

Capacity of a Class of Deterministic Relay Channels

Young-Han Kim, *Member, IEEE*

Abstract—The capacity of a class of deterministic relay channels with transmitter input X , receiver output Y , relay output $Y_1 = f(X, Y)$, and separate noiseless communication link of capacity R_0 from the relay to the receiver, is shown to be

$$C(R_0) = \sup_{p(x)} \min\{I(X; Y) + R_0, I(X; Y, Y_1)\}.$$

Roughly speaking, every bit from the relay is worth one bit to the receiver until saturation at capacity.

Index Terms—Capacity, deterministic relay channel, primitive relay channel.

We consider a class of relay channels, which we call *primitive relay channels*, as depicted in Fig. 1. Here the channel input signal X is received by the relay Y_1 and the receiver Y through a channel $p(y, y_1|x)$, and the relay can communicate to the receiver over a separate noiseless link of rate R_0 . We wish to communicate a message index $W \in [2^{nR}] := \{1, 2, \dots, 2^{nR}\}$ reliably over this relay channel.

A $(2^{nR}, 2^{nR_0}, n)$ code is specified by an encoding function $X^n : [2^{nR}] \rightarrow \mathcal{X}^n$, a relay function $J : \mathcal{Y}_1^n \rightarrow [2^{nR_0}]$, and a decoding function $\hat{W} : \mathcal{Y}^n \times [2^{nR_0}] \rightarrow [2^{nR}]$. The probability of error is defined by $P_e^{(n)} = \Pr\{W \neq \hat{W}(Y^n, J(Y_1^n))\}$, with the message W distributed uniformly over $[2^{nR}]$. We say that a rate pair (R, R_0) is achievable if there exists a sequence of $(2^{nR}, 2^{nR_0}, n)$ codes such that $P_e^{(n)}$ tends to zero as $n \rightarrow \infty$. The capacity $C(R_0)$ is defined as the supremum of all rates R for which (R, R_0) is achievable.

With relay-to-receiver communication decoupled from the broadcasting of X over $p(y, y_1|x)$, the primitive relay channel problem is considerably simpler than the general relay channel (see van der Meulen [12], Cover and El Gamal [4], Kramer *et al.* [10], El Gamal *et al.* [8], and the references therein). Nonetheless, the primitive relay channel captures essential challenges of relaying, and a complete characterization of the capacity $C(R_0)$ remains elusive; refer to [9] for further discussion.

In this correspondence, we focus on a special class of primitive relay channels, in which the channel output $Y_1 = f(X, Y)$ at the relay is a deterministic function of the channel input X and the channel output Y at the ultimate receiver. Compared to the semideterministic relay model studied by El Gamal and Aref [7], this model is more deterministic in the relay-to-receiver link and less deterministic in the transmitter-to-relay link.

We present the following main result.

Theorem 1: The capacity $C(R_0)$ of the primitive relay channel, in which the relay output $Y_1 = f(X, Y)$ is a deterministic function of the input X and the receiver output Y , is given by

$$C(R_0) = \sup_{p(x)} \min\{I(X; Y) + R_0, I(X; Y, Y_1)\}. \quad (1)$$

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The author is with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093 USA (e-mail: yhk@ucsd.edu).

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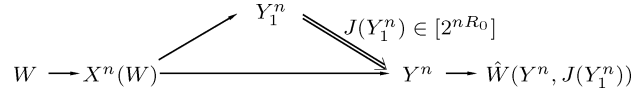


Fig. 1. The relay channel with noiseless link of capacity R_0 .

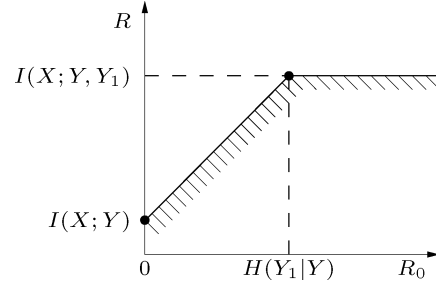


Fig. 2. The set of achievable (R, R_0) pairs for a fixed $p(x)$.

Proof: The converse follows immediately from the cut-set upper bound on information flows [6, Sec. 15.10].

For the achievability, it suffices to show that given the input distribution $p(x)$ (and the corresponding joint distribution $p(x, y, y_1)$), any rate pair (R, R_0) such that

$$R < I(X; Y) + R_0 \quad (2a)$$

$$R < I(X; Y, Y_1) = I(X; Y) + H(Y_1|Y) \quad (2b)$$

is achievable, as depicted in Fig. 2. (Here the equality in (2b) follows since the channel determinism $Y_1 = f(X, Y)$ implies $I(X; Y_1|Y) = H(Y_1|Y)$.) Without loss of generality, suppose the channel is discrete. Then we can use the Slepian–Wolf rate $R_0 > H(Y_1|Y)$ to describe Y_1^n to the receiver [11] and achieve $R < I(X; Y, Y_1)$. On the other hand, by ignoring the relay ($R_0 = 0$), we can achieve $R < I(X; Y)$. Time-sharing these two extreme points, we can achieve the entire region described by (2), which completes the proof. \square

There are alternative coding schemes that achieve the capacity (1) without time-sharing, as sketched in the following examples.

Example 1 (Hash-and-Forward [5]): Suppose the input signal under power constraint P is transmitted over a Gaussian primitive relay channel

$$\begin{aligned} Y &= X + Z \\ Y_1 &= X + Z_1 \end{aligned}$$

where Z and Z_1 are jointly Gaussian with zero mean, equal variance $EZ^2 = EZ_1^2 = N$, and correlation coefficient ρ . Suppose $\rho = 1$. Then $Y_1 = Y$, the relay is useless, and the capacity of the relay channel is $C(R_0) = (1/2) \log(1 + P/N) = C(0)$ for all $R_0 \geq 0$.

If $\rho = 0$, the relay furnishes an independent look at X . What should the relay say to Y ? This is the most interesting case, but the capacity $C(R_0)$, mentioned in [3], remains unsolved and typifies the primary open problem of the relay channel. As a partial converse, Zhang [14] proved that a strict inequality $C(R_0) < C(0) + R_0$ holds for all $R_0 > 0$.

If $\rho = -1$ (i.e., $Z_1 = -Z$), the relay, while having no more information than the receiver Y , has much to say, since combined knowledge of Y and Y_1 allows perfect determination of X . Indeed, Theorem 1 shows that

$$C(R_0) = C(0) + R_0.$$

Every bit sent by the relay counts as one bit of information, despite the fact that the relay cannot decode the message at all.

To achieve this rate, the receiver checks $2^{n(C(R_0)-\epsilon)}$ codewords $X^n(W)$, $W \in [2^{n(C(R_0)-\epsilon)}]$, one by one, with respect to the received signal Y^n and one of 2^{nR_0} hash indexes from Y_1^n . More specifically, the receiver first forms a list of $2^{n(C(R_0)-C(0)-\epsilon)} = 2^{n(R_0-\epsilon)}$ codewords $X^n(W)$ that are jointly typical with given Y^n , or equivalently, forms a list of $2^{n(R_0-\epsilon)}$ relay received sequence candidates $Y_1^n(W) = 2X^n(W) - Y^n$. Now the relay can spend nR_0 bits to communicate the pre-defined hash index of Y_1^n , which is sufficient to convey Y_1^n asymptotically error-free to the receiver and determine the correct codeword $X^n(W)$.

The above coding scheme, called “hash-and-forward”, can be shown [5, Sec. II] to achieve the capacity (1) for any primitive relay channel with $Y_1 = f(X, Y)$.

Example 2 (Compress-and-Forward [4, Th. 6]): A binary input signal $X \in \{0, 1\}$ is sent over a binary symmetric channel $Y = X \oplus S$, where the binary additive noise $S \sim \text{Bern}(p)$ is independent of X . With no information on S available at the transmitter or the receiver, the capacity is given as $C(0) = 1 - H(p)$.

Now suppose there is an intermediate node which observes S and “relays” that information to the decoder through a side channel of capacity R_0 . Since $S = X \oplus Y$ is a deterministic function of (X, Y) , Theorem 1 shows that

$$C(R_0) = 1 - H(p) + R_0$$

for $0 \leq R_0 \leq H(p)$.

To achieve this rate, the relay compresses the state sequence S^n using the binary lossy source code of rate R_0 . More specifically, we use random covering with the standard backward channel $S = \hat{S} \oplus U$ for the binary rate distortion problem [6, Sec. 10.3.1], where $\hat{S} \in \{0, 1\}$ is the reconstruction symbol and $U \sim \text{Bern}(q)$ is independent of \hat{S} (and X) with parameter q satisfying $R_0 = I(S; \hat{S}) = H(p) - H(q)$. Thus, using nR_0 bits, the relay can describe the reconstruction sequence \hat{S}^n to the ultimate receiver. Finally, decoding codewords X^n , generated randomly according to $\text{Bern}(1/2)$, based on (Y^n, \hat{S}^n) , we can achieve the rate

$$\begin{aligned} I(X; Y, \hat{S}) &= I(X; X \oplus S, S \oplus U) \\ &\geq I(X; X \oplus U) \\ &= 1 - H(q) \\ &= 1 - H(p) + R_0. \end{aligned}$$

In general, this coding scheme, called “compress-and-forward”, can achieve $I(X; Y, \hat{S}) = I(X; Y|\hat{S})$ for any $p(x)$ and $p(\hat{s}|s)$ such that the rate requirement $I(S; \hat{S}|Y) \leq R_0$ is satisfied for the Wyner–Ziv coding [13] of S with side information Y . Simple algebra [5, Section V] shows that this achievable rate

$$R^*(R_0) = \sup_{p(x)p(\hat{s}|s): I(S; \hat{S}|Y) \leq R_0} I(X; Y, \hat{S})$$

is identical to (1) when $S = f(X, Y)$, which confirms a special case of a conjecture by Ahlswede and Han [1, Sec. V] on channels with state information partially available at the receiver. A recent paper by Aleksic, Razaghi, and Yu [2] extends the above binary example into a class of nondeterministic channels (i.e., $S \neq f(X, Y)$) and obtains interesting capacity results.

As a final note, we observe that Theorem 1 can be extended to multiple relays if the channel outputs $Y_i = f_i(X, Y)$ at the relays can be determined from (X, Y) . For example, when two relays spend rates R_1 and R_2 to communicate to the receiver, the capacity is given by

$$C(R_1, R_2) \leq \sup_{p(x)} \min \left\{ \begin{array}{l} I(X; Y) + R_1 + R_2, \\ I(X; Y, Y_1) + R_2, \\ I(X; Y, Y_2) + R_1, \\ I(X; Y, Y_1, Y_2) \end{array} \right\}.$$

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