On Solving the Multirotational Timber Harvesting Problem with Stochastic Prices: A Linear Complementarity Formulation

Margaret Insley and Kimberly Rollins

This article develops a two-factor real options model of the harvesting decision over infinite rotations assuming a known stochastic price process and using a rigorous Hamilton–Jacobi–Bellman methodology. The harvesting problem is formulated as a linear complementarity problem that is solved numerically using a fully implicit finite difference method. This approach is contrasted with the Markov decision process models commonly used in the literature. The model is used to estimate the value of a representative stand in Ontario's boreal forest, both when there is complete flexibility regarding harvesting time and when regulations dictate the harvesting date.

Key words: linear complementarity problem, Markov decision process, mean reversion, optimal harvesting, real options.

The forest economics literature has long dealt with the problem of optimal harvesting under uncertainty. An overview is provided in recent bibliographies by Newman, and Brazee and Newman. A thirty-year-long strand of this literature emphasizes the importance of valuing managerial flexibility in the context of irreversible harvesting decisions when forest product prices are volatile relative to harvesting costs (Hool; Lembersky and Johnson). Failure to include the value of management options where they exist will result in an incorrect valuation of a forestry investment. Because the formulation and modeling of timber harvesting problems to accurately incorporate the value of managerial flexibility is unlikely to result in closed-form solutions, forest economics models have used a number of approaches to solve complex optimal harvesting problems under stochastic prices, including Markov decision process (MDP) models and simulation. More recently, these approaches have increasingly drawn from the burgeoning finance literature on the valuation of financial and real options (Dixit and Pindyck; Trigeorgis). These approaches explicitly incorporate into the value of the resource any opportunities for managers to adjust harvesting plans in response to stochastic events as they unfold—opportunities that may be thought of as embedded options.

The contributions of the real options literature are in the development of powerful decision models and solution techniques that can offer improvements in accuracy over methods based on MDP models, a standard in the forest economics literature; and in the formulation and interpretation of the decision problem by explicitly recognizing the parallels between financial options, such as call options on a stock, and real options, which refer to the opportunities to acquire real assets. We can view the opportunity to harvest a stand of trees as a real option, similar to an American call option, which can be exercised at any time. The exercise price is the cost of harvesting the trees and transporting them to the point of sale. The option to choose the optimal harvest time, based on wood volume and price, and the option to abandon the investment if wood prices are too

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low are embedded in the tree harvesting opportunity. A real options approach focuses on the importance of options embedded in the harvesting decision.[1]

In their 1999 review article, Brazee and Newman refer to the rather slow development of the options approach to forestry due to its mathematically demanding nature. In this article, we make a methodological contribution to implementing real options approaches by demonstrating the numerical solution of the multirotational optimal harvesting problem to a specified degree of accuracy using a technique that can handle a fairly general class of specifications for price uncertainty. The multirotational harvesting problem represents a path-dependent option, which is significantly more complex than the single rotation problem. Thus, this article extends the single rotation analysis of Insley in a nontrivial manner.

An important benefit of the approach presented in this article is the assurance that the solution obtained is accurate given the chosen inputs. The particular numerical procedure used in solving the option value problem can have a large impact on the estimated value of an uncertain investment—a point emphasized in the finance literature (Wilmott; Wilmott, Dewynne, and Howison). In instances when accuracy is important, the approach presented in this article will represent a significant improvement over other techniques in the literature for dealing with price uncertainty. In a policy-making context, accuracy will be important in decisions that involve tradeoffs—such as in an evaluation of the benefits versus costs of a regulation that will restrict forest owners' ability to freely determine the optimal harvest schedule, but that would provide environmental benefits.

We develop a model of the harvesting decision at the stand level assuming stochastic prices and deterministic volume. The objective is to maximize the net present value of the investment over an infinite stream of future rotations, where price, P, is assumed to follow an Ito process,

$$(1) \qquad dP = a(P, t)\,dt + b(P, t)\,dz.$$

In equation (1), $a(P, t)$ and $b(P, t)$ represent known functions and dz is the increment of a Wiener process. For this article, we assume mean-reverting prices (first-order autoregressive), but our technique could easily handle geometric Brownian motion or some other process with alternate specifications of $a(P, t)$ and $b(P, t)$, such as a table of discrete parameter values.

The harvesting problem is specified as a linear complementarity problem (LCP), and is solved numerically using a fully implicit finite difference approach (Wilmott, Dewynne, and Howison; Tavella). This is a rigorous technique for which the existence and uniqueness of solutions follow from a large mathematics literature. It has guaranteed convergence with easily determined error bounds. Wilmott, Dewynne, and Howison discusses American options in terms of variational inequalities and linear complementarity formulations. Proofs of the existence and uniqueness of solutions are available in the literature (Elliott and Ockendon; Friedman; Kinderlehrer and Stampacchia). To our knowledge this technique has not been used previously to solve a multirotation optimal harvesting problem with a general stochastic price process (as in equation (1)) and when land value is determined endogenously.

One of the purposes of this article is to distinguish among different approaches that can be used to incorporate managerial flexibility into optimal harvest decisions, and to describe the contributions of real options approaches to solving forestry economics problems. We derive theoretically the relationship between the MDP and LCP approaches used to solve the stochastic, optimal harvesting problem. We demonstrate that the numerical solution of the LCP provides guaranteed error bounds and permits the use of a finer level of resolution than is typically used in solving MDP models. We show that the improvement in accuracy offered by this finer level of resolution can be very significant. We apply the model to data from Ontario, Canada to estimate the opportunity costs of harvest restrictions such as those that are imposed to maintain an even annual flow of timber from public forest lands.

In the next section we provide a fairly detailed literature review that traces developments in forestry economics in dealing with optimal harvesting with uncertain prices. We situate our article within the literature and specify in more detail its particular contributions.

[1] Another innovation in the finance literature relevant to forest economics is contingent claims analysis, which allows one to avoid the conceptual difficulties of specifying an exogenous discount rate. However, for a storable commodity it is still necessary to estimate a "convenience yield" or market price of risk (Trigeorgis, Dixit and Pindyck). We do not use contingent claims analysis in this article.

Literature Review and Modeling Approaches

Almost thirty years ago, Lembersky and Johnson illustrated that a simplistic Faustmann-type approach to estimating stand value ignores managerial flexibility and hence undervalues the resource. Speaking in the context of optimal actions for a forest manager faced with uncertainty in product prices, these authors point out that "it is not advisable to predetermine the specific management action to carry out at each of the future time points at which a decision will be made. A predetermined action can turn out to be inappropriate for stand and market conditions at implementation" (Lembersky and Johnson, p. 109). The decision maker who has the ability to make decisions based on observed outcomes of random variables such as price and growth over time, instead of having to follow a prescribed rule based on expected values, can decide to harvest early to take advantage of an upswing in prices or delay harvesting if prices are depressed.[2]

Analytical Models and Closed-Form Solutions

A number of earlier studies focused on general theoretical implications of the problem of uncertainty in harvesting decisions, using stylized analytical models with closed-form solutions. These models necessarily greatly abstract from the reality of most forest harvesting problems in order to obtain analytical solutions.

For example, Brock, Rothschild, and Stiglitz; Brock and Rothschild; and Miller and Voltaire consider the harvesting problem in the general context of stochastic capital theory and optimal stopping problems. This approach focuses on deriving analytical solutions and comparative static results to determine how the value of the asset varies with the parameters that describe the stochastic growth process of the state variables. The majority of these is single rotation models, while Miller and Voltaire extend the Brock, Rothschild, and Stiglitz capital theory model to the multirotation case and develop a barrier rule for stochastic revenue. These models treat revenue as the stochastic variable; the price and quantity state variables are not separately distinguished. Lohmander (1988a) derives comparative static results for continuous harvesting under price and volume growth uncertainty.

Willassen uses the theory of stochastic impulse control to derive an explicit solution to the stochastic multirotational optimal harvesting problem with revenue as the state variable. Drift and diffusion parameters of the stochastic process are independent of time. Sødal also derives the closed-form solution to the same problem using a simplified approach based on Dixit, Pindyck, and Sødal. While allowing for closed-form solutions, these models are not practicable for more applied problems where price and quantity must be separately modeled.

Recent research in this vein focuses on adding different sources of uncertainty such as stochastic forest growth (Alvarez) and a stochastic interest rate (Alvarez and Koskela).

Models Where Prices Follow Geometric Brownian Motion

Earlier articles to model price and volume separately typically assumed price could be characterized by geometric Brownian motion. Clarke and Reed, and Reed and Clarke derive an optimal harvesting rule when price and volumetric growth are random variables. Price follows geometric Brownian motion whereas growth in volume is a function of stand age plus random Brownian motion. Under their (myopic look-ahead) rule the age at which a stand is harvested becomes a random variable that is independent of the absolute level of timber prices. This is a consequence of assuming geometric Brownian motion and ignoring harvesting costs. Insley, and Yin and Newman (1995) demonstrate that incorporating harvesting and management costs into the model would imply that the optimal harvest time is no longer independent of price.

Using the Clarke and Reed results, but incorporating land rent costs as deterministic, Yin and Newman (1997) compare the optimal harvest time where growth and price are stochastic to what would be proscribed by Faustmann or maximum sustained yield rules. In reality land rent should reflect the value of the bare land, which would equal the expected discounted net benefit from optimally managing the timber stand forever. Thus land rent should be endogenous, determined jointly with the value of the harvesting opportunity, although this considerably complicates the analysis.

[2] This is true if prices follow a stochastic process that reverts to some mean over time. If prices follow geometric Brownian motion, then the value of flexibility comes from avoiding uneconomic harvests.

Thomson provides an example of one of the earlier uses of a real options approach in a forestry application. Thomson determines land rent endogenously assuming stumpage prices follow geometric Brownian motion. He compares stand value and rotation ages (as a function of price) with a fixed price Faustmann model. Thomson solves his model using a lattice method (a binomial tree), which is commonly used in finance and is in fact a simple version of an explicit finite difference scheme. Wilmott discusses the advantages of finite difference methods, such as the one used in this article, over the binomial tree. In general, finite difference schemes offer more flexibility than binomial trees to handle complex problems. For example, with mean-reverting prices a trinomial tree with nonstandard branching would be employed, as described in Hull. As with other explicit methods, the binomial tree approach suffers from stability constraints that restrict the timestep size used in the numerical solution and can make solution very slow. Finally, Coleman, Li, and Verma demonstrate that the smooth pasting condition is only approximately satisfied with binomial trees.[3]

Morck, Schwartz, and Strangeland more explicitly bring to their model insights from financial real options methods. They use a contingent claims approach to determine the optimal harvesting rate for a firm with a ten-year lease on a mature forest. This is a problem of inventory management where growth in inventory is assumed to follow Brownian motion with a drift, and timber prices are assumed to follow geometric Brownian motion.

Models That Incorporate Alternate Price Processes

In general, the assumption of geometric Brownian motion makes solution of the tree harvesting problem more tractable. If management and harvesting costs are ignored, the problem can be solved analytically. However, the assumption of geometric Brownian motion may not be realistic for many commodities over the long term because of the implication that the expected price level and variance will rise over time without bound. Alternatively, price may be modeled as some sort of stationary process. The precise specification of the price process can have a large effect on the estimated value and optimal timing of a resource investment (Insley, Sarkar).

Some researchers of the late 1980s and early 1990s modeled price as an identically distributed random variable. For example, Lohmander (1988b) investigated the effect of stochastic prices on optimal harvesting in a single rotation problem assuming price could be modeled as a random draw from a uniform distribution. Brazee and Mendelsohn solved a multirotation model with price represented as a random draw from a normal distribution. Under this assumption the timber price in any given period is statistically independent of its level in any other period. Haight (1990, 1991) presents two other examples in this vein. Although the assumption of serially uncorrelated prices is not realistic, these articles provide useful intuition of the impact of stochastic, stationary prices. More recently stationarity or mean reversion is typically captured by assuming price follows an Ornstein–Uhlenbeck sort of process (see Dixit and Pindyck). An Ornstein–Uhlenbeck process (as specified in Section 3, equation (2)) has independent increments, meaning that the change in price between two consecutive periods is independent of the change between any two other consecutive periods.

With mean-reverting prices, the optimal rotation problem must be solved numerically. Models based on MDPs represent one possible approach. MDPs have become a standard in the forest economics literature for incorporating ecological and market risks into harvest decisions. This approach models stochastic state variables in discrete time and computes matrices of transition probabilities that reflect the probability of moving from one state to another, conditional on management decisions. The transition matrix is typically estimated by simulation, and various techniques are used to determine optimal decisions based on the possibilities offered by the transition matrix. These techniques include the policy improvement algorithm, linear programming, and successive approximation, and are described in the operations research literature (e.g., Hillier and Lieberman).

The MDP approach is somewhat limited by the use of the transition matrix, which expands dramatically as the number of possible outcomes of a stochastic variable increases. This is typically handled by grouping stochastic outcomes, such as prices, into a small number of categories, such as "high," "medium," and

[3] The smooth pasting condition must be satisfied at the early exercise of an American-type option—in our case when it is optimal to harvest a stand of trees. Dixit and Pindyck explain the smooth pasting condition.

"low." As will be shown below, the LCP approach used in this article does not require simulations to calculate transition probabilities, and stochastic variables can be specified in much finer detail. This allows for finer degrees of resolution, and therefore greater accuracy in the determination of decision criteria. However, it may be noted that if there are many stochastic factors, MDP models (or simulation, discussed below) may be the only computationally feasible method.

Examples of MDP models are numerous (Lembersky and Johnson; Norstrom; Kao; Teeter and Caulfield; Kaya and Buongiorno; Lin and Buongiorno; Buongiorno). Lin and Buongiorno incorporate natural catastrophes as well as diversity of tree species and tree size. Buongiorno provides useful insight by interpreting Faustmann's formula as a special case of a MDP model in which the probability of moving from one state to another is equal to unity.

Plantinga,[4] Haight and Holmes, and Gong solve the optimal harvesting problem with mean-reverting prices using a Markov transition matrix and a discrete stochastic dynamic programming algorithm. For simplicity, these articles treat the value of the bare land as deterministic. Brazee, Amacher, and Conway, in examining the benefits of adaptive management when prices are mean reverting, note that the gains vary directly with the level of mean reversion.

Gong compares a stochastic dynamic programming approach based on the use of a MDP, with a simulation method for finding the optimal harvest policy. He notes the difficulty of determining accurate stand values with simulation. Longstaff and Schwartz, and Andersen have recently proposed two methods to adapt a simulation approach to valuing an American option. Hull provides a useful summary of these methods. Wilmott discusses in more detail the difficulties with using simulation approaches to value an American type option, that is, one for which early exercise may be optimal.

The real options literature provides an alternative to MDP approaches and simulation in formulating the problem in terms of a partial differential equation, which can be solved numerically using techniques from the large literature on numerical analysis. Saphores, Khalaf, and Pelletier demonstrate the use of Galerkin's method (a finite-element method) to solve the problem of whether to preserve or harvest a stand of old growth forest when lumber prices follow geometric Brownian motion with jumps. Insley formulates the harvesting problem with mean-reverting prices for a single rotation as an LCP, and demonstrates a numerical solution using a fully implicit finite difference scheme.

[4] Plantinga describes the conceptual relationship between the notion of option value and the previous literature on harvesting under uncertainty.

Characterizing the Price Process for Timber

There is a large literature examining the time path of commodity prices, and this continues to be an area of active research. It has been suggested that some sort of mean-reverting process provides a better description of the price path of many commodities (see Lund, Bessembinder et al. Hassett and Metcalf, Schwartz.) As noted by Schwartz, in an equilibrium setting we would expect that when prices are relatively high, supply will increase since higher-cost producers of the commodity will enter the market putting downward pressure on prices. Conversely, when prices are relatively low, the higher cost producers will exit the market putting upward pressure on prices. Schwartz examines the spot prices of oil, copper, and gold. He finds strong mean reversion for oil and copper. Bessembinder et al. use the term structure of futures prices to test whether investors anticipate mean reversion in spot asset prices. They find a large degree of mean reversion in crude oil and agricultural commodities, and a lesser degree of mean reversion in metals.

Unfortunately, it is difficult to conclude definitively that the price of any particular commodity exhibits mean reversion or possesses a unit root and hence is nonstationary. Many different tests exist, none of which has been shown analytically to be uniformly most powerful (e.g., Ahrens and Sharma).

A number of studies have examined the statistical properties of stumpage prices in various markets and have obtained mixed results. Several researchers have examined pine sawtimber stumpage prices in the southern United States. Haight and Holmes, Hultkrantz (1993) and Yin and Newman (1995, 1996, 1997) all find evidence that supports stationary, autoregressive models. Prestemon extends the data series used in Hultkrantz (1993), and Yin and Newman (1996) and improves upon the statistical tests used. He finds that most of the monthly series contain nonstationary as well as stationary components and that quarterly prices are closer to pure nonstationary processes. Brazee, Amacher, and Conway test price series for pine and hardwood in Virginia and finds the unit root hypothesis is rejected for the former, but not the latter. Hultkrantz (1995) examines the behavior of timber rents in Sweden over a seventy-nine year time span and accounts for a structural break in the price level using a Perron test. He rejects the unit root hypothesis for his data series.[5] Saphores, Khalaf, and Pelletier find evidence of both jumps and ARCH effects in stumpage prices in the U.S. Pacific Northwest.

One theoretical argument that has been used in favor of a random walk process is that it implies prices that are consistent with an informationally efficient timber market. However, McGough, Plantinga, and Provencher show that stationary serially correlated prices can arise in an informationally efficient timber market even when market shocks are independent and identically distributed.

The choice of price process in modeling the optimal harvesting decision will continue to be the subject of ongoing research. A challenge of resource economists is to develop models and solution algorithms that handle various assumptions regarding price, depending on the circumstances of a particular market.

A Multirotational Real Options Model

This article extends the model of Insley to a multirotation framework with the bare land value determined endogenously. Much of the previous cited literature may be thought of in terms of different approaches to solving the LCP. Unlike the single rotation problem, the multirotation case represents a "path-dependent option." A path-dependent option is one whose value depends on the history of an underlying state variable, not just on its final value. For the multirotational optimal harvesting problem, the value of the stand today depends on the quantity of lumber which itself depends on when the stand was last harvested. Path dependency significantly complicates the solution of the valuation problem (see Wilmott).

The importance of solving the complete multirotational problem will vary on a case-by-case basis. In areas where trees are fast growing and rotations are fairly short, the harvesting decision proscribed by the multirotational problem will be expected to be quite different from the single rotation case. As another example, if recreational value of the standing forest depends on population growth, then the value of the harvesting opportunity will be time dependent, making the multirotational specification important. We will examine the empirical significance of solving the multi- versus single rotation problem in an empirical example.

The solution of the LCP can be directly linked to the transition probability matrix estimated in MDP models. We will show that the MDP approach is, in fact, an indirect method of solving the LCP and that it is unnecessary to use simulation to calculate transition probabilities. The transition probability densities are the solutions of the forward Kolmogorov equations, which are embedded in the solution of the LCP. The explicit connection between the MDP model and the numerical solution of the LCP is discussed in more detail in Appendix B.

Our empirical application examines the cost of harvesting restrictions on the value of harvesting a stand in Ontario's boreal forests. It is only by correctly modeling the impact of stochastic prices on the optimal harvesting decision that the impact of regulatory restrictions can be fully described.

Formulation of the Model

The tree harvesting decision applicable to a publicly owned forest is modeled from the point of view of a social planner. Hence, taxes and stumpage payments are ignored. The price of timber sold to the mill is assumed to follow a known stochastic process. The value of the stand of trees is estimated assuming the harvesting decision will be determined optimally in the future whatever the price path turns out to be. In a world without taxes and stumpage payments, the estimated value at the beginning of the first rotation is the maximum amount that a private firm would be willing to pay for the right to harvest the trees at some time in the future, providing the firm has complete flexibility to determine the harvest date and that markets exist for the logs.

The mean-reverting price process is a special case of the general Ito process (equation (1)). We specify the process as follows:

$$(2) \qquad dP = \eta(\bar{P} - P)\,dt + \sigma P\,dz$$

[5] Perron found that the exclusion of a break in trend can bias the augmented Dickey–Fuller test and the Leybourne and McCabe test toward acceptance of a unit root.

where P is the price of saw logs, η is the mean reversion parameter, σ is the constant variance rate, and dz is an increment of a Wiener process. According to equation (2), price reverts to a long-run average of $\bar{P}$. The variance rate grows with P, so that the variance is zero if P is zero.[6]

The age of the stand, or time since the last harvest, α is given as

$$(3) \qquad \alpha = t - t_h$$

where t is the current time and t_h is the time of the last harvest. Wood volume is assumed to be a deterministic function of age

$$(4) \qquad Q = g(\alpha).$$

Age is used as a state variable, along with price, P. It follows that

$$(5) \qquad d\alpha = dt.$$

The decision to harvest the stand of trees can be formulated as an optimal stopping problem where the owner must decide in each period whether it is better to harvest immediately or delay until the next period. This decision process can be expressed as a Hamilton–Jacobi–Bellman equation,

$$(6) \qquad V(t, P, \alpha) = \max\{(P - C)Q + V(t, P, 0); A\Delta t + (1 + \rho\Delta t)^{-1} \times E[V(t + \Delta t, P + \Delta P, \alpha + \Delta\alpha)]\}$$

where E is the expectation operator, V is the value of the opportunity to harvest, C is the per unit harvesting cost, A is the per period amenity value of standing forest less any management costs, and ρ is the annual discount rate.

The first expression in the braces ({ }) represents the return if harvesting occurs in the current period, t. It includes the net revenue from harvesting the trees plus the value of the land after harvesting, $V(t, P, 0)$. This is the value that could be attained if the land were sold subsequent to the harvest, assuming that the land will remain in forestry. This value is ignored in the single rotation problem.

The second expression in the braces is the *continuation region* and represents the value of delaying the decision to harvest for another period. It includes any amenity value of the standing forest, such as its value as a recreation area, less any forest management costs, A. It also includes the expected value of the option to harvest in the next period, discounted to the current period.

In the empirical application that follows, amenity value is not considered. However, it would be easy to include some expression for amenity value as a function of stand age, population growth, or some other variable. Investment values and harvesting rules would be changed accordingly. If A were expressed as a function of population growth, then V would be dependent on calendar time, as well as stand age, and we would no longer be characterizing a steady-state solution.

Following standard arguments (Dixit and Pindyck; Wilmott, Dewynne, and Howison) we can derive a partial differential equation that describes V in the continuation region

$$(7) \qquad V_t + \tfrac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + V_\alpha = 0.$$

In contrast to the single rotation problem, we now have the term V_α in the partial differential equation.

The full optimal stopping problem can be formulated as an LCP, which is equivalent to the optimal stopping problem of equation (6) (Wilmott, Dewynne, and Howison; Tavella). T denotes the terminal time. Let τ be defined as time remaining in the option's life, that is, $\tau \equiv T - t$. Rearranging equation (7) and substituting τ for t, we define an expression, HV, as follows:

$$(8) \qquad HV \equiv \rho V - \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P) \times V_P + A + V_\alpha - V_\tau\right].$$

In equation (8), ρV represents the return required on the investment opportunity for the rational investor to continue to hold the option. The expression within brackets ([ ]) represents the actual return over the infinitesimal time interval dt. The actual return has terms reflecting how V changes with changes in P and α. It also includes the flow of amenity value less management costs, A.

[6] This format is more appealing than the simple Ornstein–Uhlenbeck process in which the variance rate is $\sigma\, dz$. In the simple Ornstein–Uhlenbeck process, as price becomes small, the constant volatility could cause prices to become negative.

Then, the LCP is given as

$$(9) \quad (i) \quad HV \geq 0$$
$$(ii) \quad V(\tau, P, \alpha) - [(P - C)Q + V(\tau, P, 0)] \geq 0$$
$$(iii) \quad HV[V(\tau, P, \alpha) - [(P - C)Q + V(\tau, P, 0)]] = 0.$$

The LCP expresses the rational individual's strategy with regard to holding versus killing the option to harvest the stand of trees. Part (i) of equation (9) states that the required return for holding the option must be at least as great as the actual return. We would not expect a situation in which the required return is less than the actual return to persist in competitive markets. Part (ii) states that the value of the option, V, must be at least as great as the return from harvesting immediately. The return from harvesting immediately is the sum of the net revenue from selling the logs $(P - C)Q$ plus the value of the land immediately after harvesting, $V(t, P, 0)$. V would never drop below the return from harvesting immediately because the rational investor would harvest before that could happen. Finally, part (*iii*) states that at least one of statements (*i*) or (*ii*) must hold as a strict equality. If $HV = 0$, it is worthwhile to continue to hold the asset. If $V - (P - C)Q - V(t, P, 0) = 0$, it is worthwhile harvesting the asset. If both expressions hold as strict equalities, then the investor is indifferent between harvesting and continuing to hold the asset.

The LCP is solved numerically as is described in Appendix A.[7] This involves discretizing the relevant partial differential equation including a penalty term that enforces the American constraint (equation (9), (*ii*)). We are left with a series of nonlinear algebraic equations that must be solved iteratively. The implied Markov matrix can be found through manipulation of the discretized version of equation (9). Details are provided in Appendix B.

We can contrast the LCP for the multirotation case with that for the single rotation as in Insley. There are two state variables: α and P, as opposed to only P in the single rotation case. In general, having more than one state variable considerably complicates the estimation of V. However, through the "method of characteristics" described in Appendix A we are able to simplify the solution. The other difference of note with the multirotation problem is that part (*ii*) of equation (9), the so-called American constraint, now contains the value of the bare land after harvesting. Of course, it is the value of the bare land that is being solved for in the first place. Hence, solving the LCP will require an iterative procedure that starts with an initial guess for $V(t, P, 0)$ and then updates that guess through successive iterations. In the single rotation case, $V(t, P, 0)$ is set to zero, ignoring the value of the land after harvesting.

Boundary conditions for the problem are specified as follows:

1. As $P \to 0$, we need no special boundary condition to prevent negative prices. Referring back to equation (2), we see that as $P \to 0, dP \to \eta\bar{P}$, which is positive.
2. As $P \to \infty$, we follow Wilmott and set $V_{PP} = 0$.
3. As $\alpha \to 0$, we require no boundary condition since the partial differential equation is first-order hyperbolic in the α direction, with outgoing characteristic in the negative α direction.
4. As $\alpha \to \infty$, we assume $V_\alpha \to 0$, and hence no boundary condition is required. This means that as stand age gets very large, the value of the option to harvest, V, does not change with α. In essence, we are presuming the wood volume in the stand has reached some sort of steady state.
5. Terminal condition. As T gets large, it is assumed that $V = 0$. T is made large enough that this assumption has a negligible effect on V today.

We use this model to examine the opportunity costs of harvesting restrictions in Ontario's provincially owned forests. Almost 90% of the timber volume consumed by Ontario mills in 2000–2001 was from public land. Firms with licenses to harvest in Ontario's public forests are constrained by allowable cut regulations that aim to maintain constant annual wood flows within a region, and hence are limited in their ability to manage for price risk.[8] Such constraints reduce the return to the firm holding a

[7] The pseudocode is available on request from the authors.

[8] At one point in Ontario's history, a firm risked forfeiting its harvesting license if it failed to harvest the allowable cut. In recent years it appears that firms are not penalized for harvesting less than the allowable cut, and there is little documented evidence of firms exceeding the allowable cut. There appears to be a trend toward allowing firms holding forestry licenses more flexibility, within limits. Firms are also affected by industry structure, developed over years of forestry regulation, which tends to favor even wood flows.

harvesting license and thus affect a firm's willingness to make investments in forest management. The real options model developed here is used to compare the value of a license to harvest a stand of trees with and without allowable cut restrictions on harvesting time. The difference between the two gives some indication of the cost of current allowable cut restrictions.

To fully address the efficiency and other economic implications of allowable cut regulations would require consideration of all benefits and costs of these policies, including environmental impacts and possible effects on employment in logging communities. This would also require modeling the optimal harvesting decisions at the forest level, rather than the stand level as is done in this article. Such a complete analysis is beyond the scope of this article. Using a stand-level model and a real options approach, we estimate the value of the commercial harvest and the opportunity cost of policies to maintain annual sustained yields.

Data and Parameter Estimates

The case examined is for a stand of Jack Pine (Site Class 1) in the Romeo Malette Forest Unit, which is managed under a Sustainable Forest License by Tembec, Inc. The Romeo Malette forest consists of 477,109 hectares of productive forestland and is located northeast of the town of Timmins, Ontario. For this article we will examine the economics of a so-called basic level of silvicultural investment that represents the current level of spending on many stands in Ontario's boreal forest. Basic management involves assisted natural and artificial regeneration, including site preparation and removal of competing species. The focus is on manipulating species composition and achieving full site occupancy. Silvicultural costs estimated by Tembec (in Canadian$/hectare) are $200 for site preparation and $360 to purchase nursery stock in year 1, $360 for planting in year 2, $120 for tending in year 5, and finally $10 for monitoring in year 35. Amenity value is assumed to be zero, so that A in equation (8) reflects only silvicultural costs.

Yield curves for Jack Pine saw logs and pulp consistent with basic management in the Romeo Malette Forest Unit were provided by Tembec.[9] Yield of the most valuable class of saw logs peaks at about 300 cubic meters per hectare at around an age of 100 years.

We did not have a historical time series for mill gate lumber prices, and instead used monthly data (1980–2003) for the price of spruce-pine-fir random length 2 × 4's in Toronto.[10] The data, converted to Canadian dollars, deflated by the consumer price index, and seasonally adjusted, are shown in figure 1. We performed an augmented Dickey–Fuller (ADF) test on the price series to investigate whether the data-generating process appears to be a random walk (and hence nonstationary). As in Prestemon, the lag length for the ADF test was chosen by minimizing the Schwartz information criterion. The optimal lag length by this criterion is zero. The Ljung–Box Q-statistic at twelve lags was 12.8, which is not significant (p-value of 0.38), meaning that we do not reject the null of no serial correlation in the residuals. The results of the Dickey–Fuller test, given in table 1, indicate that we can reject the null hypothesis of a unit root at the 1% significance level.

Two other tests of stationarity were carried out, and the results are also reported in table 1. Using the Leybourne–McCabe test (Leybourne and McCabe), we do not reject the null hypothesis at the 5% level that the series is stationary. Using the variance ratio test (Lo and MacKinlay), we are able to reject the null of a unit root but only at the 10% level.

It may also be noted that an ARCH LM test was done on the residuals of the regression: $P_t - P_{t-1} = c(1) + c(2)P_{t-1}$ and there was evidence of the ARCH effect. When we move from discrete to continuous time, ARCH and GARCH models translate into complex stochastic volatility models, which are beyond the scope of this article.[11] However, the model and solution algorithm presented in this article could easily be adapted to time-dependent volatility.

Based on these test results it seems reasonable to adopt a mean-reverting stochastic process for lumber price in our optimal harvesting model. Of course, as noted above, none of the tests for a unit root is considered definitive. However, the purpose of this article is not to examine in detail the price path of timber, but rather to demonstrate a methodology that can

[9] The yield curves were estimated by M. Penner of Forest Analysis Ltd., Huntsville, Ontario, for Tembec. These yield curves are available from the authors on request.

[10] Data were purchased from Madison's Canadian Lumber Reporter Ltd., P.O. Box 2486, Vancouver, British Columbia V6B 3W7 Canada, Phone: 604-984-6838.

[11] Chuan Duan derives the diffusion limit for a class of GARCH(1,1) models and also extends to the GARCH(p,q) specification.

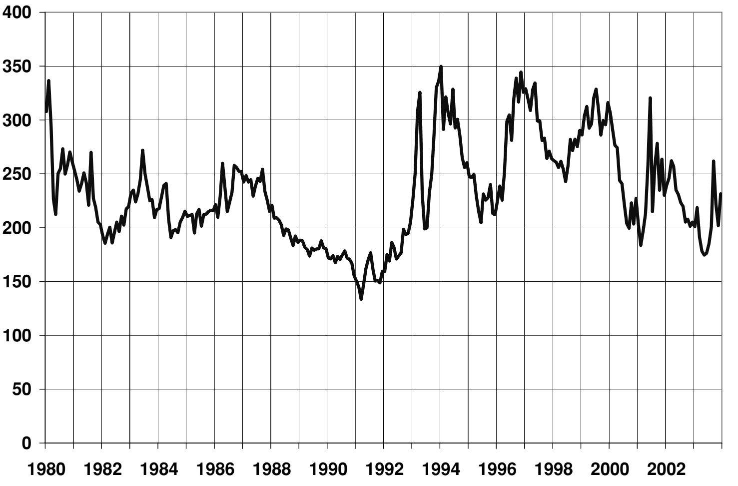

Figure 1. Real price of softwood lumber, Toronto, Ontario, 1993 Canadian dollars per cubic meter (Data source: Madison's Canadian Lumber Reporter, Monthly data from January 1980 to December 2003, First Friday of each month, Eastern Spruce-Pine-Fir Std #2& Better, Kiln-dried, Random Length = 2 × 4, Deflated by the Canadian consumer price index, converted to Canadian dollars, and seasonally adjusted.)

solve the optimal harvesting problem under an autoregressive or other chosen stochastic price path.

We assume lumber prices follow a mean-reverting process as is described in equation (2). A discrete time approximation is

$$(10) \quad P_t - P_{t-1} = \eta \bar{P} \Delta t - \eta \Delta t P_{t-1} + \sigma P_{t-1} \sqrt{\Delta t} \epsilon_t$$

where ϵ_t is $N(0, 1)$. Dividing through by P_{t-1} and using the notation

$$(11) \quad c(1) \equiv -\eta \Delta t; \quad c(2) \equiv \eta \Delta t \bar{P}; \quad e_t \equiv \sigma \sqrt{\Delta t} \epsilon_t,$$

the relevant parameters can be estimated by ordinary least squares on the following equation:

$$(12) \quad \frac{P_t - P_{t-1}}{P_{t-1}} = c(1) + c(2) \frac{1}{P_{t-1}} + e_t.$$

Regression estimates are shown in table 2. From the definitions of $c(1)$, $c(2)$, and e_t in equation (11) and given that Δt is one month, the parameter estimates are $\eta = 0.8$, $\bar{P} = \$230/m^3$, and $\sigma = 0.27$.

Harvesting costs and product prices were provided to the authors on a confidential basis. Representative harvesting costs for Ontario are reported in Rollins et al. as $31/m^3. The price of saw logs at the millgate is approximately $50/m^3. The $\bar{P}$ estimate given above refers to the real price at Toronto. This had to be translated into the price of raw logs purchased by the mill. The value of spruce-pine-fir lumber in Toronto in 2003 is close to this value for $\bar{P}$. Hence, the 2003 price going into the mill reported (in confidence) by Tembec was chosen as $\bar{P}$ for the analysis. We use 3% and 5% real discount rates for the analysis.

Empirical Results

Flexible Harvesting Time

This section presents the estimated value of the option to harvest the representative stand of trees on public forest land (V in equation (9)) given that the goal is to maximize the net

Table 1. Tests for Stationarity of the Ontario Softwood Lumber Price Series

Test	Statistic	Critical Value (Signif. Level)	Conclusion
Dickey–Fuller	−3.88	−3.45 (1%)	Reject H_0 of unit root
Leybourne–McCabe	0.075	0.148 (5%)	Do not reject H_0 of stationarity
Variance ratio (6 lags)	1.77	1.645 (10%)	Reject H_0 of unit root

Table 2. Parameter Estimates of Equation (12)

Variable	Coefficient	*t*-statistic		
c(1)	−.069	−2.95	Sample:	1980:02 to 2003:12
c(2)	15.9	3.10	Number of observations:	287
			$R^2 = 0.03$	SE of regression: 0.077

present value of the commercial value of the timber. Consumer surplus is ignored, implying the wood harvested is destined for the export market. We assume that adequate log markets exist, and ignore any values other than commercial timber value. The decision variable is whether to harvest the stand given the market price for any given time period. This implies that there are no restrictions (regulatory or otherwise) on firms as to when a stand could be harvested. This is called the social perspective to distinguish it from the perspective of a regulated firm, which is constrained as to harvesting times.

Solving equation (9) using the prices, costs, and discount rates given in the previous section, we estimate, for any given stand age, the threshold price above which it is optimal to harvest the stand. Appendix A provides details about the solution of the numerical model. Figure 2 illustrates these results for a real discount rate of 3%. Each individual graph in the figure represents a different stand age, and indicates the net merchantable volume in cubic meters per hectare (NMV) achieved by that age, as well as the critical price, P^*, at which it is worthwhile harvesting. The solid curve in each of the graphs represents the value of the opportunity to harvest the stand of trees, V, after the silvicultural treatments are completed. The dashed line represents the payout from harvesting immediately. When the value of the opportunity to harvest, V, is above the payout line, the value of delaying the harvest exceeds the value of harvesting immediately, and it is worthwhile waiting. Once V touches the payout line, it is worthwhile harvesting immediately. The point of tangency determines the critical price (and demonstrates the smooth pasting condition.)

Figure 2 indicates that it would be worth harvesting a thirty-six-year-old stand if the price of SPF1 reached $90/m^3. Although this is a fairly young age by boreal forest standards, with a

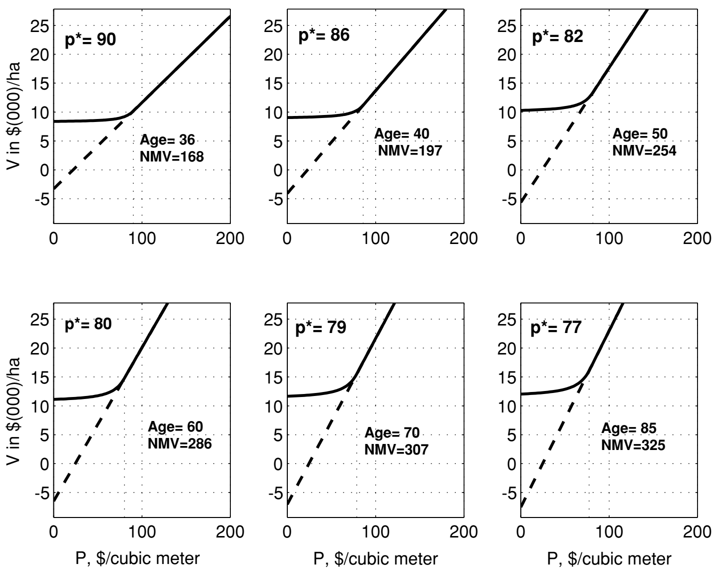

Figure 2. Value of the opportunity to harvest a stand at different stand ages, discount rate = 3% (Heavy solid line: value of the option to harvest. Heavy dashed line: payout from harvesting immediately. P^*: critical price at which harvesting is worthwhile. NMV: net merchantable volume, m^3/ha.)

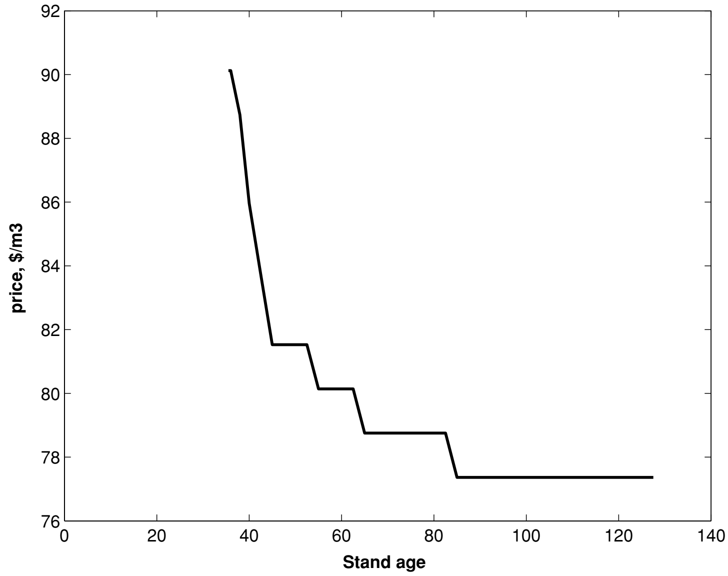

Figure 3. Critical price versus stand age, discount rate = 3%

mean-reverting price, it makes sense to harvest early to take advantage of a temporarily high price. By the age of forty years the critical price has dropped to $86. As the stand ages, the critical price continues to drop as the opportunity cost of delaying the harvest is reduced.

Figure 3 depicts the whole spectrum of critical prices for stand ages from zero to 130 years. Critical prices begin at the age of thirty-five years; harvesting is not allowed in the model prior to that date, by which time all silvicultural expenditures have been incurred. The critical price drops rapidly to about the age of eighty-five years and then reaches a steady state of $77.

Of significant interest is the value of the opportunity to harvest the stand of trees at the beginning of the first rotation shown in table 3 (second column). This value represents the maximum amount that a firm would be willing to pay for harvesting rights to the stand, ignoring taxes and other charges, and assuming the firm has complete flexibility as to the timing of the harvest and that markets exist for the logs. For comparison the value using a single rotation analysis is also shown, as well the value calculated using the simple Faustmann formula, assuming a constant price equal to the long-run mean-reverting level used in the stochastic model. The magnitude by which valuation estimates from the real options approach exceed those from the Faustmann analysis reflects the value of having the flexibility to optimally manage in the face of price volatility. The analysis is very sensitive to the discount rate chosen, since the silvicultural costs occur in the first thirty-five years, and the benefits, in terms of higher volumes, occur after thirty-five years.

In table 3, we show a single amount for land value that is unrelated to price. This contrasts with stands of the age of thirty-six years and above, shown in figure 2, for which value increases with the current price. Given our mean-reverting price process, no matter what the price is at the beginning of a rotation, by the time the stand achieves harvestable volumes, we expect the price to have reverted toward the long-run mean. Thus, land values at the beginning of a rotation are insensitive to today's product prices given the parameters we have chosen for our mean-reverting process. If we had assumed that price follows a process of geometric Brownian motion, then the value of the bare land would be dependent on today's price.

Table 3. Value of the Land at the Beginning of the First Rotation, $/Hectare

Discount Rate	Real Options: Multirotation	Real Options: Single Rotation	Faustmann
3%	$1,978	$1,520	$305
5%	$61	$54	−$512

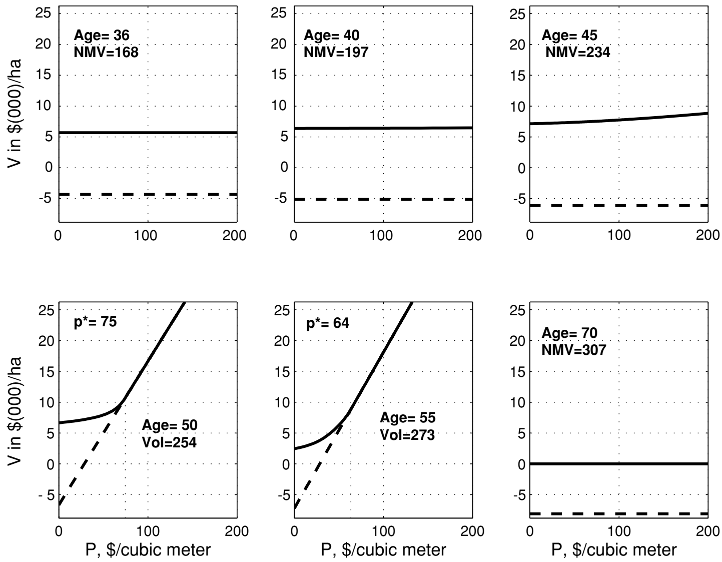

Figure 4. Value of the opportunity to harvest when harvesting must occur between ages 50 and 55, 3% discount rate (Heavy solid line: value of the option to harvest. Heavy dashed line: Payout from harvesting immediately. P^*: critical price at which harvesting is worthwhile. NMV: net merchantable volume, m^3/ha.)

Harvesting Restrictions

Wood flow in Ontario's public forests is typically determined over an entire region (forest unit) consisting of many individual stands. With our stand-level model we cannot fully describe the impact of harvesting restrictions; however, we can mimic their impact to some extent. In particular we consider the effect of minimum harvesting requirements, which would force a firm to harvest a certain quantity of wood even in times of depressed markets. In the stand-level model we mimic this restriction by requiring harvesting at a particular stand age.

Figure 4 shows the value of the option to harvest when the restriction is imposed that harvesting must occur when the stand is between the ages of fifty and fifty-five years. If harvesting does not occur within that period it is assumed the firm loses its rights to harvest. Comparing figure 4 with figure 2, we note that in the restricted case, value is now fairly insensitive to price at the age of forty years, since it is still ten years before harvesting can occur. We also note that at the ages of fifty and fifty-five years the critical prices are lower than for the unrestricted case. By the age of fifty-five years the critical price has dropped to $64 (compared to $80 for the unrestricted case), reflecting the fact that if harvesting does not occur in that year the land will be worthless to the firm as it will have to give up its license. Under this restriction, the value of the land at the beginning of the rotation is $960/ha, significantly lower than in the unrestricted case of $1,978/ha.

It is already well known in the forestry literature that the pursuit of an even flow of timber can significantly change the economics of commercial forestry due to the impact on the ability of the forest manager to respond to price volatility. Our modeling approach offers an improved ability to estimate the magnitude of the costs of these types of restrictions.

Accuracy of Results and MDP Approaches

The value estimates reported above are computed using a numerical solution methodology. As with any numerical method, we must be concerned with truncation error in the discretizations of time, age, and price (Tavella and Randall). The finer the grid used in the numerical solution, the more accurate will be our estimated value for V. In Appendix C, we show that the grid size with which we have computed the results of table 3 gives us results to an acceptable degree of accuracy. Refining the grid further changes the solution by only 0.4%.

We do not do a direct comparison with MDP models. However, we do show that we can get

Table 4. Comparing Solutions for Very Coarse Grid and Medium Grid. Value of the Land at the Beginning of the First Rotation, $/Hectare, 3% Discount Rate

	Very Coarse Grid	Medium Grid
Price $\$/m^3$ $P = [0, \ldots, 277]$	11 nodes	73 nodes
Quantity $\$/ha$ $\alpha = [0, \ldots, 125]$	13 nodes	107 nodes
Timestep size	0.25 years	0.125 years
Value at $t = 0$	$4,123	$ 1,978

a very large change in our answer when we use a grid size that is much coarser than that used to compute our base case results. This is an important point when comparing with the MDP models. We demonstrate in Appendix B that in theory the MDP model and the numerical solution of the LCP are equivalent. However, a numerical solution of the LCP using a finite difference approach permits use of a finer grid scheme. In an effort to handle more stochastic variables and keep the solution tractable, MDP models are often solved with a very coarse grid. In addition, convergence studies are typically not reported with MDP models.

In table 4 we demonstrate how our value estimate changes when we significantly reduce the number of nodes at which a solution is estimated. We observe a huge change in our estimated V values as we move from a medium to a very coarse grid. In any given example, it is clearly important to determine whether limiting the number of nodes has resulted in large inaccuracies.

Conclusion

This article has presented a two-factor multirotation model of the tree harvesting decision. The problem is specified as an LCP, which is solved using a fully implicit finite difference approach—an approach that is commonly used in the finance literature for valuing real and financial options. We contrast our methodology with other approaches used in the forestry literature to handle optimal rotation with stochastic prices, such as MDP models. We note that the LCP and the MDP models are in theory equivalent. An important benefit of the LCP approach is that we are assured that the solution will converge to the correct answer (based on a large numerical analysis literature), and we can easily check the accuracy by solving for successively finer grids. We demonstrated that the value of the harvesting option varies widely when a coarser solution grid is used. We have not solved an MDP model for comparison. However, our analysis does suggest that care should be taken when reporting results without carrying out numerical convergence studies, since the value of the option to harvest can be very sensitive to the number of discrete levels of the stochastic variables.

We used our model to address a policy issue in the Ontario boreal forest. This article has shown that the value of an investment in forestry can be significantly affected when a firm's ability to react to volatile prices is constrained. Constraints may be due to government regulations, such as allowable cut requirements, or may reflect the structural realities of an industry in which vertically integrated firms having invested in mill capacity want to maintain a reasonable capacity utilization. There are, no doubt, costs to a mill if input is highly variable, but these costs should be balanced with the benefits of being able to react optimally to price swings. The value of the option to harvest a stand of trees should be an important consideration in any review of forest management regulations, with the goal of designing regulations that continue to meet environmental constraints, but offer firms the maximum flexibility to manage license areas in the face of price risk. The true costs of regulations that limit flexibility can only be fully understood using a model that correctly values the option to harvest under uncertainty.

A direction for future research is to extend the methodologies from the finance literature to consider harvesting and other constraints at a regional forest unit level under stochastic prices. This would involve a multistand approach, as well as consideration of incremental mill costs with swings in capacity utilization. Biological and catastrophic risk are other avenues of research using a real options approach.

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Appendix A: Numerical Solution of the Linear Complementarity Problem

Method of Characteristics

The solution of equation (9) is accomplished by discretizing the term $V_\alpha - V_\tau$ in HV by the method of characteristics (Morton and Mayers).

Consider some function $U(X, \tau)$. τ refers to time to expiry of the option, or $T - t$. Then, we can write

$$\text{(A.1)} \quad \frac{dU}{d\tau} = U_X X_\tau + U_\tau.$$

If U satisfies the equation

$$\text{(A.2)} \quad U_\tau + a(X, \tau) U_X = 0$$

then, from equation (A.1), if we let $a(X, \tau) = X_\tau$, then $dU = 0$ along the characteristic curves defined by

$$\text{(A.3)} \quad \frac{dX}{d\tau} = a(X, \tau).$$

If we consider the simple case where $a(X, \tau) =$ constant $= \hat{a}$, then the solution to equation (A.2) is

$$\text{(A.4)} \quad U(X, \tau) = U(X - \hat{a}\tau, 0).$$

This can be verified by taking the total derivative of $U(X - \hat{a}\tau, \tau)$ and observing that $dU = 0$ when $X_\tau = \hat{a}$ and $\tau = 0$. In the case when $a(X, \tau) \neq$ constant, we can still approximate equation (A.4) in discrete time by

$$\text{(A.5)} \quad \frac{U(X_i, \tau^{n+1}) - U(X_i - a(X_i, \tau^{n+1})\Delta\tau, \tau^n)}{\Delta\tau} + O(\Delta\tau) = 0.$$

The basic PDE in the continuation regions for the tree harvesting problem, equation (7), can be written in terms of τ as follows:

$$\text{(A.6)} \quad V_\tau - V_\alpha = \frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A$$

where the left-hand side is a function of α and τ for a fixed P, and the right-hand side is a function of P for a fixed α and τ.

The left hand side of equation (A.6) looks like the left-hand side of equation (A.2), if $a(X, \Delta\tau) = -1$, recalling that $d\alpha/dt = 1$, and replacing X with α and U with V. As will be shown below, this observation allows us to approximate the two-factor problem by solving a set of one-dimensional PDEs and employing an interpolation operation at each time step to exchange information between the one-dimensional PDEs.

We now consider the numerical solution of the LCP, equation (9), using the characteristic approach and a fully implicit differencing scheme. Define nodes on the axes for P, α, and τ by

$$\text{(A.7)} \quad \begin{aligned} \mathbf{P} &= [P_1, P_2, \ldots, P_i, \ldots, P_M] \\ \boldsymbol{\alpha} &= [\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots, \alpha_J] \\ \boldsymbol{\tau} &= [\tau_1, \tau_2, \ldots, \tau_n, \ldots \tau_N]. \end{aligned}$$

We concern ourselves with the value of the option to harvest at points defined by the three-dimensional grid $(P, \alpha, \tau) = (P_i, \alpha_j, \tau_n)$. At any point on the grid the value of the option is $V = V(P_i, \alpha_j, \tau_n) \equiv V_{ij}^n$.

To impose the conditions of the LCP we define a function $\Pi(V)$ to be a penalty term that will prevent the value of the option V from ever falling below the payout from harvesting immediately, $(P - C)Q + V(\tau, P, 0)$. Zvan, Forsyth, and Vetzal discuss the penalty method. The penalty term, Π, equals 0 in the continuation region (i.e., when $HV = 0$ and $V > (P - C)Q + V(\tau, P, 0)$) and $\Pi > 0$ when it is optimal to harvest (i.e., when $HV > 0$ and $V = (P - C)Q + V(\tau, P, 0)$). If we include the penalty term in equation (A.6), equation (9), the LCP, can be approximated by

$$\text{(A.8)} \quad V_\tau - V_\alpha = \tfrac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V).$$

Using equation (A.5), our difference scheme for equation (A.8) can be written as

$$\text{(A.9)}\qquad \frac{V(P_i, \alpha_j, \tau^{n+1}) - V(P_i, \alpha_j + \Delta\tau, \tau^n)}{\Delta\tau} = \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V)\right]_{ij}^{n+1}.$$

Let

$$\text{(A.10)}\qquad V^*(P_i, \alpha_j)^n \equiv V\left(P_i, \alpha_j + \Delta\tau, \tau^n\right).$$

Then equation (A.9) can be written as (Bermejo)

$$\text{(A.11)}\qquad \frac{V(P_i, \alpha_j)^{n+1} - V^*(P_i, \alpha_j)^n}{\Delta\tau} = \left[\frac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \Pi(V)\right]_{ij}^{n+1}.$$

Note that the right-hand side of equation (A.11) has derivatives with respect to P only. Therefore, equation (A.11) defines a set of discretized one-dimensional partial differential equations, one for each α_j. These can be solved independently within each time step. After a time step is completed, these one-dimensional PDEs exchange information through the interpolation operation,

$$\text{(A.12)}\qquad V^*(P_i, \alpha_j)^n = V\left(P_i, \alpha_j + \Delta\tau, \tau^n\right).$$

We used linear interpolation.

Within each time step the LCP, equation (9), is solved for fixed α_j using a fully implicit finite difference method and the penalty method, as described in Insley. The right-hand side of equation (A.11) is discretized using a central, forward, or backward difference scheme as appropriate. Note that if we solve first for $V(P_i, \alpha = 0)^{n+1}$ at each time step, then we have the information required to determine the value of the penalty term implicitly. Hence, we do not need to iterate within each time step to determine the value of the bare land.

Discretization for i = [2, . . . , M − 1]

We now concern ourselves with the discretization of the RHS of equation (A.11) and make use of the following definitions:

$$\text{(A.13)}\qquad \alpha_i \equiv \frac{\sigma^2 P_i^2}{\Delta P_{i+1/2} + \Delta P_{i-1/2}};$$
$$\beta_i \equiv \frac{\eta(\bar{P} - P_i)}{\Delta P_{i+1/2} + \Delta P_{i-1/2}}$$

where $\Delta P_{i+1/2} \equiv P_{i+1} - P_i$ and $\Delta P_{i-1/2} \equiv P_i - P_{i-1}$. Using a fully implicit approach and central differencing along the P axis, our difference scheme when $i = 2, \ldots, M - 1$ for P_i and $j = 1, \ldots, J$ for α_j is

$$\text{(A.14)}\qquad \frac{V\left(P_i, \alpha_j, \tau^{n+1}\right) - V\left(P_i, \alpha_j + \Delta\tau, \tau^n\right)}{\Delta\tau} = \left\{\frac{\sigma^2 P^2}{2}\left[\frac{\frac{V_{i+1,j} - V_{ij}}{\Delta P_{i+1/2}} - \frac{V_{ij} - V_{i-1,j}}{\Delta P_{i-1/2}}}{\frac{\Delta P_{i+1/2} - \Delta P_{i-1/2}}{2}}\right] + \eta(\bar{P} - P)\left[\frac{V_{i+1,j} - V_{i-1,j}}{\Delta P_{i+1/2} + \Delta P_{i-1/2}}\right] - \rho V_{ij} + A + \frac{\pi_{ij}}{\Delta\tau}[(P_i - C)Q_j + V_{i0} - V_{ij}]\right\}^{n+1}.$$

The superscript $n + 1$ on the right-hand side means that all variables within the braces are evaluated at $\tau = n + 1$.

Extensive manipulation of the right-hand side of equation (A.14) results in

$$\text{(A.15)}\qquad \text{RHS} \equiv \left[a_i V_{i-1,j} + b_i V_{i+1,j} - (a_i + b_i + \delta)V_{ij} + A + \frac{\pi_{ij}}{\Delta\tau}[(P_i - C)Q_j + V_{i0} - V_{ij}]\right]^{n+1}$$

where

$$\text{(A.16)}\qquad a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}} - \beta_i; \quad b_i \equiv \frac{\alpha_i}{\Delta P_{i+1/2}} + \beta_i.$$

The last term on the right-hand side of equation (A.15) is the discretized penalty function $\Pi(V)$, where

$$\text{(A.17)}\qquad \begin{aligned} \pi_{ij}^{n+1} &= 0 \quad \text{if } V_{ij}^{n+1} \geq (P_i - C)Q_j + V_{i0} \\ \pi_{ij}^{n+1} &= L \quad \text{if } V_{ij}^{n+1} < (P_i - C)Q_j + V_{i0} \end{aligned}$$

where L is a suitably large number. In this article, L is set to 10^6.

To avoid an oscillatory solution it is necessary that a and b from equation (A.16) both be nonnegative. Since β can be either positive or negative, a and b can be either positive or negative, but if one is positive then the other is negative. If $a < 0$, a forward difference scheme for P is used to avoid oscillations rather than the central scheme shown in equation (A.14). If b is negative, then a backward difference scheme must be used for P (Zvan, Forsyth, and Vetzal).

Using a forward difference scheme a and b in equation (A.15) are defined as

$$\text{(A.18)} \qquad a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}}; \quad \equiv \frac{\alpha_i}{\Delta P_{i+1/2}} + \gamma_i$$
$$\text{where } \gamma_i = \frac{\eta(\bar{P} - P)}{\Delta P_{i+1/2}}.$$

Using a backward difference scheme a and b in equation (A.15) are defined as

$$\text{(A.19)} \qquad a_i \equiv \frac{\alpha_i}{\Delta P_{i-1/2}} - \theta_i; \quad b_i \equiv \frac{\alpha_i}{\Delta P_{i+1/2}}$$
$$\text{where } \theta_i = \frac{\eta(\bar{P} - P)}{\Delta P_{i-1/2}}.$$

Central differencing is used as much as possible to ensure the most accurate solution. Forward or backward differencing is used only at nodes where negative values of a or b are obtained using central differencing.

Discretization of Boundary Conditions for P

When $i = 1$ no special boundary condition is required for P, and a forward discretization scheme can be used. As $P \to 0$, equation (A.11) becomes

$$\text{(A.20)} \qquad \frac{V(P_i, \alpha_j)^{n+1} - V^*(P_i, \alpha_j)^n}{\Delta\tau} = [\eta \bar{P} V_P - \rho V + A + \Pi(V)]^{n+1}_{i=1,j}.$$

A forward discretization is then

$$\text{(A.21)} \qquad \frac{V(P_i, \alpha_j)^{n+1} - V^*(P_i, \alpha_j)^n}{\Delta\tau} = \Bigg[b_i V_{i+1} - (b_i + \rho)V_i + A + \frac{\pi_{ij}}{\Delta\tau} \times [(P_i - C)Q_j + V_{i0} - V_{ij}] \Bigg]^{n+1}_{1,j}$$

with b defined as in equation (A.18). In equation (A.18) as $P \to 0$ we see that $\alpha = 0$ and hence $a = 0$ and also, $b = \gamma$.

When $i = M$ we set $V_{PP} = 0$ in equation (A.11) and use a backward difference scheme that gives

$$\text{(A.22)} \qquad \frac{V(P_i, \alpha_j)^{n+1} - V^*(P_i, \alpha_j)^n}{\Delta\tau} = \Bigg[\frac{\eta(\bar{P} - P)}{\Delta P_{i-1/2}} V_{i-1} + \Big(\frac{\eta(\bar{P} - P)}{\Delta P_{i-1/2}} - \rho\Big) V_i + A + \frac{\pi_{ij}}{\Delta\tau}[(P_i - C)Q_j + V_{i0} - V_{ij}] \Bigg]^{n+1}_{M,j}.$$

Iterative Solution

Equations (A.14), (A.17), (A.21), and (A.24) represent a system of nonlinear algebraic equations, due to the penalty term and hence, must be solved iteratively. A description of the iterative solution for a single rotation problem is given in Insley. The iterative solution can best be understood if we write the system of equations in matrix form. To this end, define the vectors

$$\text{(A.23)} \qquad \boldsymbol{V}^{n+1} \equiv \begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ \vdots \\ V_M^{n+1} \end{bmatrix}; \quad \boldsymbol{V}^{*n} \equiv \begin{bmatrix} V_1^{*n} \\ V_2^{*n} \\ \vdots \\ V_M^{*n} \end{bmatrix};$$

$$\boldsymbol{P}^n \equiv \begin{bmatrix} P_1^n \\ P_2^n \\ \vdots \\ P_M^n \end{bmatrix}; \quad \boldsymbol{A}^n \equiv \begin{bmatrix} A_1^n \\ A_2^n \\ \vdots \\ A_M^n \end{bmatrix}.$$

Also, define a diagonal matrix as

$$\text{(A.24)}$$
$$[\bar{\boldsymbol{Q}}(\boldsymbol{V}^{n+1})]_{ii} = L \quad \text{if } V_i^{n+1} < \left(P_i^{n+1} - C\right) Q_j - V_{i0}^{n+1}$$
$$= 0 \quad \text{otherwise}$$
$$[\bar{\boldsymbol{Q}}(\boldsymbol{V}^{n+1})]_{ij} = 0 \quad \text{if } i \neq j.$$

Let $\boldsymbol{B}$ be an $M \times M$ matrix. Multiplying $\boldsymbol{B}$ by the vector $\boldsymbol{V^{n+1}}$ gives a vector $\boldsymbol{B}\boldsymbol{V}^{n+1}$ with the following elements:

$$\text{(A.25)} \qquad \begin{bmatrix} \left[\frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}} + \rho\right] V_1^{n+1} - \left[\frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}}\right] V_2^{n+1} \\ -\Delta\tau b_2 V_1^{n+1} + \Delta\tau(a_2 + b_2 + \rho)V_2^{n+1} - \Delta\tau a_2 V_3^{n+1} \\ \vdots \\ -\Delta\tau b_i V_{i-1}^{n+1} + \Delta\tau(a_i + b_i + \rho)V_i^{n+1} - \Delta\tau a_i V_{i+1}^{n+1} \\ \vdots \\ -\Delta\tau b_{M-1} V_{M-2}^{n+1} + \Delta\tau(a_{M-1} + b_{M-1} + \rho_{M-1})V_{M-1}^{n+1} - \Delta\tau a_{M-1} V_M^{n+1} \\ \left[\Delta\tau \frac{\eta(\bar{P} - P_M)}{\Delta P_{M-1/2}}\right] V_{M-1}^{n+1} - \left[\Delta\tau \frac{\eta(\bar{P} - P)}{\Delta P_{M-1/2}} - \rho\right] V_M^{n+1} \end{bmatrix}.$$

Equations (A.14), (A.17), (A.21), and (A.24) can be written as

$$\text{(A.26)}\qquad \boldsymbol{B}\boldsymbol{V}^{n+1} + [\boldsymbol{I} + \bar{\boldsymbol{Q}}(\boldsymbol{V}^{n+1})]\boldsymbol{V}^{n+1} = \boldsymbol{V}^{*n} + \Delta\tau \boldsymbol{A}^{n} + \bar{\boldsymbol{Q}}(\boldsymbol{V}^{n+1})$$

where $\boldsymbol{I}$ is the identity matrix. We can express $\boldsymbol{V}^{*n}$ as $\boldsymbol{V}^{*n} = \boldsymbol{F}\boldsymbol{V}^{n}$ where $\boldsymbol{F}$ is an interpolation matrix. For linear interpolation, $\boldsymbol{F}$ has the properties that its entries are nonnegative and all row sums are 1. Equation (A.26) is solved iteratively as described in Insley.

Appendix B: Markov Decision Process Models and the Linear Complementarity Problem

The key to a Markov decision model is the Markov matrix, or transition probability matrix of the model. (Karlin and Taylor, and Hillier and Lieberman discuss Markov transition probability matrices.) Consider a random variable X. We refer to X_n as the outcome of the nth trial. The state space is labeled by nonnegative integers (0, 1, 2, 3, …). The probability of X_{n+1} being in state j, given that X_n is in state i (a one-step transition probability) is denoted by $G_{ij}^{n,n+1}$, that is

$$\text{(B.1)}\qquad G_{ij}^{n,n+1} = Pr\{X_{n+1} = j \mid X_n = i\}.$$

$$\text{(B.7)}\qquad \hat{\boldsymbol{B}} = \begin{bmatrix} \frac{\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}} & \frac{-\Delta\tau\eta\bar{P}}{\Delta P_{i+1/2}} & 0 & 0 & 0 & \dots & 0 & 0 \\ -\Delta\tau b_2 & \Delta\tau(a_2+b_2) & -\Delta\tau a_2 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -\Delta\tau b_{M-1} & \Delta\tau(a_{M-1}+b_{M-1}) & -\Delta\tau a_{M-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{\Delta\tau\eta(\bar{P}-P_M)}{\Delta P_{M-1/2}} & \frac{-\Delta\tau\eta(\bar{P}-P_M)}{\Delta P_{M-1/2}} \end{bmatrix}.$$

When the one-step transition probabilities are independent of time, the Markov process is said to have stationary transition probabilities. In this case $G_{ij}^{n,n+1} = G_{ij}$. G_{ij} is the probability that the state value goes from state i to j in one trial. All the G_{ij} can be arranged in a matrix referred to as the Markov matrix or the transition probability matrix of the process, denoted by P. An example of G for four different states is

$$\text{(B.2)}\qquad \boldsymbol{G} = \begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{10} & G_{11} & G_{12} & G_{13} \\ G_{20} & G_{21} & G_{22} & G_{23} \\ G_{30} & G_{31} & G_{32} & G_{33} \end{pmatrix}.$$

The $(i+1)$st row of $\boldsymbol{G}$ is the probability distribution of the values of X_{n+1} under the condition $X_n = i$. The order of the matrix (number of rows) is equal to the number of states. Given four different states the quantities G_{ij} satisfy

$$\text{(B.3)}\qquad G_{ij} \geq 0, \quad i, j = 0, 1, 2, 3$$
$$\sum_{j=0}^{3} G_{ij} = 1, \qquad i = 0, 1, 2, 3.$$

We can show how a matrix with all the properties of the Markov matrix is derived in the solution of the LCP. For convenience we define a variable W such that $V = e^{-\rho\tau} W$. Thus,

$$\text{(B.4)}\qquad V_\tau = -e^{-\rho\tau}\rho W + e^{-\rho\tau} W_\tau.$$

Substituting these expressions into equation (A.8) and setting $A = 0$, we get

$$\text{(B.5)}\qquad W_\tau - W_\alpha = \tfrac{1}{2}\sigma^2 P^2 W_{PP} + \eta(\bar{P} - P)W_P + e^{\rho\tau}\Pi(e^{-\rho\tau} W).$$

We can discretize equation (B.5) as shown in Appendix A. We end up with an equation similar to equation (A.26). We are ignoring the penalty term here for simplicity.

$$\text{(B.6)}\qquad [\hat{\boldsymbol{B}} + \boldsymbol{I}]\boldsymbol{W}^{n+1} = \boldsymbol{W}^{*n}$$

The matrix $[\hat{\boldsymbol{B}}]$ has the following elements:

It follows that

$$\text{(B.8)}\qquad \boldsymbol{W}^{n+1} = [\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{W}^{*n}.$$

We know that $V^{n+1} = e^{-\rho\Delta\tau} W^{n+1}$. It follows that

$$\text{(B.9)}\qquad \boldsymbol{V}^{n+1} = e^{-\rho\Delta\tau}[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{V}^{*n}.$$

Recall that $\boldsymbol{V}^{*n} = \boldsymbol{F}\boldsymbol{V}^{n}$ where $\boldsymbol{F}$ is the interpolation matrix. We note that $[\hat{\boldsymbol{B}} + \boldsymbol{I}]$ will have positive diagonals, nonpositive off-diagonals, and rows that sum to 1.

LEMMA 1. *The row sums of* $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ *are unity.*

Proof: Let e be an $[\,\boldsymbol{I} \times 1]$ column vector of 1's. We can observe that $[\hat{\boldsymbol{B}} + \boldsymbol{I}]\boldsymbol{F}\boldsymbol{e} = \boldsymbol{F}\boldsymbol{e}$ (since $\hat{\boldsymbol{B}}\boldsymbol{e} = 0$ and $\boldsymbol{F}\boldsymbol{e} = \boldsymbol{e}$). This implies that $\boldsymbol{F}\boldsymbol{e} = [\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}\boldsymbol{e}$. The lemma follows since

(B.10) $$\text{rowsum}_i[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F} = [[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}\boldsymbol{e}]_i = [\boldsymbol{F}\boldsymbol{e}]_i = 1. \quad \blacksquare$$

LEMMA 2. *All elements of* $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ *are nonnegative.*

Proof: $[\hat{\boldsymbol{B}} + \boldsymbol{I}]$ has positive diagonals, nonpositive off-diagonals, and is diagonally dominant. It may therefore be classified as an M-matrix, which has the property that all elements of its inverse are nonnegative (Varga). We stated previously that the elements of $\boldsymbol{F}$ are all nonnegative. ■

THEOREM 1: $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ *satisfies* (B.4) *and hence is a Markov matrix.*

Proof: By Lemma 2, all elements of $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ are nonnegative. By Lemma 1, the row sums of $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ are all unity. Hence, all elements must be less than or equal to 1. ■

We can see that $[\hat{\boldsymbol{B}} + \boldsymbol{I}]^{-1}\boldsymbol{F}$ is equivalent to a Markov transition matrix showing the probability of moving from one state at τ_n to another at τ_{n+1}. From equation (B.9), the value of the option at period $n + 1$ is equal to the value in period n multiplied by the Markov transition matrix and a discount factor.

Solving the Markov chain decision model is an alternative method to solving the partial differential equation, equation (A.8). The Markov matrix is frequently estimated through simulation. This is unnecessary with the approach used in this article. Instead, we discretize the PDE and solve the resulting system of equations iteratively using well-established numerical techniques. There is no need to directly estimate the Markov matrix. Note that the elements of $(\boldsymbol{B} + \boldsymbol{I})^{-1}\boldsymbol{F}$ are to $O(\Delta\tau^2)$ equal to the discretized Green's function that must solve the PDE,

(B.11) $$V_\tau - V_\alpha = \tfrac{1}{2}\sigma^2 P^2 V_{PP} + \eta(\bar{P} - P)V_P - \rho V + A + \delta(P - P')\delta(\tau - \tau')$$

where $\delta(\tau - \tau')$ represents the delta function (see Wilmott, Dewynne, and Howison). The Green's function of this PDE is simply the transition probability density function, $P(P, t, P', t')$ relating the probability of a transition from state $(P, t) \rightarrow (P', t')$. This is also the solution of the forward Kolmogorov equation.

Although the Green's function (hence the transition density function) can be determined in some simple cases, it is not in general possible to obtain an analytic solution. However, we can always discretize the PDE directly and hence obtain an approximate transition matrix to $O(\Delta\tau^2)$. Note that since we solve the LCP at each time step we are directly enforcing the optimal control, which ensures that the solution satisfies the smooth pasting condition. The usual stochastic dynamic programming approach applies a control in explicit fashion, hence the solution is in an inconsistent state after each timestep. This will be an issue only in nonautonomous problems, but will not matter when we are solving for a steady state, as in this article.

Note that we do not compute $(\hat{\boldsymbol{B}} + \boldsymbol{I})^{-1}$, which is a dense matrix, but solve $(\hat{\boldsymbol{B}} + \boldsymbol{I})\boldsymbol{V}^{n+1} = \boldsymbol{V}^{*n}$, which is considerably more efficient, since $(\boldsymbol{B} + \boldsymbol{I})$ is tridiagonal.

Appendix C: Convergence of the Finite Difference Scheme

The accuracy of the numerical solution to the PDE depends critically on the number of nodes used for each factor: price (P), age (α), and time (τ) (Wilmott, Tavella and Randall). The more the nodes, the more accurate the solution, but the longer the solution time. An upper limit on the number of nodes will eventually be reached based on the memory limits of the computer. It is important to check whether an acceptable number of nodes has been included by noting how much the solution changes as the grid is refined. Specifically, we observe the change in the answer as we double the total number of nodes.

More formally, let $\Delta\tau = c_1 h$, $\Delta P = c_2 h$, $\Delta\alpha = c_3 h$, where c_1, c_2, c_3 are constants independent of h. Since first-order timestepping is being used, and the cumulative effect of linear interpolation in the α direction will be $O(\frac{\Delta\alpha^2}{\Delta\tau}) = O(h)$, we expect that the computed solution V^{comp} is related to the exact solution V^{exact} by

(C.1) $$V_h^{\text{comp}} = V^{\text{exact}} + c_4 h, \quad h \rightarrow 0$$

where c_4 is a constant independent of h. This implies that

(C.2) $$\frac{V_h - V_{h/2}}{V_{h/2} - V_{h/4}} \sim 2, \quad h \rightarrow 0.$$

In table C.1 we see that this ratio is about 1.5, indicating that we are near the asymptotic convergence range. In particular, we can be confident that the solution on the finest grid is 1,986 ± 7.5 or accurate to within about 0.4%.

Table C.1. Convergence of Solution as Grid Is Refined for the Basic Regime, 3% Discount Rate

Grid	No. of α Nodes	No. of P Nodes	Time Step Size (Years)	V ($/ha)	Change in V	Ratio
Coarse	55	37	0.25	1,967.5		
Medium	110	73	0.125	1,978.6	11.1	
Fine	220	145	0.0625	1,986.1	7.5	1.5