Math. Z. 229, 601–606 (1998)

Mathematische Zeitschrift


# Representation type and stable equivalence of Morita type for finite dimensional algebras

**Henning Krause**

Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
(e-mail: henning@mathematik.uni-bielefeld.de)

Received January 3, 1997; in final form August 20, 1997

In this note we show that two finite dimensional algebras have the same representation type if they are stably equivalent of Morita type.

Stable equivalences of Morita type were introduced for blocks of group algebras by Broué [2], see also [7]. The concept is motivated by a result of Rickard. In [10], he proved that any derived equivalence between finite dimensional self-injective algebras $\Lambda$ and $\Gamma$ induces a stable equivalence of a particulary nice form. More precisely, he constructs bimodules ${}_{\Lambda}B_{\Gamma}$ and ${}_{\Gamma}C_{\Lambda}$ such that the corresponding tensor functors induce mutually inverse equivalences between the stable categories $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Gamma$ of finite dimensional modules for $\Lambda$ and $\Gamma$. Stable equivalences of this form are said to be of Morita type. They occur frequently, for instance in the theory of blocks of group algebras. Other examples are obtained from two finite dimensional algebras $\Lambda$ and $\Gamma$ which are tilted from each other. The corresponding trivial extensions $T\Lambda$ and $T\Gamma$ are derived equivalent self-injective algebras [9], and therefore stably equivalent of Morita type. In fact, Assem and de la Peña proved in [1] that $T\Lambda$ and $T\Gamma$ have the same representation type. It seems to be an interesting project to understand the precise relation between the geometric approach of de la Peña [8] which is used in [1], and the approach presented in this paper.

The representation type of a finite dimensional algebra $\Lambda$ is traditionally defined using the concept of a continuous one-parameter family in the category $\mathrm{mod}\,\Lambda$ of finite dimensional $\Lambda$-modules. For instance, $\Lambda$ is of tame representation type if for every $n \in \mathbb{N}$ only finitely many such families are needed to parametrize all indecomposable $\Lambda$-modules of dimension $n$. An alternative approach uses so-called generic modules. This was suggested by Crawley-Boevey and he established for tame algebras a correspondence between continuous one-parameter families and generic modules [3]. In [6], we used generic modules to show that a stable equivalence $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Gamma$ induces a bijection between continuous one-parameter families in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$. However, without any extra assumption on the stable equivalence it remains an open question how the dimensions of the modules in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma$ are related. In this paper we settle the problem for stable equivalences which are induced by an appropriate functor $\mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$. Stable equivalences of Morita type are of this form, and therefore we can prove that two finite dimensional algebras have the same representation type if they are stably equivalent of Morita type.

I would like to thank Thorsten Holm for drawing my attention to the representation type problem for derived equivalent algebras.

We now begin with some notation and recall briefly the definitions which are needed in this paper. Let $\Lambda$ be an associative ring with identity. Denote by $\mathrm{Mod}\,\Lambda$ the category of (right) $\Lambda$-modules and let $\mathrm{mod}\,\Lambda$ be the full subcategory of finitely presented $\Lambda$-modules. A sequence of morphisms $0 \to L \to M \to N \to 0$ in $\mathrm{Mod}\,\Lambda$ is *pure-exact* if the induced sequence $0 \to \mathrm{Hom}_\Lambda(X, L) \to \mathrm{Hom}_\Lambda(X, M) \to \mathrm{Hom}_\Lambda(X, N) \to 0$ is exact for every $X$ in $\mathrm{mod}\,\Lambda$. In this case the morphism $L \to M$ is called a *pure monomorphism*, and $L$ is *pure-injective* if every pure monomorphism $L \to M$ splits. We shall always assume that every finitely presented $\Lambda$-module is pure-injective. For example, every artin algebra has this property. We denote by $\mathrm{Pinj}\,\Lambda$ the full subcategory of pure-injective $\Lambda$-modules and the *Ziegler spectrum* $\mathrm{Zsp}\,\Lambda$ is the set of isomorphism classes of indecomposable pure-injective $\Lambda$-modules. A $\Lambda$-module $M$ is *endofinite* if the length $\ell_{\mathrm{end}}(M)$ of $M$, when regarded in the natural way as an $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module, is finite. Note that every endofinite module is pure-injective. An endofinite module is called *generic* if it is indecomposable but not finitely presented [3]. Given a $\Lambda$-module $X$ we denote by $\mathrm{Add}\,X$ the smallest full subcategory of $\mathrm{Mod}\,\Lambda$ which contains $X$ and is closed under taking arbitrary coproducts and direct summands. The *stable category* $\underline{\mathrm{Mod}}_X\,\Lambda$ has the same objects as $\mathrm{Mod}\,\Lambda$ but $\underline{\mathrm{Hom}}_X(M, N)$ is the group of all $\Lambda$-morphisms $M \to N$ modulo the subgroup of those which factor through an object in $\mathrm{Add}\,X$. For example, one obtains the usual stable module category $\underline{\mathrm{Mod}}\,\Lambda$ by taking $X = \Lambda$. Analogously, the categories $\underline{\mathrm{Pinj}}_X\,\Lambda$ and $\underline{\mathrm{mod}}_X\,\Lambda$ are defined.

We shall need the following lemma.

**Lemma.** Let $X$ be endofinite. Then every $\Lambda$-module $M$ has a decomposition $M = M_X \coprod X_M$ such that $X_M$ is a maximal direct summand of $M$ which belongs to $\mathrm{Add}\,X$.

*Proof.* The union of a chain of pure submodules is again a pure submodule. Also, $\mathrm{Add}\,X$ is closed under taking direct limits and every module in $\mathrm{Add}\,X$

is pure-injective [5, Corollary 9.8]. Thus, by Zorn's lemma there is a maximal pure submodule $X_M$ of $M$ contained in $\operatorname{Add} X$ which is a direct summand. $\square$

The next proposition collects the essential properties of a stable equivalence $\underline{\operatorname{mod}}_X \Lambda \to \underline{\operatorname{mod}}_Y \Gamma$ which is induced by some functor $\operatorname{Mod} \Lambda \to \operatorname{Mod} \Gamma$.

**Proposition.** *Let $f\colon \operatorname{Mod} \Lambda \to \operatorname{Mod} \Gamma$ be a functor commuting with direct limits and products. Suppose there are endofinite modules $X$ in $\operatorname{mod} \Lambda$ and $Y$ in $\operatorname{mod} \Gamma$ such that $f$ induces an equivalence $\underline{\operatorname{mod}}_X \Lambda \to \underline{\operatorname{mod}}_Y \Gamma$. Then the following holds:*

(1) *$f$ induces an equivalence $\underline{\operatorname{Pinj}}_X \Lambda \to \underline{\operatorname{Pinj}}_Y \Gamma$.*

(2) *$f$ induces a bijection $\operatorname{Zsp} \Lambda \setminus \operatorname{Add} X \to \operatorname{Zsp} \Gamma \setminus \operatorname{Add} Y$, $M \mapsto M_f$, such that $fM = M_f \coprod Y_{fM}$ and $Y_{fM}$ is a maximal direct summand of $fM$ in $\operatorname{Add} Y$.*

(3) *There exists $c \in \mathbb{N}$ such that $\ell_{\text{end}}(fM) \le c \cdot \ell_{\text{end}}(M)$ for all $M$ in $\operatorname{Mod} \Lambda$.*

(4) *If $M$ is a generic $\Lambda$-module, then $M_f$ is generic.*

*Proof.* (1) The functor $f$ induces the following commutative diagram of functors:

$$\begin{array}{ccc} \operatorname{mod} \Lambda & \xrightarrow{f} & \operatorname{mod} \Gamma \\ \downarrow & & \downarrow \\ \underline{\operatorname{mod}}_X \Lambda & \xrightarrow{\sim} & \underline{\operatorname{mod}}_Y \Gamma \end{array}$$

The canonical functor $\operatorname{mod} \Lambda \to \underline{\operatorname{mod}}_X \Lambda$ extends uniquely to a functor $p_\Lambda\colon \operatorname{Mod} \Lambda \to \varinjlim \underline{\operatorname{mod}}_X \Lambda$ which commutes with direct limits. Here, $\varinjlim \underline{\operatorname{mod}}_X \Lambda$ denotes the category $\operatorname{Flat}((\underline{\operatorname{mod}}_X \Lambda)^{\text{op}}, \operatorname{Ab})$ of flat functors $(\underline{\operatorname{mod}}_X \Lambda)^{\text{op}} \to \operatorname{Ab}$ which is characterized by the following three properties [6]:

(i) $\varinjlim \underline{\operatorname{mod}}_X \Lambda$ contains, up to equivalence, $\underline{\operatorname{mod}}_X \Lambda$ as a full subcategory.

(ii) $\varinjlim \underline{\operatorname{mod}}_X \Lambda$ is an additive category with direct limits.

(iii) Every object in $\varinjlim \underline{\operatorname{mod}}_X \Lambda$ is a direct limit of objects in $\underline{\operatorname{mod}}_X \Lambda$.

We obtain therefore the following commutative diagram of functors which commute with direct limits:

$$\begin{array}{ccc} \operatorname{Mod} \Lambda & \xrightarrow{f} & \operatorname{Mod} \Gamma \\ \downarrow{\scriptstyle p_\Lambda} & & \downarrow{\scriptstyle p_\Gamma} \\ \varinjlim \underline{\operatorname{mod}}_X \Lambda & \xrightarrow{\sim} & \varinjlim \underline{\operatorname{mod}}_Y \Gamma \end{array}$$

It follows from Proposition 2.2 and Proposition 3.2 in [6] that $p_\Lambda$ induces an equivalence $\underline{\operatorname{Pinj}}_X \Lambda \to \operatorname{Pinj}(\varinjlim \underline{\operatorname{mod}}_X \Lambda)$ where the notion of purity in $\varinjlim \underline{\mathrm{mod}}_X \Lambda$ is defined as for $\mathrm{Mod}\,\Lambda$. The assertion in (1) is now a consequence since $\varinjlim \underline{\mathrm{mod}}_X \Lambda \to \varinjlim \underline{\mathrm{mod}}_Y \Gamma$ is an equivalence.

(2) Every indecomposable pure-injective module has a local endomorphism ring. Therefore the canonical functor $\mathrm{Mod}\,\Lambda \to \underline{\mathrm{Mod}}_X \Lambda$ induces an injective map from $\mathrm{Zsp}\,\Lambda \setminus \mathrm{Add}\,X$ into the set of isomorphism classes of indecomposable objects in $\underline{\mathrm{Pinj}}_X \Lambda$. The map is surjective by the preceding lemma. Keeping the notation from this lemma, it follows from (1) that $M \mapsto M_f = (fM)_Y$ induces a bijection between $\mathrm{Zsp}\,\Lambda \setminus \mathrm{Add}\,X$ and $\mathrm{Zsp}\,\Gamma \setminus \mathrm{Add}\,Y$.

(3) This has been shown in [4]. However, we sketch the argument for the convenience of the reader. The composition $F = \mathrm{Hom}_\Gamma(\Gamma, -)\circ f\colon \mathrm{Mod}\,\Lambda \to \mathrm{Ab}$ has a presentation $\mathrm{Hom}_\Lambda(B, -) \to \mathrm{Hom}_\Lambda(A, -) \to F \to 0$ with $A$ and $B$ in $\mathrm{mod}\,\Lambda$, and there is a homomorphism $\varphi\colon \Gamma \to \mathrm{End}(F)$ such that the $\Gamma$-action on $fM$ coincides with the $\Gamma$-action on $FM$ via $\varphi$ [4, Corollary 12.2]. Choose an epimorphism $\Lambda^c \to A$ in $\mathrm{mod}\,\Lambda$. Then $c \cdot \ell_{\mathrm{end}}(M)$ bounds the length of the $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module $FM$ and therefore also the endolength of $fM$ since the $\mathrm{End}_\Lambda(M)^{\mathrm{op}}$-module structure on $FM$ is induced by that of $\mathrm{End}_\Gamma(fM)^{\mathrm{op}}$ via the canonical homomorphism $\mathrm{End}_\Lambda(M) \to \mathrm{End}_\Gamma(fM)$.

(4) It follows from the decomposition $fM = M_f \coprod Y_{fM}$ and part (3) that $M_f$ is endofinite. We claim that $M_f$ is not finitely presented. Otherwise we find a finitely presented $\Lambda$-module $L$ such that $fL$ and $M_f$ are isomorphic in $\underline{\mathrm{mod}}_Y \Gamma$ since $f$ induces an equivalence $\underline{\mathrm{mod}}_X \Lambda \to \underline{\mathrm{mod}}_Y \Gamma$. Therefore $fM$ and $fL$ are isomorphic in $\underline{\mathrm{Pinj}}_Y \Gamma$ since we assume that every finitely presented module is pure-injective. Using (1) it follows that $M$ and $L$ are isomorphic in $\underline{\mathrm{Pinj}}_X \Lambda$. Thus there are morphisms $\varphi\colon M \to L$ and $\psi\colon L \to M$ in $\mathrm{Mod}\,\Lambda$ such that $\mathrm{id}_M - \psi\circ\varphi$ factors through an object in $\mathrm{Add}\,X$. Therefore $\mathrm{id}_M - \psi\circ\varphi \in \mathrm{rad}\,\mathrm{End}_\Lambda(M)$ and $\varphi$ is a split monomorphism, a contradiction. We conclude that $M_f$ is generic. $\square$

Given a ring $\Lambda$ and $n \in \mathbb{N}$, we denote by $g_\Lambda(n)$ the number of isomorphism classes of generic $\Lambda$-modules of endolength $n$, and $\gamma_\Lambda(n) = \sum_{i=1}^n g_\Lambda(i)$.

**Corollary.** *Suppose there is an equivalence $\underline{\mathrm{mod}}_X \Lambda \to \underline{\mathrm{mod}}_Y \Gamma$ which is induced by a functor $\mathrm{Mod}\,\Lambda \to \mathrm{Mod}\,\Gamma$ commuting with direct limits and products. Then there exists $c \in \mathbb{N}$ such that $\gamma_\Lambda(n) \le \gamma_\Gamma(cn)$ for all $n$. In particular, the following holds:*

(1) *If $\Gamma$ is generically tame, i.e. $\gamma_\Gamma(n) < \infty$ for all $n$, then $\Lambda$ is generically tame.*

(2) *If $\Gamma$ is generically of polynomial growth, i.e. there exists a polynomial $p$ such that $\gamma_\Gamma(n) \le p(n)$ for all $n$, then $\Lambda$ is generically of polynomial growth.*

(3) *If $\Gamma$ is generically domestic, i.e. there exists $N \in \mathbb{N}$ such that $\gamma_\Gamma(n) \leq N$ for all $n$, then $\Lambda$ is generically domestic.*

*Proof.* The formula $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$ is an immediate consequence of the proposition, and (1) – (3) follow from this formula. □

Suppose that $\Lambda$ is a finite dimensional algebra over an algebraically closed field and denote by $\mu_\Lambda(n)$ the minimal number of continuous one-parameter families which is needed to parametrize all but finitely many indecomposable $\Lambda$-modules of dimension $n$. In [3, Theorem 4.4 and 5.6], Crawley-Boevey has shown that $\mu_\Lambda(n)$ is finite for all $n$ iff $\Lambda$ is generically tame; moreover in this case $\mu_\Lambda(n) = \sum_{i|n} g_\Lambda(i)$. Therefore the 'generical' definition of the representation type coincides with the 'classical' one. More precisely:

(i) $\Lambda$ is of tame representation type (i.e. $\mu_\Lambda(n) < \infty$ for all $n$) iff $\Lambda$ is generically tame.

(ii) $\Lambda$ is of polynomial growth (i.e. there exists a polynomial $p$ such that $\mu_\Lambda(n) \leq p(n)$ for all $n$) iff $\Lambda$ is generically of polynomial growth.

(iii) $\Lambda$ is of domestic representation type (i.e. there exists $N \in \mathbb{N}$ such that $\mu_\Lambda(n) \leq N$ for all $n$) iff $\Lambda$ is generically domestic.

We now obtain our promised result. To this end we call two finite dimensional algebras over a field $k$ *stably equivalent of Morita type* if there are bimodules $B$ in $\operatorname{mod} \Lambda^{\mathrm{op}} \otimes_k \Gamma$ and $C$ in $\operatorname{mod} \Gamma^{\mathrm{op}} \otimes_k \Lambda$ such that the functors

$$- \otimes_\Lambda B \colon \operatorname{mod} \Lambda \longrightarrow \operatorname{mod} \Gamma \qquad \text{and} \qquad - \otimes_\Gamma C \colon \operatorname{mod} \Gamma \longrightarrow \operatorname{mod} \Lambda$$

induce mutually inverse equivalences between $\underline{\operatorname{mod}}\, \Lambda$ and $\underline{\operatorname{mod}}\, \Gamma$.

**Corollary.** *Let $\Lambda$ and $\Gamma$ be finite dimensional algebras over an algebraically closed field. Suppose that $\Lambda$ and $\Gamma$ are stably equivalent of Morita type. Then there exists $c \in \mathbb{N}$ such that $\gamma_\Lambda(n) \leq \gamma_\Gamma(cn)$ and $\gamma_\Gamma(n) \leq \gamma_\Lambda(cn)$ for all $n$. In particular, the following holds:*

(1) *$\Lambda$ is of tame representation type if and only $\Gamma$ is of tame representation type.*

(2) *$\Lambda$ is of polynomial growth if and only $\Gamma$ is of polynomial growth.*

(3) *$\Lambda$ is of domestic representation type if and only $\Gamma$ is of domestic representation type.*

*Proof.* Denote by ${}_\Lambda B_\Gamma$ and ${}_\Gamma C_\Lambda$ the bimodules which induce the stable equivalence of Morita type. The corresponding tensor functors $- \otimes_\Lambda B \colon \operatorname{Mod} \Lambda \to \operatorname{Mod} \Gamma$ and $- \otimes_\Gamma C \colon \operatorname{Mod} \Gamma \to \operatorname{Mod} \Lambda$ commute with direct limits and products since $B$ is finitely presented over $\Lambda$ and $C$ is finitely presented over $\Gamma$. The assertion therefore follows from the preceding corollary and Crawley-Boevey's result. □

## References

1. I. Assem, J.A. de la Peña, On the tameness of trivial extension algebras, Fund. Math. **149** (1996) 171–181.
2. M. Broué, Equivalences of blocks of group algebras, in: V. Dlab, L.L. Scott (eds.), Finite dimensional algebras and related topics, Kluwer (1994) 1–26.
3. W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. **63** (1991) 241–264.
4. H. Krause, Functors on locally finitely presented categories, Colloq. Math. **75** (1998) 105–131.
5. H. Krause, Exactly definable categories, J. Algebra **201** (1998) 456–492.
6. H. Krause, Stable equivalence preserves representation type, Comment. Math. Helv. **72** (1997) 266–284.
7. M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and $p$-groups, Math. Z. **223** (1996) 87–100.
8. J.A. de la Peña, Constructible functors and the notion of tameness, Comm. Algebra **24** (1996) 1939-1955.
9. J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra, **61** (1989) 303–317.
10. J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **43** (1991) 37–48.