INTERPOLATION OF COMPACT OPERATORS: THE MULTIDIMENSIONAL CASE

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ABSTRACT

We investigate how compact operators behave under J and K interpolation methods for N spaces and two parameters. First we study those methods: relationship with those already existing in the literature, estimates for the norms of interpolated operators, examples, characterization as Aronszajn–Gagliardo functors, ... We also describe the relationship between Sparr and Fernandez methods and we derive sharp estimates for the norms of interpolated operators in Fernandez' case. Then we investigate the behaviour of compact operators. We begin with the case when one of the N-tuples reduces to a single Banach space, and later we treat the general case by means of the approach developed in **[8]**.

0. *Introduction*

The behaviour of compact operators under interpolation is a question that has received much attention during the last few years. We refer, for example, to the articles by Cobos, Edmunds and Potter **[5]**, Cobos and Fernandez **[6]**, Cobos and Peetre **[8]**, Cwikel **[9]**, and Cobos **[3, 4]**. All these papers deal with interpolation methods for two spaces. In the present article we discuss a question which we left open in **[8]**: the multidimensional case.

We restrict our attention to interpolation methods of the type used in the classical real method but having two parameters (t and s instead of the classical t).

In the literature there are essentially four such methods: the J- and K-methods developed by Sparr **[19]** for, in the first non-trivial case, three spaces, and in general, n spaces; and the other two studied by Fernandez **[12]** for four, respectively 2^n, spaces. The relationship between them has not been satisfactorily described yet. Moreover, although Fernandez' methods were introduced ten years ago, it seems that only rough estimates are known for the norms of the interpolated operators.

We start by studying a J- and a K-method for N spaces that use two parameters (t, s). The N spaces should be thought of as sitting on the vertices of a convex polygon. The idea of carrying out such a generalization was suggested by Peetre in **[18]**. In particular, for the case of the simplex we recover Sparr's spaces, and for the case of the square we get Fernandez' ones. Our approach explains the restriction on parameters in Fernandez' case.

This is done in § 1, where we also describe the relationship between the spaces of Sparr and of Fernandez.

Then, in § 2, we establish estimates for the norms of interpolated operators. In particular, we obtain a sharp estimate in Fernandez' case.

In § 3, we apply our interpolation methods to certain vector-valued sequence spaces. We also describe the J- and K-methods as Aronszajn–Gagliardo functors.

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In the rest of the paper, we discuss the behaviour of compact operators under these methods. We first derive Lions–Peetre type compactness results in this multidimensional setting (§ 4). These theorems require that one of the N-tuples reduces to a single Banach space. To get rid of this assumption we use the representation of the J- and K-methods as Aronszajn–Gagliardo functors. This description allows us to use certain projections on the N-tuples of sequence spaces that define the functors. We must be able to combine the information given by the projections with our estimates for the norms of interpolated operators. This is achieved by imposing a certain geometrical condition on the polygon upon which the N-tuples are sitting.

We call *admissible* those polygons having such a geometrical property. They form a wide class including regular polygons. The investigation of admissible polygons is carried out in § 5.

Finally, we combine all these results together with the approach developed in [8] to obtain the general compactness results (§ 6).

1. *J- and K-methods for N spaces and two parameters*

Let $\Pi = \overline{P_1 P_2 \ldots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$ $(j = 1, \ldots, N)$, and let $\bar{A} = \{A_1, \ldots, A_n\}$ be a Banach N-tuple, that is, a family of N Banach spaces A_j $(j = 1, \ldots, N)$ all continuously embedded in some Hausdorff topological vector space $\mathscr{A}$. Each space A_j should be thought of as sitting on the vertex P_j (see Fig. 1.1).

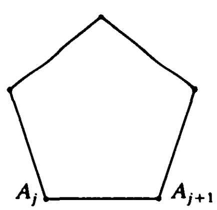

FIG. 1.1

Given any couple of positive numbers t, s, we define the K- and J-functionals by

$$K(t, s; a) = K(t, s; a; \bar{A}) = \inf\left\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} :\ a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\right\},$$

$$J(t, s; a) = J(t, s; a; \bar{A}) = \max_{1 \leq j \leq N} \{t^{x_j} s^{y_j} \|a\|_{A_j}\}.$$

In this way we obtain a family of norms on $\Sigma(\bar{A}) = A_1 + A_2 + \ldots + A_N$ (respectively $\Delta(\bar{A}) = A_1 \cap A_2 \cap \ldots \cap A_N$), any two of them being equivalent.

Let $1 \leq q \leq \infty$ and let (α, β) be any point in the interior of Π ($(\alpha, \beta) \in \operatorname{Int} \Pi$). We define $\bar{A}_{(\alpha,\beta),q;K}$ as the collection of all elements $a \in \Sigma(\bar{A})$ having a finite norm

$$\|a\|_{(\alpha,\beta),q;K} = \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} K(t, s; a))^q \frac{dt}{t} \frac{ds}{s}\right)^{1/q}.$$

On the other hand, we let $\bar{A}_{(\alpha,\beta),q;J}$ be the space of all those elements $a \in \Sigma(\bar{A})$

which can be represented in the form

$$a=\int_0^\infty\int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s},\tag{1}$$

where $u(t, s)$ is a strongly measurable $\Delta(\bar{A})$-valued function and satisfies

$$\left(\int_0^\infty\int_0^\infty (t^{-\alpha}s^{-\beta}J(t,s\,;u(t,s)))^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}<\infty.\tag{2}$$

The norm $\|\cdot\|_{(\alpha,\beta),q;J}$ in $\bar{A}_{(\alpha,\beta),q;J}$ is given by the infimum of the values of the integral (2) over all such representations (1) of a.

EXAMPLE 1.1. In the special case where Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$ and $\alpha>0$, $\beta>0$ with $\alpha+\beta<1$, we recover Sparr spaces $\bar{A}^S_{(\alpha,\beta),q;K}$ and $\bar{A}^S_{(\alpha,\beta),q;J}$ (see **[19]**).

EXAMPLE 1.2. If Π coincides with the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$, then we obtain Fernandez spaces $\bar{A}^F_{(\alpha,\beta),q;K}$ and $\bar{A}^F_{(\alpha,\beta),q;J}$ (see **[12, 13]**). Now $0<\alpha$, $\beta<1$.

In contrast to the classical case of Banach couples, where K- and J-methods coincide to within equivalence of norms (see **[2]** or **[20]**), it is not true in general that $\bar{A}_{(\alpha,\beta),q;K}$ coincides with $\bar{A}_{(\alpha,\beta),q;J}$. Counter-examples for the cases of Sparr and Fernandez spaces can be found in **[19]** and **[10]**, respectively. We now have only the following inclusion:

THEOREM 1.3. *Let Π, $\bar{A}$, q and (α,β) be as above. Then $\bar{A}_{(\alpha,\beta),q;J}$ is continuously embedded in $\bar{A}_{(\alpha,\beta),q;K}$.*

Proof. Let $a\in\bar{A}_{(\alpha,\beta),q;J}$ and $\varepsilon>0$, and let

$$a=\int_0^\infty\int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s}$$

be a representation of a such that

$$\left(\int_0^\infty\int_0^\infty (t^{-\alpha}s^{-\beta}J(t,s\,;u(t,s)))^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}<\|a\|_{(\alpha,\beta),q;J}+\varepsilon.$$

For any positive numbers w, z we have

$$\begin{aligned}K(w,z\,;a)&\le\int_0^\infty\int_0^\infty K(w,z\,;u(t,s))\frac{dt}{t}\frac{ds}{s}\\&\le\int_0^\infty\int_0^\infty \min_{1\le j\le N}\{(w/t)^{x_j}(z/s)^{y_j}\}J(t,s\,;u(t,s))\frac{dt}{t}\frac{ds}{s}\\&=\int_0^\infty\int_0^\infty \min_{1\le j\le N}\{t^{-x_j}s^{-y_j}\}J(tw,sz\,;u(tw,sz))\frac{dt}{t}\frac{ds}{s}.\end{aligned}$$

Applying Minkowski's inequality, we get

$$\|a\|_{(\alpha,\beta),q;K} \leqslant \left[\int_0^\infty \int_0^\infty \left(\int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{t^{\alpha-x_j} s^{\beta-y_j}\} (tw)^{-\alpha} (sz)^{-\beta} \times J(tw, sz\,; u(tw, sz)) \frac{dt}{t}\frac{ds}{s}\right)^q \frac{dw}{w}\frac{dz}{z}\right]^{1/q}$$

$$\leqslant \left[\int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{t^{\alpha-x_j} s^{\beta-y_j}\} \frac{dt}{t}\frac{ds}{s}\right] (\|a\|_{(\alpha,\beta),q;J} + \varepsilon).$$

So it only remains to check that the integral factor is finite. Writing it in exponential notation, we have

$$I = \int_0^\infty \int_0^\infty \min_{1\leqslant j\leqslant N} \{t^{\alpha-x_j} s^{\beta-y_j}\} \frac{dt}{t}\frac{ds}{s}$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \min_{1\leqslant j\leqslant N} \{e^{u(\alpha-x_j)+v(\beta-y_j)}\}\, du\, dv$$

$$\leqslant \sum_{j=1}^N \iint_{\Gamma_j} e^{-u(\alpha-x_j)-v(\beta-y_j)}\, du\, dv,$$

where

$$\Gamma_j = \{(u, v) \in \mathbb{R}^2\colon\ u(\alpha-x_j)+v(\beta-y_j) \geqslant u(\alpha-x_k)+v(\beta-y_k),\ k=1, ..., N\}.$$

Put $P=(\alpha, \beta)$ and recall that $P_j=(x_j, y_j)$. For $(u, v) \in \Gamma_j$, it follows from the fact that

$$u(\alpha-x_j)+v(\beta-y_j) \geqslant u(\alpha-x_{j+1})+v(\beta-y_{j+1})$$

that

$$\langle (u, v), P_{j+1}-P_j \rangle \geqslant 0.$$

In the same way, we see that

$$\langle (u, v), P_{j-1}-P_j \rangle \geqslant 0.$$

On the other hand, the convexity of the polygon Π, together with the fact that $P \in \operatorname{Int} \Pi$, gives that there are two positive numbers λ_j, μ_j such that

$$P - P_j = \lambda_j(P_{j+1}-P_j) + \mu_j(P_{j-1}-P_j)$$

(see Fig. 1.2).

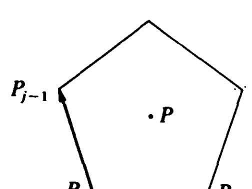

FIG. 1.2

Now making the change of variables

$$u^* = \langle (u, v), P_{j+1}-P_j \rangle, \quad v^* = \langle (u, v), P_{j-1}-P_j \rangle$$

and taking into account the fact that

$$\begin{aligned}-u(\alpha-x_j)-v(\beta-y_j)&=-\langle (u,v),P-P_j\rangle\\&=-\lambda_j\langle (u,v),P_{j+1}-P_j\rangle-\mu_j\langle (u,v),P_{j-1}-P_j\rangle\\&=-\lambda_j u^*-\mu_j v^*,\end{aligned}$$

we derive

$$I\leqslant\sum_{j=1}^{N}\frac{1}{|J_j|}\int_0^\infty\int_0^\infty e^{-\lambda_j u^*-\mu_j v^*}\,du^*\,dv^*=\sum_{j=1}^{N}\frac{1}{|J_j|}\frac{1}{\lambda_j\mu_j}<\infty.$$

Here

$$J_j=\begin{vmatrix}x_{j+1}-x_j & y_{j+1}-y_j\\ x_{j-1}-x_j & y_{j-1}-y_j\end{vmatrix}.$$

This completes the proof.

Let us go back again to the question of when the K- and J-Fernandez spaces coincide. According to an argument due to Milman [**16**, Theorem 4.1],

$$(\Xi)\qquad ((A_1,A_2,)_{\alpha,q},(A_3,A_4)_{\alpha,q})_{\beta,q}\subseteq\bar{A}^F_{(\alpha,\beta),q;K}$$

and

$$\bar{A}^F_{(\alpha,\beta),q;J}\subseteq((A_1,A_2)_{\alpha,q},(A_3,A_4)_{\alpha,q})_{\beta,q},$$

where $(\cdot,\cdot)_{\theta,p}$ stands for the classical real method with parameters θ and p.

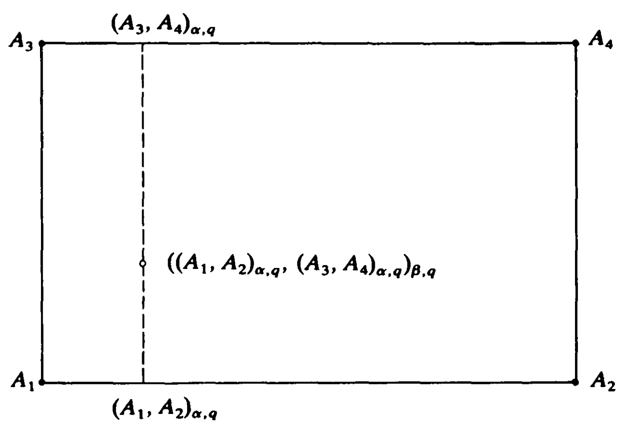

FIG. 1.3

On the other hand, it follows directly from the definitions of the Fernandez spaces that

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;K}=(A_1,A_3,A_2,A_4)_{(\beta,\alpha),q;K}$$

and

$$(A_1,A_2,A_3,A_4)_{(\alpha,\beta),q;J}=(A_1,A_3,A_2,A_4)_{(\beta,\alpha),q;J}.$$

Consequently, in order that the K- and J-spaces coincide, the following condition is necessary:

$$(*) \qquad ((A_1, A_2)_{\alpha,q}, (A_3, A_4)_{\alpha,q})_{\beta,q} = ((A_1, A_3)_{\beta,q}, (A_2, A_4)_{\beta,q})_{\alpha,q}.$$

Equality (*) seems very natural from the geometrical point of view and one might think that it is satisfied for any Banach 4-tuple. Nevertheless, this is not the case. Consider the scalar sequence spaces

$$A_1 = \left\{(\xi_n): \sum_{n=-\infty}^{\infty} \xi_n = 0 \text{ and } \|(\xi_n)\|_{A_1} = \sum_{n=-\infty}^{\infty} \max(1, 2^n)\,|\xi_n| < \infty\right\},$$

$$A_2 = \left\{(\xi_n): \sum_{n=-\infty}^{\infty} \xi_n = 0 \text{ and } \|(\xi_n)\|_{A_2} = \sum_{n=-\infty}^{\infty} \max(1, 2^{-n})\,|\xi_n| < \infty\right\},$$

$$A_3 = \left\{(\xi_n): \|(\xi_n)\|_{A_3} = \sum_{n=-\infty}^{\infty} \min(1, 2^n)\,|\xi_n| < \infty\right\},$$

$$A_4 = \left\{(\xi_n): \|(\xi_n)\|_{A_4} = \sum_{n=-\infty}^{\infty} \min(1, 2^{-n})\,|\xi_n| < \infty\right\}.$$

Then we have

$$((A_1, A_2)_{\frac{1}{2},1}, (A_3, A_4)_{\frac{1}{2},1})_{\frac{1}{2},1} \neq ((A_1, A_3)_{\frac{1}{2},1}, (A_2, A_4)_{\frac{1}{2},1})_{\frac{1}{2},1},$$

as can be checked by adapting the arguments given by Dore, Guidetti and Venni **[11]** for the case of the complex method.

We close this section by discussing the relationship between Sparr and Fernandez spaces.

Let $\bar{A} = \{A_1, A_2, A_3, A_4\}$ be a Banach 4-tuple and let

$$\bar{A}_K^F = \bar{A}_{(\alpha,\beta),q;K}^F \quad \text{and} \quad \bar{A}_J^F = \bar{A}_{(\alpha,\beta),q;J}^F$$

be the Fernandez spaces as described in Example 1.2.

In order to interpolate the 4-tuple $\bar{A}$ by Sparr methods, we need three independent parameters (t_1, t_2, t_3) and three positive numbers $(\theta_1, \theta_2, \theta_3)$ such that $\theta_1 + \theta_2 + \theta_3 < 1$; then spaces

$$\bar{A}_K^S = \bar{A}_{(\theta_1,\theta_2,\theta_3),q;K}^S \quad \text{and} \quad \bar{A}_J^S = \bar{A}_{(\theta_1,\theta_2,\theta_3),q;J}^S$$

are defined similarly as in Example 1.1 but this time using the functionals

$$\hat{K}(t_1, t_2, t_3; a) = \inf\left\{\|a_1\|_{A_1} + t_1\|a_2\|_{A_2} + t_2\|a_3\|_{A_3} + t_3\|a_4\|_{A_4}: a = \sum_{j=1}^{4} a_j\right\},$$

$$\hat{J}(t_1, t_2, t_3; a) = \max\{\|a\|_{A_1}, t_1\|a\|_{A_2}, t_2\|a\|_{A_3}, t_3\|a\|_{A_4}\}.$$

THEOREM 1.4. *Let, in this context,* $0 < \alpha, \beta < 1$ *and put*

$$\theta_1 = \alpha(1-\beta), \quad \theta_2 = \beta(1-\alpha) \quad \textit{and} \quad \theta_3 = \alpha\beta.$$

Then the following continuous inclusions hold:

$$\bar{A}_J^F \hookrightarrow \bar{A}_J^S \hookrightarrow \bar{A}_K^S \hookrightarrow \bar{A}_K^F.$$

Proof. Take $t_3 = t_1 t_2$. We have

$$(3) \qquad t_1^{-\theta_1} t_2^{-\theta_2} t_3^{-\theta_3} = t_1^{-\alpha(1-\beta)} t_2^{-\beta(1-\alpha)} (t_1 t_2)^{-\alpha\beta} = t_1^{-\alpha} t_2^{-\beta}.$$

Thus, for $a = \sum_{j=1}^{4} a_j$,

$$\|a\|_{\bar{A}_K^F} = \left[\int_0^\infty \int_0^\infty \left(t_1^{-\alpha} t_2^{-\beta} \inf\{\|a_1\|_{A_1} + t_1\|a_2\|_{A_2} + t_2\|a_3\|_{A_3} + t_1 t_2\|a_4\|_{A_4}\}\right)^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\right]^{1/q}$$

$$\leq c\left[\int_0^\infty \int_0^\infty \int_0^\infty \left(t_1^{-\theta_1} t_2^{-\theta_2} t_3^{-\theta_3} \inf\{\|a_1\|_{A_1} + t_1\|a_2\|_{A_2} + t_2\|a_3\|_{A_3} + t_3\|a_4\|_{A_4}\}\right)^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}\right]^{1/q}$$

$$= \|a\|_{\bar{A}_K^S}.$$

Here $c = (\alpha\beta q)^{1/q}$. This shows that $\bar{A}_K^S \hookrightarrow \bar{A}_K^F$. The inclusion $\bar{A}_J^S \hookrightarrow \bar{A}_K^S$ is known (see [**19**, Proposition 5.1]). Let us check the remaining embedding $\bar{A}_J^F \hookrightarrow \bar{A}_J^S$.

First note that it is possible to give discrete characterizations for spaces $\bar{A}_J^F$ and $\bar{A}_J^S$ using sums instead of integrals, and then the following norms are equivalent:

$$\|a\|_{\bar{A}_J^F} \sim \inf\left\{\left[\sum_{(n_1,n_2)\in\mathbb{Z}^2} (2^{-n_1\alpha-n_2\beta} J(2^{n_1}, 2^{n_2}; u_{n_1,n_2}))^q\right]^{1/q} : a = \sum_{(n_1,n_2)\in\mathbb{Z}^2} u_{n_1,n_2}\right\},$$

$$\|a\|_{\bar{A}_J^S} \sim \inf\left\{\left[\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} (2^{-n_1\theta_1-n_2\theta_2-n_3\theta_3} \hat{J}(2^{n_1}, 2^{n_2}, 2^{n_3}; v_{n_1,n_2,n_3}))^q\right]^{1/q} : a = \sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} v_{n_1,n_2,n_3}\right\}.$$

Let $a \in \bar{A}_J^F$ and let $a = \sum_{(n_1,n_2)\in\mathbb{Z}^2} u_{n_1,n_2}$ be any representation of a as above. Then we obtain a representation of a in the Sparr way by writing

$$v_{n_1,n_2,n_3} = \begin{cases} u_{n_1,n_2} & \text{if } n_3 = n_1 + n_2, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, by (3), we get

$$\left[\sum_{(n_1,n_2,n_3)\in\mathbb{Z}^3} (2^{-n_1\theta_1-n_2\theta_2-n_3\theta_3} \hat{J}(2^{n_1}, 2^{n_2}, 2^{n_3}; v_{n_1,n_2,n_3}))^q\right]^{1/q} = \left[\sum_{(n_1,n_2)\in\mathbb{Z}^2} (2^{-n_1\alpha-n_2\beta} J(2^{n_1}, 2^{n_2}; u_{n_1,n_2}))^q\right]^{1/q}.$$

This gives the inclusion.

2. *Interpolation of operators*

Let $\bar{B} = \{B_1, \ldots, B_N\}$ be another Banach N-tuple which we imagine as sitting on the vertices of (another copy of) our convex polygon $\Pi = \overline{P_1 \ldots P_N}$. We write $T: \bar{A} \to \bar{B}$ to mean that T is a linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j ($j = 1, \ldots, N$). We put

$$\|T\|_{\bar{A},\bar{B}} = \max\{\|T\|_{A_1,B_1}, \ldots, \|T\|_{A_N,B_N}\}.$$

Again let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $1 \leq q \leq \infty$. If T: $\bar{A} \to \bar{B}$, then clearly the restriction of T to $\bar{A}_{(\alpha,\beta),q;K}$ defines a bounded operator

$$T\colon\ \bar{A}_{(\alpha,\beta),q;K} \to \bar{B}_{(\alpha,\beta),q;K}$$

and

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;K},\bar{B}_{(\alpha,\beta),q;K}} \leq \|T\|_{\bar{A},\bar{B}}. \tag{4}$$

A similar estimate holds for the J-method, namely

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;J},\bar{B}_{(\alpha,\beta),q;J}} \leq \|T\|_{\bar{A},\bar{B}}. \tag{4'}$$

In this section we shall obtain estimates which are sharper than (4) and (4′).

For the case of the classical real method for Banach couples $(A_0, A_1)_{\theta,q}$, the convexity inequality

$$\|T\|_{\bar{A}_{\theta,q},\bar{B}_{\theta,q}} \leq \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}$$

is an indispensible tool. This estimate extends to the case of K- and J-Sparr spaces (Example 1.1). In fact, we have

$$\|T\|_{\bar{A}^S_{(\alpha,\beta),q;K},\bar{B}^S_{(\alpha,\beta),q;K}} \leq \|T\|_{A_1,B_1}^{1-(\alpha+\beta)} \|T\|_{A_2,B_2}^{\alpha} \|T\|_{A_3,B_3}^{\beta}$$

and a similar estimate holds for J-spaces. But the situation is not so clear for Fernandez spaces (Example 1.2). Theorem 1.4 and its connection with the iterated real method (Ξ) suggest that

$$\|T\|_{\bar{A}^F_{(\alpha,\beta),q;K},\bar{B}^F_{(\alpha,\beta),q;K}} \leq \|T\|_{A_1,B_1}^{(1-\alpha)(1-\beta)} \|T\|_{A_2,B_2}^{\alpha(1-\beta)} \|T\|_{A_3,B_3}^{(1-\alpha)\beta} \|T\|_{A_4,B_4}^{\alpha\beta} \tag{5}$$

and

$$\|T\|_{\bar{A}^F_{(\alpha,\beta),q;J},\bar{B}^F_{(\alpha,\beta),q;J}} \leq \|T\|_{A_1,B_1}^{(1-\alpha)(1-\beta)} \|T\|_{A_2,B_2}^{\alpha(1-\beta)} \|T\|_{A_3,B_3}^{(1-\alpha)\beta} \|T\|_{A_4,B_4}^{\alpha\beta}. \tag{5'}$$

Nevertheless, both estimates (5) and (5′) fail, as can easily be shown from the fact that for any 'diagonally equal' 4-tuple $\bar{E} = (E_0, E_1, E_1, E_0)$ one has (see [**10**, Example 1.25])

$$\bar{E}^F_{(\frac{1}{2},\frac{1}{2}),\infty;K} = E_0 + E_1$$

and

$$\bar{E}^F_{(\frac{1}{2},\frac{1}{2}),1;J} = E_0 \cap E_1.$$

This gives an idea of the kind of difficulty we are facing.

To proceed to our improvement of (4) and (4′), call $M_j = \|T\|_{A_j,B_j}$ and consider the K-case first. We have

$$\begin{aligned} K(t, s\,; Ta) &\leq \inf\Big\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \|Ta_j\|_{B_j}\colon\ a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\Big\} \\ &\leq \inf\Big\{\sum_{j=1}^{N} \lambda^{x_j}\mu^{y_j} M_j (t/\lambda)^{x_j} (s/\mu)^{y_j} \|a_j\|\colon\ a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\Big\} \\ &\leq \max_{1\leq j\leq N} \{\lambda^{x_j}\mu^{y_j} M_j\} K(t/\lambda, s/\mu\,; a). \end{aligned}$$

Thus integrating and changing variables we derive

$$\begin{aligned}
&\|Ta\|_{(\alpha,\beta),q;K}\\
&\leq \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha}s^{-\beta} \max_{1\leq j\leq N} \{\lambda^{x_j}\mu^{y_j}M_j\} K(t/\lambda, s/\mu\,; a)\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\
&= \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha}s^{-\beta} \max_{1\leq j\leq N} \{\lambda^{x_j-\alpha}\mu^{y_j-\beta}M_j\} K(t, s\,; a)\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q}\\
&= \max_{1\leq j\leq N} \{\lambda^{x_j-\alpha}\mu^{y_j-\beta}M_j\}\, \|a\|_{(\alpha,\beta),q;K}.
\end{aligned}$$

Consequently,

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;K},\bar{B}_{(\alpha,\beta),q;K}} \leq \inf_{\lambda>0,\mu>0} \left[\max_{1\leq j\leq N} \{\lambda^{x_j-\alpha}\mu^{y_j-\beta}M_j\}\right] \tag{6}$$

and the same estimate remains true for J-spaces. This leads to the following definition.

DEFINITION 2.1. Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$ for $j = 1, \dots, N$, and let $(\alpha, \beta) \in \operatorname{Int}\Pi$. Then for any N non-negative real numbers $M_1, \dots, M_N$, we put

$$D_{\alpha,\beta}(M_1, \dots, M_N) = \inf_{t>0,s>0} \left[\max_{1\leq j\leq N} \{t^{x_j-\alpha}s^{y_j-\beta}M_j\}\right].$$

According to (6), in order to estimate the norm of the interpolated operator for the K- (respectively J-) method we only need to study the function $D_{\alpha,\beta}$. We do that first for the case of Fernandez spaces.

THEOREM 2.2. *Let Π be the unit square, let $0<\alpha, \beta<1$, and let $D_{\alpha,\beta}$ be the function associated to them.*

For each 4-tuple of non-negative numbers (M_1, M_2, M_3, M_4), let $\mathscr{C} = \mathscr{C}(M_1, M_2, M_3, M_4)$ be the set formed by

$$N_1 = M_1^{1-\beta}M_3^{1-\alpha}M_4^{\alpha+\beta-1},$$

$$N_2 = M_1^{1-\beta}M_3^{\beta-\alpha}M_4^{\alpha},$$

$$N_3 = M_1^{1-\alpha}M_2^{\alpha-\beta}M_4^{\beta},$$

$$N_4 = M_1^{1-\alpha-\beta}M_2^{\alpha}M_3^{\beta},$$

and let $\mathscr{C}^*$ be the subset of $\mathscr{C}$ formed by those numbers N_j which only have positive exponents (in other words, if, say, $\alpha>\beta$ then N_3 belongs to $\mathscr{C}^*$ but N_2 does not).

Then

$$D_{\alpha,\beta}(M_1, M_2, M_3, M_4) = \max\{N_j\colon\ N_j \in \mathscr{C}^*\}.$$

Proof. Observe that

$$\begin{aligned}
D_{\alpha,\beta}(M_1, M_2, M_3, M_4) &= \inf_{t>0,s>0} [\max\{M_1, tM_2, sM_3, tsM_4\}t^{-\alpha}s^{-\beta}] \\
&\leqslant \inf_{s>0}\left[\inf_{t>0}\{\max[(M_1+sM_3)t^{-\alpha}, (M_2+sM_4)t^{1-\alpha}]\}s^{-\beta}\right] \\
&= \inf_{s>0}[(M_1+sM_3)^{1-\alpha}(M_2+sM_4)^{\alpha}s^{-\beta}] \\
&= M_3^{1-\alpha}M_4^{\alpha}\inf_{s>0}\left[\left(\frac{M_1}{M_3}+s\right)^{1-\alpha}\left(\frac{M_2}{M_4}+s\right)^{\alpha}s^{-\beta}\right] \\
&= M_3^{1-\alpha}M_4^{\alpha}\inf_{s>0} f(s),
\end{aligned}$$

where

$$f(s) = (x+s)^{1-\alpha}(y+s)^{\alpha}s^{-\beta}$$

and

$$x = M_1/M_3, \quad y = M_2/M_4.$$

Now we distinguish two cases.

Case 1: $y \leqslant x$. We have

$$f(s) \leqslant \begin{cases} x^{1-\alpha}y^{\alpha}s^{-\beta} & \text{if } s \leqslant y, \\ x^{1-\alpha}s^{\alpha}s^{-\beta} = x^{1-\alpha}s^{\alpha-\beta} & \text{if } y \leqslant s \leqslant x, \\ s^{1-\alpha}s^{\alpha}s^{-\beta} = s^{1-\beta} & \text{if } x \leqslant s. \end{cases}$$

Since the function $s^{-\beta}$ is decreasing, $s^{1-\beta}$ is increasing and $s^{\alpha-\beta}$ is increasing if $\alpha \geqslant \beta$ and decreasing if $\alpha < \beta$, we see that

$$\inf_{s>0} f(s) \leqslant \begin{cases} x^{1-\alpha}y^{\alpha-\beta} & \text{if } \alpha \geqslant \beta, \\ x^{1-\beta} & \text{if } \alpha < \beta. \end{cases}$$

Hence

$$\begin{aligned}
&D_{\alpha,\beta}(M_1, M_2, M_3, M_4) \\
&\leqslant \begin{cases} M_3^{1-\alpha}M_4^{\alpha}(M_1/M_3)^{1-\alpha}(M_2/M_4)^{\alpha-\beta} = M_1^{1-\alpha}M_2^{\alpha-\beta}M_4^{\beta} = N_3 & \text{if } \alpha \geqslant \beta \\ M_3^{1-\alpha}M_4^{\alpha}(M_1/M_3)^{1-\beta} = M_1^{1-\beta}M_3^{\beta-\alpha}M_4^{\alpha} = N_2 & \text{if } \alpha < \beta \end{cases} \\
&\leqslant \max\{N_j\colon N_j \in \mathscr{C}^*\}.
\end{aligned}$$

Case 2: $x \leqslant y$. This case is completely analogous. This time we have

$$f(s) \leqslant \begin{cases} x^{1-\alpha}y^{\alpha}s^{-\beta} & \text{if } s \leqslant x, \\ y^{\alpha}s^{1-\alpha-\beta} & \text{if } x \leqslant s \leqslant y, \\ s^{1-\beta} & \text{if } y \leqslant s, \end{cases}$$

so

$$\inf_{s>0} f(s) \leqslant \begin{cases} x^{1-\alpha-\beta}y^{\alpha} & \text{if } \alpha+\beta \leqslant 1, \\ y^{1-\beta} & \text{if } \alpha+\beta > 1, \end{cases}$$

and therefore

$$
\begin{aligned}
&D_{\alpha,\beta}(M_1, M_2, M_3, M_4)\\
&\leqslant \begin{cases} M_3^{1-\alpha}M_4^{\alpha}(M_1/M_3)^{1-\alpha-\beta}(M_2/M_4)^{\alpha} = M_1^{1-\alpha-\beta}M_2^{\alpha}M_3^{\beta} = N_4 & \text{if } \alpha+\beta \leqslant 1\\ M_3^{1-\alpha}M_4^{\alpha}(M_2/M_4)^{1-\beta} = M_2^{1-\beta}M_3^{1-\alpha}M_4^{\alpha+\beta-1} = N_1 & \text{if } \alpha+\beta > 1 \end{cases}\\
&\leqslant \max\{N_j\colon N_j \in \mathscr{C}^*\}.
\end{aligned}
$$

This shows that

$$D_{\alpha,\beta}(M_1, M_2, M_3, M_4) \leqslant \max\{N_j\colon N_j \in \mathscr{C}^*\}.$$

The reverse inequality can easily be checked by substituting each M_j in terms of $\max\{M_1, tM_2, sM_3, tsM_4\}$ and taking into account the fact that the sum of all exponents in each N_j is equal to 1.

As a direct consequence, we have

Corollary 2.3. *Let Π be the unit square and $\alpha = \beta = \frac{1}{2}$. For any 4-tuple of non-negative numbers (M_1, M_2, M_3, M_4) we have*

$$D_{\frac{1}{2},\frac{1}{2}}(M_1, M_2, M_3, M_4) = \max\{\sqrt{M_1M_4}, \sqrt{M_2M_3}\}.$$

Corollary 2.4. *Let Π be the unit square, let P_j, P_{j+1} be two fixed adjacent vertices of Π, and let $0<\alpha$, $\beta<1$. Then*

$$D_{\alpha,\beta}(M_1, M_2, M_3, M_4) \to 0 \quad \text{as } M_j \to 0 \text{ and } M_{j+1} \to 0.$$

Next we extend Corollary 2.4 to any convex polygon Π.

Theorem 2.5. *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon, let P_j, P_{j+1} be two fixed adjacent vertices of Π, and let $(\alpha, \beta) \in \operatorname{Int}\Pi$. Then*

$$D_{\alpha,\beta}(M_1, \dots, M_N) \to 0 \quad \text{as } M_k \to 0 \text{ for all } 1 \leqslant k \leqslant N \text{ with } k \neq j, j+1.$$

Proof. Recall that $P_k = (x_k, y_k)$. We have

$$
\begin{aligned}
D_{\alpha,\beta}(M_1, \dots, M_N) &\leqslant \inf_{t>0,s>0} \sum_{k=1}^{N} t^{x_k-\alpha}s^{y_k-\beta}M_k\\
&= \inf_{u,v\in\mathbb{R}} \sum_{k=1}^{N} e^{u(x_k-\alpha)+v(y_k-\beta)}M_k.
\end{aligned}
$$

Now consider the equation $px + qy = r$ of the line through P_j and P_{j+1} (see Fig. 2.1).

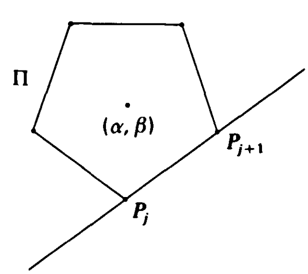

Fig. 2.1

Since $(\alpha, \beta) \in \operatorname{Int} \Pi$, it follows that

$$p\alpha + q\beta > r \quad \text{and} \quad px_j + qy_j = r,$$

so that $p(x_j - \alpha) + q(y_j - \beta) < 0$. In the same way we see that

$$p(x_{j+1} - \alpha) + q(y_{j+1} - \beta) < 0.$$

Hence, given any $\varepsilon > 0$, we can choose real numbers u, v such that

$$e^{u(x_j-\alpha)+v(y_j-\beta)} M_j + e^{u(x_{j+1}-\alpha)+v(y_{j+1}-\beta)} M_{j+1} \leqslant \tfrac{1}{2}\varepsilon.$$

(Take u, v proportional to p, q.) Then, taking all M_k for $k \neq j, j+1$ sufficiently small, we get

$$\sum_{\substack{1 \leqslant k \leqslant N \\ k \neq j, j+1}} e^{u(x_k-\alpha)+v(y_k-\beta)} M_k \leqslant \tfrac{1}{2}\varepsilon.$$

Combining the last two inequalities, we finally obtain

$$D_{\alpha,\beta}(M_1, \ldots, M_N) \leqslant \sum_{k=1}^{N} e^{u(x_k-\alpha)+v(y_k-\beta)} M_k \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

For later use, we close this section with the following consequence of Theorem 2.5.

COROLLARY 2.6. *Let $\Pi = \overline{P_1 \ldots P_N}$ be a convex polygon, let P_j, P_{j+1} be two fixed adjacent vertices of Π, and let $(\alpha, \beta) \in \operatorname{Int} \Pi$. Assume further that $\bar{A} = \{A_1, \ldots, A_N\}$ and $\bar{B} = \{B_1, \ldots, B_N\}$ are Banach N-tuples and let $(T_m)_{m=1}^{\infty}$ be a sequence of bounded operators*

$$T_m \colon \bar{A} \to \bar{B} \quad \textit{for } m = 1, 2, \ldots.$$

If $\sup_{m \in \mathbb{N}} \{\|T_m\|_{A_j, B_j}, \|T_m\|_{A_{j+1}, B_{j+1}}\} < \infty$ and $\|T_m\|_{A_k, B_k} \to 0$ as $m \to \infty$ for all $1 \leqslant k \leqslant N$ with $k \neq j$, $j+1$, then

$$\|T_m\|_{\bar{A}_{(\alpha,\beta),q;K}, \bar{B}_{(\alpha,\beta),q;K}} \to 0 \quad \textit{as } m \to \infty$$

and

$$\|T_m\|_{\bar{A}_{(\alpha,\beta),q;J}, \bar{B}_{(\alpha,\beta),q;J}} \to 0 \quad \textit{as } m \to \infty.$$

3. *Some examples*

Let $\Pi = \overline{P_1 \ldots P_N}$ be again our convex polygon with vertices $P_j = (x_j, y_j)$, and let $(\alpha, \beta) \in \operatorname{Int} \Pi$. Consider further a sequence of Banach spaces $(F_{m,n})_{(m,n) \in \mathbb{Z}^2}$.

For $j = 1, \ldots, N$, we write $F^j_{m,n}$ to mean the space $F_{m,n}$ normed by $2^{-mx_j - ny_j} \|\cdot\|_{F_{m,n}}$, that is,

$$F^j_{m,n} = 2^{-mx_j - ny_j} F_{m,n} = (F_{m,n}, 2^{-mx_j - ny_j} \|\cdot\|_{F_{m,n}}).$$

Moreover, if $1 \leqslant q \leqslant \infty$, we denote by $l_q(F^j_{m,n})$ the vector-valued l_q space, that is to say,

$$l_q(F^j_{m,n}) = \Bigg\{ (x_{m,n}) \colon x_{m,n} \in F_{m,n} \text{ and}$$

$$\|(x_{m,n})\|_{l_q(F^j_{m,n})} = \Big(\sum_{m,n} (2^{-mx_j - ny_j} \|x_{m,n}\|_{F_{m,n}})^q \Big)^{1/q} < \infty \Bigg\}.$$

In order to derive the interpolation properties of these spaces, we shall use the following discrete representation of K- and J-spaces: $a \in \bar{A}_{(\alpha,\beta),q;K}$ if and only if

$$\|a\|_{(\alpha,\beta),q;K} = \left[\sum_{(m,n)\in\mathbb{Z}^2} (2^{-mx_j-ny_j}K(2^m, 2^n; a))^q\right]^{1/q} < \infty;$$

$a \in \bar{A}_{(\alpha,\beta),q;J}$ if and only if

$$a = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n} \quad (\text{convergence in } \Sigma(\bar{A}))$$

with $(u_{m,n}) \subset \Delta(\bar{A})$ and

$$|||(u_{m,n})|||_{(\alpha,\beta),q;J} = \left[\sum_{(m,n)\in\mathbb{Z}^2} (2^{-mx_j-ny_j}J(2^m, 2^n; u_{m,n}))^q\right]^{1/q} < \infty.$$

The discrete norm of the J-space is given by

$$\|a\|_{(\alpha,\beta),q;J} = \inf |||(u_{m,n})|||_{(\alpha,\beta),q;J},$$

where the infimum is taken over all representations $(u_{m,n})$ of a as above.

We remark that although we denote discrete and continuous norms by the same symbol, in fact they are only equivalent. This will cause no confusion.

THEOREM 3.1. *Let* $1 \leq q_1, \ldots, q_N, q \leq \infty$. *Then*

$$(l_{q_j}(F^j_{m,n}))_{(\alpha,\beta),q;K} = (l_{q_j}(F^j_{m,n}))_{(\alpha,\beta),q;J} = l_q(2^{-\alpha m-\beta n}F_{m,n})$$

(*with equivalence of norms*).

Proof. We start by checking that

(7) $$(l_\infty(F^j_{m,n}))_{(\alpha,\beta),q;K} \hookrightarrow l_q(2^{-\alpha m-\beta n}F_{m,n}).$$

Let $(b_{m,n}) \in (l_\infty(F^j_{m,n}))_{(\alpha,\beta),q;K}$, and let

$$(b_{m,n}) = \sum_{j=1}^{N} (b^j_{m,n})$$

be any decomposition of $(b_{m,n})$ with $(b^j_{m,n}) \in l_\infty(F^j_{m,n})$. We have

$$\begin{aligned}
&\min_{1\leq j\leq N} \{2^{(v-m)x_j+(\mu-n)y_j} \|b_{m,n}\|_{F_{m,n}}\} \\
&\leq \sum_{k=1}^{N} \min_{1\leq j\leq N} \{2^{(v-m)x_j+(\mu-n)y_j} \|b^k_{m,n}\|_{F_{m,n}}\} \\
&\leq \sum_{k=1}^{N} 2^{vx_k+\mu y_k}(2^{-mx_k-ny_k} \|b^k_{m,n}\|_{F_{m,n}}) \\
&\leq \sum_{k=1}^{N} 2^{vx_k+\mu y_k} \|(b^k_{m,n})\|_{l_\infty(F^K_{m,n})}.
\end{aligned}$$

Thus

$$\sup_{(m,n)\in\mathbb{Z}^2} \left[\min_{1\leq j\leq N} \{2^{(v-m)x_j+(\mu-n)y_j} \|b_{m,n}\|_{F_{m,n}}\}\right] \leq K(2^v, 2^\mu; (b_{m,n})).$$

Hence

$$\|(b_{m,n})\|^q_{(l_\infty(F^j_{m,n}))_{(\alpha,\beta),q;K}}$$
$$= \sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m-\beta n}K(2^m, 2^n; (b_{m,n})))^q$$
$$\geqslant \sum_{(m,n)\in\mathbb{Z}^2} \left[2^{-\alpha m}2^{-\beta n} \sup_{(\nu,\mu)\in\mathbb{Z}^2} \left(\min_{1\leqslant j\leqslant N} \{2^{(m-\nu)x_j+(n-\mu)y_j} \|b_{m,n}\|_{F_{m,n}}\}\right)\right]^q$$
$$\geqslant \sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m}2^{-\beta n} \|b_{m,n}\|_{F_{m,n}})^q$$
$$= \|(b_{m,n})\|^q_{l_q(2^{-\alpha m-\beta n}F_{m,n})}.$$

Next we establish that

$$l_q(2^{-\alpha m-\beta n}F_{m,n}) \hookrightarrow (l_1(F^j_{m,n}))_{(\alpha,\beta),q;J}. \tag{8}$$

Given any $b=(b_{m,n})\in l_q(2^{-\alpha m-\beta n}F_{m,n})$, define $u_{m,n}=\bar{b}_{m,n}$ as the double sequence having all coordinates equal to zero except for the (m, n)th one which is $b_{m,n}$. Then

$$b = \sum_{(m,n)\in\mathbb{Z}^2} u_{m,n} \quad \text{(convergence in the sum).}$$

Moreover,

$$J(2^m, 2^n; u_{m,n}) = \max_{1\leqslant j\leqslant N} \{2^{mx_j+ny_j} \|u_{m,n}\|_{l_1(F^j_{m,n})}\}$$
$$= \max_{1\leqslant j\leqslant N} \{2^{mx_j+ny_j}2^{-mx_j-ny_j} \|b_{m,n}\|_{F_{m,n}}\}$$
$$= \|b_{m,n}\|_{F_{m,n}}.$$

Therefore

$$\sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m-\beta n}J(2^m, 2^n; u_{m,n}))^q = \sum_{(m,n)\in\mathbb{Z}^2} (2^{-\alpha m-\beta n} \|b_{m,n}\|_{F_{m,n}})^q$$
$$= \|b\|^q_{l_q(2^{-\alpha m-\beta n}F_{m,n})},$$

and this implies (8).

Now take any $1\leqslant q_1, \ldots, q_N\leqslant\infty$. According to (7), (8) and Theorem 1.3, the following continuous inclusions hold:

$$l_q(2^{-\alpha m-\beta n}F_{m,n}) \hookrightarrow (l_1(F^j_{m,n}))_{(\alpha,\beta),q;J}$$
$$\hookrightarrow (l_{q_j}(F^j_{m,n}))_{(\alpha,\beta),q;J}$$
$$\hookrightarrow (l_{q_j}(F^j_{m,n}))_{(\alpha,\beta),q;K}$$
$$\hookrightarrow (l_\infty(F^j_{m,n}))_{(\alpha,\beta),q;K}$$
$$\hookrightarrow l_q(2^{-\alpha m-\beta n}F_{m,n}).$$

The proof is complete.

In order to see a concrete example, take all $F_{m,n}$ equal to the scalar field $\mathbb{K}$. Then $l_q(F^j_{m,n})$ is nothing but $l_q(2^{-mx_j-ny_j})$, that is, the scalar sequence space l_q with weight $2^{-mx_j-ny_j}$ on the (m,n)th coordinate, where $(m, n)\in\mathbb{Z}^2$, and we obtain

COROLLARY 3.2. *Let* $1 \leqslant q \leqslant \infty$. *Then*

$$(l_\infty(2^{-mx_j-ny_j}))_{(\alpha,\beta),q;K} = l_q(2^{-\alpha m-\beta n})$$

and

$$(l_1(2^{-mx_j-ny_j}))_{(\alpha,\beta),q;J} = l_q(2^{-\alpha m-\beta n}).$$

Next we shall show that the interpolation formulae stated in Corollary 3.2 characterize K- and J-interpolation methods in a certain way. For this object, we need the multidimensional analogues of the two Aronszajn–Gagliardo functors [1] (cf. [14, 17]).

In what follows, $\bar{Z} = \{Z_1, \ldots, Z_N\}$ stands for a fixed Banach N-tuple, while Z denotes a fixed intermediate space for $\bar{Z}$, that is,

$$\Delta(\bar{Z}) \hookrightarrow Z \hookrightarrow \Sigma(\bar{Z}).$$

Given any Banach N-tuple $\bar{B} = \{B_1, \ldots, B_N\}$, let

$$\mathscr{V} = \mathscr{V}(\bar{B}) = \{R\colon\ R\colon\ \bar{B} \to \bar{Z},\ \|R\|_{\bar{B},\bar{Z}} \leqslant 1\}$$

be the collection of all bounded linear operators R from the N-tuple $\bar{B}$ to $\bar{Z}$ having norm

$$\|R\|_{\bar{B},\bar{Z}} = \max_{1\leqslant j\leqslant N} \{\|R\|_{B_j,Z_j}\}$$

less than or equal to 1.

If W is any of the spaces $Z, Z_1, \ldots, Z_N, \Delta(\bar{Z}), \Sigma(\bar{Z})$, we write $l_\infty[W] = l_\infty(\mathscr{V}(\bar{B}), W)$ to denote the Banach space formed by all bounded W-valued families $w = \{w_R\}$ with $\mathscr{V}$ as indexed set. The norm of $l_\infty[W]$ is given by

$$\|w\|_{l_\infty[W]} = \sup_{R\in\mathscr{V}} \|w_R\|_W.$$

For $b \in \Sigma(\bar{B})$, we write

$$ib = \{Rb\}_{R\in\mathscr{V}}.$$

Note that the family ib belongs to $l_\infty[\Sigma(\bar{Z})]$.

Now we are ready to introduce the Aronszajn–Gagliardo maximal ('co-orbit') functor. Define

$$H(\bar{B}) = H[\bar{Z}\,; Z](\bar{B}) = \{b\colon\ b \in \Sigma(\bar{B}),\ ib \in l_\infty[Z]\}.$$

The space $H(\bar{B})$ becomes a Banach space when endowed with the natural induced norm. Moreover, H is an interpolation functor. That is to say, if $T\colon \bar{A} \to \bar{B}$, then $T\colon H(\bar{A}) \to H(\bar{B})$.

Next we turn to the dual construction: the Aronszajn–Gagliardo minimal ('orbit') functor.

Given any Banach N-tuple $\bar{A} = \{A_1, \ldots, A_N\}$, put

$$\mathscr{U} = \mathscr{U}(\bar{A}) = \{S\colon\ S\colon\ \bar{Z} \to \bar{A},\ \|S\|_{\bar{Z},\bar{A}} \leqslant 1\},$$

and let $l_1[W] = l_1(\mathscr{U}(\bar{A}), W)$, the Banach space of all absolutely summable families $v = \{v_S\}$ of elements of W indexed by $\mathcal{U}$. The norm of $l_1[W]$ is given by

$$\|v\|_{l_1[W]} = \sum_{S \in \mathcal{U}} \|v_S\|_W.$$

For $z = \{z_S\} \in l_1[\Sigma(\bar{Z})]$, let $\pi z = \sum_{S \in \mathcal{U}} S z_S$.
We define

$$G(\bar{A}) = G[\bar{Z}\,;Z](\bar{A}) = \{a\colon\; a \in \Sigma(\bar{A}),\; \exists z \in l_1[Z],\, a = \pi z\},$$

and we endow $G(\bar{A})$ with the natural quotient norm (as a quotient of $l_1[Z]$). In this way, we obtain another interpolation functor:

$$T\colon\; \bar{A} \to \bar{B} \quad \text{implies} \quad T\colon\; G(\bar{A}) \to G(\bar{B}).$$

Now we are in a position to show the announced relationship between formulae in Corollary 3.2 and K- and J-functors.

THEOREM 3.3. *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and let $1 \leq q \leq \infty$. Then for any Banach N-tuple $\bar{A} = \{A_1, \dots, A_N\}$ we have*

$$H[\{l_\infty(2^{-mx_j-ny_j})\}\,;l_q(2^{-\alpha m-\beta n})](\bar{A}) = \bar{A}_{(\alpha,\beta),q;K},$$

and

$$G[\{l_1(2^{-mx_j-ny_j})\}\,;l_q(2^{-\alpha m-\beta n})](\bar{A}) = \bar{A}_{(\alpha,\beta),q;J}$$

with equality of (discrete) norms.

Proof. The result can be checked by adapting the arguments used in the case of the classical real method (see **[14]** and **[7]**).

4. *Lions–Peetre type compactness results*

In 1964 Lions and Peetre **[15]** established compactness theorems for general functors on Banach couples assuming that one of the couples reduces to a single Banach space. These theorems turned out to be essential tools in the proofs of all (classical and modern) compactness results. See, for example, **[5, 6, 8, 9]** (see also, however, **[3]**). In this section we derive multidimensional compactness results of Lions–Peetre type.

THEOREM 4.1. *Let $\Pi = \overline{P_1 \dots P_N}$ be a convex polygon with $P_k = (x_k, y_k)$, let P_j, P_{j+1} be two fixed adjacent vertices of Π, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and, finally, let $1 \leq q \leq \infty$. Assume that $\bar{A} = \{A_1, \dots, A_N\}$ is a Banach N-tuple, that B is a Banach space and that T is a linear operator $T\colon \bar{A} \to B$.*
If $T\colon A_k \to B$ is compact for all $1 \leq k \leq N$ with $k \neq j, j+1$, then

$$T\colon\; \bar{A}_{(\alpha,\beta),q;K} \to B$$

is also compact.

Proof. Since

$$\bar{A}_{(\alpha,\beta),q_0;K} \hookrightarrow \bar{A}_{(\alpha,\beta),q_1;K} \quad \text{for } 1 \leq q_0 \leq q_1 \leq \infty,$$

we may assume that $q = \infty$.

Let D be any bounded subset of $\bar{A}_{(\alpha,\beta),\infty;K}$. We are going to show that $T(D)$ is a precompact subset of B, from which follows the compactness of $T\colon \bar{A}_{(\alpha,\beta),\infty;K} \to B$.

Put

$$M_k = \|T\|_{A_k,B} \quad \text{and} \quad C = \sup\{\|a\|_{(\alpha,\beta),\infty;K}\colon\ a \in D\},$$

where this time we are using the continuous norm of $\bar{A}_{(\alpha,\beta),\infty;K}$ given by

$$\|a\|_{(\alpha,\beta),\infty;K} = \sup_{t>0,s>0} \{t^{-\alpha}s^{-\beta}K(t, s\,; a)\}.$$

For each t and s, we can decompose any $a \in D$ as $a = \sum_{k=1}^{N} a_k$ with $a_k \in A_k$ and $\|a_k\|_{A_k} \le 2Ct^{\alpha-x_k}s^{\beta-y_k}$. Proceeding as in the proof of Theorem 2.5, we see that given any $\varepsilon > 0$, we can choose t and s such that

$$\|Ta_j\|_{A_j} \le 2CM_j t^{\alpha-x_j}s^{\beta-y_j} \le \varepsilon/N$$

and

$$\|Ta_{j+1}\|_{A_{j+1}} \le 2CM_{j+1} t^{\alpha-x_{j+1}}s^{\beta-y_{j+1}} \le \varepsilon/N.$$

With these t and s fixed, define

$$D_k = \left\{a_k \in A_k\colon\ \exists a \in D \text{ with } a = \sum_{k=1}^{N} a_k \text{ and } \|a_k\|_{A_k} \le 2Ct^{\alpha-x_k}s^{\beta-y_k}\right\}.$$

Since D_k is bounded in A_k, we can use the compactness assumption on T to find finite subsets

$$\{b_{k,v}\}_{v=1}^{m_k} \subset B, \quad \text{with } 1 \le k \le N,\ k \ne j, j+1,$$

satisfying

$$T(D_k) \subset \bigcup_{v=1}^{m_k} \{b_{k,v} + \{b \in B\colon\ \|b\|_B \le \varepsilon/N\}\}.$$

Thus, if $a \in D$, we have

$$\begin{aligned}\Big\|Ta - \sum_{\substack{1\le k\le N\\ k\ne j,j+1}} b_{k,v_k}\Big\|_B &= \Big\|\sum_{1\le k\le N} Ta_k - \sum_{\substack{1\le k\le N\\ k\ne j,j+1}} b_{k,v_k}\Big\|_B \\ &\le \|Ta_j\|_B + \|Ta_{j+1}\|_B + \sum_{\substack{1\le k\le N\\ k\ne j,j+1}} \|Ta_k - b_{k,v_k}\|_B \\ &\le 2\varepsilon/N + \sum_{\substack{1\le k\le N\\ k\ne j,j+1}} \|Ta_k - b_{k,v_k}\|_B.\end{aligned}$$

Choosing b_{k,v_k} for $1 \le k \le N$, $k \ne j, j+1$, such that

$$\|Ta_k - b_{k,v_k}\|_B \le \varepsilon/N$$

now gives

$$\Big\|Ta - \sum_{\substack{1\le k\le N\\ k\ne j,j+1}} b_{k,v_k}\Big\|_B \le \varepsilon.$$

This shows the precompactness of $T(D)$.

By Theorem 1.3, $\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K}$, so that we also have

COROLLARY 4.2. *Under the same assumptions as in Theorem* 4.1, *the operator*

$$T\colon\ \bar{A}_{(\alpha,\beta),q;J} \to B$$

is compact.

Our next results refer to the case when the Banach space is the domain of the operator.

THEOREM 4.3. *Let* $\Pi = \overline{P_1 \ldots P_N}$ *be a convex polygon, let* P_j, P_{j+1} *be two fixed adjacent vertices of* Π, *let* $(\alpha, \beta) \in \operatorname{Int}\Pi$ *and* $1 \leqslant q \leqslant \infty$. *Assume that* $\bar{B} = \{B_1, \ldots, B_N\}$ *is a Banach N-tuple, that A is a Banach space and that T is a linear operator* $T\colon A \to \bar{B}$.

If $T\colon A \to B_k$ *is compact for all* $1 \leqslant k \leqslant N$ *with* $k \neq j, j+1$, *then*

$$T\colon\ A \to \bar{B}_{(\alpha,\beta),q;J}$$

is also compact.

Proof. This time we may assume that $q = 1$ because

$$\bar{B}_{(\alpha,\beta),q_0;J} \hookrightarrow \bar{B}_{(\alpha,\beta),q_1;J} \quad \text{for } 1 \leqslant q_0 \leqslant q_1 \leqslant \infty.$$

Note also that there is a constant $C > 0$ such that, for any $b \in \Delta(\bar{B})$ and any $t > 0$, $s > 0$,

$$\|b\|_{(\alpha,\beta),1;J} \leqslant C t^{-\alpha} s^{-\beta} J(t, s\,; b). \tag{9}$$

This follows easily from the discrete characterization of $\bar{B}_{(\alpha,\beta),1;J}$.

Let (a_n) be any bounded sequence in A. We are going to show that (Ta_n) has a convergent subsequence in $\bar{B}_{(\alpha,\beta),1;J}$.

By the compactness assumption, we can find a subsequence $(a_{n'})$ of (a_n) such that $(Ta_{n'})$ converges in B_k for all $1 \leqslant k \leqslant N$ with $k \neq j, j+1$. Let us see that $(Ta_{n'})$ is a Cauchy sequence in $\bar{B}_{(\alpha,\beta),1;J}$.

Given any $\varepsilon > 0$, using the same argument as in the proof of Theorem 2.5, we can choose $t > 0$ and $s > 0$ such that

$$\max\{2CLt^{x_j-\alpha}s^{y_j-\beta},\ 2CLt^{x_{j+1}-\alpha}s^{y_{j+1}-\beta}\} \leqslant \varepsilon,$$

where

$$L = \sup\{\|Ta_n\|_{B_j}\colon\ n \in \mathbb{N},\ 1 \leqslant j \leqslant N\}.$$

Then, by (9), we get

$$\begin{aligned}
&\|Ta_{n'} - Ta_{m'}\|_{(\alpha,\beta),1;J} \\
&\qquad \leqslant Ct^{-\alpha}s^{-\beta}J(t, s\,; Ta_{n'} - Ta_{m'}) \\
&\qquad = \max\Big[ct^{x_j-\alpha}s^{y_j-\beta}\,\|Ta_{n'} - Ta_{m'}\|_{B_j},\ Ct^{x_{j+1}-\alpha}s^{y_{j+1}-\beta}\,\|Ta_{n'} - Ta_{m'}\|_{B_{j+1}}, \\
&\qquad\qquad \max_{\substack{1\leqslant k\leqslant N\\ k\neq j,j+1}} \{Ct^{x_k-\alpha}s^{y_k-\beta}\,\|Ta_{n'} - Ta_{m'}\|_{B_k}\}\Big] \\
&\qquad \leqslant \max\Big\{\varepsilon,\ \max_{\substack{1\leqslant k\leqslant N\\ k\neq j,j+1}} \{Ct^{x_k-\alpha}s^{y_k-\beta}\,\|Ta_{n'} - Ta_{m'}\|_{B_k}\}\Big\}.
\end{aligned}$$

Choosing n' and m' big enough so that

$$\max_{\substack{1\leq k\leq N\\ k\neq j,j+1}} \{Ct^{x_k-\alpha}s^{y_k-\beta}\,\|Ta_{n'}-Ta_{m'}\|_{B_k}\}\leq\varepsilon$$

now gives that

$$\|Ta_{n'}-Ta_{m'}\|_{(\alpha,\beta),1;J}\leq\varepsilon.$$

This shows the compactness of $T\colon A\to\bar{B}_{(\alpha,\beta),1;J}$.

Combining Theorem 4.3 and Theorem 1.3, we derive

COROLLARY 4.4. *Under the same assumption as in Theorem* 4.3, *the operator*

$$T\colon A\to\bar{B}_{(\alpha,\beta),q;K}$$

is also compact.

5. *Admissible polygons*

Our next aim is to get rid of the assumption that one of the N-tuples reduces to a single Banach space.

For the case of the classical real method for couples, this was done in **[8]** using the description of the real method as an Aronszajn–Gagliardo functor. The key of this approach is the fact that on the couple $(l_1, l_1(2^{-j}))$ (and $(l_\infty, l_\infty(2^{-j}))$) we can consider the operators $\{P_n\}_{n=1}^\infty$, $\{Q_n^+\}_{n=1}^\infty$ and $\{Q_n^-\}_{n=1}^\infty$ defined by

$$P_n\xi=(\ldots,0,0,\xi_{-n},\ldots,\xi_{-1},\xi_0,\xi_1,\ldots,\xi_n,0,\ldots),$$
$$Q_n^+\xi=(\ldots,0,0,\xi_{n+1},\xi_{n+2},\ldots),$$

and

$$Q_n^-\xi=(\ldots,\xi_{-n-2},\xi_{-n-1},0,0,\ldots),$$

for any sequence $\xi=(\ldots,\xi_{-2},\xi_{-1},\xi_0,\xi_1,\xi_2,\ldots)$, and these operators satisfy the following conditions:

(I) they are uniformly bounded and P_n maps $l_1+l_1(2^{-j})$ to $l_1\cap l_1(2^{-j})$ for each $n\in\mathbb{N}$;

(II) the identity operator I on $(l_1, l_1(2^{-j}))$ can be decomposed as

$$I=P_n+Q_n^++Q_n^-,\quad \text{for } n=1,2,\ldots;$$

(III) the sequence of norms

$$\{\|Q_n^+\|_{l_1,l_1(2^{-j})}\}_{n=1}^\infty,\quad \{\|Q_n^-\|_{l_1(2^{-j}),l_1}\}_{n=1}^\infty$$

converges to 0 when $n\to\infty$.

The characterization of J- and K-spaces in terms of Aronszajn–Gagliardo functors, given in Theorem 3.3, suggests that in our multidimensional context we should work with the N-tuples

$$\{l_1(2^{-mx_j-ny_j})\}_{j=1}^N\quad(\text{and }\{l_\infty(2^{-mx_j-ny_j})\}_{j=1}^N).$$

Moreover, in order to be able to use Corollary 2.6, given any $\nu\in\mathbb{N}$ we should construct projections $\{Q_j^\nu\}_{1\leq j\leq N}$ such that a condition of the following type is satisfied: given any Q_j^ν and any vertex P_k of Π with $k \neq j, j+1$ (P_j, P_{j+1} being again adjacent vertices) one has (writing $L_1(i)$ for $l_1(2^{-mx_i-ny_i})$)

$$\|Q_j^\nu\|_{L_1(k),L_1(j)} \to 0 \quad \text{as } \nu \to \infty,$$

or

$$\|Q_j^\nu\|_{L_1(k),L_1(j+1)} \to 0 \quad \text{as } \nu \to \infty.$$

Assuming that each projection Q_j^ν is defined in the more natural way, that is, making all coordinates of ξ equal to 0 except for those $\xi_{m,n}$ when (m, n) belongs to a certain subset R_j^ν of $\mathbb{R}^2$, we are led to the following question:

(T) Is it true that for each $\nu \in \mathbb{N}$ there is a covering $\{R_j^\nu\}_{1\leqslant j\leqslant N}$ of a neighbourhood of the point ∞ in $\mathbb{R}^2$ such that, given any $1 \leqslant j \leqslant N$ and any $1 \leqslant k \leqslant N$ with $k \neq j, j+1$, then one of the following conditions holds:

(1°) $\langle (m, n), P_k - P_j \rangle \leqslant -\nu$ for any $(m, n) \in R_j^\nu$, or

(2°) $\langle (m, n), P_k - P_{j+1} \rangle \leqslant -\nu$ for any $(m, n) \in R_j^\nu$?

Let us check that the answer to (T) is 'yes' in some important cases.

EXAMPLE 5.1. If Π is the simplex with vertices $P_1 = (0, 0)$, $P_2 = (1, 0)$, $P_3 = (0, 1)$, then given any $\nu \in \mathbb{N}$ we write

$$R_1^\nu = \{(u, w) \in \mathbb{R}^2\colon\ w \leqslant -\nu\}, \quad R_2^\nu = \{(u, w) \in \mathbb{R}^2\colon\ u \geqslant \nu\}$$

and

$$R_3^\nu = \{(u, w) \in \mathbb{R}^2\colon\ u - w \leqslant -\nu\}.$$

Then, given any $j = 1, 2, 3$ and taking $k = j - 1$, we have

$$\langle (u, w), P_k - P_j \rangle \leqslant -\nu \quad \text{for any } (u, w) \in R_j^\nu.$$

Moreover, $\{R_j^\nu\}_{1\leqslant j\leqslant 3}$ is a covering of the point ∞ as the picture in Fig. 5.1 shows.

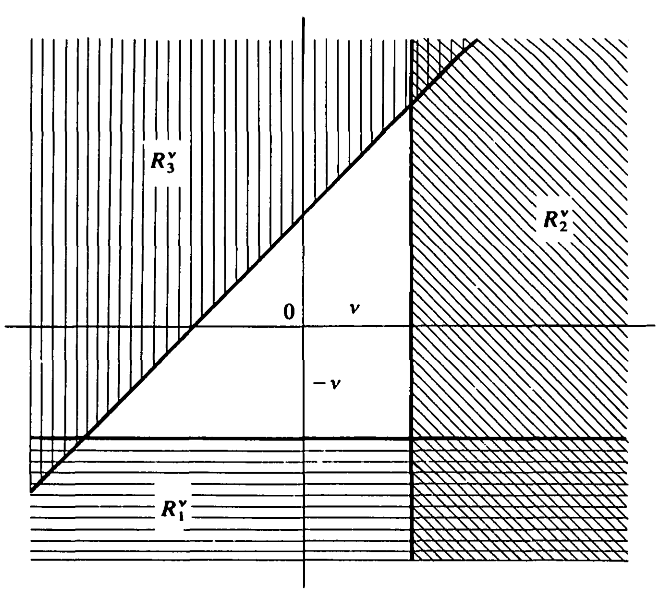

FIG. 5.1

EXAMPLE 5.2. Now let Π be the unit square with vertices $P_1=(0,0)$, $P_2=(1,0)$, $P_3=(1,1)$, $P_4=(0,1)$. Given any $\nu\in\mathbb{N}$ we put

$$R_1^\nu=\{(u,w)\in\mathbb{R}^2\colon\ w\leqslant-\nu\},\quad R_2^\nu=\{(u,w)\in\mathbb{R}^2\colon\ u\geqslant\nu\},$$
$$R_3^\nu=\{(u,w)\in\mathbb{R}^2\colon\ w\geqslant\nu\},\quad R_4^\nu=\{(u,w)\in\mathbb{R}^2\colon\ u\leqslant-\nu\}.$$

Then it is not hard to check that for any two vertices P_j, P_k with $k\neq j, j+1$, one has (1°) or (2°). Again $\{R_j^\nu\}_{1\leqslant j\leqslant 4}$ is a covering of ∞ as the picture in Fig. 5.2 indicates.

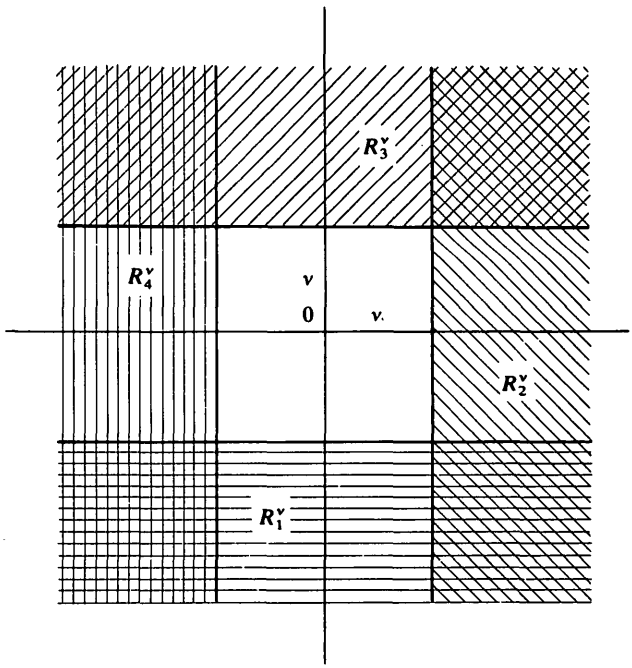

FIG. 5.2

Next we show that the answer to (T) is also 'yes' for a wide class of polygons.

In what follows, $\Pi=P_1\ldots P_N$ denotes a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $\{P_j\}_{j=1}^N$. For $j>N$ or $j<1$, we put

$$P_j=P_{j_0}\quad\text{if } j\equiv j_0\ (\mathrm{mod}\ N),\ 1\leqslant j_0\leqslant N.$$

DEFINITION 5.3. The convex polygon Π is said to be *admissible* if for each edge $\overline{P_jP_{j+1}}$ $(j=1,\ldots,N)$ there is another $\overline{P_kP_{k+1}}$ satisfying the following two conditions:

(a) the extension of the segment $\overline{P_jP_{k+1}}$ in the direction of P_j meets the extension of $\overline{P_{j+1}P_{j+2}}$ in the direction of P_{j+1}; and

(b) the extension of the segment $\overline{P_{j+1}P_k}$ in the direction of P_{j+1} meets the extension of $\overline{P_{j-1}P_j}$ in the direction of P_j.

See Fig. 5.3.

EXAMPLE 5.4. If Π is a regular polygon with at least five edges, then it is clear that Π is admissible.

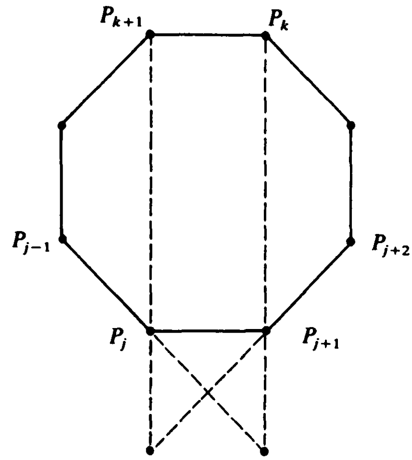

FIG. 5.3

For admissible polygons, (T) has also a positive answer as the next theorem shows.

THEOREM 5.5. *Let Π be an admissible polygon. Then for each $\nu \in \mathbb{N}$ there is a covering $\{R_j^\nu\}_{1 \leqslant j \leqslant N}$ of a neighbourhood of the point ∞ in $\mathbb{R}^2$, formed by closed sets such that given any vertices $P_j, P_k \in \Pi$ with $k \neq j, j+1$, then one of the two following conditions holds:*

(1°) $\langle (u, w), P_k - P_j \rangle \leqslant -\nu$ *for any* $(u, w) \in R_j^\nu$, *or*

(2°) $\langle (u, w), P_k - P_{j+1} \rangle \leqslant -\nu$ *for any* $(u, w) \in R_j^\nu$.

Proof. Given $\nu \in \mathbb{N}$, put

$$R_j^\nu = \{(u, w)\colon \langle (u, w), P_{j-1} - P_j \rangle \leqslant -c\nu\} \cap \{(u, w)\colon \langle (u, w), P_{j+2} - P_{j+1} \rangle \leqslant -c\nu\},$$

where c is a sufficiently large positive constant that will be fixed later.

Since Π is admissible, given $\overline{P_jP_{j+1}}$ there is another edge $\overline{P_kP_{k+1}}$ satisfying Conditions (a) and (b). We may assume that, for example, $k > j$. Now let $1 \leqslant h \leqslant N$ with $h \neq j, j+1$. Then if $j+1 < h \leqslant k$, the vector $P_h - P_{j+1}$ belongs to

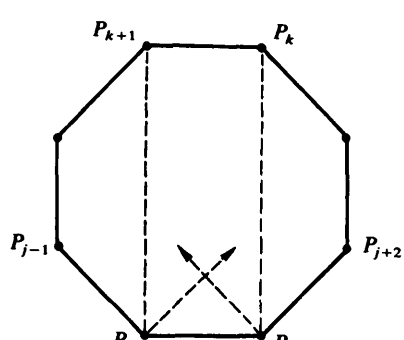

FIG. 5.4

the positive cone generated by $P_{j-1} - P_j$ and $P_{j+2} - P_{j+1}$, while if $h \geqslant k+1$, then it is the vector $P_h - P_j$ which belongs to such a cone. See Fig. 5.4.

Hence there are non-negative real numbers $\alpha_h = \alpha_{h,j}$ and $\beta_h = \beta_{h,j}$ with $\alpha_h + \beta_h > 0$ and

$$P_h - P_{j+1} = \alpha_h(P_{j-1} - P_j) + \beta_h(P_{j+2} - P_{j+1}) \quad \text{if } h \leqslant k$$

or

$$P_h - P_j = \alpha_h(P_{j-1} - P_j) + \beta_h(P_{j+2} - P_{j+1}) \quad \text{if } h \geqslant k+1.$$

Let

$$c \geqslant \frac{1}{\min_{h,j}(\alpha_{h,j} + \beta_{h,j})}.$$

Then we have if, for example, $h \leqslant j$ and $(u, w) \in R_j^\nu$,

$$\begin{aligned}\langle (u, w), P_h - P_j \rangle &= \alpha_h \langle (u, w), P_{j-1} - P_j \rangle + \beta_h \langle (u, w), P_{j+2} - P_{j+1} \rangle \\ &\leqslant -\alpha_h c\nu - \beta_h c\nu \leqslant -\nu.\end{aligned}$$

It remains to check that $\{R_j^\nu\}_{1 \leqslant j \leqslant N}$ is a covering of the point ∞. Let R_j^ν and R_{j+1}^ν be any two adjacent sets. Let γ be the angle between the edges $\overline{P_j P_{j+1}}$ and $\overline{P_{j+1} P_{j+2}}$. Since R_j^ν has an edge which is orthogonal to $\overline{P_{j+1} P_{j+2}}$, and R_{j+1}^ν another one orthogonal to $\overline{P_j P_{j+1}}$, it follows that the angle between such edges is also γ (see Fig. 5.5).

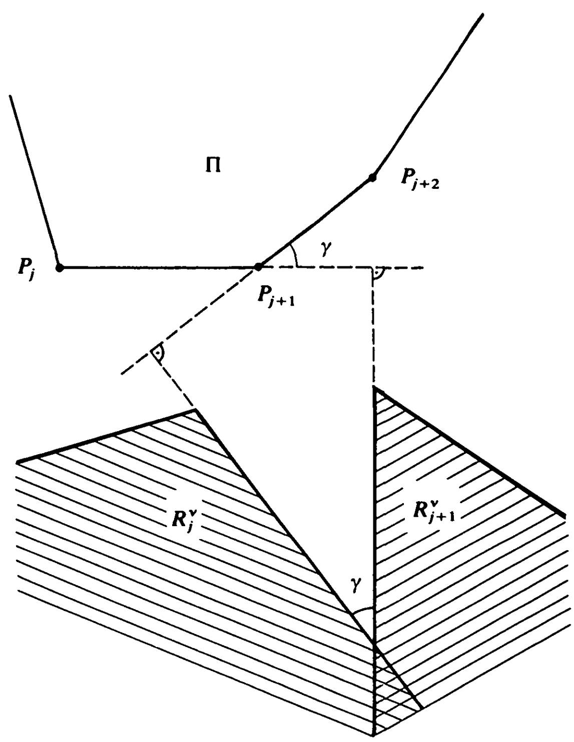

FIG. 5.5

Therefore the intersection between R_j^ν and R_{j+1}^ν is not empty. This completes the proof.

EXAMPLE 5.6. In the case of a regular hexagon or a regular octagon we have drawn the boundary of the neighbourhood of ∞ in question with the help of the computer program *Mathematica* using the general construction in Theorem 5.5. The results are shown in Figs 5.6 and 5.7.

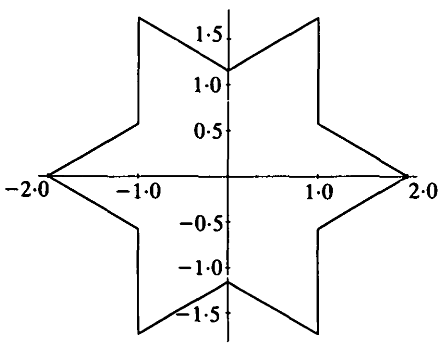

FIG. 5.6

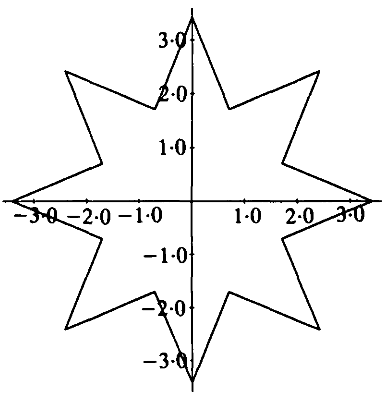

FIG. 5.7

Let S^ν be the *bounded* subset of $\mathbb{R}^2$ which we should add to the sets $\{R_j^\nu\}_{1 \le j \le N}$ in Theorem 5.5 to obtain a covering of the whole of $\mathbb{R}^2$,

$$S^\nu = \mathbb{R}^2 \setminus \bigcup_{j=1}^{N} R_j^\nu.$$

Also, put

$$K_1^\nu = R_1^\nu \quad \text{and} \quad K_j^\nu = R_j^\nu \setminus \bigcup_{i<j} K_i^\nu, \quad \text{for } 2 \le j \le N.$$

Then, for each $\nu \in \mathbb{N}$, $\{S^\nu, K_1^\nu, \ldots, K_N^\nu\}$ is a partition of $\mathbb{R}^2$.

Next, given any $\nu \in \mathbb{N}$ and any sequence $\xi = (\xi_{m,n})$, we define

$$F^\nu \xi = (\mu_{m,n}) \quad \text{where } \mu_{m,n} = \begin{cases} \xi_{m,n} & \text{if } (m,n) \in S^\nu, \\ 0 & \text{otherwise,} \end{cases}$$

and, for $j = 1, \ldots, N$,

$$Q_j^\nu \xi = (\rho_{m,n}) \quad \text{where } \rho_{m,n} = \begin{cases} \xi_{m,n} & \text{if } (m,n) \in K_j^\nu, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Theorem 5.5 and the previous discussion that the sequences of operators $\{F^\nu\}_{\nu \in \mathbb{N}}$ and $\{Q_j^\nu\}_{1 \le j \le N, \nu \in \mathbb{N}}$ satisfy the desired conditions (the multi-dimensional analogues of (I), (II) and (III)). Let us state them with precision.

THEOREM 5.7. *Let* $\Pi = \overline{P_1 \ldots P_N}$ *be the simplex, the unit square, or any admissible polygon, let* $\{F^\nu\}_{\nu \in \mathbb{N}}$ *and* $\{Q_j^\nu\}_{1 \le j \le N, \nu \in \mathbb{N}}$ *be the sequences defined above, and consider the Banach N-tuple* $\bar{l}_1 = \{l_1(2^{-mx_j - ny_j})\}_{j=1}^N$ *where* $P_j = (x_j, y_j)$, $1 \le j \le N$. *Then these sequences of operators have the following properties on* $\bar{l}_1$.

(i) *They are uniformly bounded in* $\bar{l}_1$,

$$\sup\{\|F^\nu\|_{\bar{l}_1,\bar{l}_1}, \|Q_j^\nu\|_{\bar{l}_1,\bar{l}_1}: 1 \leqslant j \leqslant N, \nu \in \mathbb{N}\} < \infty$$

and F^ν *maps* $\Sigma(\bar{l}_1)$ *to* $\Delta(\bar{l}_1)$ *for each* $\nu \in \mathbb{N}$.

(ii) *The identity operator* I *on* $\bar{l}_1$ *can be decomposed as*

$$I = F^\nu + \sum_{j=1}^{N} Q_j^\nu, \quad \textit{for } \nu = 1, 2, \ldots.$$

(iii) *Let* $1 \leqslant j \leqslant N$ *and let* $1 \leqslant k \leqslant N$ *with* $k \neq j, j+1$. *Then one has either*

(a) *for any* $\nu \in \mathbb{N}$, Q_j^ν *maps* $l_1(2^{-mx_k-ny_k})$ *to* $l_1(2^{-mx_j-ny_j})$ *and the sequence of norms* $\{\|Q_j^\nu\|_{L_1(k),L_1(j)}\}_{\nu=1}^\infty$, *where* $L_1(i) = l_1(2^{-mx_i-ny_i})$, *converges to* 0 *when* $\nu \to \infty$; *or*

(b) *for any* $\nu \in \mathbb{N}$, Q_j^ν *maps* $l_1(2^{-mx_k-ny_k})$ *to* $l_1(2^{-mx_{j+1}-ny_{j+1}})$ *and the sequence of norms* $\{\|Q_j^\nu\|_{L_1(k),L_1(j+1)}\}_{\nu=1}^\infty$ *converges to* 0 *when* $\nu \to \infty$.

Conditions (i), (ii), (iii) *remain true if we replace the N-tuple* $\bar{l}_1$ *by* $\bar{l}_\infty = \{l_\infty(2^{-mx_j-ny_j})\}_{j=1}^N$.

6. *General compactness results*

In this final section we prove the compactness results for general N-tuples without any approximation condition on them. We shall follow the approach developed in [8]. We start with the J-method.

THEOREM 6.1. *Let* $\Pi = \overline{P_1 \ldots P_N}$ *be the simplex, the unit square, or any admissible polygon, let* $(\alpha, \beta) \in \operatorname{Int} \Pi$ *and* $1 \leqslant q \leqslant \infty$. *Assume that* $\bar{A} = \{A_1, \ldots, A_N\}$ *and* $\bar{B} = \{B_1, \ldots, B_N\}$ *are Banach N-tuples, and that* $T\colon \bar{A} \to \bar{B}$ *is a linear operator such that, for any* $1 \leqslant j \leqslant N$, $T\colon A_j \to B_j$ *is compact. Then*

$$T\colon \bar{A}_{(\alpha,\beta),q;J} \to \bar{B}_{(\alpha,\beta),q;J}$$

is also compact.

Proof. By Theorem 3.3 we have

$$G(\bar{A}) = G[\{l_1(2^{-mx_j-ny_j})\}\,;\, l_q(2^{-\alpha m-\beta n})](\bar{A}) = \bar{A}_{(\alpha,\beta),q;J}$$

and

$$G(\bar{B}) = G[\{l_1(2^{-mx_j-ny_j})\}\,;\, l_q(2^{-\alpha m-\beta n})](\bar{B}) = \bar{B}_{(\alpha,\beta),q;J}.$$

Here (x_j, y_j) are the coordinates of the vertex P_j for each $1 \leqslant j \leqslant N$.

According to Theorem 5.7, there exist sequences of projections $\{F^\nu\}_{\nu\in\mathbb{N}}$ and $\{Q_j^\nu\}_{1\leqslant j\leqslant N,\nu\in\mathbb{N}}$ on the N-tuple $\{l_1(2^{-mx_j-ny_j})\}_{j=1}^N$ satisfying Conditions (i), (ii) and (iii). These operators can be extended in a natural way to the N-tuple $\bar{l}_1 = \{l_1[l_1(2^{-mx_j-ny_j})]\}$ (in the notation of § 3) by defining the image of a summable family (ξ_S) as the family $(Q_j^\nu \xi_S)$ formed by the images of its elements. The new maps (denoted by the same letters) still preserve Properties (i), (ii) and (iii), as can easily be checked.

Let π be the operator introduced in the definition of the functor G. Then clearly

$$T\colon \bar{A}_{(\alpha,\beta),q;J} \to \bar{B}_{(\alpha,\beta),q;J} \quad \text{is compact}$$

if and only if

$$\tilde{T} = T\pi\colon\ l_1[l_q(2^{-\alpha m-\beta n})] \to G(\bar{B}) \quad \text{is compact.}$$

In order to show the compactness of $\tilde{T}$ consider the following diagram of bounded operators (again we write $L_1(j)$ for $l_1(2^{-mx_j-ny_j})$):

$$\begin{array}{ccccc} & & & & l_1[L_1(1)] \xrightarrow{\tilde{T}} B_1 \\ & & & \nearrow & \cdots\cdots\cdots \\ G(\bar{l}_1) \xrightarrow{F^\nu} \bigcap_{j=1}^{N} l_1[L_1(j)] & & \hookrightarrow & & l_1[L_1(j)] \xrightarrow{\tilde{T}} B_j \\ & & & \searrow & \cdots\cdots\cdots \\ & & & & l_1[L_1(N)] \xrightarrow{\tilde{T}} B_N \end{array}$$

The assumption on T and Theorem 4.3 imply that the sequence ($\tilde{T}F^\nu$: $G(\bar{l}_1) \to G(\bar{B})$) is formed by compact operators. we claim that

$$\tilde{T}\colon\ G(\bar{l}_1) = G(\{l_1[l_1(2^{-mx_j-ny_j})]\}) \to G(\bar{B})$$

is the limit of a subsequence $(\tilde{T}F^{\nu'})$ of $(\tilde{T}F^\nu)$.

In fact,

$$\|\tilde{T} - \tilde{T}F^\nu\|_{G(\bar{l}_1),G(\bar{B})} \leq \sum_{j=1}^{N} \|\tilde{T}Q_j^\nu\|_{G(\bar{l}_1),G(\bar{B})}$$

and so to prove our claim, we must show that there is a subsequence (ν') such that, for each $1 \leq j \leq N$, one has

$$(10) \qquad \|\tilde{T}Q_j^{\nu'}\|_{G(\bar{l}_1),G(\bar{B})} \to 0 \quad \text{as } \nu' \to \infty.$$

Fix $1 \leq j \leq N$ and choose any k such that $1 \leq k \leq N$ and $k \neq j, j+1$. It follows from the fact that

$$\sup_\nu \{\|\tilde{T}Q_j^\nu\|_{l_1[L_1(k)],B_k}\} < \infty,$$

that there are a subsequence $(\tilde{T}Q_j^{\nu_1})$ of $(\tilde{T}Q_j^\nu)$ and a bounded sequence $(\xi_{\nu_1}) \subset l_1[l_1(2^{-mx_k-ny_k})]$ such that the sequences

$$(\|\tilde{T}Q_j^{\nu_1}\|_{l_1[L_1(k)],B_k}) \quad \text{and} \quad (\|\tilde{T}Q_j^{\nu_1}\xi_{\nu_1}\|_{B_k})$$

both converge to the same number, say λ.

Since $(Q_j^{\nu_1}\xi_{\nu_1})$ is a bounded sequence in $l_1[l_1(2^{-mx_k-ny_k})]$, the compactness of

$$\tilde{T}\colon\ l_1[l_1(2^{-mx_k-ny_k})] \to B_k$$

implies, by passing to another subsequence if necessary, that $(\tilde{T}Q_j^{\nu_2}\xi_{\nu_2})$ converges to some element, say b, in B_k. Thus $\|b\|_{B_k} = \lambda$.

By Condition (iii), we derive that $(\tilde{T}Q_j^{\nu_2}\xi_{\nu_2})$ converges to 0 in $\Sigma(\bar{B})$. Whence $\lambda = 0$.

Consequently, we can find a subsequence $(\bar{T}Q_j^{\nu'})$ of $(\bar{T}Q_j^{\nu})$ such that

$$\|\bar{T}Q_j^{\nu'}\|_{l_1[L_1(k)],B_k} \to 0 \quad \text{as } \nu' \to \infty$$

for any $1 \leq k \leq N$, $k \neq j, j+1$. Applying Corollary 2.6, we see that such a subsequence satisfies

$$\|\bar{T}Q_j^{\nu'}\|_{G(\bar{l}_1),G(\bar{B})} \to 0 \quad \text{as } \nu' \to \infty,$$

and this proves (10).

So far, we have established that

$$\bar{T}\colon G(\{l_1[l_1(2^{-mx_j-ny_j})]\}) \to G(\bar{B})$$

is compact. To complete the proof, we have only to realize that

$$l_1[l_q(2^{-\alpha m-\beta n})] \hookrightarrow G(\{l_1[l_1(2^{-mx_j-ny_j})]\}).$$

This embedding can be verified by applying the same argument as in [**8**, Lemma 2.1].

Combining Theorem 6.1 and Theorem 1.3, we get

COROLLARY 6.2. *Under the same assumption as in Theorem* 6.1, *the operator*

$$T\colon \bar{A}_{(\alpha,\beta),q;J} \to \bar{B}_{(\alpha,\beta),q;K}$$

is also compact.

In what follows, we will deal with the K-method exclusively.

Given any Banach N-tuple $\bar{A} = \{A_1, \ldots, A_N\}$, we denote by $\bar{A}^0 = \{A_1^0, \ldots, A_N^0\}$ the Banach N-tuple formed by the closures of $A_0 \cap A_1$ in A_j, for $1 \leq j \leq N$.

In the case of the classical real method, it follows easily, from the equivalence between the J- and K-methods, that

$$(A_0, A_1)_{\theta,q} = (A_0^0, A_1^0)_{\theta,q}.$$

In our multidimensional context, such equivalence fails, so it is not clear that

$$\bar{A}_{(\alpha,\beta),q;K} = \bar{A}^0_{(\alpha,\beta),q;K}$$

holds in general. In fact, this is not the case. Consider the 'diagonally equal' 4-tuple $\bar{A} = \{l_1, l_\infty, l_\infty, l_1\}$. Clearly $\bar{A}^0 = \{l_1, c_0, c_0, l_1\}$. But applying the Fernandez K-method (Example 1.2) with $\alpha = \beta = \frac{1}{2}$ and $q = \infty$, one has (see [**10**, Example 1.25])

$$\bar{A}_{(\frac{1}{2},\frac{1}{2}),\infty;K} = l_\infty \neq c_0 = \bar{A}^0_{(\frac{1}{2},\frac{1}{2}),\infty;K}.$$

We end the paper by describing the behaviour of compact operators under the multidimensional K-method. We again use the notation introduced in § 3.

THEOREM 6.3. *Let* $\Pi = \overline{P_1 \ldots P_N}$ *be the simplex, the unit square, or any admissible polygon, let* $(\alpha, \beta) \in \operatorname{Int}\Pi$ *and* $1 \leq q \leq \infty$. *Assume that* $\bar{A} = \{A_1, \ldots, A_N\}$ *and* $\bar{B} = \{B_1, \ldots, B_N\}$ *are Banach N-tuples, and that* $T\colon \bar{A} \to \bar{B}$ *is a linear operator such that, for any* $1 \leq j \leq N$,

$$T\colon A_j \to B_j \quad \text{is compact.}$$

Then

$$T\colon\ \bar{A}^0_{(\alpha,\beta),q;K} \to \bar{B}^0_{(\alpha,\beta),q;K}$$

is also compact.

Proof. First of all, note that, for any $1 \leq j \leq N$,

$$T\colon\ A^0_j \to B^0_j$$

is still compact.

According to Theorem 3.3, we have this time

$$H(\bar{A}^0) = H[\{l_\infty(2^{-mx_j-ny_j})\}\ ;\ l_q(2^{-\alpha m-\beta n})](\bar{A}^0) = \bar{A}^0_{(\alpha,\beta),q;K}$$

and

$$H(\bar{B}^0) = H[\{l_\infty(2^{-mx_j-ny_j})\}\ ;\ l_q(2^{-\alpha m-\beta n})](\bar{B}^0) = \bar{B}^0_{(\alpha,\beta),q;K}.$$

Let $\{F^\nu\}_{\nu\in\mathbb{N}}$ and $\{Q^\nu_j\}_{1\leq j\leq N,\nu\in\mathbb{N}}$ be the sequences of operators on the N-tuple $\{l_\infty(2^{-mx_j-ny_j})\}_{j=1}^N$ given by Theorem 5.7. Extend them in the natural way to the N-tuple $\bar{l}_\infty = \{l_\infty[l_\infty(2^{-mx_j-ny_j})]\}_{j=1}^N$, and observe that Properties (i), (ii), (iii) still hold.

Put $\hat{T} = iT$. We have that

$$T\colon\ \bar{A}^0_{(\alpha,\beta),q;K} \to \bar{B}^0_{(\alpha,\beta),q;K} \quad \text{is compact}$$

if and only if

$$\hat{T} = iT\colon\ G(\bar{A}^0) \to l_\infty[l_q(2^{-\alpha m-\beta n})] \quad \text{is compact.}$$

With the aim of showing that $\hat{T}$ is compact, and letting $L_\infty(j) = l_\infty(2^{-mx_j-ny_j})$, consider the diagram

$$\begin{array}{lll}
A^0_1 \xrightarrow{\hat{T}} l_\infty[L_\infty(1)] & \searrow & \\
\cdots\cdots\cdots\cdots & & \\
A^0_j \xrightarrow{\hat{T}} l_\infty[L_\infty(j)] & \hookrightarrow & \sum_{j=1}^N l_\infty[L_\infty(j)] \xrightarrow{F^\nu} H(\bar{l}_\infty) \\
\cdots\cdots\cdots\cdots & & \\
A^0_N \xrightarrow{\hat{T}} l_\infty[L_\infty(N)] & \nearrow &
\end{array}$$

Applying Theorem 4.1, we get that, for any $\nu \in \mathbb{N}$,

$$F^\nu\hat{T}\colon\ H(\bar{A}^0) \to H(\bar{l}_\infty) \quad \text{is compact.}$$

Next we show that the sequence $(F^\nu\hat{T})$ converges to

$$\hat{T}\colon\ H(\bar{A}^0) \to H(\bar{l}_\infty).$$

Since

$$\|\hat{T} - F^\nu\hat{T}\|_{H(\bar{A}^0),H(\bar{l}_\infty)} \leq \sum_{j=1}^N \|Q^\nu_j\hat{T}\|_{H(\bar{A}^0),H(\bar{l}_\infty)},$$

it suffices to see that each term to the right goes to 0 as $\nu \to \infty$.

Take any $1 \leq j \leq N$ and let $1 \leq k \leq N$ with $k \neq j, j+1$. Since $T\colon\ A^0_k \to B^0_k$ is compact, given any $\varepsilon > 0$, we can find a finite subset $\{a_1, \ldots, a_p\} \subset \Delta(\bar{A}^0)$ with

$\|a_r\|_{A_k^0} \leq 1$, $1 \leq r \leq p$, and such that for any $a \in A_k^0$ with $\|a\|_{A_k^0} \leq 1$, we have

$$\min_{1 \leq r \leq p} \|Ta - Ta_r\|_{B_k^0} \leq \varepsilon/2C,$$

where C is the constant of Condition (i). Hence, given any $a \in A_k^0$ with $\|a\|_{A_k^0} \leq 1$, we obtain

$$\|Q_j^\nu \hat{T}a\|_{l_\infty[L_\infty(k)]} \leq \|Q_j^\nu(\hat{T}a - \hat{T}a_r)\|_{l_\infty[L_\infty(k)]} + \|Q_j^\nu \hat{T}a_r\|_{l_\infty[L_\infty(k)]}$$
$$\leq \tfrac{1}{2}\varepsilon + \|Q_j^\nu\|_{l_\infty[L_\infty(k)], l_\infty[L_\infty(j)]} \|\hat{T}a_r\|_{l_\infty[L_\infty(j)]},$$

and the last term can also be made smaller than $\frac{1}{2}\varepsilon$ by taking ν sufficiently large (Condition (iii)).

Then

$$\|Q_j^\nu \hat{T}\|_{A_k^0, l_\infty[L_\infty(k)]} \to 0 \quad \text{as } \nu \to \infty$$

for all $1 \leq k \leq N$ with $k \neq j, j+1$, and consequently, by Corollary 2.6,

$$\|Q_j^\nu \hat{T}\|_{H(\bar{A}^0), H(\bar{l}_\infty)} \to 0 \quad \text{as } \nu \to \infty.$$

Thus

$$\hat{T}\colon\ H(\bar{A}^0) \to H(\bar{l}_\infty) \quad \text{is compact.}$$

A reasoning similar to the one in [**8**, Lemma 3.1] now gives that

$$H(\bar{l}_\infty) \hookrightarrow l_\infty[l_q(2^{-\alpha m - \beta n})].$$

Whence

$$\hat{T}\colon\ H(\bar{A}^0) \to l_\infty[l_q(2^{-\alpha m - \beta n})]$$

is also compact. This completes the proof.

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