leq (\nu_A\nu_B)(x) \vee (\nu_A\nu_B)(-y) \\ &= (\nu_A\nu_B)(x) \vee (\nu_A\nu_B)(y).\end{aligned}$$

Let $x, y \in R$. Then

$$(\nu_A\nu_B)(xy)$$

$$= \wedge\{ \vee_{i=1}^{n} \vee_{j=1}^{m}(\nu_A(x_iy_j) \vee \nu_B(x_i'y_j'))/x_i, x_i', y_j, y_j' \in R, \sum_{i=1}^{n}\sum_{j=1}^{m} x_ix_i'y_jy_j' = xy, i, j \in \mathbb{N}\}$$

$$\leq \wedge\{ \vee_{i=1}^{n} \vee_{j=1}^{m}((\nu_A(x_i) \vee \nu_A(y_j)) \vee (\nu_B(x_i') \vee \nu_B(y_j')))/x_i, x_i', y_j, y_j' \in R, \sum_{i=1}^{n}\sum_{j=1}^{m} x_ix_i'y_jy_j' = xy, i, j \in \mathbb{N}\}$$

$$= \wedge\{ \vee_{i=1}^{n} \vee_{j=1}^{m}((\nu_A(x_i) \vee \nu_B(x_i')) \vee (\nu_A(y_j) \vee \nu_B(y_j')))/x_i, x_i', y_j, y_j' \in R, \sum_{i=1}^{n}\sum_{j=1}^{m} x_ix_i'y_jy_j' = xy, i, j \in \mathbb{N}\}$$

$$= (\wedge\{ \vee_{i=1}^{n} (\nu_A(x_i) \vee \nu_B(x_i'))/x_i, x_i' \in R, i \in \mathbb{N}, \sum_{i=1}^{n} x_ix_i' = x\})$$

$$\vee (\wedge\{\vee_{j=1}^{m}(\nu_A(y_j) \vee \nu_B(y_j'))/y_j, y_j' \in R, j \in \mathbb{N}, \sum_{j=1}^{m} y_jy_j' = y\})$$

$$= (\nu_A\nu_B)(x) \vee (\nu_A\nu_B)(y)$$

Therefore AB is an ILFSR. □

Proposition 3.8. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$ *and*
$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in R\}$ *be ILFSR. Then*
(i) $(\mu_A\mu_B)(0) \geq (\mu_A\mu_B)(x)$
(ii) $(\nu_A\nu_B)(0) \leq (\nu_A\nu_B)(x)$ *for all* $x \in R$.

Definition 3.9. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$ *be an ILFSR of* R*. Then* A *is called an Intuitionistic L-fuzzy ideal of* R *(ILFI) if,*
(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$
(ii) $\mu_A(xy) \geq \mu_A(x)$
(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$
(iv) $\nu_A(xy) \leq \nu_A(x)$*, for all* $x, y \in R$.

Definition 3.10. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$ *be an ILFI of* R*. Then we define*

$$(\mu_A)_* = \{x \in R/\mu_A(x) = \mu_A(0)\}$$
$$(\nu_A)_* = \{x \in R/\nu_A(x) > \nu_A(0)\}$$

Proposition 3.11. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI. If* $\mu_A(x-y) = \mu_A(0)$ *then* $\mu_A(x) = \mu_A(y)$ *and if* $\nu_A(x-y) = \nu_A(0)$ *then* $\nu_A(x) = \nu_A(y)$, *for all* $x, y \in R$.

Theorem 3.12. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI. Then*
(i) $(\mu_A)_*$ *is an ideal of* R,
(ii) $(\nu_A)_*$ *need not be an ideal of* R.

Proposition 3.13. *Let* R *be a ring with identity 1. Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI of* R. *Then*

(i) Let $x \in R$ *be a unit. Then*

$$\mu_A(x) = \mu_A(x^{-1}) = \mu_A(1) \quad \text{and}$$
$$\nu_A(x) = \nu_A(x^{-1}) = \nu_A(1)$$

(ii) Let x *and* y *be associates then*

$$\mu_A(x) = \mu_A(y) \text{ and } \nu_A(x) = \nu_A(y)$$

Theorem 3.14. *If* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI. Then* $A \cap B$ *is an ILFI. Moreover for any* $j \in I$, I*-index set,* $A_j = \{\langle x, \mu_{A_j}(x), \nu_{A_j}(x)\rangle / x \in R\}$ *be an ILFI then* $\cap_{j\in I} A_j$ *is an ILFI.*

Proposition 3.15. *Every ILFI is an ILFSR.*

Theorem 3.16. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *and* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in R\}$ *be ILFI. Then*

$$\mu_A \circ \mu_B \subseteq \mu_C \text{ iff } \mu_A\mu_B \subseteq \mu_C \text{ and}$$
$$\nu_A \circ \nu_B \supseteq \nu_C \text{ iff } \nu_A\nu_B \supseteq \nu_C$$

Theorem 3.17. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI. Then*
(i) $\mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B$
(ii) $\nu_A \circ \nu_B \supseteq \nu_A \cap \nu_B$.

Theorem 3.18. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI. Then*
(i) $\mu_A\mu_B \subseteq \mu_A \cap \mu_B$
(ii) $\nu_A\nu_B \supseteq \nu_A \cap \nu_B$.

Theorem 3.19. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be ILFI and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFSR. Then* AB *is an ILFI.*

Proof. Clearly AB is an ILFSR. To prove AB is an ILFI, for the non-membership function, with respect to product. For $x, y \in R$,

$$\begin{aligned}(\nu_A\nu_B)(xy) &= \wedge\{\vee_{i=1}^{n}(\nu_A(xu_i) \vee \nu_B(v_i))/u_i, v_i \in R, 1 \leq i \leq n, n \in \mathbb{N}, \\ &\qquad y = \sum_{i=1}^{n} u_i v_i\} \\ &\leq (\nu_A\nu_B)(y).\end{aligned}$$

Hence AB is an ILFI. □

4. Some operations on Intuitionistic L-fuzzy ideals

Here we introduce and study Intuitionistic L-fuzzy ideal generated by an ILFI. Further residual quotient are constructed and studied.

Theorem 4.1. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI. Then* $A + B$ *is an ILFI.*

Theorem 4.2. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI. Then* AB *is an ILFI.*

Definition 4.3. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFS. Let* $\langle A\rangle = \{\langle x, \langle\mu_A\rangle, \langle\nu_A\rangle\rangle / x \in R\}$ *where*

$$\langle\mu_A\rangle = \cap\{\mu : \mu_A \subseteq \mu, \quad \mu \in \textit{ILFI}\}$$
$$\langle\nu_A\rangle = \cap\{\nu : \nu_A \subseteq \nu, \quad \nu \in \textit{ILFI}\}$$

Then $\langle A\rangle$ *is called the ILFI generated by* A.

Proposition 4.4. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFS. Then* $\langle A\rangle = \{\langle x, \langle\mu_A\rangle, \langle\nu_A\rangle\rangle / x \in R\}$ *is an ILFI and also the smallest ILFI containing* A.

Theorem 4.5. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFS. Then* A *is an ILFI iff* $A = \langle A\rangle$ *where* $\langle A\rangle = \{\langle x, \langle\mu_A\rangle, \langle\nu_A\rangle\rangle / x \in R\}$.

Proof. Let A be an ILFI. Then $\langle A\rangle \supseteq A$ and $\langle A\rangle \subseteq A$. Therefore $A = \langle A\rangle$. Conversely, $A = \langle A\rangle$ is also an ILFI. □

Theorem 4.6. *Let* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in R\}$ *and* $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle / x \in R\}$ *be an ILFS. If* $C \subseteq D$ *then* $\langle C\rangle \subseteq \langle D\rangle$.

Proof. Let $C \subseteq D$ then $\nu_D(x) \leq \nu_C(x)$, for $x \in R$.
For $K_1 \in \langle\nu_C\rangle$. Then

$$\begin{aligned}K_1 &\in \cap\{\nu : \nu_C \subset \nu, \nu \in \text{ILFI}\} \\ &\Rightarrow \nu_C \subseteq K_1 \\ &\Rightarrow \nu_C(x) \geq K_1(x) \text{ for } x \in R\end{aligned}$$

Hence, $\nu_C(x) = K_1(x) \vee \nu_D(x) \geq K_1(x)$

$$\begin{aligned} &\Rightarrow \nu_D(x) \geq K_1(x) \\ &\Rightarrow \nu_D \subseteq K_1 \\ &\Rightarrow K_1 \in \langle \nu_D \rangle \\ &\Rightarrow \langle \nu_C \rangle \subseteq \langle \nu_D \rangle \end{aligned}$$

Theorem 4.7. *Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *and* $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle / x \in R\}$ *be ILFI where* $A(0) = B(0)$. *Then* $A + B = \langle A \cup B \rangle$.

Proof. For $x \in R$,

$$\begin{aligned} (\nu_A + \nu_B)(x) &= \wedge\{\nu_A(y) \vee \nu_B(z)/y, z \in R, y + z = x\} \\ &\leq \nu_A(x) \vee \nu_B(0) \\ &= \nu_A(x) \vee \nu_A(0) \\ &= \nu_A(x) \end{aligned}$$

which implies that $\nu_A + \nu_B \supseteq \nu_A$.
Similarly $\nu_A + \nu_B \supseteq \nu_B$. Hence $\nu_A + \nu_B \supseteq \nu_A \cap \nu_B$.
Therefore $A \cup B \subseteq A + B$.
Let $C = \{\langle x, \mu_c(x), \nu_c(x)\rangle / x \in R\}$ be an ILFI such that $A \cup B \subseteq C$. Then for $x \in R$,

$$\begin{aligned} (\nu_A + \nu_B)(x) &= \wedge\{\nu_A(y) \vee \nu_B(z)/y, z \in R,\ y + z = x\} \\ &\geq \wedge\{\nu_c(y) \vee \nu_c(z)/y, z \in R,\ y + z = x\} \\ &= \nu_c(x) \end{aligned}$$

Then $A + B \subseteq C$. Hence $A + B$ is the smallest ILFI such that $A + B \subseteq C$. Thus $A + B = \langle A \cup B \rangle$. □

Definition 4.8. *Let* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in R\}$ *and* $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle / x \in R\}$ *be ILFS. Let* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ *be an ILFI. Then*

$$C : D = \{\langle x, \mu_C : \mu_D, \nu_C : \nu_D\rangle / x \in R\}$$

is called the residual quotient of C *by* D *where*

$$\begin{aligned} \mu_C : \mu_D &= \cup\{\mu_A / \mu_A \circ \mu_D \subseteq \mu_C\} \\ \nu_C : \nu_D &= \cap\{\nu_A / \nu_A \circ \nu_D \supseteq \nu_C\} \end{aligned}$$

Theorem 4.9. *Let* $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle / x \in R\}$ *and* $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle / x \in R\}$ *be ILFI. Then*
(i) $C \subseteq C : D$
(ii) $C : D$ *is an ILFI.*

Proof. (i) Now,

$$\nu_C \circ \nu_D \supseteq \nu_C \nu_D \supseteq \nu_C \cap \nu_D \supseteq \nu_C$$

Hence $C \subseteq C : D$.
(ii) Let $x, y \in R$. Also $(\nu_C : \nu_D)(-x) = (\nu_C : \nu_D)(x)$. Now,

$$\begin{aligned}
&(\nu_C : \nu_D)(x) \vee (\nu_C : \nu_D)(y) \\
&\quad = (\wedge\{\nu_A(x)/A \in \text{ILFI}, \nu_A \circ \nu_D \supseteq \nu_C\}) \vee (\wedge\{\nu_B(y)/B \in \text{ILFI}, \nu_B \circ \nu_D \supseteq \nu_C\}) \\
&\quad = \wedge\{\nu_A(x) \vee \nu_B(y)/A, B \in \text{ILFI}, (\nu_A \circ \nu_D) \cup (\nu_B \circ \nu_D) \supseteq \nu_C\} \\
&\quad \geq \wedge\{(\nu_A + \nu_B)(x+y)/A, B \in \text{ILFI}, (\nu_A + \nu_B) \circ \nu_D \supseteq \nu_C\} \\
&\quad = (\nu_C : \nu_D)(x+y).
\end{aligned}$$

Hence $(\nu_C : \nu_D)(x-y) \leq (\nu_C : \nu_D)(x) \vee (\nu_C : \nu_D)(y)$. Also for $x, y \in R$,

$$\begin{aligned}
(\nu_C : \nu_D)(xy) &= \wedge\{\nu_A(xy)/A \in \text{ILFI}, \nu_A \circ \nu_D \supseteq \nu_C\} \\
&\leq \wedge\{\nu_A(x)/A \in \text{ILFI}, \nu_A \circ \nu_D \supseteq \nu_C\} \\
&= (\nu_C : \nu_D)(x)
\end{aligned}$$

Therefore $C : D$ is an ILFI. □

Theorem 4.10. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle/x \in R\}$, $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle/x \in R\}$ and $C = \{\langle x, \mu_C(x), \nu_C(x)\rangle/x \in R\}$ be ILFI. If $\mu_A \subseteq \mu_B$, $\nu_A \supseteq \nu_B$ then*

$$\mu_A : \mu_C \subseteq \mu_B : \mu_C,\ \nu_A : \nu_C \supseteq \nu_B : \nu_C \text{ and}$$
$$\mu_C : \mu_B \subseteq \mu_C : \mu_A,\ \nu_C : \nu_B \supseteq \nu_C : \nu_A$$

Proof. Let $D = \{\langle x, \mu_D(x), \nu_D(x)\rangle/x \in R\}$ be an ILFI such that $\mu_D \circ \mu_C \subseteq \mu_A$, $\nu_D \circ \nu_C \supseteq \nu_A$. Then $\nu_D \circ \nu_C \supseteq \nu_A \supseteq \nu_B$. Hence

$$\begin{aligned}
\nu_B : \nu_C &= \cap\{\nu_D/\nu_D \circ \nu_C \supseteq \nu_B\} \\
&\subseteq \cap\{\nu_D/\nu_D \circ \nu_C \supseteq \nu_A\} \\
&= \nu_A : \nu_C
\end{aligned}$$

If $\nu_D \circ \nu_B \supseteq \nu_C$ then

$$\nu_D \circ \nu_A \supseteq \nu_D \circ \nu_B \supseteq \nu_C.$$

Hence

$$\begin{aligned}
\nu_C : \nu_A &= \cap\{\nu_D/\nu_D \circ \nu_A \supseteq \nu_C\} \\
&\subseteq \cap\{\nu_D/\nu_D \circ \nu_B \supseteq \nu_C\} \\
&= \nu_C : \nu_B
\end{aligned}$$

□

5. Quotient ring

Here we introduce Intuitionistic L-fuzzy coset and study Quotient ring of R by the membership and non-membership functions of an ILFI.

Definition 5.1. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFSR. Let $x \in R$. Then*

$$C = \{\langle x, (\mu_A(0)_{\{x\}} + \mu_A)(x), (\nu_A(0)_{\{x\}} + \nu_A)(x)\rangle / x \in R\}$$

is called an Intuitionistic L-fuzzy coset (ILFC) of A and denoted as

$$C = \{\langle x, (x + \mu_A)(x), (x + \nu_A)(x)\rangle / x \in R\}$$

Proposition 5.2. *Let $x, y \in R$. Let $C = \{\langle x, (\mu_A(0)_{\{x\}} + \mu_A)(x), (\nu_A(0)_{\{x\}} + \nu_A)(x) / x \in R\}$ be an ILFC of an ILFSR A. Then*
(i) $(\mu_A(0)_{\{x\}} + \mu_A)(y) = \mu_A(x - y)$
(ii) $(\nu_A(0)_{\{x\}} + \nu_A)(y) = \nu_A(x - y)$

Proof. For $y \in R$,

$$\begin{aligned}(\nu_A(0)_{\{x\}} + \nu_A)(y) &= \wedge[\nu_A(0)_{\{x\}}(u) \vee \nu_A(v) / u, v \in R, u + v = y] \\ &= \nu_A(0)_{\{x\}}(x) \vee \nu_A(y - x) \\ &= \nu_A(0) \vee \nu_A(y - x) \\ &= \nu_A(y - x) \\ &= \nu_A(x - y)\end{aligned}$$

Remark 5.3. *The above result becomes*

$$(x + \mu_A)(y) = \mu_A(x - y)$$
$$\text{and } (x + \nu_A)(y) = \nu_A(x - y)$$

Lemma 5.4. *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI and $x, y \in R$. Then*
(i) $x + \mu_A = y + \mu_A$ iff $\mu_A(x - y) = \mu_A(0)$
(ii) $x + \nu_A = y + \nu_A$ iff $\nu_A(x - y) = \nu_A(0)$

Proof. Suppose $x + \nu_A = y + \nu_A$. Then

$$(x + \nu_A)(x) = (y + \nu_A)(x)$$
$$\nu_A(0) = \nu_A(x - y).$$

Conversely, if $\nu_A(x - y) = \nu_A(0)$, then for $z \in R$,

$$\begin{aligned}(x + \nu_A)(z) &= \nu_A(x - z) \\ &= \nu_A(x - y + y - z) \\ &\leq \nu_A(x - y) \vee \nu_A(y - z) \\ &= \nu_A(0) \vee \nu_A(y - z) \\ &= \nu_A(y - z).\end{aligned}$$

Similarly,

$$\nu_A(y-z) = (y+\nu_A)z \leq \nu_A(x-z)$$

Therefore

$$\nu_A(x-z) = \nu_A(y-z)$$

and hence $x+\nu_A = y+\nu_A$.
Let $A = \{x, \langle \mu_A(x), \nu_A(x)\rangle / x \in R\}$ be an ILFI. Let $R/A = \{(x+\mu_A, x+\nu_A)/x \in R\}$. Define $+$ and $\cdot$ on R/A by
(i) $(x+\mu_A)+(y+\mu_A) = x+y+\mu_A$,
(ii) $(x+\nu_A)+(y+\nu_A) = x+y+\nu_A$, for all $x, y \in R$,
and
(i) $(x+\mu_A)\cdot(y+\mu_A) = xy+\mu_A$,
(ii) $(x+\nu_A)\cdot(y+\nu_A) = xy+\nu_A$, for all $x, y \in R$.

Then addition on R/A is well defined. For this let $u, v \in R$ such that

$$x+\nu_A = u+\nu_A$$

and $y+\nu_A = v+\nu_A$, for $x, y \in R$. Therefore $\nu_A(x-u) = \nu_A(0)$ and $\nu_A(y-v) = \nu_A(0)$. Hence

$$\begin{aligned} \nu_A(x+y-u-v) &= \nu_A(x-u+y-v) \\ &\leq \nu_A(x-u) \vee \nu_A(y-v) \\ &= \nu_A(0) \end{aligned}$$

Also

$$\nu_A(x+y-u-v) \geq \nu_A(0)$$

Hence $\nu_A(x+y-u-v) = \nu_A(0)$ which implies that $x+y+\nu_A = u+v+\nu_A$.
Then product on R/A is well-defined. For this, let $u, v \in R$ such that $x+\nu_A = u+\nu_A$ and $y+\nu_A = v+\nu_A$, for $x, y \in R$.
Therefore $\nu_A(x-u) = \nu_A(0)$ and $\nu_A(y-v) = \nu_A(0)$.
Hence

$$\begin{aligned} \nu_A(xy-uv) &= \nu_A(xy-uy+uy-uv) \\ &\leq \nu_A(xy-uy) \vee \nu_A(uy-uv) \\ &\leq \nu_A(x-u) \vee \nu_A(y-v) \\ &= \nu_A(0) \end{aligned}$$

which implies that

$$xy+\nu_A = uv+\nu_A$$

□

Remark 5.5. *(i) $0+\mu_A = \mu_A$, $0+\nu_A = \nu_A$ is the zero element of R/A.*

(ii) *For all* $x \in R$

$$-(x + \mu_A) = (-x) + \mu_A$$
$$-(x + \nu_A) = (-x) + \nu_A$$

Additive inverse of $x + \mu_A$, $x + \nu_A \in R/A$ *is* $(-x) + \mu_A$, $(-x) + \nu_A$.

(iii) *For all* $x, y, z \in R/A$

$$(x + y) + z = x + (y + z)$$

(iv) *For all* $x, y \in R/A$

$$x + y = y + x$$

(v) *For* $x, y, z \in R/A$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

(vi) *For* $x, y \in R/A$

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Hence $R/A = \{(x + \mu_A, x + \nu_A)/x \in R\}$ *is a ring with respect to* $+$ *and* $\cdot$ *and is called Quotient ring of* R *by* μ_A *and* ν_A.

Bibliography

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87–96.

[2] K. T. Atanassov, New Operations defined over the Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61 (1994), 137–142.

[3] K. T. Atanassov, Fuzzy sets, Theory and Applications, *Studies in fuzziness and soft computing*, 35, Physica-Verlag, Heidelberg, 1999.

[4] R. Biswas, Intuitionistic fuzzy subgroups, *Math. Forum*, **10**, (1989), 37–46.

[5] D. S. Malik and J. N. Mordeson, Fuzzy Commutative Algebra, 1998, World Scientific.

[6] D. S. Malik and J. N. Mordeson, Extensions of fuzzy subrings and fuzzy ideals, *FSS*, 45 (1992), 245–251.

[7] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8 (1965), 338–353.

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