Houston Journal of Mathematics
© 2002 University of Houston
Volume 28, No. 4, 2002

MODULES WITH MORITA-EQUIVALENT ENDOMORPHISM RINGS

ULRICH ALBRECHT

Communicated by Johnny A. Johnson

Abstract. Let A and B be modules, which are faithfully flat over their endomorphism ring. The categories of A-solvable and B-solvable modules coincide if and only if A and B are similar. While similar modules have Morita equivalent endomorphism rings, the failure of the converse raises the question which module-theoretic properties are shared by modules with equivalent endomorphism rings. This paper addresses this question by investigating equivalences between full subcategories of the categories of A- and B-solvable modules, respectively. In particular, every equivalence between the category of A-solvable and the category of B-solvable modules is induced by a Morita equivalence between E_A and E_B if A and B are faithfully flat as modules over their endomorphism ring. Several examples show that these results may fail without the faithfulness condition.

1. Introduction

Any attempt to obtain a satisfactory structure theory for large classes of torsion-free abelian groups is hindered by the existence of pathological direct sum decompositions, examples of which can be found in [9, Chapters 90 and 91]. Nevertheless, many properties of an abelian group A can be described in terms of its endomorphism ring $E = E_A$. This description frequently involves the functors $H_A = Hom(A, -)$ and $T_A = - \otimes_E A$ between the category $\mathcal{A}b$ of abelian groups and the category $\mathcal{M}_E$ of right E-modules. These functors form an adjoint pair, and can be used most effectively when considering full subcategories of the category of abelian groups, on which they induce a category equivalence with a suitable full subcategory of $\mathcal{M}_E$. The largest full subcategory of $\mathcal{A}b$ with this property is $\mathcal{C}_A$, the category of *A-solvable abelian groups*. The concepts involved in the discussion of A-solvable abelian groups readily extend to categories of right R-modules.

Since the class of torsion-free A-solvable groups need not be closed with respect to quasi-isomorphism (see [4]), the perhaps most natural way to extend the concept of A-solvability to the quasi-category of torsion-free abelian groups is to consider the class $\mathcal{AC}_A$ of *almost A-solvable groups*, whose elements are the groups G for which the natural map $\theta_G : T_A H_A(G) \to G$ is a quasi-isomorphism. Strongly indecomposable groups A and B of finite rank are quasi-isomorphic if and only if $\mathcal{A}C_A = \mathcal{A}C_B$ [3]. Naturally, the question arises whether similar conclusions can be drawn from the fact that $\mathcal{C}_A = \mathcal{C}_B$. Unfortunately, this is not the case since $\mathcal{C}_A = \mathcal{C}_B$ if A and B are near-isomorphic torsion-free abelian groups of finite rank. Instead, the investigation of R-modules A and B for which $\mathcal{C}_A = \mathcal{C}_B$ leads to the discussion of similar modules (see [5] and [11] for details). Two right R-modules A and B are *similar* if A is a direct summand of B^n and B is a direct summand of A^m for some $m, n < \omega$. One obtains that two R-modules A and B, which are faithfully flat as modules over their endomorphism ring, are similar if and only if $\mathcal{C}_A = \mathcal{C}_B$. An example is given that this equivalence fails without the faithfulness condition.

The fact that similar R-modules have Morita-equivalent endomorphism rings raises the question which additional properties are shared by modules with equivalent endomorphism rings. Section 3 describes when this equivalence occurs (Proposition 3.1), and presents additional results related to this characterization (Theorem 3.2 and Corollary 3.3). The final section addresses the question under which conditions $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent categories. It is shown that every such equivalence is induced by a Morita-equivalence between E_A and E_B (Theorem 4.1). However, faithfully flat R-modules A and B for which $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent need not be similar since there exist torsion-free abelian groups A and B of rank 1 such that $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent categories, but A and B are not similar (Theorem 4.6).

2. Similar modules

Associated with H_A and T_A are natural maps $\theta^A_N : T_A H_A(N) \to N$ and $\phi^A_M : M \to H_A T_A(M)$ defined by $\theta^A_N(\alpha \otimes a) = \alpha(a)$ and $[\phi^A_M(x)](a) = x \otimes a$ for all $a \in A$, $x \in M$, and $\alpha \in H_A(M)$. The superscripts referring to A are omitted unless it is not clear from the context which module is considered. If A is self-small, then $\mathcal{C}_A$ contains the class $\mathcal{P}_A$ of A-projective modules. Here, an R-module P is *A-projective* if it is a direct summand of an A-free module of the form $\oplus_I A$.

The smallest cardinality possible for I is the *A-rank of P*. A R-module M is *(finitely) A-generated* if it is an epimorphic image of an A-projective module (of finite A-rank).

It is a well-known fact ([5] and [11]) that similar modules have Morita-equivalent endomorphism rings, but the converse fails in general since there exist non-isomorphic subgroups A and B of $\mathbb{Q}$ with endomorphism ring $\mathbb{Z}$. Clearly, such groups A and B cannot similar. For torsion groups however, one obtains the surprising

Proposition 2.1. *The following are equivalent for torsion groups A and B:*

a) *A and B are similar.*
b) *A and B have Morita-equivalent endomorphism rings*

PROOF. It remains to show that A and B are similar if E_A and E_B are Morita equivalent. There are $n < \omega$ and an idempotent $e \in M_n(E_A)$ such that $E_B \cong eM_n(E_A)e$. Since $M_n(E_A)$ is the endomorphism ring of A^n, the group $B' = e(A^n)$ is a direct summand of A^n with endomorphism ring $eM_n(E_A)e$. Hence, $B' \cong B$ by the Baer-Kaplansky-Theorem [9, Theorem 108.1], and B is A-projective of finite A-rank. By symmetry, A and B are similar. $\square$

Similar modules A and B share many homological properties. For instance, if A is flat [(fully) faithful] as an E_A-module, then the same holds for B. To see this, suppose that A is flat as an E_A-module, and consider a map $\phi : B^n \to B$. Since B is A-solvable, $ker\ \phi$ is A-solvable by the flatness of A. But the class of A-generated modules coincides with the class of B-generated modules. Hence, B is flat as an E_B-module by Ulmer's Theorem [12]. The case that A is faithful is treated in a similar way using the fact that A is faithful as an E_A-module if and only if every exact sequence $\oplus_I A \to A \to 0$ splits.

Moreover, $\mathcal{C}_A = \mathcal{C}_B$ if A and B are similar. To show this, observe that $M \in \mathcal{C}_A$ if and only if there exists an A-balanced exact sequence $0 \to U \to \oplus_I A \to M \to 0$ with $S_A(U) = U$. Since A and B are similar, B is A-projective, and this sequence is B-balanced. Moreover, $\oplus_I A$ is a B-projective module, and $S_B(M) = M$.

The next result summarizes the basic properties of similar modules, and relates the question which R-modules are similar to a given R-module A to the existence of projective generators in $\mathcal{M}_{E_A}$. As in [5], an R-module P is *a progenerator of* $\mathcal{M}_R$ if it is a finitely generated projective generator of $\mathcal{M}_R$. Moreover, $A^\perp$ denotes the collection of all *A-balanced* exact sequences, i.e. all sequences of R-modules, with respect to which A is projective.

Proposition 2.2. *The following are equivalent for self-small R-modules A and B:*

a) *A and B are similar.*
b) $\oplus_\omega A \cong \oplus_\omega B$.
c) $B \cong T_A(P)$ *for some progenerator* P *of* $\mathcal{M}_{E_A}$.
d) $\mathcal{C}_A = \mathcal{C}_B$ *and* $A^\perp = B^\perp$.

Proof. *a)* $\Rightarrow$ *c)*: Let $P = H_A(B)$. Since B is A-projective, it remains to show that $H_A(B)$ is a generator of $\mathcal{M}_{E_A}$. However, $B^n = A \oplus A_1$ for some $n < \omega$ yields that E_A is a direct summand of the right E_A-module $H_A(B)^n$, and hence $H_A(B)$ is a progenerator.

c) $\Rightarrow$ *b)*: Since P is finitely generated and projective, $P \cong H_A T_A(P)$. Hence, $H_A(B)$ is a progenerator of $\mathcal{M}_{E_A}$, and one obtains $\oplus_{I_0} H_A(B) = E_A \oplus M_1$ for some finite index-set I_0 and suitable E_A-module M_1. An application of T_A gives a decomposition of the form $\oplus_{I_0} B \cong A \oplus A_1$.

Moreover, $P \cong H_A(B)$ yields that there is a finite set J_0 such that $H_A(B) \oplus N_1 \cong \oplus_{J_0} E_A$ for some E_A-module N_1. Another application of T_A yields a decomposition $B \oplus B_1 \cong \oplus_{J_0} A$.

Since A_1 is a direct summand of $\oplus_{I_0} B$ and B_1 is a direct summand of $\oplus_{J_0} A$ with $|I_0|, |J_0| < \aleph_0$, there is a countable index-set J_1 with $A_1 \oplus A_2 \cong \oplus_{J_1} A$ for some R-module A_2. By symmetry, there is a countable index-set I_1 with $A_2 \oplus A_3 \cong \oplus_{I_1} B$. Then, $A \oplus (A_1 \oplus A_2) \oplus A_3 \cong (\oplus_{I_0} B) \oplus (\oplus_{I_1} B)$. Inductively, one obtains modules $A_1, A_2, \ldots$ and countable index-sets $I_1, I_2, \ldots$ and $J_1, J_2, \ldots$ such that

i) $A_{2n-1} \oplus A_{2n} \cong \oplus_{J_n} A$
ii) $A_{2n} \oplus A_{2n+1} \cong \oplus_{I_n} B$.

Thus, $U_n = A \oplus (A_1 \oplus \ldots A_{2n-1})$ is B-free, and $V_n = U_n \oplus A_{2n}$ is A-free for all n. Since $V_{n+1}/V_n \cong \oplus_{J_n} A$ and $U_{n+1}/U_n \cong \oplus_{I_n} B$, one obtains that $A \oplus (\oplus_{n=1}^\infty A_n)$ is isomorphic to both $\oplus_\omega A$ and $\oplus_\omega B$.

b) $\Rightarrow$ *d)*: Since the class of objects projective with respect to a given sequence is closed under direct sums and direct summands, $A^\perp = B^\perp$. If $M \in \mathcal{C}_A$, then there is an A-balanced exact sequence $0 \to U \to \oplus_I A \to M \to 0$ with $S_A(U) = U$. Since A is B-projective, $S_B(U) = U$, and the sequence is B-balanced.

d) $\Rightarrow$ *a)* : Since A is B-solvable, there is a B-balanced exact sequence $\oplus_I B \to A \to 0$ for some index-set I. But $B \in \mathcal{C}_A$ yields $S_A(B) = B$, and the sequence splits since $A^\perp = B^\perp$. By symmetry, B is A-solvable, too. Write $\oplus_I B = A \oplus A_1$ for some index-set I, and let $\alpha : A \to \oplus_I B$ be the embedding associated with this decomposition. Furthermore, denote the projection of $\oplus_I B$ onto its i^{th}-component by π_i, and let $\delta_i : A \to \Pi_I A$ be the embedding into the i^{th}-coordinate.

If $\alpha(A) \not\subseteq \oplus_J B$ for all finite subsets J of I, then $\pi_i\alpha(A) \neq 0$ for infinitely many $i \in I$. Without loss of generality, $\pi_i\alpha(A) \neq 0$ for all $i \in I$. For each $i \in I$, choose $a_i \in A$ with $\pi_i\alpha(a_i) \neq 0$. Since B is A-projective, there is a map $\phi_i : B \to A$ with $\phi_i\pi_i\alpha(a_i) \neq 0$. Define $\lambda : A \to \Pi_I A$ by $\lambda(a) = (\delta_i\phi_i\pi_i\alpha(a))_{i\in I}$. Since $\alpha(a) \in \oplus_I B$, one has $\pi_i\alpha(a) = 0$ for almost all $i \in I$, and $\lambda \in Hom(A, \oplus_I A)$. By the self-smallness of A, there is a finite subset J' of I with $\lambda(A) \subseteq \oplus_{J'} A$, which contradicts the fact that $\lambda(a_i)$ not contained in $\oplus_{J'} A$ for all $i \in I \setminus J'$. Hence, there is a finite $J \subseteq I$ with $\alpha(A) \subseteq \oplus_J B$, and A has finite B-rank. □

Corollary 2.3. *Let A and B be self-small R-modules which are faithful as modules over their endomorphism ring. Then, A and B are similar if and only if $\mathcal{C}_A = \mathcal{C}_B$.* □

However, the requirement that $H_A(B)$ is a progenerator of M_{E_A} is not strong enough to guarantee that A and B are similar unless one also requires that B is A-solvable. Moreover, $\mathcal{C}_A$ and $\mathcal{C}_B$ may coincide without A and B being similar if A or B are not fully faithful over their endomorphism ring:

Example 2.4. a) *Consider a torsion-free abelian group G of finite rank whose endomorphism ring is the ring of lower triangular 2×2-matrices over $\mathbb{Z}$. Then, $G = A \oplus B$ with $E_A = E_B = \mathbb{Z}$. Since $Hom_{\mathbb{Z}}(A, B) \cong \mathbb{Z}$, it is a projective generator of $\mathcal{A}b$, but A and B are not similar since $Hom_{\mathbb{Z}}(B, A) = 0$.*

b) *Let A be an abelian group with $Hom(A, \mathbb{Z}) \neq 0$. Then, every abelian group is A-generated, and hence A-solvable. Moreover, A is flat as an E-module by Ulmer's Theorem* [12]. *If A and B are two abelian groups with $Hom(A, \mathbb{Z}) \neq 0 \neq Hom(B, \mathbb{Z})$ such that E_A and E_B are not Morita-equivalent, then $\mathcal{C}_A = \mathcal{A}b = \mathcal{C}_B$. Hence, Theorem 4.1 and the equivalence of a) and d) in Proposition 2.2 may fail if A and B are not faithful.*

□

3. Morita-Equivalence

The first result of this section describes when two modules, which are faithfully flat over their endomorphism ring, have equivalent endomorphism rings.

Proposition 3.1. *Let R and S be rings. The following are equivalent for self-small modules $A \in \mathcal{M}_R$ and $B \in \mathcal{M}_S$ which are faithfully flat as modules over their endomorphism ring:*

a) *E_A and E_B are equivalent rings.*

b) *There exist left-exact additive functors $\mathcal{F} : \mathcal{M}_R \to M_S$ and $\mathcal{G} : \mathcal{M}_S \to \mathcal{M}_R$ which commute with direct sums such that*

i) *There are natural transformations $\sigma : T_A H_A \to \mathcal{G}\mathcal{F}$ and $\psi : T_B H_B \to \mathcal{F}\mathcal{G}$.*

ii) *$\mathcal{F}$ and $\mathcal{G}$ induce a category equivalence between $\mathcal{P}_A$ and $\mathcal{P}_B$.*

Proof. *a) $\Rightarrow$ b)*: Since E_A and E_B are equivalent rings, select a progenerator P of $\mathcal{M}_{E_B}$ such that $E_{E_B}(P) = E_A$. Then, $\tilde{\mathcal{F}} = - \otimes_{E_A} P$ defines an equivalence between $\mathcal{M}_{E_A}$ and $\mathcal{M}_{E_B}$ whose inverse $\tilde{\mathcal{G}}$ can also be presented as a tensor-product involving a progenerator of $\mathcal{M}_{E_A}$. Set $\mathcal{F} = T_B \tilde{\mathcal{F}} H_A$ and $\mathcal{G} = T_A \tilde{\mathcal{G}} H_B$. For every $M \in \mathcal{M}_R$, one obtains a natural isomorphism $\lambda_{H_A(M)} : H_A(M) \to \tilde{\mathcal{G}}\tilde{\mathcal{F}} H_A(M)$ and a natural morphism $\tilde{\phi_M} : \tilde{\mathcal{F}} H_A(M) \to H_B T_B \tilde{\mathcal{F}} H_A(M)$ induced by the transformation $\phi : 1 \to H_B T_B$. Then $\sigma_M = T_A \tilde{G}(\tilde{\phi}_M) T_A(\lambda_{H_A(M)})$ is the desired transformation. Observe that $\tilde{\phi}_M$ is an isomorphism whenever M is projective.

Since A is flat as an E_A-module, T_A is an exact functor. Hence, $\mathcal{F}$ is left-exact as a composition of the left-exact functor H_A with two exact functors. Finally, observe that H_A, $\tilde{\mathcal{F}}$, and T_B are equivalences between $\mathcal{P}_A$, $\mathcal{P}_{E_A}$, $\mathcal{P}_{E_B}$, and $\mathcal{P}_B$ respectively since A and B are self-small.

b) $\Rightarrow$ a): In the first step, one shows that $\mathcal{F}(A)$ is self-small. Consider a map $\alpha : \mathcal{F}(A) \to \oplus_{n<\omega} \mathcal{F}(A)$, and denote the embeddings into the n^{th}-coordinate by δ_n, while the projection onto the n^{th}-coordinate is denoted by π_n. Then, $\mathcal{G}(\oplus_{n<\omega}\mathcal{F}(A))$ together with the maps $\{\mathcal{G}(\delta_n)\}_{n<\omega}$ is the coproduct of countably many copies of $\mathcal{G}\mathcal{F}(A)$ in P_A since equivalences preserve coproducts. On the other hand, the R-module $\oplus_\omega \mathcal{G}\mathcal{F}(A)$ together with the coordinate embeddings η_n also is a P_A-coproduct of countably many copies of $\mathcal{G}\mathcal{F}(A)$ since P_A is a full subcategory of $\mathcal{M}_R$, which is closed with respect to direct sums. Hence, there is an R-module isomorphism $\lambda : \oplus_\omega \mathcal{G}\mathcal{F}(A) \to \mathcal{G}(\oplus_\omega \mathcal{F}(A))$. In particular, the R-module coproducts are the coproducts in $\mathcal{P}_A$. Furthermore, $\mathcal{G}(\delta_n) = \lambda\eta_n$ yields $\mathcal{G}(\oplus_\omega \mathcal{F}(A)) = \oplus_{n<\omega} im\, \mathcal{G}(\delta_n)$. Since A is self-small, there is $m < \omega$ with $\mathcal{G}(\lambda) \subseteq \oplus_{n=1}^m \mathcal{G}(A)$, and hence $\mathcal{G}(\pi_n)\mathcal{G}(\lambda) = 0$ for all $n > m$.

Because $\mathcal{G}$ is an equivalence between $\mathcal{P}_B$ and $\mathcal{P}_A$, the module $\mathcal{G}(B)$ is A-projective, and there is a split-exact sequence $0 \to U \to \oplus_I A \to \mathcal{G}(B) \to 0$. In the last paragraph, it was shown that the coproducts in P_A are the coproducts of R-modules. Consequently, $\mathcal{F}(\oplus_I A) \cong \oplus_I \mathcal{F}(A)$, and the induced sequence $0 \to \mathcal{F}(U) \oplus_I \mathcal{F}(A) \to \mathcal{F}\mathcal{G}(B) \to 0$ is exact. Therefore, $B \cong \mathcal{F}\mathcal{G}(B)$ is $\mathcal{F}(A)$-projective. Since $\mathcal{F}(A)$ is self-small, one obtains that B and $\mathcal{F}(A)$ are similar by arguing as in the proof of Proposition 2.2. However, similar modules have Morita equivalent endomorphism rings. $\square$

Let R and S be rings, and consider modules $A \in \mathcal{M}_R$ and $B \in \mathcal{M}_S$, which are faithfully flat as modules over their endomorphism ring. A module $M \in \mathcal{C}_A$ is *A-B-Morita-invariant* if $H_A(M) \otimes_E P \in \mathcal{M}_B$ whenever P is a progenerator of M_{E_B} such that $End_{E_B}(P) = E_A$. The module $M \in \mathcal{C}_A$ is *A-Morita-invariant* R-modules if it is A-B-Morita-invariant for all possible B. The class of A-Morita-invariant modules is denoted by MI_A.

A R-module M is *locally A-projective* if every finite subset of M is contained in an A-projective direct summand of M. The module M is *κ-A-projective* if every subset X of M with $|X| < \kappa$ is contained in an A-projective submodule of M. Finally, M is *A-torsion-free* if every finitely A-generated submodule of M can be embedded into an A-projective module of finite A-rank. Again, the references to A are omitted if $A = R$.

Theorem 3.2. *Let $A \in \mathcal{M}_R$ and $B \in \mathcal{M}_S$ be self-small modules, which are faithfully flat as modules over their endomorphism ring.*

a) *The class of A-B-Morita invariant modules is closed with respect to finite direct sums, extensions and A-generated submodules.*
b) *If E_A is right and left Noetherian, then locally A-projective modules are A-B-Morita-invariant.*
c) *If κ is a regular cardinal with $|A|, |B| < \kappa$, then every κ-A-projective R-module is A-B-Morita-invariant.*
d) *If E_A is an integral domain, then all A-torsion-free R-modules are A-A-Morita-invariant.*

Proof. Let P a progenerator of $\mathcal{M}_{E_B}$ with $E_{E_B}(P) = E_A$.

a) Observe that all functors involved commute with finite direct sums, and that $\mathcal{C}_A$ is closed with respect to finite direct sums.

Consider an exact sequence $0 \to U \to M \to N \to 0$ in which $U, N \in MI_A$. Since A is faithfully flat as an E_A-module, this sequence is A-balanced, and an application of $- \otimes_{E_A} P$ yields the exact sequence $0 \to H_A(U) \otimes_{E_A} P \to H_A(M) \otimes_{E_A} P \to H_A(N) \otimes_{E_A} P \to 0$ in which the outer terms are elements of $\mathcal{M}_B$. An application of T_B gives the exact sequence $0 \to T_B(H_A(U) \otimes_{E_A} P) \to T_B(H_A(M) \otimes_{E_A} P) \to T_B(H_A(N) \otimes_{E_A} P) \to 0$, in which $T_B(H_A(M) \otimes_{E_A} P)$ is B-generated and $T_B(H_A(N) \otimes_{E_A} P)$ is B-solvable because of $H_A(B) \otimes_{E_A} P \in \mathcal{M}_B$. Hence, the last sequence is B-balanced by the faithful flatness of A. The standard commutative diagram induced by the natural transformation ϕ yields $H_A(M) \otimes_{E_A} P \in \mathcal{M}_B$.

Let U be an A-generated submodule of a module $M \in MI_A$. Then, $H_A(U) \otimes_{E_A} P$ is isomorphic to a submodule of $H_A(M) \otimes_{E_A} P \in \mathcal{M}_B$. Since B is faithfully flat as an E_B-module, $\mathcal{M}_B$ is closed with respect to submodules.

b) Observe that being right or left Noetherian is a Morita-invariant property. It suffices to establish that a right and left Noetherian endomorphism ring is *discrete in the finite topology*, i.e. that there is a finite subset X of A such that $\phi(X) = 0$ yields $\phi = 0$ for all $\phi \in E$. Once this has been shown, H_A and T_A induce a category equivalence between the locally A-projective groups and the locally projective E_A-modules [7], and the same holds for B. Then, $H_A(M) \otimes_{E_A} P$ is a locally projective E_B-module because a Morita equivalence sends locally projective E_A-modules to locally projective E_B-modules.

Suppose that, for all finite subsets X of A, there is a non-zero $\phi \in E_A$ with $\phi(X) = 0$. Then, there exist an ascending chain $X_1 \subseteq X_2 \subseteq \ldots$ of finite subsets of A such that $(X_{n+1})_*$ is a proper subset of $(X_n)_*$ where $X_* = \{\phi \mid \phi(X) = 0\}$. Since E_A is left Noetherian, $(X_n)_*$ is finitely generated, say by $\alpha_1, \ldots \alpha_m$. Then, $(X_n)_{**} = \cap\{ker\ \alpha \mid \alpha \in (X_n)_*\}$ coincides with $ker\ \alpha_1 \cap \ldots \cap ker\ \alpha_m$. However, the latter is the kernel of the map $\sigma : A \to A^m$ defined by $\sigma(a) = (\alpha_1(a), \ldots, \alpha_m(a))$. Since A is flat as an E_A-module, $ker\ \sigma$ is A-generated, and hence $(X_1)_{**} \subseteq (X_2)_{**} \subseteq \ldots$ forms a strictly ascending chain of A-solvable subgroups of A. If this chain does not become stationary, it induces an infinite strictly ascending chain $H_A((X_1)_{**}) \subseteq H_A((X_2)_{**}) \subseteq \ldots$ of right ideals of E_A, whose existence contradicts the fact that E_A is right Noetherian.

c) Since $|A| < \kappa$, one obtains that every κ-A-projective module is A-solvable, and $H_A(M)$ is a κ-projective E_A-module. Because Morita-equivalence preserves κ-projectivity, $H_A(M) \otimes_{E_A} P$ is a κ-projective E_B-module. Then, the S-module $T_B(H_A(M) \otimes_{E_A} P)$ is κ-B-projective and hence B-solvable. Since A is faithfully flat as an E_A-module, the natural map from $H_A(M) \otimes_{E_A} P$ into $T_B H_A(H_A(M) \otimes_{E_A} P)$ is a monomorphism. Moreover, the latter is an element of $\mathcal{M}_B$, and $\mathcal{M}_B$ is closed with respect to submodules.

d) Let Q be the field of quotients of E, and consider a progenerator P of $\mathcal{M}_E$ whose E-endomorphism ring is E. The injective hull $\hat{P}$ is a finitely generated vector-space over the field Q, say $\hat{P} \cong Q^n$ for some $n < \omega$. Since E is the E-endomorphism ring of P, and P is finitely generated, one obtains that Q is the E-endomorphism ring of $\hat{P}$. On the other hand, the latter ring is isomorphic to $Mat_n(Q)$. Therefore, $n = 1$, and P is isomorphic to an ideal of E. Without loss of generality, one may assume that P is an ideal of E. Since P is a progenerator of $\mathcal{M}_E$, one obtains that P is a faithfully balanced $E_E(P_E)$-E-bimodule. If $\alpha : P_E \to P_E$ is an E-map, then $\alpha\beta(x) = \beta\alpha(x)$ for all $x \in P$ and $\beta \in E$ since E is commutative. Thus, α is an endomorphism of the left $End_E(P_E)$-module P. Hence, there is $r \in E$ such that $\alpha(x) = xr$ for all $x \in P$. In particular, the inclusion map $\iota : P \to E$ is a bimodule-morphism.

Let M be an A-torsion-free R-module. Then, M is A-solvable, and $H_A(M)$ is a torsion-free E-module. The map ι induces a sequence $0 \to Tor_1^E(H_A(M), R/P) \to H_A(M) \otimes_E P \to H_A(M) \otimes_E E$ in which the last map is a right E-module map. Since P is flat, $H_A(M) \otimes_E P$ is torsion-free, and $Tor_1^E(H_A(M), R/P) = 0$ because E is an integral domain. Therefore, $H_A(M) \otimes_E P$ is isomorphic to a submodule of $H_A(M) \in \mathcal{M}_A$. Since A is faithfully flat as an E-module, $\mathcal{M}_A$ is closed with respect to submodules. Therefore, $G \in MI(A)$. □

However, the class of A-B-Morita-invariant modules need not be closed under infinite direct sums of A-small families as the following example shows: Let $A = \mathbb{Z}$ and $B = \langle \frac{1}{p} \mid p \text{ } a \text{ } prime \rangle \subseteq \mathbb{Q}$. Since $E_B = \mathbb{Z}$, choose $P = \mathbb{Z}$ as a progenerator of $\mathcal{A}b$. Observe that $\mathbb{Z}/p\mathbb{Z} \in \mathcal{M}_B$ for all primes p. However, $\oplus_p \mathbb{Z}/p\mathbb{Z} \notin \mathcal{M}_B$ since $T_B(\oplus_p \mathbb{Z}/p\mathbb{Z}) \cong \oplus_p \mathbb{Z}/p\mathbb{Z}$ is not B-solvable because $\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ } a \text{ } prime\}$ is not B-small.

Theorem 3.2 yields that the class of A-Morita-invariant modules is closed with respect to finite direct sums, A-generated submodules, and A-generated extensions. Clearly, A-projective modules are A-Morita-invariant. However, there may exist Morita-invariant modules which are not A-generated submodules of A-projective since every locally A-projective module is A-Morita-invariant if E_A is right Noetherian.

Corollary 3.3. *Every A-Morita-invariant group is torsion-free if A is a torsion-free reduced abelian group whose endomorphism ring is a subring of a finite dimensional $\mathbb{Q}$-algebra.*

Proof. Suppose that there exist an A-Morita-invariant group G such that $G[p] \neq 0$ for some prime p. Then, G has a cocyclic summand C which is A-solvable, and one may assume that G is cocyclic. Select an A-balanced exact sequence $0 \to U \xrightarrow{\alpha} P \xrightarrow{\beta} G \to 0$ in which P is A-projective and $S_A(U) = U$. If $A = pA$, then both U and P are p-divisible. If $0 \neq x \in G[p]$, then $x = \beta(y)$ for some $y \in P$. Then, $\beta(py) = px = 0$, and there is $z \in U$ with $py = \alpha(z) = p\alpha(z')$ for some suitable $z' \in U$. Since A is torsion-free, $y = \alpha(z')$ which is not possible. Hence, $A \neq pA$. Therefore, $G[p]$ is an A-generated subgroup of G. In particular, $\mathbb{Z}/p\mathbb{Z}$ is A-Morita-invariant by the remarks preceding the corollary.

By Corner's Theorem, there is a torsion-free abelian group B of rank 2n with $E_B = E$ where $n = r_0(E)$. Then, $H_A(\mathbb{Z}/p\mathbb{Z}) \in \mathcal{M}_B$. In particular, $T_B H_A(\mathbb{Z}/p\mathbb{Z})$ is a p-bounded abelian group. Consequently, $\mathbb{Z}/p\mathbb{Z} \in \mathcal{C}_A$. Since $B/pB \cong (\mathbb{Z}/p\mathbb{Z})^n$, the group B/pB is B-solvable. Because B is faithfully flat as an E-module, the sequence $0 \to B \xrightarrow{p} B \to B/pB \to 0$ is A-balanced. Hence, $0 \to H_B(B) \xrightarrow{p} H_B(B) \to H_B(B/pB) \to 0$. Therefore, $[r_p(B)]^2 = r_p(E) < \infty$. On the other hand, $r_p(B) = r_p(E)$ yields $r_p(E) = 1$.

On the other hand, there is a torsion-free abelian group C with $E(C) = E$ such that $r_p(C)$ is infinite and C is faithfully flat as an E-module. Using the same arguments as before, one obtains that $\mathbb{Z}/p\mathbb{Z}$ is C-solvable. There is a subgroup W of C with $C/W \cong \mathbb{Z}/p\mathbb{Z}$. In particular, W is C-solvable, and one has an exact sequence $0 \to H_C(W) \to H_C(C) \to H_C(\mathbb{Z}/p\mathbb{Z}) \to 0$. However, $pC \subseteq W$ yields $pH_C(C) \subseteq H_C(W)$ and hence $H_C(C)/H_C(W)$ is an epimorphic image of E/pE. Consequently, $r_p(H_C(\mathbb{Z}/p\mathbb{Z})) = r_p(E) = 1$. On the other hand,

$$r_p(H_C(\mathbb{Z}/p\mathbb{Z})) = r_p(Hom(C/pC, \mathbb{Z}/p\mathbb{Z})) = 2^{r_p(C)} \geq 2^{\aleph_0},$$

a contradiction. □

4. Category Equivalences

In this section, $\mathcal{F}$ and $\mathcal{G}$ denote additive functors, which are defined on full subcategories of module categories. If $\mathcal{F}$ and $\mathcal{G}$ are mutually inverses, then $\mathcal{F}$ and $\mathcal{G}$ commute with respect to arbitrary direct sums provided they exist in the subcategories.

Theorem 4.1. *Let R and S be rings. The following conditions are equivalent for self-small modules $A \in \mathcal{M}_R$ and $B \in \mathcal{M}_S$ which are faithfully flat as modules over their endomorphism ring.*

a) *$\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent categories.*
b) *There exists a Morita-equivalence between $\mathcal{M}_{E_A}$ and $\mathcal{M}_{E_B}$, which restricts to an equivalence between $\mathcal{M}_A$ and $\mathcal{M}_B$.*

Proof. *a) ⇒ b)*: Suppose that the equivalence is given by two functors $\mathcal{F} : \mathcal{C}_A \to \mathcal{C}_B$ and $\mathcal{G} : \mathcal{C}_B \to \mathcal{C}_A$ with corresponding natural transformations $\phi : \mathcal{G}\mathcal{F} \to 1_{\mathcal{C}_A}$ and $\psi : \mathcal{F}\mathcal{G} \to 1_{\mathcal{C}_B}$.

The first step shows that $\mathcal{F}$ transforms exact sequences of R-modules with entries from $\mathcal{C}_A$ into exact sequences of S-modules. For this consider an exact sequences $0 \to C \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ of R-modules with $C, M, N \in \mathcal{C}_A$. Then, α is the $\mathcal{C}_A$-kernel of β because A is flat over its endomorphism ring (see [2]). Since $\mathcal{F}$ preserves kernels, $\mathcal{F}(\alpha)$ is the $\mathcal{C}_B$-kernel of $\mathcal{F}(\beta)$. However, the C_B-kernel of $F(\beta)$

coincides with its kernel as a S-module since B is flat over its endomorphism ring. Hence, $\mathcal{F}(\alpha)$ is one-to-one, and the sequence $0 \to \mathcal{F}(C) \stackrel{\mathcal{F}(\alpha)}{\to} \mathcal{F}(M) \stackrel{\mathcal{F}(\beta)}{\to} \mathcal{F}(N)$ is exact. Since S- and $\mathcal{C}_B$-cokernels need not coincide, it remains to show that $\mathcal{F}(\beta)$ is onto as a S-module map. If $H = im\ \mathcal{F}(\beta)$, then H is B-solvable because it is a B-generated submodule of the B-solvable module $\mathcal{F}(N)$ and B is flat over its endomorphism ring. Moreover, $\mathcal{F}(\beta)$ induces an S-epimorphism $\overline{\beta} : \mathcal{F}(M) \to H$. Observe that $H \cong \mathcal{F}(M)/im\ \mathcal{F}(\alpha)$ yields that $\overline{\beta}$ is a $\mathcal{C}_B$-cokernel of $\mathcal{F}(\alpha)$. Hence, there is a $\mathcal{C}_B$-isomorphism $\delta : H \to \mathcal{F}(N)$ with $\delta\overline{\beta} = \mathcal{F}(\beta)$. Since δ is an S-isomorphism, $\mathcal{F}(\beta)$ is onto as a S-module map.

In the next step, one establishes that B and $\mathcal{F}(A)$ are similar. For this, consider an exact sequence $0 \to U \to \oplus_I B \to \mathcal{F}(A) \to 0$ of S-modules in which U is B-solvable. It induces the exact sequence $0 \to \mathcal{G}(U) \to \mathcal{G}(\oplus_I B) \to \mathcal{G}\mathcal{F}(A) \to 0$, which splits since $\mathcal{G}\mathcal{F}(A) \cong A$ is faithfully flat over its endomorphism ring and $\mathcal{G}\mathcal{F}(\oplus_I B)$ is A-solvable. Hence, $\mathcal{F}\mathcal{G}\mathcal{F}(A) \cong \mathcal{F}(A)$ is B-projective as a direct summand of $\mathcal{F}\mathcal{G}(\oplus_I B) \cong \oplus_I B$. A similar argument shows that B is $\mathcal{F}(A)$-projective using an exact sequence of the form $0 \to V \to \oplus_J A \to \mathcal{G}(B) \to 0$. Then, B and $\mathcal{F}(A)$ are similar by Proposition 2.2. Hence, $H_B(\mathcal{F}(A))$ is a progenerator of $\mathcal{M}_{E_B}$ once one has established that $\mathcal{F}(A)$ is self-small:

Let $\lambda : \mathcal{F}(A) \to \oplus_\omega \mathcal{F}(A)$ be a S-module-map such that $\pi_n\lambda \neq 0$ for infinitely many $n < \omega$ where $\pi_n : \oplus_\omega \mathcal{F}(A) \to \mathcal{F}(A)$ is the projection onto the n^{th}-coordinate. Then, $\mathcal{G}(\lambda) : \mathcal{G}\mathcal{F}(A) \to \mathcal{G}(\oplus_\omega \mathcal{F}(A))$ satisfies $\mathcal{G}(\pi_n)\mathcal{G}(\lambda) \neq 0$ for infinitely many $n < \omega$. Observe that the embeddings $\delta_n : \mathcal{F}(A) \to \oplus_\omega \mathcal{F}(A)$ into the n^{th}-coordinate have the property $\mathcal{G}(\oplus_\omega \mathcal{F}(A)) = \Sigma_{n<\omega} im\ \mathcal{G}(\delta_n)$ since $\mathcal{G}(\oplus_\omega \mathcal{F}(A))$ is the $\mathcal{C}_B$-coproduct of $\{\mathcal{G}(\delta_n)\}_{n<\omega}$. This sum is direct because $\mathcal{G}(\pi_n)\mathcal{G}(\delta_m) = \delta_{nm}$ where $\delta_{nm} = 0$ if $m \neq n$ and $\delta_{nm} = 1$ if $n = m$. Since $A \cong \mathcal{G}\mathcal{F}(A)$, there is $n_0 < \omega$ with $\mathcal{G}(\lambda)(A) \subseteq \Sigma_{k=0}^{n_0} im\ \mathcal{G}(\delta_k)$ by the self-smallness of A. Therefore, $\mathcal{G}(\pi_n)\mathcal{G}(\lambda) = 0$ for all $n > n_0$.

The desired equivalence between $\mathcal{M}_{E_A}$ and $\mathcal{M}_{E_B}$ is obtained by observing that E_A operates on $P = H_B(\mathcal{F}(A))$ by $\phi * x = [H_B(\mathcal{F}(\phi))](x)$, and that the assignment $\phi \to H_B(\mathcal{F}(\phi))$ defines a ring isomorphism between E_A and $E_{E_B}(P)$ since $H_B\mathcal{F}$ is an equivalence between $\mathcal{C}_A$ and $\mathcal{M}_B$ with inverse $T_B\mathcal{G}$. For reasons of simplicity, identify E_A with the E_B-endomorphism ring of P. Since P is a progenerator of $\mathcal{M}_{E_B}$, one obtains from [5, Theorem 17.8] that P is a faithfully balanced E_A-E_B-bimodule which is finitely generated and projective as a left E_A-module. Moreover, P is a generator of ${}_{E_A}\mathcal{M}$ by [5, Lemma 17.7]. By Morita's Theorem [5, Theorem 22.4], the functors $\tilde{\mathcal{F}} : \mathcal{M}_{E_A} \to \mathcal{M}_{E_B}$ and $\tilde{\mathcal{G}} : \mathcal{M}_{E_B} \to$

$\mathcal{M}_{E_A}$ defined by $\tilde{\mathcal{F}} = - \otimes_{E_A} P$ and $\tilde{\mathcal{G}} = Hom_{E_B}(P, -)$ are an equivalence between $\mathcal{M}_{E_A}$ and $\mathcal{M}_{E_B}$.

Let $\mathcal{F}_A$ be the full subcategory of $\mathcal{C}_A$ whose objects are the A-free modules. For $F_1 = \oplus_I A$ and $F_2 = \oplus_J A$ in $\mathcal{F}_A$, one has

$$T_B \tilde{\mathcal{F}} H_A(F_i) = T_B(H_A(F_i) \otimes_{E_A} P) =$$

$$H_A(F_i) \otimes_{E_A} T_B H_B(\mathcal{F}(A)) \cong_{nat} H_A(F_i) \otimes_{E_A} \mathcal{F}(A)$$

where the E_A-module-structure of $\mathcal{F}(A)$ is defined by $r * x = \mathcal{F}(r)(x)$ for all $x \in \mathcal{F}(A)$ and $r \in E_A$. If $e_i : A \to F_1$ and $f_j : A \to F_2$ denote the embeddings into the i^{th}- and j^{th}-coordinate respectively, then $\{e_i\}_{i \in I} \subseteq H_A(F_1)$ and $\{f_j\}_{j \in J} \subseteq H_A(F_2)$ are E_A-bases. Define a map

$$\eta_1 : H_A(F_1) \otimes_{E_A} \mathcal{F}(A) \to \mathcal{F}(F_1)$$

by $\eta_1((\Sigma_{i \in I} e_i r_i) \otimes x) = \Sigma_{i \in I} \mathcal{F}(e_i)[\mathcal{F}(r_i)](x)$. Observe that η_1 and η_2 are S-module maps since $\mathcal{F}(A)$ and $\mathcal{F}(F_i)$ carry a right S-module-structure, see e.g. [10]. Since every $x \in H_A(F_1) \otimes_{E_A} \mathcal{F}(A)$ can be written as $x = \Sigma_{j=1}^n (\Sigma_{i \in I} e_i t_{ij}) \otimes x_j = \Sigma_{i \in I} e_i \otimes y_i$ where $y_i = \Sigma_{j=1}^n \mathcal{F}(t_{ij})(x_i)$, one obtains $\eta_1(x) = \Sigma_{i \in I} \mathcal{F}(e_i) y_i$. Since $\mathcal{F}(e_1)$ is one-to-one, η_1 is an isomorphism. In a similar way, one obtains an isomorphism $\eta_2 : H_A(F_2) \otimes_{E_A} \mathcal{F}(A) \to \mathcal{F}(F_2)$.

If $\phi : F_1 \to F_2$, then, for each $i \in I$, there are $r_{ij} \in E_A$ with $r_{ij} = 0$ for almost all $j \in J$ such that $H_A(\phi)(e_i) = \Sigma_{j \in J} f_j r_{ij}$. Furthermore, $\phi(\Sigma_{i \in I} e_i(a_i)) = \Sigma_{j \in J} f_j(\Sigma_{i \in I} r_{ij}(a_i))$ for all $a_i \in A$ with $a_i = 0$ for almost all i. Since $\mathcal{F}(F_1) = \Sigma_{i \in I}\ im\ \mathcal{F}(e_i)$, one obtains $F(\phi)(\Sigma_{i \in I} \mathcal{F}(e_i)(x_i)) = \Sigma_{j \in J} \mathcal{F}(f_j)(\Sigma_{i \in I} \mathcal{F}(r_{ij})(x_i))$ for all $x_i \in \mathcal{F}(A)$ such that $x_i = 0$ for almost all $i \in I$. Moreover, the diagram

$$\begin{array}{ccc} H_A(F_1) \otimes_{E_A} \mathcal{F}(A) & \xrightarrow{\phi \otimes 1_{\mathcal{F}(A)}} & H_A(F_2) \otimes_{E_A} \mathcal{F}(A) \\ \wr \downarrow \eta_1 & & \wr \downarrow \eta_2 \\ \mathcal{F}(F_1) & \xrightarrow{\mathcal{F}(\phi)} & \mathcal{F}(F_2) \end{array}$$

commutes since

$$\mathcal{F}(\phi)\eta_1(e_i \otimes x_i) = F(\phi)(\mathcal{F}(e_i)(x)) = \Sigma_{j \in J} \mathcal{F}(f_j)[\mathcal{F}(r_{ij})](x_i).$$

while

$$\eta_2(H_A(\phi) \otimes 1_{\mathcal{F}(A)})(e_i \otimes x_i) = \eta_2(\Sigma_{j \in J} f_j r_{ij} \otimes x_i) = \Sigma_{j \in J} \mathcal{F}(f_j)[F(r_{ij})](x_i).$$

The standard arguments from homological algebra show that η_1 and η_2 do not depend on the chosen embeddings $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$. Hence, $T_B \tilde{\mathcal{F}} H_A$ and $\mathcal{F}$ are naturally equivalent when restricted to $\mathcal{F}_A$.

But $\tilde{\mathcal{F}}H_A(F)$ is B-projective for each A-free module F, and H_BT_B is naturally equivalent to $1_{P_{E_B}}$ where P_{E_B} is the category of projective right E_B-modules. Therefore, $\tilde{\mathcal{F}}H_A$ is naturally equivalent to $H_B\mathcal{F}$ on $\mathcal{F}_A$. Once it has been shown that this equivalence can be extended to $\mathcal{C}_A$, then for all $M \in \mathcal{M}_A$, the module $T_A(M)$ is A-solvable, and

$$M \otimes_{E_A} \mathcal{F}(A) \cong H_AT_A(M) \otimes_{E_A} \mathcal{F}(A) \cong H_B\mathcal{F}(T_A(M)) \in \mathcal{M}_B.$$

By symmetry, $\tilde{\mathcal{G}} : \mathcal{M}_B \to \mathcal{M}_A$ is the inverse to $\tilde{\mathcal{F}}$.

Given an A-solvable module M, there are A-balanced exact sequences $0 \to U \to F_1 \to M \to 0$ and $0 \to V \to F_2 \to U \to 0$ such that F_1 and F_2 are A-free, and U and V are A-solvable since A is faithfully flat as an E_A-module. By what has been shown, $\mathcal{F}$ carries these sequences into exact sequences of S-modules which are B-balanced because B is faithfully flat as an E_B-module. Hence, the exact sequence $F_2 \to F_1 \to M \to 0$ remains exact after an application of $H_B\mathcal{F}$ or $(- \otimes_{E_A} P)H_A$ since P is a projective E_A-module. One obtains the commutative diagram

$$\begin{array}{ccccccc}
H_B\mathcal{F}(F_2) & \longrightarrow & H_B\mathcal{F}(F_1) & \longrightarrow & H_B\mathcal{F}(M) & \longrightarrow & 0 \\
\wr\downarrow \eta_2 & & \wr\downarrow \eta_1 & & \wr\downarrow \eta_M & & \\
\tilde{\mathcal{F}}H_A(F_2) & \longrightarrow & \tilde{\mathcal{F}}H_A(F_1) & \longrightarrow & \tilde{\mathcal{F}}H_A(M) & \longrightarrow & 0
\end{array}$$

with induced isomorphism η_M. Observe that the bottom-row is exact since the functor $- \otimes_{E_A} \mathcal{F}(A)$ is exact because P is a projective E_A-module. To see that η is naturally, choose $N \in \mathcal{C}_A$ and a morphism $\phi : M \to N$. There is an exact sequence $P_2 \to P_1 \to N \to 0$ with P_1 and P_2 A-free. By the faithful flatness of A as an E_A-module, one obtains a commutative diagram

$$\begin{array}{ccccccc}
F_2 & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi & & \\
P_2 & \longrightarrow & P_1 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

for suitable maps ϕ_1 and ϕ_2. The standard arguments can now be used to show that η_M is natural and independent of the chosen A-free resolutions.

b) $\Rightarrow$ *a)*: Set $\mathcal{F} = T_B(- \otimes_{E_A} P)H_A$ where $- \otimes_{E_A} P$ is the given Morita-equivalence inducing an equivalence between $\mathcal{M}_A$ and $\mathcal{M}_B$. □

Corollary 4.2. *Let A and B as in Theorem 4.1. If $\mathcal{F} : C_A \to \mathcal{C}_B$ is a category equivalence, then there is a progenerator P of $\mathcal{M}_{E_B}$ with $E_A \cong E_{E_B}(P)$ such that* *the diagram*

$$\begin{array}{ccc} \mathcal{C}_A & \xrightarrow{\mathcal{F}} & \mathcal{C}_B \\ \downarrow{\scriptstyle H_A} & & \downarrow{\scriptstyle H_B} \\ \mathcal{M}_{E_A} & \xrightarrow{-\otimes_{E_A} P} & \mathcal{M}_{E_B} \end{array}$$

commutes.

In particular, the Morita equivalence of E_A and E_B in the last result guarantees the following:

- If A is a generalized rank 1 group, then so is B.
- If E_A is right Noetherian (right hereditary, right semi-hereditary, right strongly non-singular), then the same holds for E_B.

On the other hand, $\mathbb{Z}[x]$ contains an ideal I which is generated by 2 elements, but is not projective. Therefore, $Mat_2(\mathbb{Z}[x])$ is not a right p.p.-ring, although $\mathbb{Z}[x]$ is. Select a self-small group A with $E_A = \mathbb{Z}[x]$ which is faithfully flat as an E_A module to obtain examples of groups A and $B = A \oplus A$ such that E is right p.p., but $E(B)$ is not. Therefore, being a right p.p. ring is a property which is not preserved under Morita-equivalence.

Corollary 4.3. *Let R and S be ring. Suppose $A \in \mathcal{M}_R$ and $B \in \mathcal{M}_S$ are self-small modules which are faithfully flat over their endomorphism ring and have the property that $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent via a functor $\mathcal{F}$.*

a) *The classes of finitely A-generated, A-torsionless, locally A-projective, A-presented, A-torsion-free, and weakly A-solvable groups are equivalent to their B-counterparts under $\mathcal{F}$.*
b) *If $M \in \mathcal{C}_A$, then A-p.d.$M = B$-p.d.$F(M)$.*

□

Example 4.4. *If R and S are Morita-equivalent rings with associated category equivalences $\mathcal{F} : \mathcal{M}_R \to \mathcal{M}_S$ and $\mathcal{G} : \mathcal{M}_S \to \mathcal{M}_R$, then $\mathcal{C}_A$ and $\mathcal{C}_{\mathcal{F}(A)}$ are equivalent for all $A \in \mathcal{M}_R$.*

Proof. Let M be A-solvable module, and consider an A-balanced exact sequence $0 \to U \to \oplus_I A \xrightarrow{\beta} M \to 0$ in which U is A-generated. There is an epimorphism $\oplus_J A \to U \to 0$. Applications of $\mathcal{F}$ induce the exact sequences $0 \to \mathcal{F}(U) \to \mathcal{F}(\oplus_I A) \to \mathcal{F}(M) \to 0$ and $\mathcal{F}(\oplus_J A) \to \mathcal{F}(U) \to 0$. Moreover, if $\phi : \mathcal{F}(A) \to \mathcal{F}(M)$, then there is $\psi : \mathcal{G}\mathcal{F}(A) \to \mathcal{G}\mathcal{F}(\oplus_I A)$ such that $\mathcal{G}\mathcal{F}(\beta)\psi = \mathcal{G}(\phi)$. Apply $\mathcal{F}$ once more. □

Clearly, Theorem 4.1 applies if A and B are similar R-modules, as is for instance demonstrated by the next result:

Corollary 4.5. *Let A and B be torsion abelian groups. Then, $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent if and only if A and B are similar.*

PROOF. By Proposition 2.1, torsion groups with Morita-equivalent endomorphism rings are similar. □

However, there exist R-modules A and B which are not similar, but for which $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent categories. The construction of such modules is based on the following description of $\mathcal{M}_A$ in case that A is a torsion-free abelian group of rank 1. Let $\pi(A) = \{p \mid A = pA\}$, and consider a non-zero $a \in A$. The set $\sigma_a(A) = \{p \mid 0 < h_p^A(a) < \infty\}$ is uniquely determined up to equivalence of sets by A [9]. Denote the equivalence class of $\sigma_a(A)$ by $\sigma(A)$. Observe that $E = \mathbb{Z}[\frac{1}{p} \mid p \in \pi(A)]$.

Theorem 4.6. *Let A be a subgroup A of $\mathbb{Q}$, and $0 \neq a \in A$. An E-module M is in $\mathcal{M}_A$ if and only if $M[p] = 0$ for all but finitely many $p \in \sigma_a(A)$.*

PROOF. Firstly, observe that every E-module M has the property that $M[p] = 0$ for all primes $p \in \pi(A)$. Otherwise, write $M \cong F_1/F_2$ where F_1 and F_2 are free E-modules. There is $x \in F_1 \setminus F_2$ such that $px \in F_2$ for some $p \in \pi(A)$. Since $E = pE$, there is $z \in F_2$ with $px = pz$ since $p \in \pi(A)$. One obtains $x = z$, which is not possible. Hence, $tM = \oplus_{p \notin \pi(A)} M_p$.

Now suppose that $M \in \mathcal{M}_A$, and assume that $M[p] \neq 0$ for infinitely many $p \in \sigma_a(A)$. Since A is flat as an E-module, $T_A(M_p)$ is the p-torsion subgroup of the A-solvable group $T_A(M)$ for all primes p. Furthermore, $A/pA \cong \mathbb{Z}/p\mathbb{Z}$ for all primes $p \notin \pi(A)$ yields that $T_A(tM)$ is A-generated, and hence A-solvable. Since $T_A(tM) \cong \oplus_{p \notin \pi(A)} T_A(M_p)$, the family $\{T_A(M_p) \mid p \notin \pi(A)\}$ is A-small, i.e. for every map $\alpha : A \to \oplus_{p \notin \pi(A)} T_A(tM)$, one has $\pi_p \alpha = 0$ for almost all $p \notin \pi(A)$ where $\pi_p : T_A(tM) \to T_A(M_p)$ is the projection induced by $tM = \oplus_{p \notin \pi(A)} M_p$. However, $0 < h_p^A(a) < \infty$ for all primes $p \in \sigma_a(A)$ yields that A contains a subgroup U such that $A/U \cong \oplus_{p \in \sigma_a(A)} \mathbb{Z}/p\mathbb{Z}$. Therefore, one can find a monomorphism $\alpha' : A/U \to T_A(tM)$ such that $\pi_p \alpha' \neq 0$ for all $p \in \sigma_a(A)$ for which $M[p] \neq 0$, a contradiction. Consequently, $T_A(M_p) = 0$ for almost all primes $p \in \sigma_a(A)$. Since A is faithfully flat as an E-module, $M_p = 0$ for all but finitely many of these primes.

Conversely, consider an E-module M such that $M[p] = 0$ for all but finitely many primes $p \in \sigma_a(A)$. Then, there is an exact sequence $0 \to T_A(tM) \to$

$T_A(M) \to T_A(M/tM) \to 0$. Since the class of A-solvable groups is closed with respect to A-generated extensions, $T_A(M)$ is an A-solvable group once one has shown that $T_A(tM)$ and $T_A(M/tM)$ are A-solvable. Then, $H_A T_A(M) \in \mathcal{M}_A$, and the natural map $\phi_M : M \to H_A T_A(M)$ is a monomorphism since A is faithfully flat as an E-module [2]. Furthermore, $\mathcal{M}_A$ is closed with respect to submodules whenever A is a faithfully flat E-module, and hence $M \in \mathcal{M}_A$ [2].

Write $tM = N \oplus M_{p_1} \oplus \ldots M_{p_m}$ where $p_1, \ldots, p_m$ are the primes $p \in \sigma_a(A)$ with $M[p] \neq 0$. It suffices to show that a p-group is A-solvable whenever $p \notin \pi(A)$. By [6, Theorem 1.4], $h_p^A(a) < \infty$ for all $p \notin \pi(A)$ yields $A/p^nA \cong \mathbb{Z}/p^n\mathbb{Z}$ for all $n < \omega$. Hence, every p-group K is A-generated. Moreover, $\phi(A) \cong \mathbb{Z}/p^n\mathbb{Z}$ for some $n < \omega$ whenever $\phi \in H_A(K)$. Therefore, every finitely A-generated subgroup of K is finite, and hence A-solvable. Consequently, K itself is A-solvable. Hence, every $T_A(M_p)$ is A-solvable for all primes p. Since $\mathcal{C}_A$ is closed with respect to direct sums of A-small families, $T_A(tM)$ is A-solvable once one has shown that $\{T_A(M_p) \mid p \notin \sigma_a(A)\}$ is A-small.

Consider a map $\alpha : A \to \oplus_{p \notin \sigma_a(A)} T_A(M_p)$, and let $\pi_1 = \{p \notin \sigma_a(A) \mid \pi_p\alpha \neq 0\}$. Then, $ker\ \alpha$ contains a non-zero b. For each $p \in \pi_1$, there is $c_p \in A$ such that $0 \neq \alpha(c_p) \in M[p]$. Hence, $pc_p \in ker\ \alpha$, and there are relatively prime integers m_p and n_p with $m_p p c_p = n_p b$. However, $n_p = pk_p$ yields $m_p c_p = k_p b$. Consequently, $m_p \alpha(c_p) = 0$ from which one obtains the contradiction $\alpha(c_p) = 0$ since $\alpha(c_p) \in M[p]$ and p does not divide m_p. Thus, p does not divide n_p, and $h_p^A(b) > 0$ whenever $p \in \pi_1$. Consequently, $\pi_1 \subseteq \sigma_b(A) \setminus \sigma_a(A)$. Since the latter set is finite, the same holds for π_1. Furthermore, if H is any torsion-free A-generated group, and U is a finitely A-generated subgroup of H, then there is an exact sequence $0 \to V \to A^m \to U \to 0$ for some $m < \omega$. Because of [9, Lemma 86.8], V is a direct summand of A^m, and U is A-projective. Therefore, every finitely A-generated subgroup of H is A-solvable, and the same holds for H. □

Corollary 4.7. *Let A and B be subgroups of $\mathbb{Q}$.*

a) *A and B are similar if and only if $A \cong B$.*
b) *$\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent if and only if $\pi(A) = \pi(B)$ and $\sigma(A) = \sigma(B)$.*

Proof. a) is obvious.

b) Suppose that $\mathcal{C}_A$ and $\mathcal{C}_B$ are equivalent. Since E_A and E_B are Morita-equivalent subrings of $\mathbb{Q}$, they are equal, and hence $\pi(A) = \pi(B)$. Moreover, every Morita-equivalence between M_{E_A} and M_{E_B} has to be the identity functor. By Theorem 4.1, $\mathcal{M}_A = \mathcal{M}_B$. But then $\sigma(A) = \sigma(B)$. Conversely, observe $\mathcal{C}_A \tilde{\mathcal{M}}_A = \mathcal{M}_B \tilde{\mathcal{C}}_B$. □

It is easy to construct subgroups A and B of $\mathbb{Q}$ that either have incomparable types or satisfy $type(A) < type(B)$ and have the properties $\sigma(A) = \sigma(B)$ and $\pi(A) = \pi(B) = \emptyset$: For instance, let π_1 and π_2 be infinite disjoint subsets of the set of primes, and consider the subgroups $A_1 = \mathbb{Z}1 + \langle \frac{1}{p} \mid p \in \pi_1 \cup \pi_2 \rangle$, $A_2 = \mathbb{Z}1 + \langle \frac{1}{p} \mid p \in \pi_1 \rangle + \langle \frac{1}{p^3} \mid p \in \pi_2 \rangle$ and $B = \mathbb{Z}1 + \langle \frac{1}{p^2} \mid p \in \pi_1 \cup \pi_2 \rangle$ of $\mathbb{Q}$. Then, $\sigma(A_1) = \sigma(A_2) = \sigma(B)$ and $\pi(A_1) = \pi(A_2) = \pi(B) = \emptyset$. Observe that $type(A_1) < type(B)$, while A_2 and B have incomparable types.

References

[1] Albrecht, U.; Abelian groups, A, such that the category of A-solvable groups is preabelian; Contemporary Mathematics 87 (1989); 179 - 201.
[2] Albrecht, U.; Endomorphism rings, tensor products and Fuchs' Problem 47; Contemporary Mathematics 130 (1992); 17 - 31.
[3] Albrecht, U.; Finite extensions of A-solvable abelian groups; to appear.
[4] Albrecht, U., and Goeters, P.: Strong S-groups; to appear in Colloquium Mathematicum.
[5] Anderson, F., and Fuller, K.; Rings and Categories of Modules; Graduate Texts in Mathematics 13; Springer Verlag (1992).
[6] Arnold, D.; Finite Rank Torsion-Free Abelian Groups and Rings; Springer LNM 931 (1982).
[7] Arnold, D., and Murley, C.; Abelian groups, A, such that $Hom(A, -)$ preserves direct sums of copies of A; Pac. J. of Math. 56 (1975); 7 - 20.
[8] Faticoni, T., and Goeters, P.; Examples of torsion-free groups of finite rank flat as modules over their endomorphism ring; Communications in Algebra 19(1) (1991); 1 - 27.
[9] Fuchs, L.; Infinite Abelian Groups, Vols. I and II; Academic Press (1970/73).
[10] Rotman, J.; An Introduction to Homological Algebra; Academic Press (1989)
[11] Stenström, B.; Rings of Quotients; Springer Verlag (1975).
[12] Ulmer, F.; A flatness criterion in Grothendieck categories; Inv. Math 19 (1973); 331 - 336.

Received April 14, 2000

Department of Mathematics, Auburn University, Auburn, AL 36849, U.S.A.
E-mail address: albreuf@mail.auburn.edu