Weighted convergence of Lagrange interpolation based on Gauss-Kronrod nodes

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Dedicated to Professor P. L. Butzer on the occasion of his 70th birthday

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Abstract

The Gauss-Kronrod quadrature scheme, which is based on the zeros of Legendre polynomials and Stieltjes polynomials, is the standard method for automatic numerical integration in mathematical software libraries. For a long time, very little was known about the underlying Lagrange interpolation processes. Recently, the authors proved new bounds and asymptotic properties for the Stieltjes polynomials, and subsequently applied these results to investigate the associated interpolation processes. The purpose of this paper is to give a survey on the quality of these interpolation processes, with additional results that extend and complete the existing ones. The principal new results in this paper are necessary and sufficient conditions for weighted convergence. In particular, we show that the Lagrange interpolation polynomials are equivalent to the polynomials of best approximation in certain weighted Besov spaces.

Key Words and Phrases:

Lagrange interpolation, Gauss-Kronrod nodes, weighted convergence, Besov spaces, Stieltjes polynomials

1 INTRODUCTION

Interpolation processes which are based on the zeros of orthogonal polynomials typically converge very rapidly for smooth functions. Their main advantage over spline approximation operators, in particular, is that their error is not saturated for functions of a certain low order of smoothness. Polynomial interpolation often leads to exponential convergence rates. However, orthogonal polynomials of different degrees most often do not have many zeros in common. Hence, in practical implementations of such interpolation processses most of the function values computed in one step cannot be used in the following ones. The idea of extended interpolation is to construct a practically efficient approximation method by adding further nodes to the existing ones. This leads to a sequence of "refined" interpolation nodes. This new type of interpolation matrix leads to several problems concerning convergence and error bounds.

There exists an extensive literature on interpolation processes based on the zeros of orthogonal polynomials. Furthermore, there exist results on interpolation processes based on the zeros of products of classical orthogonal polynomials with respect to different weight functions (cf., e.g., [4, 5, 6, 17, 25, 26, 42]). Presently, the most important practical example of refined interpolation nodes is the Gauss-Kronrod quadrature routine, which uses nested quadrature formulas based on the Lagrange interpolation polynomials with respect to the zeros of Legendre and associated Stieltjes polynomials. Gauss-Kronrod routines are used in the automatic integration routines in the NAG library [36], the IMSL library [21] and the Mathematica software package [48]. The purpose of this paper is to give a survey on recent results on the error of the interpolation processes based on Gauss-Kronrod nodes. We include several new results in order to generalise and complete the existing ones.

Gauss-Kronrod formulas were introduced in 1964 by A. S. Kronrod [23, 24] in order to estimate the error of Gaussian quadrature formulas. Based on the n nodes $x_{1,n}, \ldots, x_{n,n}$ of the Gaussian formula Q_n^G, the Gauss-Kronrod formula

$$Q_{2n+1}^{GK}[f] \;=\; \sum_{\nu=1}^{n} A_{\nu,n} f(x_{\nu,n}) + \sum_{\mu=1}^{n+1} B_{\mu,n+1} f(\xi_{\mu,n+1})$$

is constructed by choosing $n+1$ additional nodes $\xi_{\mu,n+1}$ and weights $A_{\nu,n}$, $B_{\mu,n+1}$ which are chosen in such a way that polynomials of a degree as high as possible are integrated exactly. The additional nodes $\xi_{1,n+1}, \ldots, \xi_{n+1,n+1}$ are the zeros of the Stieltjes polynomials E_{n+1}, defined up to a multiplicative constant by the orthogonality relations (see §2)

$$\int_{-1}^{1} P_n(x) E_{n+1}(x)\, x^k\, dx \;=\; 0, \qquad k = 0, \ldots, n.$$

Here, P_n is the nth Legendre polynomial whose zeros are the nodes of Q_n^G. The polynomials E_{n+1} were first studied by T. J. Stieltjes in 1894 [1], who conjectured that its zeros, for all $n \in \mathbb{N}$, are real, inside $(-1, 1)$, and that they interlace with the zeros of P_n. G. Szegö proved these properties in 1935 [46]. After Szegö's paper, for a long time no new results on the Stieltjes polynomials appeared in the literature. Also Kronrod's work contains no references to Stieltjes' and Szegö's work. The connection was independently pointed out in the Eastern literature by I. P. Mysovskih (1964, [35]) and in the Western literature by P. Barrucand (1970, [2]). G. Monegato proved in 1976 [31], that the positivity of the quadrature weights $B_{\mu,n+1}$ associated with the additional nodes $\xi_{\mu,n+1}$ is equivalent to the interlacing property of the nodes. The positivity of all quadrature weights was proved by Monegato in 1978 [32]. Many authors have considered the location of the zeros and the positivity of the quadrature weights for more general weight functions. In particular, cf. Gautschi and Notaris [18], Gautschi and Rivlin [19], Monegato [33], Peherstorfer [44], as well as the survey papers of Monegato [33, 34], Gautschi [16] and Notaris [41].

The Gauss-Kronrod formula is based on the Lagrange interpolation process $\mathcal{L}_{2n+1}$ with respect to the zeros of $P_n E_{n+1}$ which can efficiently be used for practical computations in connection with the interpolation process which is based on the zeros of P_n and with the interpolation process L_{n+1} which is based on the zeros of E_{n+1}. Monegato conjectured in [33] on the basis of numerical results that the interpolation process $\mathcal{L}_{2n+1}$ has Lebesgue constants of the optimal order $\log n$. This conjecture remained open for a long time.

The reason for this was a lack of precise knowledge on the Stieltjes polynomials E_{n+1} and its zeros. While Szegö proved the interlacing property of the nodes in [46], for a long time no sharper results on the asymptotic behaviour of the zeros or lower bounds for the differences were known. Peherstorfer in [44] considered Stieltjes polynomials for weight functions of the kind $w(x) = \frac{W(x)}{\sqrt{1-x^2}}$, where $W \in C^2[-1,1]$ and $W \geq \mathcal{C} > 0$ for some real constant $\mathcal{C}$. For weight functions of this kind, Peherstorfer proved results on the asymptotic behaviour of the associated Stieltjes polynomials. The paper [44] generalised earlier works on Bernstein-Szegö weight functions in [40, 43]. However, the case of the Legendre weight function and the abovementioned problems remained open (see also [44, p. 186]).

Essential progress was made in the paper [10], which contains results on the asymptotic behaviour of Stieltjes polynomials for ultraspherical weight functions w_λ, $\lambda \in [0,1]$. This includes the Legendre case for $\lambda = \frac{1}{2}$. The asymptotic formulas have been used to obtain error estimates for Gauss-Kronrod quadrature formulas in many important function spaces, and comparisons with other quadrature formulas, see [12]. The proof of pointwise bounds for the Stieltjes polynomials, which are precise in the whole interval $[-1,1]$, and respective lower bounds for the distances of the zeros led to a proof of Monegato's conjecture on the optimal order of the Lebesgue constants of $\mathcal{L}_{2n+1}$ in [13]. It is well known that the Lebesgue constants associated with the zeros of P_n are of the order $\mathcal{O}(\sqrt{n})$. Hence, adding the nodes $\xi_{1,n+1}, \ldots, \xi_{n+1,n+1}$ does not only lead to an efficient error estimation, but improves at the same time the interpolation process. A surprising result in [13] is that also the Lebesgue constants associated with L_{n+1} are of optimal order $\mathcal{O}(\log n)$. The aim of the papers [13, 14, 15] was to obtain more results on the convergence of these interesting interpolation processes. In particular, the boundedness of the operators in suitable subspaces of $L^p[-1,1]$ was investigated in these papers. Recently in [15], Marcinkiewicz-Zygmund inequalities have been proved for the Gauss-Kronrod nodes and for the Stieltjes zeros alone. These inequalities can be used to revisit the existing results and to deduce new error bounds for many function spaces in L^p weighted norms, which is the subject of this paper. The principal new results in this paper are necessary and sufficient conditions for the L^p weighted convergence of the interpolation processes. In particular, we show that the Lagrange interpolation polynomials with respect to both the Gauss-Kronrod nodes and the Stieltjes zeros alone are equivalent to the polynomials of best approximation in certain weighted Besov spaces. In §2, we state the fundamental properties of Stieltjes polynomials as well as asymptotic relations and inequalities. Section 3 contains results on the Lagrange interpolation processes and some numerical examples.

2 STIELTJES POLYNOMIALS

Let P_n be the Legendre polynomial defined by

$$\int_{-1}^{1} P_n(x)\, x^k\, dx = 0, \qquad k = 0, 1, \ldots, n-1, \tag{1}$$

and $P_n(1) = 1$. For $n \geq 0$, the Stieltjes polynomial E_{n+1} is defined by

$$\int_{-1}^{1} P_n(x) E_{n+1}(x) x^k\, dx = 0, \qquad k = 0, 1, \ldots, n, \tag{2}$$

and the normalisation

$$E_{n+1}(x) \;=\; \frac{2^n}{\gamma_n}\, x^{n+1} + p(x), \qquad \gamma_n = \frac{2^{2n}\, n!^2}{(2n+1)!}, \quad p \in \Pi_n. \tag{3}$$

Here and in the following, Π_n is the space of all algebraic polynomials of degree $\leq n$. Up to a multiplicative constant, the polynomial E_{n+1} is defined uniquely by (2). G. Szegö proved in 1935 [46] that the zeros of E_{n+1} are real and in $(-1, 1)$ for all $n \in \mathbb{N}$, and that they interlace with the zeros of P_n. However, no sharper results on Stieltjes polynomials and its zeros have been known for a long time.

A classical approximation to the Legendre polynomials is given by Laplace's formula,

$$P_n(\cos\theta) \;=\; \sqrt{\frac{2}{\pi\, n\, \sin\theta}}\; \cos\left\{ \left(n + \frac{1}{2}\right)\theta - \frac{\pi}{4} \right\} \;+\; \mathcal{O}(n^{-3/2}), \tag{4}$$

which holds uniformly for $\epsilon \leq \theta \leq \pi - \epsilon$, $\epsilon \in (0, \frac{\pi}{2})$ arbitrary but fixed. The asymptotic behaviour of the Stieltjes polynomials was studied numerically in [33, p. 235]. This reference contains the observation that $P_n E_{n+1}$ numerically behaves like the Chebyshev polynomial of the first kind T_{2n+1}. In [44] Peherstorfer proved asymptotic formulas for the case weight functions of the kind $w(x) = \frac{W(x)}{\sqrt{1-x^2}}$, $W \in C^2[-1, 1]$, and $W \geq \mathcal{C} > 0$ (cf. [44, §4]), but the question remained open for the Legendre weight [44, p. 186]. This problem was solved in [10],

$$E_{n+1}(\cos\theta) \;= 2\sqrt{\frac{2n\sin\theta}{\pi}}\; \cos\left\{ \left(n + \frac{1}{2}\right)\theta + \frac{\pi}{4} \right\} \;+\; \mathcal{O}(1), \tag{5}$$

uniformly for $\epsilon < \theta < \pi - \epsilon$, $\epsilon \in (0, \frac{\pi}{2})$ fixed. While this formula gives the precise behaviour inside the interval $(-1, 1)$, one cannot deduce the boundary behaviour near the endpoints ± 1. For Legendre polynomials, a well-known and more precise result is (cf. [47, Theorem 8.21.13])

$$P_n(\cos\theta) = \sqrt{\frac{2}{\pi\, n\, \sin\theta}}\ \left\{\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] \ +\ (n\sin\theta)^{-1}\mathcal{O}(1)\right\}, \qquad (6)$$

uniformly for $cn^{-1} \le \theta \le \pi - cn^{-1}$, $c > 0$ arbitrary but fixed. The analogous result for Stieltjes polynomials is (cf. [11, Lemma 1]).

$$\begin{aligned} &E_{n+1}(\cos\theta) \\ &= \ C_n(\theta)\sqrt{n\,\sin\theta}\ \left\{\ \cos\left[\left(n+\frac{1}{2}\right)\theta + \frac{\pi}{4}\right]\ +\ (n\sin\theta)^{-1}A_n(\theta)\ \right\}\ +\ B_n(\theta), \end{aligned}$$

where for every c which is independent of n we have

$$\max(|A_n(\theta)|, |B_n(\theta)|, |C_n(\theta)|) < \mathcal{C}, \qquad \theta \in [cn^{-1}, \pi - cn^{-1}],$$

$\mathcal{C} = \mathcal{C}(c)$ independent of n. For $c \ge \pi$, we have $1 \le \frac{1}{2}\sqrt{\frac{\pi}{2}}C_n(\theta) \le 1.0180\ldots$. In [10, Theorem (ii)], an asymptotic approximation was also given for the derivative of Stieltjes polynomials,

$$E'_{n+1}(\cos\theta)\ =\ 2n\sqrt{\frac{2\,n}{\pi\,\sin\theta}}\ \sin\ \left\{\left(n+\frac{1}{2}\right)\theta\ +\ \frac{\pi}{4}\right\}\ +\ \mathcal{O}(\sqrt{n}),$$

uniformly for $\varepsilon < \theta < \pi - \varepsilon$, $\varepsilon \in (0, \frac{\pi}{2})$. Formulas (4) and (5) show that the product $P_n E_{n+1}$ behaves asymptotically like a constant multiple of the Chebyshev polynomial T_{2n+1},

$$P_n(x)E_{n+1}(x)\ =\ \frac{2}{\pi}\ T_{2n+1}(x)\ +\ o(1), \qquad x \in [-1+\delta, 1-\delta], \qquad (7)$$

$\delta > 0$ independent of n. The following asymptotic result from [15, Lemma 4] is an application of the above results and is an important tool for the investigation of the interpolation processes which are based on the zeros of E_{n+1}.

Lemma 1 *Let $r \ge 1$. Then we have*

$$\lim_{n\to\infty}\int_{n^{-1}}^{\pi-n^{-1}} \left|\left(E_{n+1}(\cos\theta)\ \sin^{-1/2}\theta\right)^r - 2^r\left(\frac{2n}{\pi}\right)^{\frac{r}{2}}\cos^r\left\{\left(n+\frac{1}{2}\right)\theta + \frac{\pi}{4}\right\}\right|^2 d\theta\ =\ 0, \qquad (8)$$

and

$$\lim_{n\to\infty}\int_{n^{-1}}^{\pi-n^{-1}} |(P_n(\cos\theta)E_{n+1}(\cos\theta))^r - \left(\frac{2}{\pi}\right)^r \cos^r(2n+1)\theta\,|^2\, d\theta\ =\ 0.$$

In the following, for the zeros $\xi_{1,n+1}, \ldots, \xi_{n+1,n+1}$ of E_{n+1}, ordered increasingly, the cos-arguments will be denoted by $\theta_{\mu,n+1}$,

$$\xi_{\mu,n+1}\ =\ \cos\theta_{\mu,n+1}, \qquad \mu = 1, \ldots, n+1.$$

The zeros of $P_n E_{n+1}$, ordered increasingly, will be denoted by

$$y_{\nu,2n+1} \;=\; \cos\psi_{\nu,2n+1}, \qquad \nu = 1,\ldots,2n+1.$$

As an application of the formula (5), the paper [10] also contains results on the asymptotic distribution of the zeros of E_{n+1},

$$\theta_{n+2-\mu,n+1} \;=\; \frac{\mu - \frac{3}{4} \;+\; o(1)}{n+\frac{1}{2}}\pi, \tag{9}$$

uniformly for all $\theta_{n+2-\mu,n+1} \in [\epsilon, \pi-\epsilon]$, $\epsilon > 0$ fixed. An analogous result for the zeros of Legendre polynomials is well known (cf. [47, Theorem 8.9.1]),

$$\phi_{n+1-\nu,n} \;=\; \frac{\nu - \frac{1}{4} \;+\; o(1)}{n+\frac{1}{2}}\pi \tag{10}$$

uniformly for $x_{\nu,n} \in [-1+\epsilon, 1-\epsilon]$, $\epsilon > 0$ fixed.

The following uniform upper bound on Stieltjes polynomials is given in [34],

$$|E_{n+1}(x)| \;\le\; \frac{4}{\gamma_n}, \qquad x \in [-1,1].$$

However, the asymptotic formula (5) indicates that this bound is not sharp near the endpoints ± 1. A bound which gives the precise order was proved in [13],

$$|E_{n+1}(x)| \;\le\; 2\,C^* \sqrt{\frac{2n+1}{\pi}}\;\sqrt[4]{1-x^2} \;+55, \qquad n \ge 1, \tag{11}$$

where $C^* = 1.0180\ldots$, and

$$E_{n+1}(1) \ge \frac{2}{3\sqrt{\pi}}, \qquad n \ge 1.$$

The last inequality shows that (11) is of precise order also in the endpoints ± 1. Similarly as in (7), we obtain from (11) an upper bound for the product $P_n E_{n+1}$ which has the precise order in the whole interval $[-1,1]$.

$$|P_n(x) E_{n+1}(x)| \;\le\; \mathcal{C}, \qquad -1 \le x \le 1,$$

where $\mathcal{C}$ is a positive constant which is independent of n.

Formula (9) implies that the zeros of Stieltjes polynomials have a very regular distribution which is typical for orthogonal polynomials. Furthermore, (9) states that the zeros of E_{n+1} are also distributed very regularly with respect to the zeros of P_n, asymptotically they lie midway between two successive zeros of P_n. However, these statements only follow for the zeros which are inside closed subintervals of $(-1,1)$. The following result from [13, Theorem 2.4] improves the interlacing result of Szegö by stating lower bounds for the distances of all zeros.

Theorem 1 *We have*

$$\liminf_{n\to\infty} \inf_{0\le\nu\le 2n+1} (2n+1)\,(\psi_{\nu,2n+1} - \psi_{\nu+1,2n+1}) \;>\; \mathcal{C} \;>\; 0,$$

and

$$\liminf_{n\to\infty} \inf_{0\le\mu\le n+1} (n+1)\,(\theta_{\mu,n+1} - \theta_{\mu+1,n+1}) \;>\; \mathcal{C} \;>\; 0,$$

where $\psi_{0,2n+1} = \theta_{0,n+1} = \pi$, $\psi_{2n+2,2n+1} = \theta_{n+2,n+1} = 0$, *and* $\mathcal{C}$ *is a positive constant which is independent of* n, ν *and* μ.

The following result on the derivatives of Stieltjes polynomials was proved in [13] and is an important tool in our investigation of the interpolation operators.

Lemma 2 *There exists a positive constant* $\mathcal{C}$ *such that for all* $n \in \mathbb{N}$

$$\frac{\mathcal{C}^{-1}}{n\sqrt{n}}\,(1-(\xi_{\mu,n+1})^2)^{\frac{1}{4}} \;\le\; \frac{1}{|E'_{n+1}(\xi_{\mu,n+1})|} \;\le\; \frac{\mathcal{C}}{n\sqrt{n}}\,(1-(\xi_{\mu,n+1})^2)^{\frac{1}{4}}. \quad (12)$$

In the following, we use the usual notation

$$g \in L^p(I) \iff \|g\|_{L^p(I)} := \left(\int_I |g(x)|^p\,dx\right)^{\frac{1}{p}} \;<\; \infty,$$

if $1 \le p < \infty$, and

$$g \in L^\infty(I) \iff \|g\|_{L^\infty(I)} := \operatorname*{ess\,sup}_{x\in I} |g(x)| \;<\; \infty,$$

for $I \subset \mathbb{R}$. Furthermore, let $L^p := L^p((-1,1))$ and $\|g\|_p := \|g\|_{L^p((-1,1))}$. In the following we state a lower bound for the weighted L^p norm of the Stieltjes polynomials. We consider weights which belong to the class DT of Ditzian-Totik weights. These are weight functions u of the type

$$u(x) \;=\; (1+x)^\alpha(1-x)^\beta\,\tilde{\omega}_0(\sqrt{1+x})\,\tilde{\omega}_1(\sqrt{1-x}). \quad (13)$$

The functions $\tilde{\omega}_k$ are either identical to 1 or concave moduli of continuity of first order, i.e., semiadditive, nonnegative, continuous, nondecreasing on $[0,1]$ with $\tilde{\omega}_k(0)=0$ and $2\,\tilde{\omega}_k\left(\frac{a+b}{2}\right) \ge \tilde{\omega}_k(a) + \tilde{\omega}_k(b)$ for all $a,b \in [0,1]$. Furthermore, we assume that for every $\varepsilon > 0$, the functions $\frac{\tilde{\omega}_k(x)}{x^\varepsilon}$ are nonincreasing, with $\lim_{x\to 0+}\frac{\tilde{\omega}_k(x)}{x^\varepsilon} = \infty$. A special case are the classical Jacobi weights, for which we have $\tilde{\omega}_k \equiv 1$ for $k \in \{0,1\}$. For $u \in DT$ and $m \in \mathbb{N}$ we define

$$\begin{aligned} u_m(x) \;=\; & (\sqrt{1+x}+m^{-1})^{2\alpha}\,\tilde{\omega}_0(\sqrt{1+x}+m^{-1}) \\ & \times(\sqrt{1-x}+m^{-1})^{2\beta}\,\tilde{\omega}_1(\sqrt{1-x}+m^{-1}). \end{aligned} \quad (14)$$

Furthermore, we define

$$\varphi(x) = \sqrt{1-x^2}.$$

Theorem 2 *Let* $1 < p < \infty$, $u \in DT$, $u \in L^p$ *and* $r \in \mathbb{N}$. *Then there is a positive constant* $\mathcal{C} > 0$ *such that*

$$\liminf_{n\to\infty} \|\,(E_{n+1})^r u\|_p \;\ge\; \mathcal{C}\;n^{\frac{n}{2}}\;\|u\varphi^{\frac{r}{2}}\|_p \;>\; 0 \quad (15)$$

and

$$\liminf_{n\to\infty} \|\,(P_nE_{n+1})^r u\|_p \;\ge\; \mathcal{C}\;\|u\|_p \;>\; 0. \quad (16)$$

3 WEIGHTED CONVERGENCE OF EXTENDED INTERPOLATION

3.1 Results on weighted uniform convergence

Let L_{n+1} be the interpolation process based on the zeros of E_{n+1}, and let $\mathcal{L}_{2n+1}$ be the interpolation process based on the zeros of P_nE_{n+1}. For $1 \le p \le \infty$ and $u \in DT$, let

$$\mathcal{E}_k(f)_{u,p} \;=\; \inf_{p\in\Pi_k} \|[f-p]u\|_p$$

and

$$\mathcal{E}_k(f)_p \;=\; \mathcal{E}_k(f)_{1,p}.$$

In the paper [13], the authors proved the error bound

$$\|f - \mathcal{L}_{2n+1}f\|_\infty \;\le\; \mathcal{C}\;\log n\; \mathcal{E}_{2n}(f)_\infty, \tag{17}$$

and thus that the Lebesgue constants of the interpolation process $\mathcal{L}_{2n+1}$ are of optimal order. Furthermore, they proved that also the interpolation process L_{n+1} has Lebesgue constants of optimal order,

$$\|f - L_{n+1}f\|_\infty \;\le\; \mathcal{C}\;\log n\; \mathcal{E}_n(f)_\infty. \tag{18}$$

However, in numerical applications, e.g., for the numerical solution of integral equations, these bounds are often not applicable because the function may have endpoint singularities. An example is the function

$$f(x) \;=\; \log\frac{1}{1-x^2}.$$

For functions of this type, it is useful to consider the convergence in weighted norms. For $u \in DT$, define the function class

$$C_u^0 \;=\; \{f \in C^0_{\mathrm{loc}} \mid \lim_{|x|\to 1} u(x)f(x) = 0\}$$

and the norm

$$\|f\|_{C_u^0} \;=\; \|fu\|_\infty.$$

In order to obtain higher convergence rates for more regular functions, we consider the function classes

$$C_u^k \;=\; \{f \in C_u^0 \mid \|f^{(k)}\varphi^k\|_{C_u^0} < \infty\}, \qquad k \ge 1,$$

and define the norm

$$\|f\|_{C_u^k} \;=\; \|f\|_{C_u^0} + \|f^{(k)}\varphi^k\|_{C_u^0} < \infty.$$

The following result extends the uniform convergence results from [13] to weighted uniform convergence with weights $u \in DT$.

Theorem 3 *Let $f \in C_u^0$, $u \in DT$ and bounded.*

(a) *If $(u\sqrt{\varphi})^{-1} \in L^1$, then*

$$\|[f - L_{n+1}f]u\|_\infty \;\le\; \mathcal{C}\;\log n\; \mathcal{E}_n(f)_{u,\infty},$$

where $\mathcal{C}$ is a constant which is independent of n and f.

(b) *If $u^{-1} \in L^1$, then*

$$\|[f - \mathcal{L}_{2n+1}f]u\|_\infty \;\le\; \mathcal{C}\;\log n\; \mathcal{E}_{2n}(f)_{u,\infty},$$

where $\mathcal{C}$ is a constant which is independent of n and f.

Proof. We first consider the operator L_{n+1}. It is sufficient to prove

$$|u(x)\,L_{n+1}(f,x)| \;\le\; \mathcal{C}\;\log n\;\|fu\|_\infty,$$

where $\mathcal{C}$ is independent of n, x and f. Let d be chosen such that $\xi_{d,n+1} \le x \le \xi_{d+1,n+1}$. Let also $x - \xi_{d,n+1} \le \xi_{d+1,n+1} - x$ (the other case can be treated analogously). Now

$$\begin{aligned}|u(x)L_{n+1}(f,x)| \;&\le\; \left|\frac{E_{n+1}(x)f(\xi_{d,n+1})u(x)}{{E_{n+1}}'(\xi_{d,n+1})(x-\xi_{d,n+1})}\right| \\ &\quad + \left|\sum_{\substack{\mu=1\\ \mu\neq d}}^{n+1}\frac{E_{n+1}(x)f(\xi_{\mu,n+1})u(x)}{{E_{n+1}}'(\xi_{\mu,n+1})(x-\xi_{\mu,n+1})}\right| \\ &=: \; I_1 \;+\; I_2.\end{aligned}$$

Since $1 \pm \xi_{d,n+1} \sim 1 \pm x$, we have $u(\xi_{d,n+1}) \sim u(x)$. Since u is bounded, for the first part we have

$$\begin{aligned} I_1 \;&=\; \left|\frac{u(x)}{u(\xi_{d,n+1})}\right|\;\left|\frac{E_{n+1}(x)f(\xi_{d,n+1})u(\xi_{d,n+1})}{{E_{n+1}}'(\xi_{d,n+1})(x-\xi_{d,n+1})}\right| \\ &\le\; \mathcal{C}\;\|fu\|_\infty\end{aligned}$$

by the same argument as in [13, Proof of Theorem 3.1]. For the second part, we use (11) and (12) to obtain

$$\begin{aligned}&\left|\sum_{\substack{\mu=1\\ \mu\neq d}}^{n+1}\frac{E_{n+1}(x)f(\xi_{\mu,n+1})u(x)}{{E_{n+1}}'(\xi_{\mu,n+1})(x-\xi_{\mu,n+1})}\right| \\ &\quad\le\; \mathcal{C}\;\|fu\|_\infty\;\sum_{\substack{\mu=1\\ \mu\neq d}}^{n+1}\frac{\varphi(\xi_{\mu,n+1})}{n}\,\frac{u(x)\;(1-x^2)^{\frac14}}{u(\xi_{\mu,n+1})\sqrt{\varphi(\xi_{\mu,n+1})}|x-\xi_{\mu,n+1}|}.\end{aligned}$$

The last expression is a Darboux sum. Since u is bounded, we have

$$\begin{aligned} I_2 \;&\le\; \mathcal{C}\;\|fu\|_\infty\;\left(\int_{-1}^{\xi_{d-2,n+1}} + \int_{\xi_{d+2,n+1}}^{1}\right)\frac{dt}{u(t)\,\sqrt{\varphi(t)}\,|x-t|}\;u(x)\,(1-x^2)^{\frac14} \\ &\le\; \mathcal{C}\;\|fu\|_\infty\;\log n,\end{aligned}$$

since $(u\sqrt{\varphi})^{-1} \in L^1$, and using the same technique as in [25, Lemma 4.1].

The proof for $\mathcal{L}_{2n+1}$ is analogous. □

Remark 1 If $f \in C_u^k$, then we obtain from Theorem 3 with the estimates from [8, (8.2.1) and p. 92]

$$\|[f - L_{n+1}f]u\|_\infty \;\le\; \mathcal{C}\;\frac{\log n}{n^k}\;\max_{|x|\le 1-\frac{\mathcal{C}}{n^2}}|f^{(k)}(x)\varphi^k(x)u(x)|.$$

Example 1 We consider the interpolation of the function

$$f(x) = \log(1+x)$$

by the operator L_{n+1}. The bound (18) does not even yield convergence for this function. For $\alpha > 0$, let

$$v^\alpha(x) = 2^{-\alpha}(1+x)^\alpha.$$

Estimating the weighted best approximation to f by [8, (8.2.1) and p. 92], we obtain

$$|\log(1+x) - L_{n+1}(f_1, x)|\, v^\alpha(x) \;\le\; \frac{\mathcal{C}}{n^{2\alpha}}\, \log n. \tag{19}$$

Here $0 < \alpha < \frac{3}{4}$ in view of Theorem 3. The numerical results in Table 1 indicate that, in contrast to the error of best approximation, we cannot expect that (19) holds also for $\alpha > \frac{3}{4}$ in the case of the interpolation operator L_{n+1}. The numerical results were obtained on a HP-9000 using Mathematica 3.0 [48].

3.2 Weighted L^p-convergence

For many applications, it is more important to consider the convergence in the L^p mean, $1 < p < \infty$, instead of uniform convergence. The convergence of interpolation processes in the L^p mean, in particular at the zeros of orthogonal polynomials, has been considered by many authors, cf., e.g., [38] and the papers cited therein. Let w be a nonnegative weight function in $[-1,1]$ with $0 < \|w\|_1 < \infty$, and let $x_{1,m}, \ldots, x_{m,m}$ be the zeros of the orthogonal polynomial $P_m(w, \cdot)$ with respect to w. Let

$$\lambda_m(w,x) \;=\; \min_{\substack{p\in\Pi_{m-1}\\ p(x)=1}} \int_{-1}^{1} |p(t)|^2\, w(t)\, dt$$

be the mth Christoffel function with respect w. For Ditzian-Totik weights u we have (cf. [27])

$$\lambda_m(u,x) \;\sim\; \left(\frac{\sqrt{1-x^2}}{m} \;+\; \frac{1}{m^2} \right) u_m(x), \tag{20}$$

where u_m is defined as in (14). An important tool for the study of mean convergence of interpolation processes with respect to the nodes $x_{1,m}, \ldots, x_{m,m}$ are the Marcinkiewicz-Zygmund inequalities (cf. [51, §X.7] for the definitions in the trigonometric apces)

$$\int_{-1}^{1} |q(x)u(x)|^p\, dx \;\le\; \mathcal{C} \sum_{\nu=1}^{m} \lambda_m(u^p, x_{\nu,m})\, |q(x_{\nu,m})|^p, \qquad q \in \Pi_{2m-1}, \tag{21}$$

and

$$\sum_{\nu=1}^{m} \lambda_m(u^p, x_{\nu,m})\, |q(x_{\nu,m})|^p \;\le\; \mathcal{C} \int_{-1}^{1} |q(x)u(x)|^p\, dx, \qquad q \in \Pi_{2m-1}.$$

The study of necessary and sufficient conditions for the existence of these inequalities, in particular for the zeros of orthogonal polynomials, has attracted much interest in the literature (see for instance [7, 27, 28, 30, 49, 50]). The following results from [15] give the Marcinkiewicz-Zygmund inequalities for the zeros of E_{n+1} and for the zeros of $P_n E_{n+1}$.

Theorem 4 *Let $1 < p < \infty$, $u \in DT$, $u \in L^p$. The following assertions are equivalent.*

1. *For all* $P \in \Pi_n$ *we have*

$$\mathcal{C}^{-1}\,\|Pu\|_p \le \left(\sum_{\mu=1}^{n+1} \lambda_{n+1}(u^p,\xi_{\mu,n+1})\,|P(\xi_{\mu,n+1})|^p\right)^{\frac{1}{p}} \le \mathcal{C}\,\|Pu\|_p, \quad (22)$$

   where $\mathcal{C}$ *is independent of* n *and* P.

2. $$(u\sqrt{\varphi})^{-1} \in L^{p'}, \qquad \textit{where} \quad \frac{1}{p}+\frac{1}{p'}=1. \quad (23)$$

Theorem 5 *Let* $1<p<\infty$, $u \in DT$, $u \in L^p$. *Let* $y_{1,2n+1},\ldots,y_{2n+1,2n+1}$ *be the zeros of* P_nE_{n+1}. *The following assertions are equivalent.*

1. *For all* $P \in \Pi_{2n}$ *we have*

$$\mathcal{C}^{-1}\,\|Pu\|_p \le \left(\sum_{\mu=1}^{2n+1} \lambda_n(u^p,y_{\mu,2n+1})\,|P(y_{\mu,2n+1})|^p\right)^{\frac{1}{p}} \le \mathcal{C}\|Pu\|_p, \quad (24)$$

   where $\mathcal{C}$ *is independent of* n *and* P.

2. $$u^{-1} \in L^{p'}, \qquad \textit{where} \quad \frac{1}{p}+\frac{1}{p'}=1. \quad (25)$$

For the mean convergence of the interpolation processes related to the zeros of Stieltjes polynomials, the following result has been proved in [13].

Theorem 6 *Let* $1<p<\infty$, *let* $u \in DT$, $u \in L^p$, *and let* $f \in L^p_u$ *be continuous.*

(a) *We have*

$$\|[f-L_{n+1}(f)]u\|_p \le \mathcal{C}\,\mathcal{E}_n(f)_\infty,$$

where $\mathcal{C}$ *is a constant which is independent of* n *and* f.

(b) *We have*

$$\|[f-\mathcal{L}_{2n+1}(f)]u\|_p \le \mathcal{C}\,\mathcal{E}_{2n}(f)_\infty,$$

where $\mathcal{C}$ *is a constant which is independent of* n *and* f.

For $p=2$ and $u \equiv 1$, Theorem 6 yields an Erdös-Turán type convergence result. However, for functions with endpoint singularities like f in Example 1, also these results are not applicable, since the error of best uniform approximation does not tend to zero when the degree of the polynomials is increased. But for functions where the interpolation polynomials will converge in a norm which is weighted by a weight u, the polynomials of best weighted approximation, with the same weight u, will converge as well. Thus it is natural to consider error estimates using the error of best weighted approximation in these situations.

Theorem 7 *For* $1<p<\infty$, *let* $f \in L^p_u \cap C^0_u$, $u \in DT$ *and bounded.*

(a) *If* $(u\sqrt{\varphi})^{-1} \in L^1$, *then*

$$\|[f - L_{n+1}(f)]u\|_p \;\le\; \mathcal{C}\; \mathcal{E}_n(f)_{u,\infty},$$

where $\mathcal{C}$ *is a constant which is independent of* n *and* f.

(b) *If* $u^{-1} \in L^1$, *then*

$$\|[f - \mathcal{L}_{2n+1}(f)]u\|_p \;\le\; \mathcal{C}\; \mathcal{E}_{2n}(f)_{u,\infty},$$

where $\mathcal{C}$ *is a constant which is independent of* n *and* f.

Proof. We prove (a), the proof of (b) is analogous. It is sufficient to prove

$$\|L_{n+1}(f)u\|_p \;\le\; \mathcal{C}\; \|fu\|_\infty.$$

Let $g = (\mathrm{sgn} L_{n+1}(f))\, |uL_{n+1}(f)|^{p-1}$ and

$$\pi(t) = \int_{-1}^{1} \frac{E_{n+1}(x) - E_{n+1}(t)}{x - t}\, u(x)\, g(x)\, dx.$$

The function π is a polynomial of degree $\le n$. Now

$$\begin{aligned}
\|L_{n+1}(f)u\|_p &= \sum_{\mu=1}^{n+1} \frac{f(\xi_{\mu,n+1})\,\pi(\xi_{\mu,n+1})}{E_{n+1}{}'(\xi_{\mu,n+1})} \\
&\le \mathcal{C}\; \|fu\|_\infty \sum_{\mu=1}^{n+1} \frac{\varphi(\xi_{\mu,n+1})}{n} \, \frac{\pi(\xi_{\mu,n+1})}{u(\xi_{\mu,n+1})\,\sqrt{n\varphi(\xi_{\mu,n+1})}},
\end{aligned}$$

by (12). From [27, Theorem 2.6] we obtain that for any set of points $-1 = y_{1,m} < \cdots < y_{m,m} = 1$ which are distributed in such a way that

$$\liminf_{n\to\infty} \inf_{1\le\nu\le m-1} \; m\,(\arccos y_{\nu,m} - \arccos y_{\nu+1,m}) \;\ge\; \mathcal{C} \; > 0,$$

for every $W \in DT$ and every polynomial $Q \in \Pi_{lm}$, l being a fixed integer, we have for some $\mathcal{C}$ which is independent of n and Q

$$\sum_{k=1}^{m} \frac{\varphi(y_{k,m})}{m}\, Q(y_{k,m})\, W(y_{k,m}) \;\le\; \mathcal{C} \int_{-1+n^{-2}}^{1-n^{-2}} |Q(t)| W(t)\, dt.$$

Applying this inequality, and defining $A_n = [-1 + n^{-2}, 1 - n^{-2}]$, we have

$$\begin{aligned}
\|L_{n+1}(f)u\|_p^p &\le \mathcal{C}\; \|fu\|_\infty \int_{A_n} \frac{\pi(t)}{u(t)\,\sqrt{n\varphi(t)}}\, dt \\
&\le \mathcal{C}\; \|fu\|_\infty \left(\int_{A_n} \frac{1}{u(t)\,\sqrt{n\varphi(t)}} |H(E_{n+1}ug, t)|\, dt \right. \qquad (26) \\
&\qquad \left. + \int_{A_n} \frac{E_{n+1}(t)}{u(t)\,\sqrt{n\varphi(t)}} |H(ug, t)|\, dt \right),
\end{aligned}$$

where H denotes the Hilbert transform

$$H(f, x) = \lim_{\varepsilon\to\infty} \left(\int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^{1} \right) \frac{f(t)}{t - x}\, dt. \qquad (27)$$

We recall that H is bounded in L^p, $1 < p < \infty$,

$$\|H(f)\|_p \;\leq\; \mathcal{C}\;\|f\|_p,$$

where $\mathcal{C}$ is independent of f. Using (11), we have

$$\begin{aligned} I_2 \;&:=\; \int_{A_n} \frac{E_{n+1}(t)}{u(t)\sqrt{n\varphi(t)}}\,|H(ug,t)|\,dt \\ &\leq\; \mathcal{C}\int_{A_n}\frac{1}{u(t)}\,|H(ug,t)|\,dt. \end{aligned}$$

We recall from [39] that if F and G have compact support K, $F \in L^p$ and $G \in L^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$, then

$$\int_K FH(G) \;=\; -\int_K GH(F). \tag{28}$$

Applying this inversion, we have

$$I_2 \;\leq\; \mathcal{C}\int_{A_n} u(t)g(t)H(G_2u^{-1},t)\,dt,$$

where

$$G_2(t) = \operatorname{sgn} H(ug,t).$$

Now

$$|H(G_2u^{-1},t)| = \left|\int_{-1}^{1}\frac{G_2(x)}{u(x)\,(x-t)}\,dx\right| \leq \left|\frac{H(G_2,t)}{u(t)}\right| + \int_{-1}^{1}\left|\frac{u^{-1}(x)-u^{-1}(t)}{x-t}\right|\,dx.$$

For any $v \in DT$ with $\alpha,\beta < 0$ and $|t| < 1$, it is easy to prove that

$$\int_{-1}^{1}\left|\frac{v(x)-v(t)}{x-t}\right|\,dx \;\leq\; \mathcal{C}\;v(t).$$

Using this, we have

$$I_2 \;\leq\; \mathcal{C}\left(\int_{A_n} g(t)H(G_2,t)\,dt + \int_{A_n} g(t)\,dt\right).$$

If $p=1$, then $|g(t)| \leq 1$, and

$$\int_{A_n} g(t)H(G_2,t)\,dt \;\leq\; \sqrt{2}\|H(G_2)\|_2 \leq \mathcal{C}\|G_2\|_2 \leq \mathcal{C},$$

using the boundedness of H. If $1 < p < \infty$, then

$$\begin{aligned} &\int_{A_n} g(t)H(G_2,t)\,dt \\ &\quad\leq\; \int_{A_n} |u(t)L_{n+1}(f,t)|^{p-1}H(G_2,t)\,dt \\ &\quad\leq\; \|L_{n+1}(f)u\|_p^{p-1}\|H(G_2)\|_p \;\leq\; \mathcal{C}\;\|L_{n+1}(f)u\|_p^{p-1}, \end{aligned}$$

using the boundedness of the Hilbert transform in L^p. Furthermore,

$$\int_{A_n} g(t)\,dt \;\leq\; \mathcal{C}\;\|L_{n+1}(f)u\|_p^{p-1}.$$

On the other hand, in (26) we have

$$\begin{aligned} I_1 \;&:=\; \int_{A_n} \frac{|H(E_{n+1}ug,t)|}{u(t)\,\sqrt{n\,\varphi(t)}}\,dt \\ &=\; \int_{A_n} \frac{G_1(t)}{u(t)\,\sqrt{n\,\varphi(t)}}\,H(E_{n+1}ug,t)\,dt, \end{aligned}$$

where

$$G_1(t) = \operatorname{sgn} H(E_{n+1}ug,t).$$

Using (28) again, we have

$$|I_1| \;\leq\; \int_{A_n} n^{-1/2}|E_{n+1}(t)|u(t)g(t)H\left(\frac{G_1}{u\,\sqrt{\varphi}},t\right)\,dt.$$

Using the same argument as above, we have

$$|I_1| \;\leq\; \mathcal{C}\;\|L_{n+1}(f)u\|_p^{p-1},$$

which leads to the result. □

3.3 Comparison with best polynomial approximation in Besov spaces

In order to study the behaviour of the interpolation processes in suitable subspaces of L^p_u, we need some preliminary definitions and results. For $u \in DT$, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, the modulus of smoothness of Ditzian and Totik is defined by

$$\Omega^k_\varphi(f,t)_{u,p} \;=\; \sup_{0<h\leq t} \|(\Delta^k_{h\varphi}f)u\|_{L^p(I_{hk})}, \tag{29}$$

where

$$\begin{aligned} \Delta_{h\varphi}f(x) \;&=\; f(x+\frac{h}{2}\varphi(x)) - f(x-\frac{h}{2}\varphi(x)) \\ \Delta^k_{h\varphi} \;&=\; \Delta_{h\varphi}\;\Delta^{k-1}_{h\varphi}, \qquad k\geq 1, \\ I_{hk} \;&:=\; [-1+2h^2k^2, 1-2h^2k^2]. \end{aligned}$$

The modulus of smoothness $\Omega^k_\varphi(f,t)_{u,p}$ is used in [8, p. 94, (8.2.1) and (8.2.2)] to characterise the best approximation by algebraic polynomials, in the following sense,

$$\mathcal{E}_n(f)_{u,p} \;\leq\; \mathcal{C}\;\int_0^{\frac{1}{n}} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t}\;dt, \tag{30}$$

$\mathcal{C}$ independent of f and n, and

$$\Omega^k_\varphi(f,t)_{u,p} \;\leq\; \mathcal{C}\;t^k\;\sum_{i=0}^{\lfloor\frac{1}{t}\rfloor}(1+i)^{k-1}\mathcal{E}_i(f)_{u,p}, \tag{31}$$

$\mathcal{C}$ independent of f and t. Furthermore, the modulus of smoothness $\Omega_\varphi^k(f,t)_{u,p}$ defined in (29) is equivalent to the $\tilde{K}$-functional from [8],

$$\Omega_\varphi^k(f,t)_{u,p} \sim \sup_{0\le h\le t} \inf_{g^{(k-1)}\in AC(I_{hk})} \left\{ \|(f-g)u\|_{L^p(I_{hk})} + h^k \|g^{(r)}\varphi^r u\|_{L^p(I_{hk})} \right\}. \tag{32}$$

The following new result is fundamental for convergence results and error estimates for many function spaces.

Theorem 8 *Assume $u \in DT$, $u \in L^p$, $1<p<\infty$, and let $p' = \frac{p}{p-1}$. Let L be one of the operators L_{n+1} or $\mathcal{L}_{2n+1}$. Then there exists a constant $\mathcal{C}$ such that for all $n \in \mathbb{N}$, and all locally continuous functions f such that*

$$\int_0^1 \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt \;<\; \infty,$$

there holds

$$\|[f-L(f)]u\|_p \;\le\; \frac{\mathcal{C}}{n^{1/p}} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,t)_{u,p}}{t^{1+1/p}}\,dt, \qquad 1\le k<n,$$

if and only if

$$\begin{cases} (u\sqrt{\varphi})^{-1} \in L^{p'} & \text{if } \; L = L_{n+1}, \\ u^{-1} \in L^{p'} & \text{if } \; L = \mathcal{L}_{2n+1}. \end{cases}$$

Proof. Let $L = L_{n+1}$; the proof for $\mathcal{L}_{2n+1}$ is analogous, using the Marcinkiewiz-Zygmund inequalities for the zeros of P_nE_{n+1}. We have to prove

$$\|L_{n+1}(f)u\|_p \;\le\; \mathcal{C}\left[\|fu\|_p + \frac{1}{n^{1/p}} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt\right]. \tag{33}$$

Once we have proved (33), we use

$$\|[f-L_{n+1}(f)]u\|_p \le \|[f-P]u\|_p + \|[L_{n+1}(f-P)]u\|_p,$$

which holds for every $P \in \Pi_n$. By the estimate for the polynomials of best approximation from (30), we have

$$\mathcal{E}_n(f)_{u,p} \;\le\; \frac{\mathcal{C}}{n^{\frac{1}{p}}} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt,$$

and the result follows for $k=1$. For $1<k<n$ the result follows with the assertion from [27, Proposition 4.2],

$$\int_0^{\frac{1}{n}} \frac{\Omega_\varphi(f-P,t)_{u,p}}{t^{1+1/p}}\,dt \;\le\; \mathcal{C}\left[n^{\frac{1}{p}}\|[f-P]u\|_p + \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,t)_{u,p}}{t^{1+1/p}}\,dt\right],$$

which holds for $1<p<\infty$, $n\in\mathbb{N}$, $1\le k<n$, and every $P\in\Pi_n$.

Using (20) we have

$$\begin{aligned} \|L_{n+1}(f)u\|_p^p \;&\le\; \mathcal{C} \sum_{\mu=1}^{n+1} \lambda_{n+1}(u^p,\xi_{\mu,n+1})\;|f(\xi_{\mu,n+1})|^p \\ &\le\; \mathcal{C} \sum_{\mu=1}^{n+1} \frac{\varphi(\xi_{\mu,n+1})}{n}\;|f(\xi_{\mu,n+1})\,u_n(\xi_{\mu,n+1})|^p. \end{aligned}$$

Introducing the notation $\xi_{\mu,n+1} = \cos\theta_{\mu,n+1}$, $I_\mu = [\theta_{\mu+1,n+1}, \theta_{\mu,n+1}]$, we have

$$\begin{aligned} |f(\xi_{\mu,n+1})| &= |f(\cos\theta_{\mu,n+1})| \le \sup_{\theta\in I_\mu} |f(\cos\theta)| \\ &\le \mathcal{C}\left[n^{\frac{1}{p}} \left(\int_{I_\mu} |f(\cos\theta)|^p \, d\theta \right)^{\frac{1}{p}} + \int_0^{\frac{1}{n}} \frac{\omega(f(\cos), t)_{L^p(I_\mu)}}{t^{1+1/p}} \, dt \right], \end{aligned}$$

where we have used an embedding inequality, cf. for instance [22] or [27, Lemma 4.1]. Since $\sin\theta_{\mu,n+1} \sim \sin\theta$ for $\theta \in I_\mu$, we have for $\mu = 1, \ldots, n$

$$\begin{aligned} \frac{\sin\theta_{\mu,n+1}}{n} |f(\cos\theta_{\mu,n+1})|^p &\le \mathcal{C} \int_{I_\mu} |f(\cos\theta)|^p \sin\theta \, d\theta \\ &\quad + \frac{\mathcal{C}}{n} \left[\int_0^{\frac{1}{n}} \frac{\omega(f(\cos), t)_{L^p(I_\mu)}}{t^{1+1/p}} \sin^{\frac{1}{p}} \theta_{\mu,n+1} \, dt \right]^p \\ &=: A + B. \end{aligned}$$

Now

$$\begin{aligned} A &= \mathcal{C} \int_{\theta_{\mu+1,n+1}}^{\theta_{\mu,n+1}} |f(\cos\theta)|^p \sin\theta \, d\theta = \mathcal{C} \int_{\xi_{\mu,n+1}}^{\xi_{\mu+1,n+1}} |f(x)|^p \, dx \\ &\le \frac{\tilde{\mathcal{C}}}{(u_n(\xi_{\mu,n+1}))^p} \int_{\xi_{\mu,n+1}}^{\xi_{\mu+1,n+1}} |f(x)u(x)|^p \, dx, \end{aligned}$$

by the mean value theorem. Let $g \in AC_{loc}[-1,1]$, i.e. $g(\cos) \in AC_{loc}[0,\pi]$. Using the usual modulus of continuity we have

$$\begin{aligned} \omega(f(\cos), t)_{L^p(I_\mu)} &\le \omega(f(\cos) - g(\cos), t)_{L^p(I_\mu)} + \omega(g(\cos), t)_{L^p(I_\mu)} \\ &\le 2\,\|f(\cos) - g(\cos)\|_{L^p(I_\mu)} + \mathcal{C}\, t \left\| \frac{d}{dt} g(\cos t) \right\|_{L^p(I_\mu)}. \end{aligned}$$

Hence

$$\begin{aligned} &\omega(f(\cos), t)_{L^p(I_\mu)} \sin^{\frac{1}{p}} \theta_{\mu,n+1} \\ &\quad\le \mathcal{C}\left\{ \left[\int_{\theta_{\mu+1,n+1}}^{\theta_{\mu,n+1}} |f(\cos\theta) - g(\cos\theta)|^p \sin\theta \, d\theta \right]^{\frac{1}{p}} \right. \\ &\qquad\quad \left. + \, t \left[\int_{\theta_{\mu+1,n+1}}^{\theta_{\mu,n+1}} |g'(\cos\theta)\sin\theta|^p \sin\theta \, d\theta \right]^{\frac{1}{p}} \right\} \\ &\quad\le \frac{\mathcal{C}}{u_n(\xi_{\mu,n+1})} \left\{ \left[\int_{\xi_{\mu,n+1}}^{\xi_{\mu+1,n+1}} |f(x) - g(x)|^p u^p(x) \, dx \right]^{\frac{1}{p}} \right. \\ &\qquad\quad \left. + t \left[\int_{\xi_{\mu,n+1}}^{\xi_{\mu+1,n+1}} |g'(x)\varphi(x)u(x)|^p \, dx \right]^{\frac{1}{p}} \right\} \\ &\quad\le \frac{\mathcal{C}}{u_n(\xi_{\mu,n+1})} \left[\|(f-g)u\|_{L^p(J_\mu)} + t\,\|g'\varphi u\|_{L^p(J_\mu)} \right], \end{aligned}$$

where $J_\mu = [\xi_{\mu,n+1}, \xi_{\mu+1,n+1}]$. Therefore, we have for $\mu = 1, \dots, n$

$$B \le \left[\frac{\mathcal{C}}{u_n(\xi_{\mu,n+1})} \int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p(J_\mu)} + t\|g'\varphi u\|_{L^p(J_\mu)}}{t^{1+1/p}} \, dt \right]^p$$

and

$$\begin{aligned} \frac{\varphi(\xi_{\mu,n+1})}{n} \, |f(\xi_{\mu,n+1})u(\xi_{\mu,n+1})|^p \ &\le \ \mathcal{C} \int_{\xi_{\mu,n+1}}^{\xi_{\mu+1,n+1}} |f(x)u(x)|^p \, dx \\ + \quad \frac{\mathcal{C}}{n} &\left[\int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p(J_\mu)} + t\,\|g'\varphi u\|_{L^p(J_\mu)}}{t^{1+1/p}} \, dt \right]^p . \end{aligned}$$

For $\mu = n+1$, we have

$$|f(\xi_{n+1,n+1})| = |f(\cos\theta_{n+1,n+1})| \le \sup_{\theta \in I_n} |f(\cos\theta)|.$$

Adding these inequalities, we obtain

$$\begin{aligned} \left(\sum_{\mu=1}^{n+1} \frac{\varphi(\xi_{\mu,n+1})}{n} \, |f(\xi_{\mu,n+1})u(\xi_{\mu,n+1})|^p \right)^{\frac{1}{p}} \ &\le \mathcal{C} \left(\int_{\xi_{1,n+1}}^{\xi_{n+1,n+1}} |f(x)u(x)|^p \, dx \right)^{\frac{1}{p}} \\ + \quad &\left(\frac{\mathcal{C}}{n} \sum_{\mu=1}^{n} \left[\int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p(J_\mu)} + t\|g'\varphi u\|_{L^p(J_\mu)}}{t^{1+1/p}} \, dt \right]^p \right)^{\frac{1}{p}} . \end{aligned}$$

Using the Minkowski inequality (see [20, p.148, Th.201]), we have

$$\begin{aligned} &\left(\frac{\mathcal{C}}{n} \sum_{\mu=1}^{n} \left[\int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p(J_\mu)} + t\|g'\varphi u\|_{L^p(J_\mu)}}{t^{1+1/p}} \, dt \right]^p \right)^{\frac{1}{p}} \\ \le \quad & \frac{\mathcal{C}}{n^{\frac{1}{p}}} \int_0^{\frac{1}{n}} \frac{\left(\sum_{\mu=1}^{n} \|(f-g)u\|^p_{L^p(J_\mu)} + t \sum_{\mu=1}^{n} \|g'\varphi u\|^p_{L^p(J_\mu)} \right)^{\frac{1}{p}}}{t^{1+1/p}} \, dt \\ \le \quad & \frac{\mathcal{C}}{n^{\frac{1}{p}}} \int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p((\xi_{1,n+1},\xi_{n+1,n+1}))} + t\|g'\varphi u\|_{L^p((\xi_{1,n+1},\xi_{n+1,n+1}))}}{t^{1+1/p}} \, dt. \end{aligned}$$

Concluding, we have

$$\begin{aligned} &\|L_{n+1}(f)u\|_p \le \ \mathcal{C} \ \|fu\|_p \\ &+ \quad \frac{\mathcal{C}}{n^{\frac{1}{p}}} \int_0^{\frac{1}{n}} \frac{\|(f-g)u\|_{L^p((\xi_{1,n+1},\xi_{n+1,n+1}))} + t\|g'\varphi u\|_{L^p((\xi_{1,n+1},\xi_{n+1,n+1}))}}{t^{1+1/p}} \, dt, \end{aligned}$$

and taking the infimum for $g \in AC_{loc}$,

$$\|L_{n+1}(f)u\|_p \ \le \ \mathcal{C}\|fu\|_p \ + \ \frac{\mathcal{C}}{n^{\frac{1}{p}}} \left(\int_0^{\frac{1}{n}} \frac{\tilde{K}_{1,\varphi}(f,t)_{u,p}}{t^{1+1/p}} \, dt \right) .$$

We obtain the first part of the theorem by (32).

To show that (33) implies $(\sqrt{\varphi}u)^{-1} \in L^{p'}$, we consider the function

$$f_d(x) = \begin{cases} \frac{x-\xi_{d-1,n+1}}{\xi_{d,n+1}-\xi_{d-1,n+1}}, & x \in [\xi_{d-1,n+1}, \xi_{d,n+1}), \\ \frac{\xi_{d+1,n+1}-x}{\xi_{d-1,n+1}-\xi_{d,n+1}}, & x \in [\xi_{d,n+1}, \xi_{d+1,n+1}], \\ 0, & \text{otherwise}, \end{cases}$$

where $\xi_{d,n+1}$ is a fixed interpolation node of L_{n+1}. We have $f_d \in AC$ and $\|f_d\|_\infty = 1$. Replacing f by f_d in (33), we have

$$\|l_{d,n+1}u\|_p \le \mathcal{C}\ [\|f_d u\|_p + \frac{1}{n}\|f_d'\varphi u\|_p], \quad l_{d,n+1}(x) = \frac{E_{n+1}(x)}{E_{n+1}{}'(\xi_{d,n+1})(x-\xi_{d,n+1})},$$

and we deduce

$$\|l_{d,n+1}u\|_p \le \mathcal{C}\ u_n(\xi_{d,n+1}) \ \left(\frac{\varphi(\xi_{d,n+1})}{n}\right)^{\frac{1}{p}}.$$

Then, by using an estimate in [15, p. 12], there results

$$\mathcal{C}\ \left\|\frac{E_{n+1}u}{\cdot - \xi_{d,n+1}}\right\| \frac{\varphi(\xi_{d,n+1})}{n} \le \|l_{d,n+1}u\|_p \le u_n(\xi_{d,n+1})\frac{\varphi(\xi_{d,n+1})^{\frac{1}{p}}}{n},$$

which implies $(\sqrt{\varphi}u)^{-1} \in L^{p'}$, as it has already been proved in [15, p. 12, (21), and the following pages]. □

Theorem 8 can be used to obtain error estimates for many function spaces. E.g., for $f \in W_k^p(u)$, $k \ge 1$, using (31), we obtain the estimate

$$\Omega_\varphi^k(f,t)_{u,p} \ \le \ \mathcal{C}\ t^k\ \|f^{(k)}\varphi^k u\|_p \ < \ \infty.$$

By Theorem 8 we obtain the optimal speed of convergence for the interpolation of $f \in W_k^p(u)$.

Corollary 1 *Under the assumptions of Theorem 8 we have for $f \in W_k^p(u)$, $k \ge 1$,*

$$\|[f - L_{n+1}(f)]u\|_p \ \le \ \mathcal{C}\ n^{-k}\ \|f^{(k)}\varphi^k u\|_p$$

and

$$\|[f - \mathcal{L}_{2n+1}(f)]u\|_p \ \le \ \mathcal{C}\ n^{-k}\ \|f^{(k)}\varphi^k u\|_p.$$

Theorem 8 shall now be used to obtain the boundedness of the interpolation operators in certain subspaces of L_u^p. Natural spaces for the study of polynomial interpolation methods are the weighted Besov spaces from [9]. For $k > r$, a seminorm is defined by

$$|f|_{u,p,q,r} \ = \ \begin{cases} \left(\int_0^1 \left(\frac{\Omega_\varphi^k(f,t)_{u,p}}{t^{r+1/q}}\right)^q dt\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup\limits_{t\ge 0} \frac{\Omega_\varphi^k(f,t)_{u,p}}{t^r}, & q = \infty. \end{cases}$$

The weighted Besov space with respect to $\Omega_\varphi^k(f,t)_{u,p}$ is defined by

$$B_{r,q}^p(u) \ = \ \{\ f \in L_u^p \ : \ \|f\|_{B_{r,q}^p(u)} \ := \ \|fu\|_p + |f|_{u,p,q,r} \ < \ \infty\ \}.$$

In the papers [9, 27], many properties of these function spaces are described. In particular, $f \in B^p_{r,q}(u)$ holds if and only if f is in the space normed by

$$\|f\|_{E^p_{r,q}(u)} := \begin{cases} \left(\sum_{i=0}^{\infty} \left[(1+i)^{r-1/q}\mathcal{E}_i(f)_{u,p}\right]^q\right)^{\frac{1}{q}}, & 1 \le q < \infty, \\ \sup_{i\ge 0}(1+i)^r \mathcal{E}_i(f)_{u,p}, & q = \infty. \end{cases}$$

The two norms are equivalent,

$$\|f\|_{B^p_{r,q}(u)} \sim \|f\|_{E^p_{r,q}(u)}. \tag{34}$$

In the following we obtain error estimates for

$$f \in \tilde{B}^p_{r,q}(u) = B^p_{r,q} \cap C^0_u,$$

$1 < p < \infty$, $1 \le q \le \infty$ and $r > \frac{1}{p}$.

Corollary 2 *Under the assumptions of Theorem 8, we have for $f \in \tilde{B}^p_{r,q}(u)$, $1 < p < \infty$, $1 \le q \le \infty$, $r > \frac{1}{p}$ the error bounds*

$$\|[f - L_{n+1}(f)]u\|_p \le \frac{\mathcal{C}}{n^r} \|f\|_{B^p_{r,q}(u)}$$

and

$$\|[f - \mathcal{L}_{2n+1}(f)]u\|_p \le \frac{\mathcal{C}}{n^r} \|f\|_{B^p_{r,q}(u)}.$$

Furthermore, Theorem 8 can be used to prove the boundedness of the interpolation operators in some weighted Besov spaces. Thus, in these spaces, the interpolation operators have the same speed of convergence as the polynomials of best approximation.

Theorem 9 *Assume $u \in DT$, $u \in L^p$, $1 < p < \infty$, and let $p' = \frac{p}{p-1}$. Let L be one of the operators L_{n+1} or $\mathcal{L}_{2n+1}$, let $s > \frac{1}{p}$ and $1 \le q \le \infty$. Then there exists a constant $\mathcal{C}$ such that for all $n \in \mathbb{N}$, all real numbers $0 \le r \le s$ and all $f \in \tilde{B}^p_{s,q}(u)$, we have*

$$\|f - L(f)\|_{B^p_{r,q}(u)} \le \frac{\mathcal{C}}{n^{s-r}} \|f\|_{B^p_{s,q}(u)},$$

if and only if

$$\begin{cases} (u\sqrt{\varphi})^{-1} \in L^{p'} & \text{if } L = L_{n+1}, \\ u^{-1} \in L^{p'} & \text{if } L = \mathcal{L}_{2n+1}. \end{cases} \tag{35}$$

Consequently,

$$\sup_{n\to\infty} \|L\|_{\tilde{B}^p_{s,q}(u)\to\tilde{B}^p_{s,q}(u)} < \infty$$

if and only if the conditions (35) *hold.*

Proof. If $(u\sqrt{\varphi})^{-1} \in L^{p'}$, by using [27, Proposition 4.3], we have

$$\begin{aligned} \|f - L_{n+1}(f)\|_{B^p_{r,q}} &\le Cn^r\|(f - L_{n+1}f)u\|_p \\ &+ \frac{C}{n^{s-r}} \begin{cases} \left(\int_0^{\frac{1}{n}} \left[\frac{\Omega^k_\varphi(f,t)_{u,p}}{t^{s+1/p}}\right]^q dt\right)^{\frac{1}{q}} & 1 \le q < \infty \\ \sup\limits_{0<t\le\frac{1}{n}} \frac{\Omega^k_\varphi(f,t)_{u,p}}{t^s} & q = \infty \end{cases} \end{aligned}$$

Since $f \in B^p_{s,q}$, the first part of the theorem follows by using Corollary 2 with $r = s$. To prove the second part, we have to repeat word by word the proof of Theorem 8. The only change is that if $r > 1$, then the function f_d has to be replaced with its j-th antiderivative $G_{d,j}$, $j > r$, with the conditions: $G^{(j)}_{d,j}(x) = f_d(x), G_{d,j}(\xi_{d,n+1}) = 1$ and $G_{d,j}(x) = 0, x \notin [\xi_{d-1,n+1}, \xi_{d+1,n+1}]$. □

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	α					
n	0.1	0.4	0.7	1	2	5
5	2.2 (0)	1.6 (−1)	7.1 (−2)	5.2 (−2)	3.8 (−2)	2.9 (−2)
10	1.9 (0)	9.4 (−2)	3.0 (−2)	2.1 (−2)	1.5 (−2)	1.1 (−2)
15	1.8 (0)	6.9 (−2)	1.8 (−2)	1.2 (−2)	8.4 (−3)	6.2 (−3)
20	1.7 (0)	5.5 (−2)	1.2 (−2)	7.7 (−3)	5.6 (−3)	4.1 (−3)
25	1.6 (0)	4.6 (−2)	8.9 (−3)	5.6 (−3)	4.0 (−3)	3.0 (−3)
30	1.7 (0)	4.0 (−2)	6.9 (−3)	4.3 (−3)	3.1 (−3)	2.3 (−3)
35	1.5 (0)	3.5 (−2)	5.6 (−3)	3.4 (−3)	2.4 (−3)	1.8 (−3)
40	1.5 (0)	2.8 (−2)	4.4 (−3)	2.8 (−3)	2.0 (−3)	1.5 (−3)

Table 1. *Maximum error* $\|[f - L_{n+1}(f)]\, v^{\alpha}\|_{\infty}$