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RIESZ TRANSFORMS, g-FUNCTIONS, AND MULTIPLIERS FOR THE LAGUERRE SEMIGROUP

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Abstract. We introduce the notions of Riesz transforms, g-functions and multipliers associated with the Laguerre differential operator in d dimensions

$$\mathcal{L}_\alpha = \sum_{i=1}^{d} y_i \frac{\partial^2}{\partial y_i^2} + (\alpha_i + 1 - y_i)\frac{\partial}{\partial y_i},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$, $y_i > 0$, and prove that they are bounded in L^p-spaces for $1 < p < \infty$ and weak type 1-1 when $2(\alpha_i + 1) \in \mathbb{N}$, $i = 1, \cdots, d$.

1. Introduction and Preliminaries

The purpose of this paper is to study the boundedness properties of some classical operators in Harmonic Analysis in the context of the multidimensional Laguerre semigroup. The one-dimensional case was studied by Muckenhoupt, see [M1] and [M2] and recently, Dinger [D] proved in higher dimensions that the maximal operator for the Laguerre semigroup is weak-type 1-1. The operators we introduce and study in this paper are: the Riesz transforms, Littlewood-Paley g-functions, and multipliers; see definitions in section (3). We prove that these operators are strong type $p - p$ for $p > 1$ and weak-type 1-1. We also obtain bounds independent of the dimension, see Theorem (3.4). To this end, we use quadratic transformations that relate the Hermite and Laguerre polynomials, see lemma (1.1), proposition (3.1), and lemma (2.1).

The paper is organized as follows. The rest of this section contains some definitions and background for Laguerre and Hermite polynomials that will be

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used later. Section (2) contains some technical lemmas. The main results are proved in section (3). Finally in (4), we study the behavior of the operators introduced when the measure is changed, see Theorem (4.3).

Given $\alpha > -1$, the one-dimensional Laguerre polynomials of type α are

$$L_k^\alpha(y) = \frac{1}{k!} e^y y^{-\alpha} \frac{d^k}{dy^k}(e^{-y} y^{k+\alpha}).$$

Each L_k^α is a polynomial of degree k and appropriately normalized they form a complete orthonormal system in $L^2\left((0,+\infty), \mu_\alpha(y)\, dy\right)$ where $\mu_\alpha(y) = y^\alpha e^{-y}$. The Laguerre differential operator of type α is

$$\mathcal{L}_\alpha = y\frac{d^2}{dy^2} + (\alpha + 1 - y)\frac{d}{dy},$$

and we have

$$\mathcal{L}_\alpha L_k^\alpha(y) = -k L_k^\alpha(y). \tag{1.1}$$

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i > -1$, the multidimensional Laguerre polynomials of type α are tensor product of one-dimensional Laguerre polynomials. Indeed, if $y \in (0, +\infty)^d$ and $k = (k_1, \ldots, k_d)$, where k_i is a non-negative integer, then the multidimensional Laguerre polynomials of type α and degree k are given by

$$L_k^\alpha(y) = L_{k_1}^{\alpha_1}(y_1) L_{k_2}^{\alpha_2}(y_2) \cdots L_{k_d}^{\alpha_d}(y_d),$$

where $L_{k_i}^{\alpha_i}(\cdot)$ is a one-dimensional Laguerre polynomial of type α_i.

The Laguerre differential operator of type α in d dimensions is

$$\mathcal{L}_\alpha = \sum_{i=1}^{d} y_i \frac{\partial^2}{\partial y_i^2} + (\alpha_i + 1 - y_i)\frac{\partial}{\partial y_i}, \tag{1.2}$$

and the differential equation (1.1) generalizes to

$$\mathcal{L}_\alpha L_k^\alpha(y) = -|k| L_k^\alpha(y),$$

where $|k| = k_1 + \cdots + k_d$. The measure $\mu_\alpha(y)dy$, where $\mu_\alpha(y) = y_1^{\alpha_1} \cdots y_d^{\alpha_d} e^{-(y_1 + \cdots + y_d)}$, makes the differential operator $\mathcal{L}_\alpha$ self-adjoint in $L^2\left((0,+\infty)^d, \mu_\alpha(y)\, dy\right)$.

Since we shall use results for the Hermite semigroup, we recall the definition of one-dimensional Hermite polynomials. They are defined by

$$H_0(x) = 1, H_k(x) = e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, k \geq 1.$$

Given a multi-index $k = (k_1, \cdots, k_d)$ with k_i a non-negative integer, and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, the multi-dimensional Hermite polynomial of degree k is defined by

$$H_k(x) = H_{k_1}(x_1) \cdots H_{k_d}(x_d),$$

where $H_{k_i}(\cdot)$ is the one-dimensional Hermite polynomial of degree k_i. The Hermite polynomials are orthogonal on $L^2(\mathbb{R}^d, \gamma_d(x)\,dx)$ where $\gamma_d(x) = \frac{1}{\pi^{d/2}} e^{-|x|^2}$. By L we denote the Ornstein-Uhlenbeck differential operator in $\mathbb{R}^d$ defined by

$$L = \frac{1}{2}\Delta - x \cdot \text{grad}. \tag{1.3}$$

The eigenvalues of L are of the form $\lambda = -|k|$, where $k = (k_1, ..., k_d)$, k_i are non-negative integers and the corresponding eigenfunctions are the multi-dimensional Hermite polynomials $H_k(x)$. The operator L is self-adjoint in $L^2(\mathbb{R}^d, \gamma_d(x)\,dx)$.

In order to define the operators considered in this paper, see section (3), we introduce the following notion of gradient associated with $\mathcal{L}_\alpha$. Let $y = (y_1, \cdots, y_d) \in (0, \infty)^d$, and $F(y) = (F_1(y), \cdots, F_d(y))$. Given $f = f(y)$ we let

$$\text{grad}_\alpha f(y) = \left(\sqrt{y_1}\, \frac{\partial f}{\partial y_1}(y), \cdots, \sqrt{y_d}\, \frac{\partial f}{\partial y_d}(y) \right), \tag{1.4}$$

and

$$\text{div}_\alpha F(y) = \sum_{i=1}^{d} \sqrt{y_i} \left(\frac{\partial F_i}{\partial y_i}(y) + \left(\frac{\alpha_i + 1/2}{y_i} - 1 \right) F_i(y) \right).$$

It is easy to check that

$$\text{div}_\alpha \text{grad}_\alpha = \mathcal{L}_\alpha,$$

and

$$\int_{(0,\infty)^d} \text{div}_\alpha F(y)\, f(y)\, d\mu_\alpha(y) = - \int_{(0,\infty)^d} F(y) \cdot \text{grad}_\alpha f(y)\, d\mu_\alpha(y),$$

for sufficiently smooth functions f and F on $(0, \infty)^d$.

If $f \geq 0$ and $\mathcal{L}_\alpha f = 0$ then

$$\mathcal{L}_\alpha(f^p) = p(p-1) f^{p-2} |\text{grad}_\alpha f|^2,$$

for $1 \leq p < \infty$; see [St, page 49]. This leads us to define the following notion of derivative:

$$\delta_i f(y) = \sqrt{y_i}\, \frac{\partial f}{\partial y_i}(y), \qquad i = 1, \cdots, d.$$

In case that $u = u(y, t)$ we define

$$\text{grad}_\alpha u = (u_t, \delta_1 u, \cdots, \delta_d u), \tag{1.5}$$

and notice that if $u \geq 0$ satisfies the equation

$$\tilde{\mathcal{L}_\alpha} u = u_{tt} + \mathcal{L}_\alpha u = 0,$$

then

$$\tilde{\mathcal{L}_\alpha}(u^p) = p(p-1)u^{p-2}|\mathrm{grad}_\alpha u|^2.$$

We also observe that for the Ornstein-Uhlenbeck operator we have

$$L = \mathrm{div}_\gamma \mathrm{grad}_\gamma,$$

with

$$\mathrm{grad}_\gamma f(x) = \left(\frac{1}{\sqrt{2}} \frac{\partial f}{\partial x_1}(x), \cdots, \frac{1}{\sqrt{2}} \frac{\partial f}{\partial x_d}(x) \right),$$

and

$$\mathrm{div}_\gamma F = \sum_{i=1}^{d} \left(\frac{1}{\sqrt{2}} \frac{\partial F_i}{\partial x_i} - \sqrt{2} x_i F_i \right),$$

where $F(x) = (F_1(x), \cdots, F_d(x))$.

We finish this section showing the connection between Laguerre and Hermite polynomials. Indeed, the following lemma shows that if α has a special form then the Laguerre polynomials of type α can be expressed by means of Hermite polynomials.

Lemma 1.1. *Let L_k^α be a one-dimensional Laguerre polynomial of type α with $\alpha = \frac{n}{2} - 1$ and $x \in \mathbb{R}^n$. Then we have the expansion:*

$$L_k^\alpha(|x|^2) = \sum_{|r|=k} a_r \, H_{2r}(x), \qquad r = (r_1, \ldots, r_n).$$

Remark 1.2. *In one dimension,*

$$L_k^{-1/2}(x^2) = \frac{(-1)^k}{2^{2k}\, k!} H_{2k}(x),$$

see [S], *formula (5.6.1).*

Proof. Note that the expression on the left is a polynomial of degree $2k$ in n variables. The significance of this formula is that the summand on the right does not involve lower order terms of the Hermite polynomials. The proof uses the orthogonality of the Hermite and Laguerre polynomials and integration in polar coordinates. Let $p(x)$ be a monomial of degree less than $2k$. We will show that $L_k^\alpha(|x|^2)$ is orthogonal to $p(x)$ in "Hermite" space, i.e.,

$$\int_{\mathbb{R}^n} L_k^\alpha(|x|^2)p(x)e^{-|x|^2}\,dx = 0.$$

First, note that the integral equals zero if $p(x)$ has any odd power factors because of the evenness of the rest of the integrand and Fubini's theorem which would allow us to isolate the variable with the odd exponent. Hence, we can assume $p(x)$ is of degree $2d$ with $d < k$. Now we evaluate the integral in polar coordinates.

$$\begin{aligned}\int_{\mathbb{R}^n} L_k^\alpha(|x|^2)p(x)e^{-|x|^2}\,dx &= \int_0^\infty \int_{S^{n-1}} L_k^\alpha(r^2)e^{-r^2}r^{2d}p(x')r^{n-1}\,d\sigma(x')dr \\ &= C_p\int_0^\infty L_k^\alpha(r^2)e^{-r^2}r^{2d}r^{n-1}\,dr \\ &= C_p\int_0^\infty L_k^\alpha(t)t^d t^{n/2-1}e^{-t}\,dt \\ &= C_p\sum_{j\le d}\int_0^\infty b_j L_k^\alpha(t)L_j^\alpha(t)t^{n/2-1}e^{-t}\,dt,\end{aligned}$$

by expanding $t^d = \sum_{j\le d} b_j L_j^\alpha(t)$. When $\alpha = \dfrac{n}{2} - 1$, the integrals are zero by the orthogonality of the Laguerre polynomials. □

2. Technical lemmas

Lemma (1.1) can be used to obtain boundedness of operators associated with Laguerre polynomials from the boundedness of the corresponding operators associated with the Hermite polynomials. This is the method used in [D] for the maximal operator of the Laguerre semigroup. To clarify and systematize this fact, we recall the notion of quadratic transformation.

Let $(n_1, \cdots, n_d)$ be a multi-index with n_i positive integers. We define the variables

$$x^i = (x_1^i, \cdots, x_{n_i}^i), \qquad i = 1, \cdots, d,$$

and the quadratic transformation

$$\phi(x^1, \cdots, x^d) = (|x^1|^2, \cdots, |x^d|^2). \tag{2.6}$$

We have the following formula of change of variables.

Lemma 2.1. *Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i = \dfrac{n_i}{2} - 1$ and $n_i \in \mathbb{N}$. Let $f(y_1, \ldots, y_d)$ be a function defined for $y = (y_1, \ldots, y_d) \in (0, +\infty)^d$. The following formula holds*

$$C(d,n) \int_{(0,+\infty)^d} f(y)\mu_\alpha(y)\, dy$$

$$= \int_{\mathbb{R}^{|n|}} f(\phi(x^1, \ldots, x^d)) e^{-(|x^1|^2+\ldots+|x^d|^2)}\, dx^1 \ldots dx^d, \qquad |n| = \sum_{i=1}^{d} n_i. \tag{2.7}$$

Proof. By Fubini's theorem, the right hand side of (2.7) equals

$$\int_{\mathbb{R}^{|n|-n_d}} e^{-(|x^1|^2+\ldots+|x^{d-1}|^2)} \Big(\int_{\mathbb{R}^{n_d}} f(|x^1|^2, \ldots, |x^{d-1}|^2, |x^d|^2) e^{-|x^d|^2} dx^d \Big) dx^1 \ldots dx^{d-1}.$$

By integration in polar coordinates with respect to x^d, the inner integral equals

$$\int_0^\infty t^{n_d-1} \int_{|x^d|=1} f(|x^1|^2, \ldots, |x^{d-1}|^2, t^2) e^{-t^2}\, d\sigma(x^d)\, dt$$
$$= \operatorname{area}(S_{n_d-1}) \int_0^\infty t^{n_d-1} e^{-t^2} f(|x^1|^2, \ldots, |x^{d-1}|^2, t^2)\, dt$$
$$= \operatorname{area}(S_{n_d-1}) \frac{1}{2} \int_0^\infty s^{\frac{n_d}{2}-1} e^{-s} f(|x^1|^2, \ldots, |x^{d-1}|^2, s)\, ds$$
$$= \operatorname{area}(S_{n_d-1}) \frac{1}{2} \int_0^\infty s^{\alpha_d} e^{-s} f(|x^1|^2, \ldots, |x^{d-1}|^2, s)\, ds, \qquad \text{since } \frac{n_d}{2} - 1 = \alpha_d.$$

Hence, by integration in polar coordinates with respect to the remaining variables, (2.7) follows with

$$C(d,n) = 2^{-d} \prod_{i=1}^{d} \operatorname{area}(S_{n_i-1}).$$

□

The following lemma connects operators defined for Laguerre polynomials with operators defined for Hermite polynomials.

Lemma 2.2. *Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i = \dfrac{n_i}{2} - 1$ and $n_i \in \mathbb{N}$. Suppose that T and T' are linear operators defined on polynomials and such that*

$$(Tf)(\phi(x)) = T'(f \circ \phi)(x), \qquad x \in \mathbb{R}^{|n|}.$$

Let B_1 and B_2 be Banach spaces. Then we have

(1) *If $1 < p < \infty$ and*

$$T' : L^p_{B_1}(\mathbb{R}^{|n|}; e^{-(|x^1|^2+\ldots+|x^d|^2)}) \to L^p_{B_2}(\mathbb{R}^{|n|}; e^{-(|x^1|^2+\ldots+|x^d|^2)})$$

is bounded then

$$T : L^p_{B_1}((0,+\infty)^d;\mu_\alpha) \to L^p_{B_2}((0,+\infty)^d;\mu_\alpha).$$

is bounded.

(2) *If*

$$T' : L^1_{B_1}(\mathbb{R}^{|n|}; e^{-(|x^1|^2+\ldots+|x^d|^2)}) \to L^{1,\infty}_{B_2}(\mathbb{R}^{|n|}; e^{-(|x^1|^2+\ldots+|x^d|^2)})$$

then

$$T : L^1_{B_1}((0,+\infty)^d;\mu_\alpha) \to L^{1,\infty}_{B_2}((0,+\infty)^d;\mu_\alpha).$$

Proof. We begin with the proof of 1. Let $|\cdot|_i$ denote the norm in B_i, $i = 1, 2$. By (2.7), we can write

$$\begin{aligned}
&\int_{(0,+\infty)^d} |Tf(y)|_2^p \,\mu_\alpha(y)\,dy \\
&= C(d,n)^{-1} \int_{\mathbb{R}^{|n|}} |Tf(\phi(x^1,\ldots,x^d))|_2^p \, e^{-(|x^1|^2+\ldots+|x^d|^2)}\,dx^1\ldots dx^d \\
&= C(d,n)^{-1} \int_{\mathbb{R}^{|n|}} |T'(f\circ\phi)(x)|_2^p \, e^{-(|x^1|^2+\ldots+|x^d|^2)}\,dx^1\ldots dx^d \\
&\le C\,C(d,n)^{-1} \int_{\mathbb{R}^{|n|}} |f\circ\phi(x)|_1^p \, e^{-(|x^1|^2+\ldots+|x^d|^2)}\,dx^1\ldots dx^d \\
&= C\,C(d,n)^{-1}\,C(d,n) \int_{(0,+\infty)^d} |f(y)|_1^p \,\mu_\alpha(y)\,dy.
\end{aligned}$$

To prove 2, we let

$$E_\lambda = \{y \in (0,+\infty)^d : |Tf(y)|_2 > \lambda\},$$

and estimate the μ_α-measure of this set. From (2.7) it follows that

$$\begin{aligned}
&\int_{(0,+\infty)^d} \chi_{E_\lambda}(y)\mu_\alpha(y)\,dy \\
&= C(d,n)^{-1} \int_{\mathbb{R}^{|n|}} \chi_{E_\lambda}(\phi(x^1,\ldots,x^d)) e^{-(|x^1|^2+\cdots+|x^d|^2)}\,dx^1\ldots dx^d \\
&= C(d,n)^{-1} \int_{\mathbb{R}^{|n|}} \chi_{F_\lambda}(x^1,\ldots,x^d) e^{-(|x^1|^2+\cdots+|x^d|^2)}\,dx^1\ldots dx^d,
\end{aligned}$$

where

$$F_\lambda = \{(x^1,\ldots,x^d) \in \mathbb{R}^{|n|} : |T'(f\circ\phi)(x^1,\ldots,x^d)|_2 > \lambda\}.$$

Since T' is of weak-type (1-1), it follows that

$$\begin{aligned}\mu_\alpha\left(E_\lambda\right) = C(d,n)^{-1}\,\gamma_{|n|}\left(F_\lambda\right) &\le C(d,n)^{-1}\,\frac{C}{\lambda}\|f(\phi(x^1,\dots,x^d))\|_{1,\gamma_{|n|}}\\ &= C(d,n)^{-1}\,C(d,n)\,\frac{C}{\lambda}\|f(y)\|_{1,\mu_\alpha}\\ &= \frac{C}{\lambda}\|f(y)\|_{1,\mu_\alpha},\end{aligned}$$

and the proof is complete. □

3. Main Results

We introduce the Hermite and Laguerre semigroups. If $H_\beta(x)$ is a multidimensional Hermite polynomial of degree β, and $0 < t < \infty$, then the Hermite semigroup is given by

$$N_t H_\beta(x) = e^{-|\beta| t} H_\beta(x).$$

Analogously, if $L_k^\alpha(x)$ is a Laguerre polynomial of degree k and type α, and $0 < t < \infty$, then the Laguerre semigroup is given by

$$M_t^\alpha L_k^\alpha(y) = e^{-|k|t} L_k^\alpha(y).$$

The connection between these semigroups is given by the following proposition, see [D].

Proposition 3.1. *Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i = \dfrac{n_i}{2} - 1$ and $n_i \in \mathbb{N}$; and $f(y_1, \dots, y_d)$ be a polynomial defined for $y = (y_1, \dots, y_d) \in (0, +\infty)^d$. Then*

$$(M_t^\alpha f)(\phi(x)) = N_{t/2}(f \circ \phi)(x)$$

for $x \in \mathbb{R}^{|n|}$; $n = (n_1, \cdots, n_d)$.

Proof. Let L_k^α be a Laguerre polynomial of type $\alpha = (\alpha_1, \cdots, \alpha_d)$, with $k = (k_1, \dots, k_d)$. Given the one-dimensional Laguerre polynomial $L_{k_i}^{\alpha_i}(z)$, let us consider $L_{k_i}^{\alpha_i}(|x^i|^2)$, and note that this is a polynomial in n_i variables of degree $2k_i$. Let $H_\beta(x^i)$ be a multidimensional Hermite polynomial of degree β in the variables

$x^i_1, \dots, x^i_{n_i}$. By lemma (1.1), we can write

$$\begin{aligned}
&(M_t^\alpha L_k^\alpha)(\phi(x^1,\dots,x^d))\\
&= e^{-|k|t} L_k^\alpha(\phi(x^1,\dots,x^d)) = \prod_{i=1}^d e^{-k_i t} L_{k_i}^{\alpha_i}(|x^i|^2)\\
&= \prod_{i=1}^d e^{-k_i t}(\sum_{|r|=k_i} a_r^i\, H_{2r}(x^i)) = \prod_{i=1}^d(\sum_{|r|=k_i} a_r^i\, e^{-k_i t}\, H_{2r}(x^i))\\
&= \prod_{i=1}^d(\sum_{|r|=k_i} a_r^i\,(N_{t/2}H_{2r})(x^i)) = \prod_{i=1}^d N_{t/2}(\sum_{|r|=k_i} a_r^i\, H_{2r})(x^i)\\
&= \prod_{i=1}^d N_{t/2}(L_{k_i}^{\alpha_i}\circ\phi_i)(x^i) = N_{t/2}(\prod_{i=1}^d L_{k_i}^{\alpha_i}\circ\phi)(x)\\
&= N_{t/2}(L_k^\alpha\circ\phi)(x).
\end{aligned}$$

Here, $\phi_i(x^i) = |x^i|^2$. □

Using the principle of subordination we define the Poisson semigroups

$$Q_t f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} N_{t^2/4u} f\, du$$

and

$$P_t^\alpha f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} M^\alpha_{t^2/4u} f\, du.$$

An immediate consequence of Proposition (3.1) is the following lemma connecting these subordinated semigroups.

Lemma 3.2. *Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i = \frac{n_i}{2} - 1$ and $n_i \in \mathbb{N}$; and $f(y_1, \dots, y_d)$ be a polynomial defined for $y = (y_1, \dots, y_d) \in (0, +\infty)^d$. Then*

(1) *We have $(P_t^\alpha f)(\phi(x)) = Q_{t/\sqrt{2}}(f\circ\phi)(x)$ for $x \in \mathbb{R}^{|n|}$; $n = (n_1, \cdots, n_d)$.*
(2) *For each $l \in \mathbb{N}$*

$$\partial_u^l P_u^\alpha f(\phi(x))|_{u=t} = \frac{1}{2^{l/2}}\, \partial_u^l Q_u(f\circ\phi)(x)|_{u=t/\sqrt{2}}$$

for $x \in \mathbb{R}^{|n|}$; $n = (n_1, \cdots, n_d)$.

We are now ready to give the notions of Riesz transforms, g-functions, and multipliers for the Laguerre case. Using the formula

$$s^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts}\, t^a \,\frac{dt}{t},$$

where $a > 0$ and $s > 0$, we define the powers of a second order differential operator $L \geq 0$ (on an appropriate class of functions) by the formula

$$L^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty T_t f(x)\, t^a \,\frac{dt}{t},$$

where T_t is the infinitesimal generator of L. We shall use this formula with the operators $-L$ and $-\mathcal{L}_\alpha$, as in (1.2) and (1.3). In these cases the class of functions f's considered are polynomials.

The remarks and the definition of gradient made in section (1) lead us to define the Riesz-Laguerre transform by

$$\mathcal{R}_\alpha = \operatorname{grad}_\alpha (-\mathcal{L}_\alpha)^{-1/2},$$

where $\operatorname{grad}_\alpha$ is given by (1.4). Writing the Riesz-Laguerre transform in coordinates yields

$$\mathcal{R}_{\alpha,i} = \delta_i (-\mathcal{L}_\alpha)^{-1/2} \qquad i = 1, \cdots, d.$$

The Littlewood-Paley function g_α is defined by

$$g_\alpha f(y) = \left(\int_0^\infty |t \operatorname{grad}_\alpha P_t^\alpha f(y)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\operatorname{grad}_\alpha$ is given by (1.5). Also, the multipliers of Laplace transform type for the Laguerre semigroup are given by

$$m(\mathcal{L}_\alpha) f(y) = -\mathcal{L}_\alpha \int_0^\infty M_s^\alpha f(y)\, a(s)\, ds,$$

for some function $a(s)$ uniformly bounded on $(0, \infty)$. Notice that these definitions are consistent with the Hermite case, see [GC-M-Sj-T], [G], and [St].

We shall denote by

$$\mathcal{R}_H, \qquad g_H, \qquad \text{and} \qquad m_H(L),$$

the Riesz-Hermite transforms, the g-function and the multipliers in the Hermite case, respectively. The connection between these transformations for the corresponding semigroups is given by the following lemma.

Lemma 3.3. *Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $\alpha_i = \frac{n_i}{2} - 1$ and $n_i \in \mathbb{N}$; and $f(y_1, \ldots, y_d)$ be a polynomial defined for $y = (y_1, \ldots, y_d) \in (0, +\infty)^d$. Then*

(1) *If* $a > 0$ *then* $(-\mathcal{L}_\alpha)^{-a} f(\phi(x)) = 2^a\,(-L)^{-a}(f \circ \phi)(x)$*;*
(2) $|\mathcal{R}_\alpha f(\phi(x))|_{\ell_2^d} = 2^{-1/2}\,|\mathcal{R}_H(f\circ\phi)(x)|_{\ell_2^{|n|}}$*;*
(3) $g_\alpha f(\phi(x)) = 2^{-1/2}\,g_H(f\circ\phi)(x)$*;*
(4) *If* $m(\mathcal{L}_\alpha)$ *is a Laplace transform type Laguerre multiplier for the function* $a(\cdot)$ *then*
$$m(\mathcal{L}_\alpha) f(\phi(x)) = m_H(L)(f\circ\phi)(x),$$
where $m_H(L)$ *is a Laplace transform type Hermite multiplier for the function* $a(2\cdot)$*;*

for $x \in \mathbb{R}^{|n|}$*;* $n = (n_1, \cdots, n_d)$*.*

We now state the main result in the paper.

Theorem 3.4. *Let* $\alpha = (\alpha_1, \cdots, \alpha_d)$ *with* $\alpha_i = \dfrac{n_i}{2} - 1$ *and* $n_i \in \mathbb{N}$*. The Riesz-Laguerre transform* $\mathcal{R}_\alpha$*, the Littlewood-Paley function* g_α *and the Laplace transform type Laguerre multipliers are all bounded in* $L^p\big((0,+\infty)^d, \mu_\alpha(x)\,dx\big)$*,* $1 < p < \infty$*, and weak-type (1-1). Moreover, if* $1 < p < \infty$ *then the type constants for the Riesz-Laguerre transform and the Littlewood-Paley function* g_α *are independent of the dimension* d*.*

Proof. For the Hermite case, the Riesz transform and the Littlewood-Paley function are bounded in $L^p(\gamma)$ with constants independent of the dimension, see [Gn], [G] and [P]. The weak type 1-1 of the Riesz-Hermite transforms is proved in [F-G-Sc]. For the Littlewood-Paley g-function in the Hermite case a similar result was proved by Scotto, [Sc]. The boundedness in L^p for the multipliers in the Hermite and Laguerre is contained in [St]. The weak-type 1-1 for the Hermite multipliers was proved in [GC-M-Sj-T]. The proof of the theorem then follows by combining these results with lemmas (2.2) and (3.3).

□

4. Some remarks about weighted inequalities

We study the behavior of the operators previously defined in other measure spaces. We shall use the following theorem due to Rubio de Francia, see [GC-R, page 554].

Theorem 4.1. *Let* (X, μ) *be a measure space,* G *a Banach space, and* T *a sublinear operator from* G *into* $L^s(X)$*, which satisfies for some* $s < p$*, the following inequality*
$$\|(\sum_j |Tf_j|^p)^{1/p}\|_{L^s(X)} \le C_{p,s}(\sum_j \|f_j\|_G^p)^{1/p}$$

where $C_{p,s}$ is a constant depending on p and s. Then there exists a positive function u such that $u^{-1} \in L^{\frac{s}{p-s}}(X)$ and

$$\int_X |Tf(x)|^p u(x) d\mu(x) \leq \|f\|_G.$$

A simple consequence this theorem is the following.

Corollary 4.2. *Let T be a sublinear operator such that*

$$\mu_\alpha\{y : (\sum_j |Tf_j(y)|^p)^{1/p} > \lambda\} \leq \frac{C}{\lambda} \int_{(0,\infty)^d} (\sum_j |f_j(y)|^p)^{1/p} \, d\mu_\alpha(y). \tag{4.1}$$

Then for any v such that $\int_{(0,\infty)^d} v^{-\frac{1}{p-1}}(y) d\mu_\alpha(y) < \infty$, and $s < p$, there exists a positive function u such that $u^{-1} \in L^{\frac{s}{p-s}}(X)$ and

$$\int_{(0,\infty)^d} |Tf|^p u(y) d\mu_\alpha(y) \leq \int_{(0,\infty)^d} |f|^p v(y) d\mu_\alpha(y).$$

Proof. Since $\mu_\alpha((0,\infty)^d)$ is finite and $s < p$, it follows from Tchebyshev's inequality that

$$\begin{aligned}
&\|(\sum_j |Tf_j|^p)^{1/p}\|_{L^s(\mu_\alpha)} \\
&\leq C_s \sup_{\lambda>0} \lambda \, \mu_\alpha\{y : (\sum_j |Tf_j(y)|^p)^{1/p} > \lambda\} \\
&\leq \int_{(0,\infty)^d} (\sum_j |f_j(y)|^p)^{1/p} \, d\mu_\alpha(y) \\
&\leq C_s \left(\int_{(0,\infty)^d} \sum_j |f_j(y)|^p v(y) \, d\mu_\alpha(y) \right)^{1/p} \left(\int_{(0,\infty)^d} v^{-\frac{1}{p-1}}(y) \, d\mu_\alpha(y) \right)^{1/p'} \\
&\leq C_s \left(\int_{(0,\infty)^d} \sum_j |f_j(y)|^p v(y) \, d\mu_\alpha(y) \right)^{1/p} \\
&= C_s (\sum_j \|f_j\|^p_{L^p(vd\mu_\alpha)})^{1/p}.
\end{aligned}$$

Therefore, the hypotheses of Theorem (4.1) are satisfied with $G = L^p(vd\mu_\alpha)$, and the corollary follows. □

Finally, Corollary (4.2) yields the following result.

Theorem 4.3. *Let* $\alpha = (\alpha_1, \cdots, \alpha_d)$ *with* $\alpha_i = \frac{n_i}{2} - 1$ *and* $n_i \in \mathbb{N}$. *Then the maximal operator for the Laguerre semigroup, the Riesz-Laguerre transforms and the Littlewood-Paley function* g_α *have a bounded extension from* $L^1_{\ell^p}(d\mu_\alpha)$ *into weak-*$L^1_{\ell^p}(d\mu_\alpha)$, $1 < p < \infty$. *Moreover, if* $v^{-1/(p-1)} \in L^1(\mu_\alpha)$ *then there exists a function* u *such that the Riesz-Laguerre transform and the Littlewood-Paley function* g_α *are bounded from* $L^p(v\mu_\alpha)$ *into* $L^p(u\mu_\alpha)$, $1 < p < \infty$.

Proof. The inequality (4.1) holds when T is either the Riesz-Hermite transform or the function g in the Hermite case, and μ_α replaced by the Gaussian, see [H-T-V]. Therefore from (2.7) and Lemma (3.3) we have that (4.1) holds for the Riesz-Laguerre transform and the Littlewood-Paley function g_α. Therefore the hypothesis in Corollary (4.2) holds and hence the theorem follows. □

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