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Contents

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On G-algebras

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Abstract In this paper, the notion of G-algebra is introduced which is a generalization of QS-algebra and a necessary and sufficient condition for a G-algebra to become QS-algebra is given. We proved that the class of all medial G-algebras forms a variety and is congruence permutable. Finally, we shown that every associative G-algebra is a group.

Keywords G-algebra, QS-algebra, 0-commutative, medial.

§1. Introduction and preliminaries

In 1966, Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2001, J. Neggers, S. S. Ahn and H. S. Kim introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002, J. Neggers and H. S. Kim introdued a new notion, called a B-algebra and obtained several results. In 2007, A. Walendziak introduced a new notion, called a BF-algebra which is a generalization of B-algebra. We introduce a new notion, called a G-algebra, which is a generalization of QS-algebra. The concept of 0-commutative, G-part and medial of a G-algebra are introduced and studied their properties. First, we recall certain definitions from [1], [7], [8], [9] that are required in the paper.

Definition 1.1.[7] A BCI-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(B_1) $(x * y) * (x * z) \leq (z * y)$.

(B_2) $x * (x * y) \leq y$.

(B_3) $x \leq x$.

(B_4) $x \leq y$ and $y \leq x$ imply $x = y$.

(B_5) $x \leq 0$ implies $x = 0$, where $x \leq y$ is defined by $x * y = 0$.

If (B_5) is replaced by $(B_6) : 0 \leq x$, then the algebra is called a BCK-algebra [5]. It is known that every BCK-algebra is a BCI-algebra but not conversely.

A BCH-algebra [7] is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_4) and (B_7) : $(x*y)*z = (x*z)*y$. It is shown that every BCI-algebra is a BCH-algebra but not conversely.

Definition 1.2.[8] A Q-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_7) and $(B_8) : x * 0 = x$.

A Q-algebra X is said to be a QS-algebra [2] if it satisfies the additional relation:

$$(B_9): \ (x * y) * (x * z) = z * y$$

for any $x,\ y,\ z \in X$. It is shown that every BCH-algebra is a Q-algebra but not conversely.

Definition 1.3.[9] A B-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying (B_3), (B_8) and $(B_{10}) : (x * y) * z = x * (z * (0 * y))$.

A B-algebra X is said to be 0-commutative if $a * (0 * b) = b * (0 * a)$ for any $a,\ b \in X$. In [8], it is shown that Q-algebras and B-algebras are different notions.

Definition 1.4.[1] A BF-algebra is an algebra $(X, *, 0)$ of type (2, 0) satisfying (B_3), (B_8) and $(B_{11}) : 0 * (x * y) = (y * x)$. Note that every B-algebra is BF-algebra but not conversely.

§2. G-algebras

In this section we define the notion of G-algebra and observe that the axioms in the definition are independent. Also, we study the properties of G-algebra and we give a necessary and sufficient condition for a G-algebra to become QS-algebra.

Definition 2.1. A G-algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying axioms:

(B_3) $\quad x * x = 0$.

(B_{12}) $\quad x * (x * y) = y$ for all $x,\ y,\ z \in A$.

Example 2.1. Let $A := \mathbb{R} - \{-n\},\ 0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define a binary operation $*$ on A by

$$x * y = \frac{n(x - y)}{n + y}.$$

Then $(A, *, 0)$ is a G-algebra.

Note that every commutative B-algebra is a G-algebra but converse need not be true and every QS-algebra is a G-algebra but converse need not be true.

Example 2.2. Let $A = \{0, 1, 2\}$ in which $*$ is defined by

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Then $(A, *, 0)$ is a G-algebra but not a QS-algebra because

$$(0 * 1) * 2 = 1 * 2 = 2 \neq 1 = 2 * 1 = (0 * 2) * 1.$$

Example 2.3. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ in which $*$ is defined by

$*$	0	1	2	3	4	5	6	7
0	0	2	1	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Then $(A, *, 0)$ is a G-algebra which is not a $BCK/BCI/BCH/Q/QS/B$-algebras.

It is easy to see that G-algebras and Q-algebras are different notions. For example, Example 2.2 is a G-algebra, but not a Q-algebra. Consider the following example. Let $A = \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(A, *, 0)$ is a Q-algebra, but not a G-algebra, since $0 * (0 * 2) = 0 * 0 = 0 \neq 2$.

We observe that the two axioms (B_3) and (B_{12}) are independent. Let $A = \{0, 1, 2\}$ be a set with the following left table.

$*$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Then the axiom(B_{12}) holds but not (B_3), since $2 * 2 \neq 0$.

Similarly, the set $A = \{0, 1, 2\}$ with the above right table satisfy the axiom (B_3) but not (B_{12}), since $1 * (1 * 2) = 1 * 1 = 0 \neq 2$.

Proposition 2.1. If $(A, *, 0)$ is a G-algebra, then the following conditions hold:

(B_{13}) $x * 0 = x$.

(B_{14}) $0 * (0 * x) = x$, for any $x,\ y \in A$.

Proof. Let $(A, *, 0)$ be a G-algebra and $x,\ y \in A$. Then $x * 0 = x * (x * x) = x$ (by B_{12}). Put $x = 0$ and $y = x$ in B_{12}, then we get B_{14}.

Proposition 2.2. Let $(A, *, 0)$ be a G-algebra. Then, for any $x,\ y \in A$, the following conditions hold:

(i) $(x * (x * y)) * y = 0$.

(ii) $x * y = 0$ implies $x = y$.

(iii) $0 * x = 0 * y$ implies $x = y$.

Proof. (i) $(x * (x * y)) * y = y * y = 0$.

(ii) Let $x * y = 0$. Then, by (B_{12}) and (B_{13}), $y = x * (x * y) = x * 0 = x$.

(iii) Let $0 * x = 0 * y$. Then $0 * (0 * x) = 0 * (0 * y)$ and hence $x = y$.

Theorem 2.1. If $(A, *, 0)$ be a G-algebra satisfying $(x * y) * (0 * y) = x$ for any $x,\ y \in A$ then $x * z = y * z$ implies $x = y$.

Proof. Let $(A, *, 0)$ be a G-algebra $(x * y) * (0 * y) = x$ for any $x,\ y \in A$. Then

$$\begin{aligned} & x * z = y * z \\ \Rightarrow\quad & (x * z) * (0 * z) = (y * z) * (0 * z) \\ \Rightarrow\quad & x = y. \end{aligned}$$

We now investigate some relations between G-algebras and $BCI/BCH/Q/BF$-algebras. The following theorems can be proved easily.

Theorem 2.2. Every G-algebra satisfying (B_9) is a BCI-algebra.

Theorem 2.3. Every G-algebra satisfying (B_9) is a BCH-algebra.

Theorem 2.4. Every G-algebra satisfying (B_9) is a Q-algebra.

Theorem 2.5. Every G-algebra satisfying (B_7) is a BF-algebra.

In the following theorem we show that the conditions (B_7) and (B_9) are equivalent.

Theorem 2.6. Let $(A, *, 0)$ be a G-algebra. Then the following are equivalent:

(i) $(x * y) * z = (x * z) * y$ for all $x,\ y,\ z \in A$.

(ii) $(x * y) * (x * z) = z * y$ for all $x,\ y,\ z \in A$.

Proof. (i)⇒(ii) Let $x,\ y,\ z \in A$ and assume (i). Then

$$(x * y) * z = (x * y) * (x * (x * z)) = (x * z) * y.$$

(ii)⇒(ii) Let $x,\ y,\ z \in A$ and assume (ii). Then

$$(x * y) * (x * z) = (x * (x * z)) * y = z * y.$$

In the following, we characterize G-algebra interms of Q-algebra. The following proposition can be proved easily.

Proposition 2.3. Let $(A, *, 0)$ be a G-algebra. Then the following are equivalent:

(i) A is a Q-algebra.

(ii) A is a QS-algebra.

(iii) A is a BCH-algebra.

Lemma 2.1. Let $(A, *, 0)$ be a G-algebra. Then $a * x = a * y$ implies $x = y$ for any $a,\ x,\ y \in A$.

Proof. Let $a,\ x,\ y \in A$. Then $a * x = a * y \Rightarrow a * (a * x) = a * (a * y) \Rightarrow x = y$.

Theorem 2.7. Let $(A, *, 0)$ be a G-algebra. Then the following are equivalent:

(i) $(x * y) * (x * z) = z * y$ for all $x,\ y,\ z \in A$.

(ii) $(x * z) * (y * z) = x * y$ for all $x,\ y,\ z \in A$.

Proof. (i)⇒(ii) Let $x,\ y,\ z \in A$ and assume (i). Then, by (B_{12}),

$$\begin{aligned} & (x*y)*(x*z) = z*y \\ \Rightarrow\quad & (x*y)*((x*y)*(x*z)) = (x*y)*(z*y) \\ \Rightarrow\quad & x*z = (x*y)*(z*y). \end{aligned}$$

(ii)⇒(i) Let $x,\ y,\ z \in A$ and assume (ii). Then, by (B_{12}) and by Lemma 2.1,

$$\begin{aligned} & (x*z)*(y*z) = x*y \\ \Rightarrow\quad & (x*z)*(y*z) = (x*z)*((x*z)*(x*y)) \\ \Rightarrow\quad & y*z = (x*z)*(x*y). \end{aligned}$$

§3. G-part of G-algebras

In this section, we define 0-commutative, medial and give a necessary and sufficient condition for a G-algebra to become a medial G-algebra. Also we investigate the properties of G-part in G-algebras.

Definition 3.1. A G-algebra $(A, *, 0)$ is said to be 0-commutative if $x*(0*y) = y*(0*x)$ for any $x,\ y \in A$. A non-empty subset S of a G-algebras, A is called a subalgebra of A if $x*y \in S$ for any $x,\ y \in S$.

Example 3.1. Let $A = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(A, *, 0)$ is a 0-commutative G-algebra.

Theorem 3.1. Let $(A, *, 0)$ be a 0-commutative G-algebra. Then $(0*x)*(0*y) = y*x$ for any $x,\ y \in A$.

Proof. Let $x,\ y \in A$. Then $(0*x)*(0*y) = y*(0*(0*x)) = y*x$.

Theorem 3.2. Let $(A, *, 0)$ be a 0-commutative G-algebra satisfying $0*(x*y) = y*x$. Then $(x*y)*(0*y) = x$ for any $x,\ y \in A$.

Proof. Let $x,\ y \in A$. Then $(x*y)*(0*y) = y*(0*(x*y)) = y*(y*x) = x$.

Definition 3.2. Let A be a G-algebra. For any subset S of A, we define

$$G(S) = \{x \in S \mid 0*x = x\}.$$

In particular, if $S = A$ then we say that $G(A)$ is the G-part of a G-algebra. For any G-algebra A, the set $B(A) = \{x \in A \mid 0*x = 0\}$ is called a p-radical of A. A G-algebra is said to be p-semisimple if $B(A) = \{0\}$.

The following property is obvious

$$G(A) \cap B(A) = \{0\}.$$

Proposition 3.1. Let $(A, *, 0)$ be a G-algebra. Then $x \in G(A)$ if and only if $0 * x \in G(A)$.

Proof. If $x \in G(A)$, then $0 * x = x$ and hence $0 * x \in G(A)$. Conversely, if $0 * x \in G(A)$, then $0 * (0 * x) = 0 * x$, and hence $x = 0 * x$. Therefore $x \in G(A)$.

Theorem 3.3. If S is a subalgebra of a G-algebra $(A, *, 0)$, then $G(A) \cap S = G(S)$.

Proof. Clearly $G(A) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 * x = x$ and $x \in S \subseteq A$. Hence $x \in G(A)$. Therefore $x \in G(A) \cap S$. Thus $G(A) \cap S = G(S)$.

The following theorem can be proved easily.

Theorem 3.4. Let $(A, *, 0)$ be a G-algebra. If $G(A) = A$ then A is p-semisimple.

Definition 3.3. A G-algebra $(A, *, 0)$ satisfying $(x * y) * (z * u) = (x * z) * (y * u)$ for any $x,\ y,\ z$ and $u \in A$, is called a medial G-algebra.

We can observe that Example 2.1 is a medial G-algebra.

Lemma 3.1. If A is a medial G-algebra, then, for any $x,\ y,\ z \in A$, the following holds:

(i) $(x * y) * x = 0 * y$.

(ii) $x * (y * z) = (x * y) * (0 * z)$.

(iii) $(x * y) * z = (x * z) * y$.

Proof. Let A be a medial G-algebra and $x,\ y,\ z \in A$. Then

(i) $(x * y) * x = (x * y) * (x * 0) = (x * x) * (y * 0) = 0 * y$.

(ii) $(x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z)$.

(iii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y$.

The following theorem can be proved easily.

Theorem 3.5. Every medial G-algebra is a QS-algebra.

Theorem 3.6. Let A be a medial G-algebra. Then the right cancellation law holds in $G(A)$.

Proof. Let $a,\ b,\ x \in G(A)$ with $a * x = b * x$. Then, for any $y \in G(A)$, $x * y = (0 * x) * y = (0 * y) * x = y * x$. Therefore

$$a = x * (x * a) = x * (a * x) = x * (b * x) = x * (x * b) = b.$$

Now we give a necessary and sufficient condition for a G-algebra to become medial G-algebra.

Theorem 3.7. A G-algebra A is medial if and only if it satisfies the following conditions:

(i) $y * x = 0 * (x * y)$ for all $x,\ y \in A$.

(ii) $x * (y * z) = z * (y * x)$ for all $x,\ y,\ z \in A$.

Proof. Suppose $(A, *, 0)$ is medial. Then

(i) $0 * (x * y) = (y * y) * (x * y) = (y * x) * (y * y) = (y * x) * 0 = y * x$.

(ii) $x * (y * z) = 0 * ((y * z) * x) = 0 * ((y * z) * (x * 0)) = 0 * ((y * x) * z) = z * (y * x)$.

Conversely assume that the conditions hold. Then $(x * y) * (z * u) = u * (z * (x * y)) = u * (y * (x * z)) = (x * z) * (y * u)$.

Corollary 3.1. The class of all of medial G-algebras forms a variety, written $\nu(MG)$.

Proposition 3.2.[3] A variety ν is congruence-permutable if and only if there is a term $p(x, y, z)$ such that

$$\nu \models p(x, x, y) \approx y \text{ and } \nu \models p(x, y, y) \approx x.$$

Corollary 3.2. The variety $\nu(MG)$ is congruence-permutable.

Proof. Let $p(x, y, z) = x * (y * z)$. Then by (B_{12}) and (B_8) we have $p(x, x, y) = y$ and $p(x, y, y) = x$, and so the variety $\nu(MG)$ is congruence permutable.

§4. Conclusion and future research

In this paper, we have introduced the concept of G-algebras and studied their properties. In addition, we have defined G-part, p-radical and medial of G-algebra and proved that the variety of medial algebras is congruence permutable. Finally, we proved that every associative G-algebra is a group.

In our future work, we introduce the concept of fuzzy G-algebra, Interval-valued fuzzy G-algebra, intuitionistic fuzzy structure of G-algebra, intuitionistic fuzzy ideals of G-algebra and Intuitionistic (T, S)-normed fuzzy subalgebras of G-algebras, intuitionistic L-fuzzy ideals of G-algebra. I hope this work would serve as a foundation for further studies on the structure of G-algebras.

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$(1,2)^*$-Q-Closed sets in Bitopological Spaces

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Abstract In this paper, New classes of sets called $(1,2)^*$-Q-closed sets and $(1,2)^*$-Q_s-closed sets are formulated in bitopological settings and some of their properties are studied. Moreover the notions of $(1,2)^*$-Q-continuity and $(1,2)^*$-Q_s-continuity are defined and some of their characterisations are investigated.

Keywords $(1,2)^*$-Q-closed sets, $(1,2)^*$-Q_s-closed sets, $(1,2)^*$-Q-continuity, $(1,2)^*$-Q_s-continuity.
2000 Mathematics Subject Classification: 54E55.

§1. Introduction

The study of generalized closed sets in a topological space was initiated by Levine [6]. In 1996, Maki, Umehara and Noiri [7] introduced the class of pre-generalized closed sets to obtain properties of pre-$T_{1/2}$-generalized closed sets and pre-$T_{1/2}$-spaces. The modified forms of generalized closed sets and generalized continuity were studied by Balachandran, Sundaram and Maki [3]. Recently Mohana and Arockiarani [8] developed $(1,2)^*$-πg-closed sets in bitopological spaces. In this paper, we introduce a new classes of sets called $(1,2)^*$-Q-closed sets in bitopological spaces and study some of their properties.

Throughout this paper (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,η_1,η_2) (or simply X, Y and Z) will always denote bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned.

§2. Preliminaries

Definition 2.1.[9] A subset S of a bitopological space X is said to be $\tau_{1,2}$-open if $S = A\cup B$ where $A \in \tau_1$ and $B \in \tau_2$. A subset S of X is said to be

(i) $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open.

(ii) $\tau_{1,2}$-clopen if S is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed.

Definition 2.2.[9] Let S be a subset of the bitopological space X. Then

(i) The $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-$int(S)$ is defined by $\bigcup\{G : G \subseteq S$ and G is $\tau_{1,2}$-open$\}$.

(ii) The $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-$cl(S)$ is defined by $\bigcap\{F : S \subseteq F$ and F is $\tau_{1,2}$-closed$\}$.

Definition 2.3. A subset A of a bitoplogical space X is called

(i) $(1,2)^*$-regular open [9] if $A = \tau_{1,2}$-int$(\tau_{1,2}$-$cl(A))$.

(ii) $(1,2)^*$-α-open [9] if $A \subseteq \tau_{1,2}$-int$(\tau_{1,2}$-$cl(\tau_{1,2}$-$int(A)))$.

(iii) $(1,2)^*$-pre open [4] if $A \subseteq \tau_{1,2}$-int$(\tau_{1,2}$-$cl(A))$.

(iv) $(1,2)^*$-semi pre open [4] if $A \subseteq \tau_{1,2}$-$cl(\tau_{1,2}$-$int(\tau_{1,2}$-$cl(A)))$.

The complement of the sets mentioned above are called their respective closed sets.

Definition 2.4.[2] Let S be a subset of the bitopological space X. Then

(i) The $(1,2)^*$-α-interior of S, denoted by $(1,2)^*$-α-$int(S)$ is defined by $\bigcup\{G : G \subseteq S$ and G is $(1,2)^*$-α-open$\}$.

(ii) The $(1,2)^*$-α-closure of S, denoted by $(1,2)^*$-α-$cl(S)$ is defined by $\bigcap\{F : S \subseteq F$ and F is $(1,2)^*$-α-closed$\}$.

Definition 2.5. A subset A of a bitopological space X is said to be

(i) $(1,2)^*$-πg-closed [8] in X if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-π-open in X.

(ii) $(1,2)^*$-$\pi g\alpha$-closed [1] in X if $(1,2)^*$-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-π-open in X.

(iii) $(1,2)^*$-αg-closed in X [5] if $(1,2)^*$-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

(iv) $(1,2)^*$-gp-closed [4] in X if $(1,2)^*$-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

(v) $(1,2)^*$-gpr-closed [4] in X if $(1,2)^*$-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^*$-regular open in X.

(vi) $(1,2)^*$-gsp-closed [4] in X if $(1,2)^*$-$spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

(vii) $(1,2)^*$-g-closed [5] in X if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X.

The complement of the sets mentioned above are called their respective open sets.

§3. $(1,2)^*$-Q-closed sets

Definition 3.1. A subset A of a bitopological space X is said to be $(1,2)^*$-Q-closed set if $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$ whenever $A \subseteq U$ and U is $(1,2)^*$-πg-open in X.

Theorem 3.1. Every $\tau_{1,2}$-open and $(1,2)^*$-α-closed subset of X is $(1,2)^*$-Q-closed, but not conversely.

Proof. Let A be $\tau_{1,2}$-open and $(1,2)^*$-α-closed subset of X. Let $A \subseteq U$ and U be $(1,2)^*$-πg-open in X. Since A is $(1,2)^*$-α-closed, $(1,2)^*$-$\alpha cl(A) = A$. $(1,2)^*$-$\alpha cl(A) = A = \tau_{1,2}$-$int(A) \subseteq \tau_{1,2}$-$int(U)$. Since A is $\tau_{1,2}$-open and $A \subseteq U$. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. This implies, A is $(1,2)^*$-Q-closed.

Remark 3.1. The converse of the above theorem need not be true as seen by the following Example.

Example 3.1. $(1,2)^*$-Q-closed sets need not be $\tau_{1,2}$-open and $(1,2)^*$-α-closed. Let

$$X = \{a, b, c, d\},$$

$$\tau_1 = \{\phi, X, \{a\}, \{a, b, d\}\},$$

$$\tau_2 = \{\phi, X, \{b\}, \{a, b\}\}.$$

Here $A = \{a, c\}$ is $(1,2)^*$-Q-closed, but A is not $\tau_{1,2}$-open and $(1,2)^*$-α-closed.

Definition 3.2. A subset A of a bitopological space X is said to be $(1,2)^*$-w-closed set if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^*$-*semi*-open in X. The complement of $(1,2)^*$-w-closed set is called $(1,2)^*$-w-open set.

Definition 3.3. A subset A of a bitopological space X is said to be $(1,2)^*$-R-closed set if $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$ whenever $A \subseteq U$ and U is $(1,2)^*$-w-open in X.

Theorem 3.2. (i) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-$\pi g\alpha$-closed set.

(ii) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-αg-closed.

(iii) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-gp-closed.

(iv) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-gpr-closed.

(v) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-gsp-closed.

(vi) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-R-closed.

Proof. (i) Let A be $(1,2)^*$-Q-closed set. Let $A \subseteq U$,

$$\begin{aligned} U \text{ is } \tau_{1,2}\text{-}\pi\text{-open set} &\implies \tau_{1,2}\text{-open} \\ &\implies (1,2)^* - \pi g\text{-open}. \end{aligned}$$

i.e., U is $(1,2)^*$-πg-open set. Since A is $(1,2)^*$-Q-closed set, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq U$. Thus, A is $(1,2)^*$-$\pi g\alpha$-closed.

(ii) Let A be $(1,2)^*$-Q-closed set. Let $A \subseteq U$, U is $\tau_{1,2}$-open $\Longrightarrow (1,2)^*$-πg-open. Since A is $(1,2)^*$-Q-closed, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. This implies, $(1,2)^*$-$\alpha cl(A) \subseteq U$. Therefore, A is $(1,2)^*$-αg-closed.

(iii) Let A be $(1,2)^*$-Q-closed. Let $A \subseteq U$, U is $\tau_{1,2}$-open. Since A is $(1,2)^*$-Q-closed, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U) \subseteq U$. This implies, $(1,2)^*$-$pcl(A) \subseteq U$. Therefore, A is $(1,2)^*$-gp-closed.

(iv) Let A be $(1,2)^*$-Q-closed. Let $A \subseteq U$, U is $(1,2)^*$-regular open. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U) \subseteq U$. This implies, $(1,2)^*$-$pcl(A) \subseteq U$. Therefore, A is $(1,2)^*$-gpr-closed.

(v) Every $(1,2)^*$-Q-closed set is $(1,2)^*$-αg-closed and every $(1,2)^*$-αg-closed set is $(1,2)^*$-gsp-closed. This implies, every $(1,2)^*$-Q-closed set is $(1,2)^*$-gsp-closed.

(vi) Let $A \subseteq U$, U is $(1,2)^*$-ω-open. Since A is $(1,2)^*$-Q-closed set, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. Thus, A is $(1,2)^*$-R-closed set.

However the converses of the above theorem are not true is shown by the following Examples.

Example 3.2. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{c\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$. Here $\{a\}$, $\{a, c\}$, $\{a, b\}$ are $(1,2)^*$-$\pi g\alpha$-closed sets, but not $(1,2)^*$-Q-closed sets.

Example 3.3. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, $\tau_2 = \{\phi, X, \{a, d\}, \{a, c, d\}, \{a, b, d\}\}$. Here $\{b\}$, $\{d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{b, c, d\}$ are $(1,2)^*$-αg-closed sets, but not $(1,2)^*$-Q-closed sets.

Example 3.4. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, b\}\}$, $\tau_2 = \{\phi, X\}$. Here $\{a\}$ is $(1,2)^*$-gp-closed set, but not $(1,2)^*$-Q-closed set.

Example 3.5. In Example 3.3, $\{b\}$ is $(1,2)^*$-gpr-closed set, but not $(1,2)^*$-Q-closed set.

Example 3.6. In Example 3.3, $\{d\}$ is $(1,2)^*$-gsp-closed set, but not $(1,2)^*$-Q-closed set.

Example 3.7. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{a\}\}$. Here $\{b\}$ is $(1,2)^*$-R-closed set, but not $(1,2)^*$-Q-closed set.

Theorem 3.3. If A subset A of a bitopological space X is $(1,2)^*$-Q-closed set then $(1,2)^*$-$\alpha cl(A)$-A contains no non empty $(1,2)^*$-πg-closed set.

Proof. Let F be a non empty $(1,2)^*$-πg-closed set suchthat $F \subseteq (1,2)^*$-$\alpha cl(A) - A$. Then $F \subseteq (1,2)^*$-$\alpha cl(A) - A$ and $A \subseteq X - F$ is $(1,2)^*$-πg-open. Since A is $(1,2)^*$-Q-closed, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(X - F) = X - \tau_{1,2}$-$cl(F)$. This implies, $\tau_{1,2}$-$cl(F) \subseteq X - (1,2)^*$-$\alpha cl(A)$. That is, $F \subseteq (1,2)^*$-$\alpha cl(A)$ and $F \subseteq X - (1,2)^*$-$\alpha cl(A)$. That is, $F \subseteq (1,2)^*$-$\alpha cl(A) \cap (X - (1,2)^*$-$\alpha cl(A)) = \phi$. This implies, $(1,2)^*$-$\alpha cl(A) - A$ contains no non empty $(1,2)^*$-πg-closed set.

Remark 3.2. The converse of the theorem 3.3 need not be true. If $(1,2)^*$-$\alpha cl(A)$-A contains no non empty $(1,2)^*$-πg-closed set, then A need not be $(1,2)^*$-Q-closed. For example Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b\}\}$, $\tau_2 = \{\phi, X, \{a, b\}\}$. Here $A = \{a, b\}$ is not $(1,2)^*$-Q-closed set, but $(1,2)^*$-$\alpha cl(A) - A = X - \{a, b\} = \{c\}$.

Theorem 3.3. If A and B are $(1,2)^*$-Q-closed set then $A \cup B$ is $(1,2)^*$-Q-closed set.

Proof. Let A and B be $(1,2)^*$-Q-closed sets. Let $A \cup B \subseteq U$, U be $(1,2)^*$-πg-open. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$, $(1,2)^*$-$\alpha cl(B) \subseteq \tau_{1,2}$-$int(U)$. Since A and B are $(1,2)^*$-α-closed set, $(1,2)^*$-$\alpha cl(A \cup B) = (1,2)^*$-$\alpha cl(A) \cup (1,2)^*$-$\alpha cl(B) \subseteq \tau_{1,2}$-$int(U)$. This implies $A \cup B$ is $(1,2)^*$-Q-closed set.

Remark 3.3. The intersection of two $(1,2)^*$-Q-closed sets need not be $(1,2)^*$-Q-closed. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$, $\tau_2 = \{\phi, X, \{a, c\}\}$. Then $\{a, b, c\}$ and $\{a, b, d\}$ are $(1,2)^*$-Q-closed, but $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$ is not $(1,2)^*$-Q-closed.

Theorem 3.4. If A is $(1,2)^*$-Q-closed and $A \subseteq B \subseteq (1,2)^*$-$\alpha cl(A)$ then B is $(1,2)^*$-Q-closed.

Proof. Let U be $(1,2)^*$-πg-open set of X, such that $B \subseteq U$. Let $A \subseteq B \subseteq (1,2)^*$-$\alpha cl(A)$. Therefore $A \subseteq U$ and U is $(1,2)^*$-πg-open. This implies $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. Also, $B \subseteq (1,2)^*$-$\alpha cl(A) \implies (1,2)^*$-$\alpha cl(B) \subseteq (1,2)^*$-$\alpha cl((1,2)^*$-$\alpha cl(A)) = (1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. Therefore, $(1,2)^*$-$\alpha cl(B) \subseteq \tau_{1,2}$-$int(U)$. Thus, B is $(1,2)^*$-Q-closed.

Theorem 3.5. If a subset A of X is $(1,2)^*$-πg-open and $(1,2)^*$-Q-closed then A is $(1,2)^*$-α-closed in X.

Proof. Let A be $(1,2)^*$-πg-open and $(1,2)^*$-Q-closed. Then $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(A) \subseteq A$. Therefore $(1,2)^*$-$\alpha cl(A) \subseteq A$. Therefore, A is $(1,2)^*$-α-closed.

Theorem 3.6. Let A be $(1,2)^*$-Q-closed in X then A is $(1,2)^*$-α-closed in X iff $(1,2)^*$-$\alpha cl(A)$-A is $(1,2)^*$-πg-closed.

Proof. Given A is $(1,2)^*$-Q-closed. Let A be $(1,2)^*$-α-closed. Therefore, $(1,2)^*$-$\alpha cl(A) = A$. i.e., $(1,2)^*$-$\alpha cl(A) - A = \phi$, which is $(1,2)^*$-πg-closed. Conversely, if $(1,2)^*$-$\alpha cl(A) - A$ is $(1,2)^*$-πg-closed, since A is $(1,2)^*$-Q-closed, $(1,2)^*$-$\alpha cl(A) - A$ does not contain any non empty $(1,2)^*$-πg-closed set. Therefore, $(1,2)^*$-$\alpha cl(A) - A = \phi$. This implies $(1,2)^*$-$\alpha cl(A) \subseteq A$. That is, A is $(1,2)^*$-α-closed set.

Theorem 3.7. An $\tau_{1,2}$-open set A of X is $(1,2)^*$-αg-closed iff A is $(1,2)^*$-Q-closed.

Proof. Let A be an $\tau_{1,2}$-open set and $(1,2)^*$-αg-closed set. Let $A \subseteq U$, U is $(1,2)^*$-πg-open. Since $A \subseteq U$, $\tau_{1,2}$-$int(A) \subseteq \tau_{1,2}$-$int(U)$. Therefore, $A \subseteq \tau_{1,2}$-$int(U)$, which is $\tau_{1,2}$-open. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$, since A is $(1,2)^*$-αg-closed. This implies A is $(1,2)^*$-Q-closed. Conversely, let A be $(1,2)^*$-Q-closed and $A \subseteq U$, U is $\tau_{1,2}$-open $\Longrightarrow U$ is $(1,2)^*$-πg-open. This implies $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U) \subseteq U \Longrightarrow (1,2)^*$-$\alpha cl(A) \subseteq U \Longrightarrow A$ is $(1,2)^*$-αg-closed.

Theorem 3.8. In a bitopological space X, for each $x \in X$, $\{x\}$ is $(1,2)^*$-πg-closed or its complement $X - \{x\}$ is $(1,2)^*$-Q-closed in X.

Proof. Let X be a bitopological space. To prove $\{x\}$ is $(1,2)^*$-πg-closed or $X - \{x\}$ is $(1,2)^*$-Q-closed in X. If $\{x\}$ is not $(1,2)^*$-πg-closed in X, then $X - \{x\}$ is not $(1,2)^*$-πg-open and the only $(1,2)^*$-πg-open set containing $X - \{x\}$ is X. Therefore, $(1,2)^*$-$\alpha cl(X - \{x\}) \subseteq X = \tau_{1,2}$-$int(X)$. Thus, $(1,2)^*$-$\alpha cl(X - \{x\}) \subseteq \tau_{1,2}$-$int(X) \Longrightarrow X - \{x\}$ is $(1,2)^*$-Q-closed.

Remark 3.4. $\tau_{1,2}$-closedness and $(1,2)^*$-Q-closedness are independent. In Example 3.2, $A = \{a, b\}$ is $\tau_{1,2}$-closed, but not $(1,2)^*$-Q-closed. In Remark 3.3, $A = \{a, b, d\}$ is $(1,2)^*$-Q-closed, but not $\tau_{1,2}$-closed.

Remark 3.5. $(1,2)^*$-Q-closedness and $(1,2)^*$-pre closed set are independent. In Remark 3.3, $A = \{a, b, d\}$ is $(1,2)^*$-Q-closed, but not $(1,2)^*$-pre-closed set. In Example 3.3, $A = \{b\}$ is $(1,2)^*$-pre closed set, but not $(1,2)^*$-Q-closed set.

Remark 3.6. From the above discussions and known results we have the following implications. $A \longrightarrow B$ ($A \not\leftrightarrow B$) represents A implies B but not conversely (A and B are independent of each other).

$$
\begin{tikzcd}
\boxed{(1,2)^*\text{-}\alpha g\text{-closed}} \arrow[r] & \boxed{(1,2)^*\text{-}\pi g\alpha\text{-closed}} & \boxed{(1,2)^*\text{-}gp\text{-closed}} \arrow[l] \\
\boxed{\tau_{1,2}\text{-closed}} \arrow[u] \arrow[d] \arrow[r, phantom, "\not\longleftrightarrow"] & \boxed{(1,2)^*\text{-}Q\text{-closed}} \arrow[u] \arrow[d] \arrow[ur] \arrow[dr] \arrow[r, phantom, "\not\longleftrightarrow"] & \boxed{(1,2)^*\text{-pre closed}} \\
\boxed{(1,2)^*\text{-}gsp\text{-closed}} & \boxed{(1,2)^*\text{-}gpr\text{-closed}} & \boxed{(1,2)^*\text{-}R\text{-closed}}
\end{tikzcd}
$$

Definition 3.3. The intersection of all $(1,2)^*$-πg-open subsets of X containing A is called the $(1,2)^*$-πg-kernal of A and denoted by $(1,2)^*$-πg-$ker(A)$.

Theorem 3.9. If a subset A of X is $(1,2)^*$-Q-closed, then $(1,2)^*$-$\alpha cl(A) \subseteq (1,2)^*$-$\pi g$-$ker(A)$.

Proof. Let A be $(1,2)^*$-Q-closed. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$, whenever $A \subseteq U$, U is $(1,2)^*$-πg-open. Let $x \in (1,2)^*$-$\alpha cl(A)$. If $x \notin (1,2)^*$-πg-$ker(A)$, then there exists a $(1,2)^*$-πg-open set containing A subset $x \notin U$. Therefore, $x \notin A \Longrightarrow x \notin (1,2)^*$-$\alpha cl(A)$. Which is contradiction to $x \in (1,2)^*$-$\alpha cl(A)$. Thus, $(1,2)^*$-$\alpha cl(A) \subseteq (1,2)^*$-$\pi g$-$ker(A)$.

Definition 3.5. A subset A of a bitopological space X is said to be $(1,2)^*$-Q_S-closed set in X if $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl((U))$ whenever $A \subseteq U$ and U is $(1,2)^*$-πg-open in X.

Theorem 3.10. Every $(1,2)^*$-Q-closed set is $(1,2)^*$-Q_S-closed.

Proof. Let A be any $(1,2)^*$-Q-closed set. Let $A \subseteq U$, U is $(1,2)^*$-πg-open in $X \Longrightarrow (1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl((U))$. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl((U))$. Thus, A is $(1,2)^*$-Q_S-closed set.

Definition 3.6. A subset A of X is said to be

(i) $(1,2)^*$-Q-open in X if its complement $X - A$ is $(1,2)^*$-Q-closed set in X.

(ii) $(1,2)^*$-Q_S-open in X if its complement $X - A$ is $(1,2)^*$-Q_S-closed set in X.

Theorem 3.11. Let X be a bitopological space and $A \subseteq X$,

(i) A is $(1,2)^*$-Q-open set in X iff $\tau_{1,2}$-$cl(U) \subseteq (1,2)^*$-$\alpha int(A)$ when ever $U \subseteq A$ and U is $(1,2)^*$-πg-closed.

(ii) A is $(1,2)^*$-Q_S-open set in X iff $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(U)) \subseteq (1,2)^*$-$\alpha int(A)$ when ever $U \subseteq A$ and U is $(1,2)^*$-πg-closed.

(iii) If A is $(1,2)^*$-Q-open set in X then, A is $(1,2)^*$-Q_S-open.

Proof. Let A be an $(1,2)^*$-Q-open set in X. Let $U \subseteq A$ and U is $(1,2)^*$-πg-closed. Then, $X - A$ is $(1,2)^*$-Q-closed and $X - A \subseteq X - U$ and $X - U$ is $(1,2)^*$-πg-open. Therefore, $(1,2)^*$-$\alpha cl(X - A) \subseteq \tau_{1,2}$-$int(X - U)$. This implies, $X - (1,2)^*$-$\alpha int(A) \subseteq X - \tau_{1,2}$-$cl(U) \Longrightarrow \tau_{1,2}$-$cl(U) \subseteq (1,2)^*$-$\alpha int(A)$ whenever $U \subseteq A$ and U is $(1,2)^*$-πg-closed then $\tau_{1,2}$-$cl(\tau_{1,2}$-$int((U)) \subseteq (1,2)^*$-$\alpha int(A)$.

(i) Let $A \subseteq V$ and V is $(1,2)^*$-πg-closed. $A \subseteq V \Longrightarrow X - A \supseteq X - V$, which is $(1,2)^*$-πg-open. Therefore, $\tau_{1,2}$-$cl(X - V) \subseteq (1,2)^*$-$\alpha int(X - A)$. This implies, $X - \tau_{1,2}$-$int(V) \subseteq X - (1,2)^*$-$\alpha cl(A)$. Therefore, $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(V)$. Thus, A is $(1,2)^*$-Q-closed.

(ii) Let A be an $(1,2)^*$-Q_S-open set. Let $F \subseteq A$ and F is $(1,2)^*$-πg-closed. Therefore, $X - A$ is $(1,2)^*$-Q_S-closed and $X - F$ is $(1,2)^*$-πg-open subset such that, $X - A \subseteq X - F$. Therefore, $(1,2)^*$-$\alpha cl(X - A) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(X - F))$. That is, $X - (1,2)^*$-$\alpha int(A) \subseteq \tau_{1,2}$-$int(X - \tau_{1,2}$-$int(F))$. This implies, $X - (1,2)^*$-$\alpha int(A) \subseteq X - \tau_{1,2}$-$cl(\tau_{1,2}$-$int(F))$. That is $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(F)) \subseteq (1,2)^*$-$\alpha int(A)$.

(iii) Let A be $(1,2)^*$-Q-open. To prove, A is $(1,2)^*$-Q_S-open. Let $K \subseteq A$ and K is $(1,2)^*$-πg-closed. This implies, $\tau_{1,2}$-$cl(K) \subseteq (1,2)^*$-$\alpha int(A)$. That is, $\tau_{1,2}$-$cl(\tau_{1,2}$-$int(K)) \subseteq \tau_{1,2}$-$cl(K) \subseteq (1,2)^*$-$\alpha int(A)$. Thus, A is $(1,2)^*$-Q_S-open.

§4. $(1,2)^*$-Q-Continuity and $(1,2)^*$-Q_S-Continuity

Let $f : X \longrightarrow Y$ be a function from a bitopological space X into a bitopological space Y.

Definition 4.1. A function $f : X \longrightarrow Y$ is said to be $(1,2)^*$-Q-continuous (respectively. $(1,2)^*$-Q_S-continuous) if $f^{-1}(V)$ is $(1,2)^*$-Q-closed (respectively. $(1,2)^*$-Q_S-closed) in X, for every $\sigma_{1,2}$-closed set V of Y.

Definition 4.2. A function $f : X \longrightarrow Y$ is said to be $(1,2)^*$-Q-irresolute (respectively. $(1,2)^*$-Q_S-irresolute) if $f^{-1}(V)$ is $(1,2)^*$-Q-closed (respectively. $(1,2)^*$-Q_S-closed) in X, for every $(1,2)^*$-Q-closed ($(1,2)^*$-Q_S-closed) set V of Y.

Example 4.1. Let $X = \{a, b, c, d\} = Y$, $\tau_1 = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$, $\tau_2 = \{X, \phi, \{c, d\}\}$, $\sigma_1 = \{Y, \phi, \{a\}, \{a, b, d\}\}$, $\sigma_2 = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \longrightarrow Y$ be defined by $f(a) = b,\ f(b) = c,\ f(c) = d,\ f(d) = a$. Then f is $(1,2)^*$-Q-irresolute.

Example 4.2. Let $X = \{a, b, c\} = Y$, $\tau_1 = \{X, \phi, \{a\}, \{a, c\}\}$, $\tau_2 = \{X, \phi, \{b\}\}$, $\sigma_1 = \{Y, \phi, \{b\}\}$, $\sigma_2 = \{Y, \phi, \{b, c\}\}$. Let $f : X \longrightarrow Y$ be defined by $f(a) = c,\ f(b) = b,\ f(c) = a$. Then f is $(1,2)^*$-Q-continuous.

Theorem 4.1. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two functions. Then

(i) gof is $(1,2)^*$-Q-continous if g is $(1,2)^*$-continuous and f is $(1,2)^*$-Q-continuous.

(ii) gof is $(1,2)^*$-Q-irresolute if g is $(1,2)^*$-Q-irresolute and f is $(1,2)^*$-Q-irresolute.

(iii) gof is $(1,2)^*$-Q-continous if g is $(1,2)^*$-Q-continuous and f is $(1,2)^*$-Q-irresolute.

Proof. The proof is obvious.

Definition 4.3. A space X is called an $(1,2)^*$-$\alpha\pi g$-space if the intersection of $(1,2)^*$-α-closed set with a $(1,2)^*$-πg-closed set is $(1,2)^*$-πg-closed.

Theorem 4.2. For a subset A of an $(1,2)^*$-$\alpha\pi g$-space X, the following are equivalent:

(i) A is $(1,2)^*$-Q-closed.

(ii) $\tau_{1,2}$-$cl\{x\} \cap A \neq \phi$, for each $x \in (1,2)^*$-$\alpha cl(A)$.

(iii) $(1,2)^*$-$\alpha cl(A) - A$ contains no non-empty $(1,2)^*$-πg-closed set.

Proof. (i) Let A be $(1,2)^*$-Q-closed. Let $x \in (1,2)^*$-$\alpha cl(A)$. If $\tau_{1,2}$-$cl\{x\} \cap A = \phi$ then $A \subseteq X - \tau_{1,2}$-$cl\{x\}$ is $\tau_{1,2}$-open and hence $X - \tau_{1,2}$-$cl\{x\}$ is $(1,2)^*$-πg-open. Let $U = X - \tau_{1,2}$-$cl\{x\}$. That is, $A \subseteq U$, U is $(1,2)^*$-πg-open $\Longrightarrow$ $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(U)$. i.e., $(1,2)^*$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(X - \tau_{1,2}$-$cl\{x\}) = X - \tau_{1,2}$-$cl(\tau_{1,2}$-$cl\{x\}) = X - \tau_{1,2}$-$cl\{x\}$. This implies, $(1,2)^*$-$\alpha cl(A) \subseteq X - \tau_{1,2}$-$cl\{x\}$. Since $x \in (1,2)^*$-$\alpha cl(A)$, $x \in X - \tau_{1,2}$-$cl\{x\}$, which is not possible. Therefore, $\tau_{1,2}$-$cl\{x\} \cap A \neq \phi$.

(ii) If $\tau_{1,2}$-$cl\{x\} \cap A \neq \phi$ for $x \in (1,2)^*$-$\alpha cl(A)$, to prove $(1,2)^*$-$\alpha cl(A) - A$ contains no non empty $(1,2)^*$-πg-closed set. Let $K \subseteq (1,2)^*$-$\alpha cl(A) - A$ is a non empty $(1,2)^*$-πg-closed set. Then $K \subseteq (1,2)^*$-$\alpha cl(A)$ and $A \subseteq X - K$. Let $x \in K$, then $x \in (1,2)^*$-$\alpha cl(A)$. Then by (ii), $\tau_{1,2}$-$cl\{x\} \cap A \neq \phi$. This implies, $\tau_{1,2}$-$cl\{x\} \cap A \subseteq K \cap A \subseteq ((1,2)^*$-$\alpha cl(A) - A) \cap A$. Which is a contradiction. Hence, $(1,2)^*$-$\alpha cl(A) - A$ contains no non empty $(1,2)^*$-πg-closed set.

(iii) If $(1,2)^*$-$\alpha cl(A) - A$ contains no non empty $(1,2)^*$-πg-closed set. Let $A \subseteq U$, U is $(1,2)^*$-πg-open. If $(1,2)^*$-$\alpha cl(A) \nsubseteq \tau_{1,2}$-$int(U)$, then $(1,2)^*$-$\alpha cl(A) \cap (\tau_{1,2}$-$int(U))^C = \phi$. Since, the space is a $(1,2)^*$-$\alpha\pi g$-space, $(1,2)^*$-$\alpha cl(A) \cap (\tau_{1,2}$-$int(U))^C$ is a non empty $(1,2)^*$-πg-closed subset of $(1,2)^*$-$\alpha cl(A) - A$ which is a contradiction. Therefore, A is $(1,2)^*$-Q-closed set.

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Intuitionistic fuzzy quasi weakly generalized continuous mappings

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Abstract The purpose of this paper is to introduce and study the concepts of intuitionistic fuzzy quasi weakly generalized continuous mappings in intuitionistic fuzzy topological space. Some of their properties are explored.

Keywords Intuitionistic fuzzy topology, intuitionistic fuzzy weakly generalized closed set, intuitionistic fuzzy weakly generalized open set and intuitionistic fuzzy quasi weakly generalized continuous mappings.
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§1. Introduction

Fuzzy set (FS) as proposed by Zadeh [16] in 1965 is a framework to encounter uncertainty, vagueness and partial truth and it represents a degree of membership for each member of the universe of discourse to a subset of it. After the introduction of fuzzy topology by Chang [2] in 1968, there have been several generalizations of notions of fuzzy sets and fuzzy topology. By adding the degree of non-membership to FS, Atanassov [1] proposed intuitionistic fuzzy set (IFS) in 1986 which appeals more accurate to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. In 1997, Coker [3] introduced the concept of intuitionistic fuzzy topological space. This paper aspires to overtly enunciate the notion of intuitionistic fuzzy quasi weakly generalized continuous mappings in intuitionistic fuzzy topological space and study some of their properties. We provide some characterizations of intuitionistic fuzzy quasi weakly generalized continuous mappings and establish the relationships with other classes of early defined forms of intuitionistic mappings.

§2. Preliminaries

Definition 2.1.[1] Let X be a non empty fixed set. An intuitionistic fuzzy set ((IFS) in short) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2.[1] Let A and B be $IFSs$ of the forms $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

(ii) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(iii) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

(iv) $A \cap B = \{\langle x, \mu_A(x) \bigwedge \mu_B(x), \nu_A(x) \bigvee \nu_B(x)\rangle : x \in X\}$.

(v) $A \cup B = \{\langle x, \mu_A(x) \bigvee \mu_B(x), \nu_A(x) \bigwedge \nu_B(x)\rangle : x \in X\}$.

For the sake of simplicity, the notation $A = \langle x, \mu_A, \nu_A\rangle$ shall be used instead of $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$. Also for the sake of simplicity, we shall use the notation $A = \langle x, (\mu_A, \mu_B), (\nu_A, \nu_B)\rangle$ instead of $A = \langle x, (A/\mu_A, B/\mu_B), (A/\nu_A, B/\nu_B)\rangle$.

The intuitionistic fuzzy sets $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$ are the empty set and the whole set of X, respectively.

Definition 2.3.[3] An intuitionistic fuzzy topology ((IFT) in short) on a non empty set X is a family τ of $IFSs$ in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$.

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$.

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called an intuitionistic fuzzy topological space ($IFTS$ in short) and any IFS in τ is known as an intuitionistic fuzzy open set ($IFOS$ in short) in X.

The complement A^c of an $IFOS$ A in an $IFTS$ (X, τ) is called an intuitionistic fuzzy closed set ($IFCS$ in short) in X.

Definition 2.4.[3] Let (X, τ) be an $IFTS$ and $A = \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the intuitionistic fuzzy interior and an intuitionistic fuzzy closure are defined by

$int(A) = \cup\{G/G \text{ is an } IFOS \text{ in } X \text{ and } G \subseteq A\}$,

$cl(A) = \cap\{K/K \text{ is an } IFCS \text{ in } X \text{ and } A \subseteq K\}$.

Note that for any IFS A in (X, τ), we have $cl(A^c) = (int(A))^c$ and $int(A^c) = (cl(A))^c$.

Definition 2.5. An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in an $IFTS$ (X, τ) is said to be

(i) Intuitionistic fuzzy semi closed set [6] ($IFSCS$ in short) if $int(cl(A)) \subseteq A$.

(ii) Intuitionistic fuzzy α-closed set [6] ($IF\alpha CS$ in short) if $cl(int(cl(A))) \subseteq A$.

(iii) Intuitionistic fuzzy pre-closed set [6] ($IFPCS$ in short) if $cl(int(A)) \subseteq A$.

(iv) Intuitionistic fuzzy regular closed set [6] ($IFRCS$ in short) if $cl(int(A)) = A$.

(v) Intuitionistic fuzzy generalized closed set [14] ($IFGCS$ in short) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $IFOS$.

(vi) Intuitionistic fuzzy generalized semi closed set [13] ($IFGSCS$ in short) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $IFOS$.

(vii) Intuitionistic fuzzy α generalized closed set [11] ($IF\alpha GCS$ in short) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $IFOS$.

(viii) Intuitionistic fuzzy γ closed set [5] ($IF\gamma CS$ in short) if $\mathrm{int}(cl(A)) \cap cl(int(A)) \subseteq A$.

An IFS A is called intuitionistic fuzzy semi open set, intuitionistic fuzzy α-open set, intuitionistic fuzzy pre-open set, intuitionistic fuzzy regular open set, intuitionistic fuzzy generalized open set, intuitionistic fuzzy generalized semi open set, intuitionistic fuzzy α generalized open set and intuitionistic fuzzy γ open set ($IFSOS$, $IF\alpha OS$, $IFPOS$, $IFROS$, $IFGOS$, $IFGSOS$, $IF\alpha GOS$ and $IF\gamma OS$) if the complement A^c is an $IFSCS$, $IF\alpha CS$, $IFPCS$, $IFRCS$, $IFGCS$, $IFGSCS$, $IF\alpha GCS$ and $IF\gamma CS$ respectively.

Definition 2.6.[7] An IFS $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle :\ x \in X\}$ in an $IFTS$ (X, τ) is said to be an intuitionistic fuzzy weakly generalized closed set ($IFWGCS$ in short) if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is an $IFOS$ in X.

The family of all $IFWGCSs$ of an $IFTS$ (X, τ) is denoted by $IFWGC(X)$.

Definition 2.7.[7] An IFS $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle :\ x \in X\}$ in an $IFTS$ (X, τ) is said to be an intuitionistic fuzzy weakly generalized open set ($IFWGOS$ in short) if the complement A^c is an $IFWGCS$ in X.

The family of all $IFWGOSs$ of an $IFTS$ (X, τ) is denoted by $IFWGO(X)$.

Result 2.1.[7] Every $IFCS$, $IF\alpha CS$, $IFGCS$, $IFRCS$, $IFPCS$, $IF\alpha GCS$ is an $IFWGCS$ but the converses need not be true in general.

Definition 2.8.[8] Let (X, τ) be an $IFTS$ and A $= \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the intuitionistic fuzzy weakly generalized interior and an intuitionistic fuzzy weakly generalized closure are defined by

$wgint(A) = \cup\{G/G$ is an $IFWGOS$ in X and $G \subseteq A\}$,

$wgcl(A) = \cap\{K/K$ is an $IFWGCS$ in X and $A \subseteq K\}$.

Definition 2.9.[3] Let f be a mapping from an $IFTS$ (X, τ) into an $IFTS$ (Y, σ). If $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle : y \in Y\}$ is an IFS in Y, then the pre-image of B under f denoted by $f^{-1}(B)$, is the IFS in X defined by $f^{-1}(B) = \{\langle x, f^{-1}(\mu_B(x)), f^{-1}(\nu_B(x))\rangle : x \in X\}$, where $f^{-1}(\mu_B(x)) = \mu_B(f(x))$.

If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is an IFS in X, then the image of A under f denoted by $f(A)$ is the IFS in Y defined by $f(A) = \{\langle y, f(\mu_A(y)), f_-(\nu_A(y))\rangle : y \in Y\}$ where $f_-(\nu_A) = 1 - f(1 - \nu_A)$.

Definition 2.10. Let f be a mapping from an $IFTS$ (X, τ) into an $IFTS$ (y, σ). Then f is said to be

(i) Intuitionistic fuzzy continuous [4] (IF continuous in short) if $f^{-1}(B)$ is an $IFOS$ in X for every $IFOS$ B in Y.

(ii) Intuitionistic fuzzy α continuous [6] ($IF\alpha$ continuous in short) if $f^{-1}(B)$ is an $IF\alpha OS$ in X for every $IFOS$ B in Y.

(iii) Intuitionistic fuzzy pre continuous [6] (IFP continuous in short) if $f^{-1}(B)$ is an $IFPOS$ in X for every $IFOS$ B in Y.

(iv) Intuitionistic fuzzy generalized continuous [14] (IFG continuous in short) if $f^{-1}(B)$ is an $IFGOS$ in X for every $IFOS$ B in Y.

(v) Intuitionistic fuzzy α generalized continuous [12] ($IF\alpha G$ continuous in short) if $f^{-1}(B)$

is an $IF\alpha GOS$ in X for every $IFOS$ B in Y.

(vi) Intuitionistic fuzzy weakly generalized continuous [9] ($IFWG$ continuous in short) if $f^{-1}(B)$ is an $IFWGOS$ in X for every $IFOS$ B in Y.

(vii) Intuitionistic fuzzy almost continuous [15] (IFA continuous in short) if $f^{-1}(B)$ is an $IFOS$ in X for every $IFROS$ B in Y.

(viii) Intuitionistic fuzzy almost weakly generalized continuous [10] ($IFAWG$ continuous in short) if $f^{-1}(B)$ is an $IFWGOS$ in X for every $IFROS$ B in Y.

Definition 2.11. An $IFTS$ (X,τ) is said to be an intuitionistic fuzzy $_wT_{1/2}$ space [7] ($IF_wT_{1/2}$ space in short) if every $IFWGCS$ in X is an $IFCS$ in X.

Definition 2.12. An $IFTS$ (X,τ) is said to be an intuitionistic fuzzy $_{wg}T_q$ space [7] ($IF_{wg}T_q$ space in short) if every $IFWGCS$ in X is an $IFPCS$ in X.

§3. Intuitionistic fuzzy quasi weakly generalized continuous mappings

In this section, we introduce intuitionistic fuzzy quasi weakly generalized continuous mappings and study some of their properties.

Definition 3.1. A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be an intuitionistic fuzzy quasi weakly generalized continuous mapping if $f^{-1}(B)$ is an $IFCS$ in (X,τ) for every $IFWGCS$ B of (Y,σ).

Theorem 3.1. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy continuous mapping but not conversely.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Hence f is an intuitionistic fuzzy continuous mapping.

Example 3.1. Let $X=\{a,b\}$, $Y=\{u,v\}$ and $T_1=\langle x,(0.3,0.4),(0.4,0.5)\rangle$, $T_2=\langle y,(0.3,0.4),(0.4,0.5)\rangle$. Then $\tau=\{0_\sim,T_1,1_\sim\}$ and $\sigma=\{0_\sim,T_2,1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f:(X,\tau)\to(Y,\sigma)$ defined as $f(a)=u$ and $f(b)=v$. This f is an intuitionistic fuzzy continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B=\langle y,(0.6,0.7),(0.2,0.1)\rangle$ is an $IFWGCS$ in Y but $f^{-1}(B)=\langle x,(0.6,0.7),(0.2,0.1)\rangle$ is not an $IFCS$ in X.

Theorem 3.2. Let $f:(X,\tau)\to(Y,\sigma)$ be a mapping from an $IFTS$ (X,τ) into an $IFTS$ (Y,σ) and (Y,σ) an $IF_wT_{1/2}$ space. Then the following statements are equivalent.

(i) f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

(ii) f is an intuitionistic fuzzy continuous mapping.

Proof. (i)⇒(ii) Is obviously true from the Theorem 3.1.

(ii)⇒(i) Let A be an $IFWGCS$ in Y. Since (Y,σ) is an $IF_wT_{1/2}$ space, A is an $IFCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Hence f is an intuitionistic fuzzy quasi weakly generalized continuous mappings.

Theorem 3.3. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy α continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IF\alpha CS$, $f^{-1}(A)$ is an $IF\alpha CS$ in X. Hence f is an intuitionistic fuzzy α continuous mapping.

Example 3.2. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.2, 0.4), (0.4, 0.6)\rangle$, $T_2 = \langle y, (0.2, 0.4), (0.4, 0.6)\rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy α continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B = \langle y, (0.5, 0.6), (0.2, 0.1)\rangle$ is an $IFWGCS$ in Y but $f^{-1}(B)= \langle x, (0.5, 0.6), (0.2, 0.1)\rangle$ is not an $IFCS$ in X.

Theorem 3.4. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy pre continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IFPCS$, $f^{-1}(A)$ is an $IFPCS$ in X. Hence f is an intuitionistic fuzzy pre continuous mapping.

Example 3.3. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.2, 0.3), (0.3, 0.6)\rangle$, $T_2 = \langle y, (0.2, 0.3), (0.3, 0.6)\rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy pre continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping , since the IFS $B = \langle y, (0.6, 0.4), (0.2, 0.1)\rangle$ is an $IFWGCS$ in Y but $f^{-1}(B)= \langle x, (0.6, 0.4), (0.2, 0.1)\rangle$ is not an $IFCS$ in X.

Theorem 3.5. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy generalized continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IFGCS$, $f^{-1}(A)$ is an $IFGCS$ in X. Hence f is an intuitionistic fuzzy generalized continuous mapping.

Example 3.4. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.2, 0.2), (0.3, 0.4)\rangle$, $T_2 = \langle y, (0.2, 0.2), (0.3, 0.4)\rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy generalized continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B = \langle y, (0.4, 0.5), (0.2, 0)\rangle$ is an $IFWGCS$ in Y but $f^{-1}(B)= \langle x, (0.4, 0.5), (0.2, 0)\rangle$ is not an $IFCS$ in X.

Theorem 3.6. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy α generalized continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IF\alpha GCS$, $f^{-1}(A)$ is an $IF\alpha GCS$ in X. Hence f is an intuitionistic fuzzy α generalized continuous mapping.

Example 3.5. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.2, 0.3), (0.4, 0.5)\rangle$, $T_2 = \langle y, (0.2, 0.3), (0.4, 0.5)\rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and

Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy α generalized continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B = \langle y, (0.3, 0.4), (0.2, 0) \rangle$ is an $IFWGCS$ in Y but $f^{-1}(B) = \langle x, (0.3, 0.4), (0.2, 0) \rangle$ is not an $IFCS$ in X.

Theorem 3.7. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy almost weakly generalized continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFRCS$ in Y. Since every $IFRCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IFWGCS$, $f^{-1}(A)$ is an $IFWGCS$ in X. Hence f is an intuitionistic fuzzy almost weakly generalized continuous mapping.

Example 3.6. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.4, 0.5), (0.5, 0.5) \rangle$, $T_2 = \langle y, (0.4, 0.5), (0.5, 0.5) \rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy almost weakly generalized continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping , since the IFS $B = \langle y, (0.6, 0.7), (0.3, 0.2) \rangle$ is an $IFWGCS$ in Y but $f^{-1}(B) = \langle x, (0.6, 0.7), (0.3, 0.2) \rangle$ is not an $IFCS$ in X.

Theorem 3.8. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy almost continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFRCS$ in Y. Since every $IFRCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Hence f is an intuitionistic fuzzy almost continuous mapping.

Example 3.7. Let $X = \{a, b\}$, $Y = \{u, v\}$ and $T_1 = \langle x, (0.3, 0.5), (0.4, 0.5) \rangle$, $T_2 = \langle y, (0.3, 0.5), (0.4, 0.5) \rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy almost continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B = \langle y, (0.5, 0.6), (0.4, 0.2) \rangle$ is an $IFWGCS$ in Y but $f^{-1}(B) = \langle x, (0.5, 0.6), (0.4, 0.2) \rangle$ is not an $IFCS$ in X.

Theorem 3.9. Every intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy weakly generalized continuous mapping but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping. Let A be an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, A is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFCS$ in X. Since every $IFCS$ is an $IFWGCS$, $f^{-1}(A)$ is an $IFWGCS$ in X. Hence f is an intuitionistic fuzzy weakly generalized continuous mapping.

Example 3.8. Let $X = \{a, b\}$, Y $= \{u, v\}$ and $T_1 = \langle x, (0.2, 0.1), (0.4, 0.5) \rangle$, $T_2 = \langle y, (0.2, 0.1), (0.4, 0.5) \rangle$. Then $\tau = \{0_\sim, T_1, 1_\sim\}$ and $\sigma = \{0_\sim, T_2, 1_\sim\}$ are $IFTs$ on X and Y respectively. Consider a mapping $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = u$ and $f(b) = v$. This f is an intuitionistic fuzzy weakly generalized continuous mapping but not an intuitionistic fuzzy quasi weakly generalized continuous mapping, since the IFS $B = \langle y, (0.6, 0.7), (0.2, 0.2) \rangle$ is an $IFWGCS$ in Y but $f^{-1}(B) = \langle x, (0.6, 0.7), (0.2, 0.2) \rangle$ is not an $IFCS$ in X.

Theorem 3.10. Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping from an $IFTS$ (X, τ) into an $IFTS$

(Y,σ). Then the following statements are equivalent:

(i) f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

(ii) $f^{-1}(B)$ is an $IFOS$ in X for every $IFWGOS$ B in Y.

Proof. (i)⇒(ii) Let B be an $IFWGOS$ in Y. Then B^c is an $IFWGCS$ in Y. By hypothesis, $f^{-1}(B^c) = (f^{-1}(B))^c$ is an $IFCS$ in X. Hence $f^{-1}(B)$ is an $IFOS$ in X.

(ii)⇒(i) Let B be an $IFWGCS$ in Y. Then B^c is an $IFWGOS$ in Y. By (ii), $f^{-1}(B^c)$ $=(f^{-1}(B))^c$ is an $IFOS$ in X. Hence $f^{-1}(B)$ is an $IFCS$ in X. Therefore f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Theorem 3.11. Let $f:(X,\tau)\to(Y,\sigma)$ be a mapping from an $IFTS$ (X,τ) into an $IFTS$ (Y,σ) and let $f^{-1}(A)$ be an $IFRCS$ in X for every $IFWGCS$ in Y. Then f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Proof. Let A be an $IFWGCS$ in Y. By hypothesis, $f^{-1}(A)$ is an $IFRCS$ in X. Since every $IFRCS$ is an $IFCS$, $f^{-1}(A)$ is an $IFCS$ in X. Hence f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Theorem 3.12. Let $f:(X,\tau)\to(Y,\sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping from an $IFTS$ (X,τ) into an $IFTS$ (Y,σ). Then $f(cl(A))\subseteq wgcl(f(A))$ for every IFS A in X.

Proof. Let A be an IFS in X. Then wgcl$(f(A))$ is an $IFWGCS$ in Y. Since f is an intuitionistic fuzzy quasi weakly generalized continuous mapping, $f^{-1}(wgcl(f(A)))$ is an $IFCS$ in X. Clearly $A\subseteq f^{-1}(wgcl(f(A)))$. Therefore $cl(A)\subseteq cl(f^{-1}(wgcl(f(A))))=f^{-1}(wgcl(f(A)))$. Hence $f(cl(A))\subseteq wgcl(f(A))$ for every IFS A in X.

Theorem 3.13. Let $f:(X,\tau)\to(Y,\sigma)$ be a mapping from an $IFTS$ (X,τ) into an $IFTS$ (Y,σ). Then the following statements are equivalent:

(i) f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

(ii) $f^{-1}(B)$ is an $IFOS$ in X for every $IFWGOS$ B in Y.

(iii) $f^{-1}(wgint(B))\subseteq int(f^{-1}(B))$ for every IFS B in Y.

(iv) $cl(f^{-1}(B))\subseteq f^{-1}(wgcl(B))$ for every IFS B in Y.

Proof. (i)⇒(ii) Is obviously true from the Theorem 3.10.

(ii)⇒(iii) Let B be an IFS in Y. Then $wgint(B)$ is an $IFWGOS$ in Y. By (ii), $f^{-1}(wgint(B))$ is an $IFOS$ in X. Therefore $f^{-1}(wgint(B))=int(f^{-1}(wgint(B)))$. Clearly $wgint(B)\subseteq B$. This implies $f^{-1}(wgint(B))\subseteq f^{-1}(B)$. Therefore $f^{-1}(wgint(B))=int(f^{-1}(wgint(B)))\subseteq int(f^{-1}(B))$. Hence $f^{-1}(wgint(B))\subseteq int(f^{-1}(B))$ for every IFS B in Y.

(iii)⇒(iv) It can be proved by taking the complement.

(iv)⇒(i) Let B be an $IFWGCS$ in Y. Then $wgcl(B)=B$. Therefore $f^{-1}(B)=f^{-1}(wgcl(B))\supseteq cl(f^{-1}(B))$. Hence $cl(f^{-1}(B))=f^{-1}(B)$. This implies $f^{-1}(B)$ is an $IFCS$ in Y. Hence f is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Theorem 3.14. The composition of two intuitionistic fuzzy quasi weakly generalized continuous mapping is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Proof. Let A be an $IFWGCS$ in Z. By hypothesis, $g^{-1}(A)$ is an $IFCS$ in Y. Since every $IFCS$ is an $IFWGCS$, $g^{-1}(A)$ is an $IFWGCS$ in Y. Then $f^{-1}(g^{-1}(A))=(gof)^{-1}(A)$ is an $IFCS$ in X, by hypothesis. Hence gof is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

Theorem 3.15. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \delta)$ be any two mappings. Then the following statements hold

(i) Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy continuous mapping and $g : (Y, \sigma) \to (Z, \delta)$ an intuitionistic fuzzy quasi weakly generalized continuous mapping. Then their composition $gof : (X, \tau) \to (Z, \delta)$ is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

(ii) Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping and $g : (Y, \sigma) \to (Z, \delta)$ an intuitionistic fuzzy continuous mapping [respectively intuitionistic fuzzy α continuous mapping, intuitionistic fuzzy pre continuous mapping, intuitionistic fuzzy α generalized continuous mapping and intuitionistic fuzzy generalized continuous mapping]. Then their composition $gof : (X, \tau) \to (Z, \delta)$ is an intuitionistic fuzzy continuous mapping.

(iii) Let $f : (X, \tau) \to (Y, \sigma)$ be an intuitionistic fuzzy quasi weakly generalized continuous mapping and $g : (Y, \sigma) \to (Z, \delta)$ an intuitionistic fuzzy weakly generalized continuous mapping. Then their composition $gof : (X, \tau) \to (Z, \delta)$ is an intuitionistic fuzzy continuous mapping.

Proof. (i) Let A be an $IFWGCS$ in Z. By hypothesis, $g^{-1}(A)$ is an $IFCS$ in Y. Since f is an intuitionistic fuzzy continuous mapping, $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$ is an $IFCS$ in X. Hence gof is an intuitionistic fuzzy quasi weakly generalized continuous mapping.

(ii) Let A be an $IFCS$ in Z. By hypothesis, $g^{-1}(A)$ is an $IFCS$ [respectively $IF\alpha CS$, $IFPCS$, $IF\alpha GCS$ and $IFGCS$] in Y. Since every $IFCS$ [respectively $IF\alpha CS$, $IFPCS$, $IF\alpha GCS$ and $IFGCS$] is an $IFWGCS$, $g^{-1}(A)$ is an $IFWGCS$ in Y. Then $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$ is an $IFCS$ in X, by hypothesis. Hence gof is an intuitionistic fuzzy continuous mapping.

(iii) Let A be an $IFCS$ in Z. By hypothesis, $g^{-1}(A)$ is an $IFWGCS$ in Y. Since f is an intuitionistic fuzzy quasi weakly generalized continuous mapping, $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$ is an $IFCS$ in X. Hence gof is an intuitionistic fuzzy continuous mapping.

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A short interval result for the Smarandache ceil function and the Dirichlet divisor function[1]

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Abstract In this paper we shall establish a short interval result for the Smarandache ceil function and the Dirichlet divisor function by the convolution method.

Keywords Smarandache Ceil function, Dirichlet divisor function, asymptotic formula, convolution method, short interval.

§1. Introduction

For a fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as

$$\{S_k(n) = \min m \in N : n \mid m^k\}.$$

This function was introduced by professor Smarandache. About this function, many scholars studied its properties. Ibstedt [2] presented the following property: $(\forall a, b \in N)(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b)$. It is easy to see that if $(a, b) = 1$, then $(S_k(a), S_k(b)) = 1$. In her thesis, Ren Dongmei [4] proved the asymptotic formula

$$\sum_{n \le x} d(S_k(n)) = c_1 x \log x + c_2 x + O(x^{\frac{1}{2}+\epsilon}), \tag{1}$$

where c_1 and c_2 are computable constants, and ϵ is any fixed positive number.
The aim of this paper is to prove the following:

Theorem 1.1. Let $d(n)$ denote the Dirichlet divisor function, $S_k(n)$ denote the Smarandache ceil function, then for $\frac{1}{4} < \theta < \frac{1}{3}$, $x^{\theta+2\epsilon} \le y \le x$, we have

$$\sum_{x < n \le x+y} d(S_k(n)) = H(x+y) - H(x) + O(yx^{-\frac{\epsilon}{2}} + x^{\theta+\epsilon}), \tag{2}$$

where $H(x) = t_1 x \log x + t_2 x$.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

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§2. Proof of the Theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1.

$$\sum_{n\le x} d(n) = x\log x + (2r-1)x + O(x^{\theta+\epsilon}). \tag{3}$$

The asymptotic formula (3) is the well-known Dirichlet divisor problem. The latest value of θ is $\theta = \frac{131}{416}$ proved by Huxley [6].

Lemma 2.2.

$$\sum_{n\le x} |g(n)| \ll x^{1-\alpha+\epsilon}.$$

Proof. It follows from $|g(n)| \ll n^{-\alpha+\epsilon}$.

Lemma 2.3. Let $k\ge 2$ be a fixed integer, $1<y\le x$ be large real numbers and

$$B(x,y;k,\epsilon) := \sum_{\substack{x\,<\,nm^k\,\le\,x+y\\ m\,>\,x^{\epsilon}}} 1.$$

Then we have

$$B(x,y;k,\epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{4}}.$$

Proof. This is Lemma 2.3 of Zhai [5].

Now we prove our theorem, which is closely related to the Dirichlet divisor problem.

Proof. Let $F(s) = \sum_{n=1}^{\infty} \frac{d(S_k(n))}{n^s} (\sigma > 1)$, here $d(S_k(n)$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{d(S_k(n))}{n^s} &= \prod_p \left(1 + \frac{d(S_k(p)}{p^s} + \frac{d(S_k(p^2)}{p^{2s}} + \frac{d(S_k(p^3)}{p^{3s}} + \cdots\right)\\
&= \prod_p \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \cdots\right)\\
&= \zeta(s)\prod_p \left(1 + \frac{1}{p^s} + \cdots\right)\\
&= \zeta^2(s)\prod_p \left(1 - \frac{1}{p^{2s}} + \cdots\right)\\
&= \frac{\zeta^2(s)}{\zeta(2s)} G(s).
\end{aligned}$$

So we get $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ and by the properties of Dirichlet series, it is absolutely convergent for $\Re s > \frac{1}{3}$.

By the convolution method, we have $d(S_k(n)) = \sum_{n=n_1n_2n_3^2} d(n_1)g(n_2)u(n_3)$, where $d(n)$ is the divisor function. Then

$$\sum_{x<n\le x+y} d(S_k(n)) = \sum_{x<n_1n_2n_3^2\le x+y} d(n_1)g(n_2)u(n_3) = \sum_1 + O(\sum_2 + \sum_3), \tag{4}$$

where

$$\sum_1 = \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} g(n_2)u(n_3) \sum_{\frac{x}{n_2 n_3^2} < n_1 \le \frac{x+y}{n_2 n_3^2}} d(n_1),$$

$$\sum_2 = \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_2 > x^\epsilon}} |d(n_1)g(n_2)u(n_3)|,$$

$$\sum_3 = \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_3 > x^\epsilon}} |d(n_1)g(n_2)u(n_3)|.$$

In view of Lemma 2.1, the inner sum in $\sum_1$ is

$$\frac{x+y}{n_2 n_3^2}\log\frac{x+y}{n_2 n_3^2} - \frac{x}{n_2 n_3^2}\log\frac{x}{n_2 n_3^2} + (2r-1)\frac{y}{n_2 n_3^2} + O(\frac{x^\theta}{n_2^\theta n_3^{2\theta}})$$
$$= \frac{(x+y)\log(x+y) - x\log x}{n_2 n_3^2} - y\frac{\log(n_2 n_3^2)}{n_2 n_3^2} + (2r-1)\frac{y}{n_2 n_3^2} + O(\frac{x^\theta}{n_2^\theta n_3^{2\theta}}).$$

Inserting the above expression into $\sum_1$ and after some easy calculations, we get

$$\sum_1 = H(x+y) - H(x) + O(yx^{-\epsilon} + yx^{-\frac{2}{3}\epsilon+\epsilon^2} + x^{\theta+\epsilon}). \tag{5}$$

For $\sum_2$, we have

$$|g(n_2)| \ll n_2^{-\frac{2}{3}+\epsilon} \ll x^{-\frac{2}{3}\epsilon+\epsilon^2},$$

if we notice that $n_2 > x^\epsilon$, and hence

$$\sum_2 \ll x^{-\frac{2}{3}\epsilon+\epsilon^2} \sum_{x<n_1 n_2 n_3^2 \le x+y} d(n_1) = x^{-\frac{2}{3}\epsilon+\epsilon^2} \sum_{x<n\le x+y} d_*(n),$$

where

$$d_*(n) = \sum_{n=n_1 n_2 n_3^2} d(n_1) \ll n^{\epsilon^2}.$$

Therefore we have

$$\sum_2 \ll x^{-\frac{2}{3}\epsilon+\epsilon^2} \sum_{x<n\le x+y} n^{\epsilon^2} \ll yx^{-\frac{2}{3}\epsilon+\epsilon^2}. \tag{6}$$

Since $d(n) \ll n^{\epsilon^2}, g(n_2) \ll 1$, by lemma 2.3 we have

$$\begin{aligned}\sum_3 &\ll x^{\epsilon^2} \sum_{\substack{x < n_1 n_2 n_3^2 \le x+y \\ n_3 > x^\epsilon}} 1 \\ &\ll x^{\epsilon^2} \sum_{\substack{x < n n_3^2 \le x+y \\ n_3 > x^\epsilon}} d(n) \\ &\ll x^{2\epsilon^2} \sum_{\substack{x < n n_3^2 \le x+y \\ n_3 > x^\epsilon}} 1 = x^{2\epsilon^2} B(x,y;2,\epsilon) \\ &\ll yx^{-\epsilon+2\epsilon^2} + x^{\frac{1}{4}+\epsilon^2}. \end{aligned} \tag{7}$$

Now our theorem follows from (4) and (7).

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Some sufficient conditions for ϕ-like functions

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Abstract In the present paper, using the technique of subordination chain, we obtain some sufficient conditions for normalized analytic functions to be ϕ-like. Certain sufficient conditions for starlike and close-to-convex functions are also given.

Keywords ϕ-like function, starlike function, close-to-convex, subordination chain.

§1. Introduction

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and are normalized by the conditions $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α which is analytically defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f(z) \in \mathcal{A} : \Re\frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathbb{E} \right\},$$

where α is a real number such that $0 \le \alpha < 1$. We shall use $\mathcal{S}^*$ to denote $\mathcal{S}^*(0)$, the class of univalent starlike functions (w.r.t. the origin).

A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E}, \tag{1}$$

for a starlike function g (not necessarily normalized). The class of close-to-convex functions is denoted by $\mathcal{C}$. It is well-known that every close-to-convex function is univalent. In case $g(z) \equiv z$, the condition (1) reduces to

$$\Re\ f'(z) > 0,\ z \in \mathbb{E} \quad \Rightarrow \quad f \in \mathcal{C}.$$

This simple but elegant result was independently proved by Noshiro [5] and Warchawski [8] in 1934/35.

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re\ \phi'(0) > 0$, then, the function $f \in \mathcal{A}$ is said to be ϕ-like in $\mathbb{E}$ if

$$\Re\ \frac{zf'(z)}{\phi(f(z))} > 0,\ z \in \mathbb{E}.$$

This concept was introduced by L. Brickman [3]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ. Later, Ruscheweyh [7] investigated the following general class of ϕ-like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$, then the function $f \in \mathcal{A}$ is called ϕ-like with respect to a univalent function $q, q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \ z \in \mathbb{E}.$$

For two analytic functions f and g in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function w analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1, \ z \in \mathbb{E}$ such that $f(z) = g(w(z)), \ z \in \mathbb{E}$. In case the function g is univalent, the above subordination is equivalent to: $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \ \Phi(p(0), 0; 0) = h(0). \tag{2}$$

A univalent function q is called a dominant of the differential subordination (2) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for each dominant q of (2), is said to be the best dominant of (2).

The main objective of this paper is to derive some sufficient conditions for ϕ-like, starlike, close-to-convex functions.

§2. Preliminaries

We shall need following definition and lemmas to prove our results.

Definition 2.1. A function $L(z, t), \ z \in \mathbb{E}$ and $t \geq 0$ is said to be a subordination chain if $L(., t)$ is analytic and univalent in $\mathbb{E}$ for all $t \geq 0$, $L(z, .)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{E}$ and $L(z, t_1) \prec L(z, t_2)$ for all $0 \leq t_1 \leq t_2$.

Lemma 2.1.[6] The function $L(z, t) : \mathbb{E} \times [0, \infty) \to \mathbb{C}$, ($\mathbb{C}$ is the set of complex numbers), of the form $L(z, t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ for all $t \geq 0$, and $\lim\limits_{t \to \infty} |a_1(t)| = \infty$, is said to be a subordination chain if and only if $\operatorname{Re}\left[\frac{z\partial L/\partial z}{\partial L/\partial t}\right] > 0$ for all $z \in \mathbb{E}$ and $t \geq 0$.

Lemma 2.2.[4] Let F be analytic in $\mathbb{E}$ and let G be analytic and univalent in $\overline{\mathbb{E}}$ except for points ζ_0 such that $\lim\limits_{z \to \zeta_0} G(z) = \infty$, with $F(0) = G(0)$. If $F \not\prec G$ in $\mathbb{E}$, then there is a point $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$ (boundary of $\mathbb{E}$) such that $F(|z| < |z_0|) \subset G(\mathbb{E})$, $F(z_0) = G(\zeta_0)$ and $z_0 F'(z_0) = m\zeta_0 G'(\zeta_0)$ for some $m \geq 1$.

§3. Main results

Throughout this paper, value of a complex power taken, is the principal one.

Theorem 3.1. Let α be a complex number and let q be a univalent function such that $\frac{zq'(z)}{(q(z))^{\alpha}}$ is starlike in $\mathbb{E}$. If an analytic function p, satisfies the differential subordination

$$\frac{zp'(z)}{(p(z))^{\alpha}} \prec \frac{zq'(z)}{(q(z))^{\alpha}}, \quad p(0) = q(0) = 1, \; z \in \mathbb{E}, \tag{3}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Define the function h as follows:

$$h(z) = \frac{zq'(z)}{(q(z))^{\alpha}}, \; z \in \mathbb{E}. \tag{4}$$

For the subordination (3) to be well-defined in $\mathbb{E}$, we, first, prove that $h(z)$ is univalent in $\mathbb{E}$. Differentiating (4) and simplifying a little, we get

$$\frac{zh'(z)}{Q(z)} = \frac{zQ'(z)}{Q(z)},$$

where $Q(z) = \frac{zq'(z)}{(q(z))^{\alpha}}$. In view of the given conditions, we obtain

$$\Re \, \frac{zh'(z)}{Q(z)} > 0.$$

Thus, $h(z)$ is close-to-convex and hence univalent in $\mathbb{E}$. We need to show that $p \prec q$. Suppose to the contrary that $p \not\prec q$ in $\mathbb{E}$. Then by Lemma 2.2, there exist points $z_0 \in \mathbb{E}$ and $\zeta_0 \in \partial\mathbb{E}$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta q'(\zeta_0), \; m \geq 1$. Then

$$\frac{z_0 p'(z_0)}{(p(z_0))^{\alpha}} = \frac{m\zeta_0 q'(\zeta_0)}{(q(\zeta_0))^{\alpha}}. \tag{5}$$

Consider a function

$$L(z,t) = (1+t)\frac{zq'(z)}{(q(z))^{\alpha}}, \; z \in \mathbb{E}. \tag{6}$$

The function $L(z,t)$ is analytic in $\mathbb{E}$ for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{E}$. Now

$$a_1(t) = \left(\frac{\partial L(z,t)}{\partial z}\right)_{(0,t)} = (1+t)q'(0).$$

Since q is univalent in $\mathbb{E}$, so $q'(0) \neq 0$ and therefore, it follows that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = (1+t)\frac{zQ'(z)}{Q(z)}.$$

Clearly

$$\Re \, z\frac{\partial L/\partial z}{\partial L/\partial t} > 0,$$

in view of given conditions. Hence, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (6), we have $L(z,0) = h(z)$, thus we deduce that $L(\zeta_0,t) \notin h(\mathbb{E})$

for $|\zeta_0| = 1$ and $t \geq 0$. In view of (5) and (6), we can write

$$\frac{z_0 p'(z_0)}{(p(z_0))^\alpha} = L(\zeta_0, m-1) \notin h(\mathbb{E}),$$

where $z_0 \in \mathbb{E}$, $|\zeta_0| = 1$ and $m \geq 1$ which is a contradiction to (3). Hence, $p \prec q$. This completes the proof of the theorem.

On writing $p(z) = \dfrac{zf'(z)}{\phi(f(z))}$ in Theorem 3.1, we obtain the best dominant for $\dfrac{zf'(z)}{\phi(f(z))}$.

Theorem 3.2. Let α be a complex number and let q be a univalent function such that $\dfrac{zq'(z)}{(q(z))^\alpha}$ is starlike in $\mathbb{E}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$\left(\frac{zf'(z)}{\phi(f(z))}\right)^{1-\alpha}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \prec \frac{zq'(z)}{(q(z))^\alpha},\ z \in \mathbb{E},$$

for some ϕ, analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$, then $\dfrac{zf'(z)}{\phi(f(z))} \prec q(z)$ and $q(z)$ is the best dominant.

Taking $p(z) = \dfrac{zf'(z)}{f(z)}$ in Theorem 3.1, we have the best dominant for $\dfrac{zf'(z)}{f(z)}$.

Theorem 3.3. Let α be a complex number and let q be a univalent function such that $\dfrac{zq'(z)}{(q(z))^\alpha}$ is starlike in $\mathbb{E}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{zq'(z)}{(q(z))^\alpha},\ z \in \mathbb{E},$$

then $\dfrac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Selecting $p(z) = f'(z)$ in Theorem 3.1, we obtain the best dominant for $f'(z)$.

Theorem 3.4. Suppose α is a complex number and q is a univalent function such that $\dfrac{zq'(z)}{(q(z))^\alpha}$ is starlike in $\mathbb{E}$. If $f \in \mathcal{A}$, $f'(z) \neq 0$, satisfies

$$\frac{zf''(z)}{(f'(z))^\alpha} \prec \frac{zq'(z)}{(q(z))^\alpha},\ z \in \mathbb{E},$$

then $f'(z) \prec q(z)$ and $q(z)$ is the best dominant.

§4. Deductions

(i) When dominant is $q(z) = \dfrac{1 + (1-2\beta)z}{1-z}$, $0 \leq \beta < 1$:

By selecting the dominant $q(z) = \dfrac{1 + (1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ in Theorem 3.2, Theorem 3.3 and Theorem 3.4. We see that this dominant satisfies the conditions of above theorems in following particular cases and consequently, we get the following results for ϕ-like, starlike and close-to-convex functions. For $\alpha = 0$ in Theorem 3.2, we obtain:

Corollary 4.1. Suppose $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$\frac{zf'(z)}{\phi(f(z))}\left(1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \prec \frac{2(1-\beta)z}{(1-z)^2},\ z \in \mathbb{E},$$

where ϕ is same as in Theorem 3.2, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\beta)z}{1-z},\ 0 \leq \beta < 1.$$

Take $\alpha = 1$ in Theorem 3.2, we get:

Corollary 4.2. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))} \prec \frac{2(1-\beta)z}{(1-z)[1+(1-2\beta)z]},\ z \in \mathbb{E},$$

where ϕ is same as in Theorem 3.2, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\beta)z}{1-z},\ 0 \leq \beta < 1.$$

Select $\alpha = 2$ in Theorem 3.2, we derive the following result:

Corollary 4.3. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$\frac{1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))}}{\frac{zf'(z)}{\phi(f(z))}} \prec \frac{2(1-\beta)z}{[1+(1-2\beta)z]^2},\ z \in \mathbb{E},$$

where ϕ is same as in Theorem 3.2, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\beta)z}{1-z},\ 0 \leq \beta < 1.$$

Select $\alpha = 0$ in Theorem 3.3, we obtain:

Corollary 4.4. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfy

$$\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{2(1-\beta)z}{(1-z)^2},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},\quad \text{i.e.,}\ \ f \in \mathcal{S}^*(\beta),\ 0 \leq \beta < 1.$$

For $\alpha = 1$ in Theorem 3.3, we get the following result of Billing [1,2]:

Corollary 4.5. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \prec \frac{2(1-\beta)z}{(1-z)[1+(1-2\beta)z]},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},\quad \text{i.e.,}\ \ f \in \mathcal{S}^*(\beta),\ 0 \leq \beta < 1.$$

Take $\alpha = 2$ in Theorem 3.3, we get:

Corollary 4.6. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{2(1-\beta)z}{[1+(1-2\beta)z]^2},\ z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},\ \text{ i.e., }\ f \in \mathcal{S}^*(\beta),\ 0 \leq \beta < 1.$$

For $\alpha = 0$ in Theorem 3.4, we obtain:

Corollary 4.7. Let $f \in \mathcal{A}$, $f'(z) \neq 0$, satisfy

$$zf''(z) \prec \frac{2(1-\beta)z}{(1-z)^2},\ z \in \mathbb{E},$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z},\ \text{ i.e., }\ \Re\ f'(z) > \beta,\ 0 \leq \beta < 1.$$

Put $\alpha = 1$ in Theorem 3.4, we obtain:

Corollary 4.8. If $f \in \mathcal{A}$, $f'(z) \neq 0$, satisfies

$$\frac{zf''(z)}{f'(z)} \prec \frac{2(1-\beta)z}{(1-z)[1+(1-2\beta)z]},\ z \in \mathbb{E},$$

then

$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z},\ \text{ i.e., }\ \Re\ f'(z) > \beta,\ 0 \leq \beta < 1.$$

(ii) When dominant is $q(z) = 1 + \lambda z,\ 0 < \lambda \leq 1:$

Take the dominant $q(z) = 1 + \lambda z,\ 0 < \lambda \leq 1$ in Theorem 3.2, Theorem 3.3 and Theorem 3.4. It is easy to check that this dominant satisfies the conditions of above theorems in following particular cases and consequently, we derive the following results.

Select $\alpha = 0$ in Theorem 3.2, we get:

Corollary 4.9. Let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfy

$$\left|\frac{zf'(z)}{\phi(f(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right)\right| < \lambda,\ 0 < \lambda \leq 1,$$

where ϕ is same as in Theorem 3.2, then

$$\left|\frac{zf'(z)}{\phi(f(z))} - 1\right| < \lambda,\ z \in \mathbb{E}.$$

write $\alpha = 1$ in Theorem 3.2, we obtain:

Corollary 4.10. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))} \prec \frac{\lambda z}{1+\lambda z},\ 0 < \lambda \leq 1,$$

where ϕ is same as in Theorem 3.2, then

$$\left|\frac{zf'(z)}{\phi(f(z))}-1\right|<\lambda,\ z\in\mathbb{E}.$$

Taking $\alpha=2$ in Theorem 3.2, we derive the following result:

Corollary 4.11. If $f\in\mathcal{A},\ \dfrac{zf'(z)}{\phi(f(z))}\neq 0$, satisfies

$$\frac{1+\frac{zf''(z)}{f'(z)}-\frac{z[\phi(f(z))]'}{\phi(f(z))}}{\frac{zf'(z)}{\phi(f(z))}}\prec\frac{\lambda z}{(1+\lambda z)^2},\ 0<\lambda\leq 1,$$

where ϕ is same as in Theorem 3.2,

$$\left|\frac{zf'(z)}{\phi(f(z))}-1\right|<\lambda,\ z\in\mathbb{E}.$$

Select $\alpha=0$ in Theorem 3.3, we get:

Corollary 4.12. Let $f\in\mathcal{A},\ \dfrac{zf'(z)}{f(z)}\neq 0$, satisfy

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<\lambda,\ 0<\lambda\leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\lambda,\ z\in\mathbb{E}.$$

Writing $\alpha=1$ in Theorem 3.3, we obtain the following result of Billing [1]:

Corollary 4.13. If $f\in\mathcal{A},\ \dfrac{zf'(z)}{f(z)}\neq 0$, satisfies

$$1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\prec\frac{\lambda z}{1+\lambda z},\ 0<\lambda\leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\lambda,\ z\in\mathbb{E}.$$

Put $\alpha=2$ in Theorem 3.3, we have the following result:

Corollary 4.14. If $f\in\mathcal{A},\ \dfrac{zf'(z)}{f(z)}\neq 0$, satisfies

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\prec 1+\frac{\lambda z}{(1+\lambda z)^2},\ 0<\lambda\leq 1,$$

then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\lambda,\ z\in\mathbb{E}.$$

For $\alpha=0$ in Theorem 3.4, we obtain:

Corollary 4.15. Let $f\in\mathcal{A},\ f'(z)\neq 0$, satisfy

$$|zf''(z)|<\lambda,\ 0<\lambda\leq 1,$$

then

$$|f'(z) - 1| < \lambda, \; z \in \mathbb{E}.$$

Take $\alpha = 1$ in Theorem 3.4, we have the following result:

Corollary 4.16. If $f \in \mathcal{A}, \; f'(z) \neq 0$, satisfies

$$\frac{zf''(z)}{f'(z)} \prec \frac{\lambda z}{1 + \lambda z}, \; 0 < \lambda \leq 1,$$

then

$$|f'(z) - 1| < \lambda, \; z \in \mathbb{E}.$$

(iii) When dominant is $q(z) = \dfrac{\gamma(1-z)}{\gamma - z}, \; \gamma > 1:$

Select the dominant $q(z) = \dfrac{\gamma(1-z)}{\gamma - z}, \; \gamma > 1$ in Theorem 3.2, Theorem 3.3 and Theorem 3.4. It is easy to check that this dominant satisfies the conditions of above theorems in following particular cases and consequently, we obtain the following results.

Write $\alpha = 1$ in Theorem 3.2, we get:

Corollary 4.17. Suppose $f \in \mathcal{A}, \; \dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))} \prec \frac{(1-\gamma)z}{(1-z)(\gamma - z)}, \; \gamma > 1,$$

where ϕ is same as in Theorem 3.2, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\gamma(1-z)}{\gamma - z}, \; z \in \mathbb{E}.$$

Take $\alpha = 2$ in Theorem 3.2, we obtain:

Corollary 4.18. If $f \in \mathcal{A}, \; \dfrac{zf'(z)}{\phi(f(z))} \neq 0$, satisfies

$$\frac{1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}}{\frac{zf'(z)}{\phi(f(z))}} \prec \frac{(1-\gamma)z}{\gamma(1-z)^2}, \; \gamma > 1,$$

where ϕ is same as in Theorem 3.2, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\gamma(1-z)}{\gamma - z}, \; z \in \mathbb{E}.$$

Writing $\alpha = 1$ in Theorem 3.3, we obtain the following result of Billing [1]:

Corollary 4.19. If $f \in \mathcal{A}, \; \dfrac{zf'(z)}{f(z)} \neq 0$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{(1-\gamma)z}{(1-z)(\gamma - z)}, \; \gamma > 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\gamma(1-z)}{\gamma - z}, \; z \in \mathbb{E}.$$

Select $\alpha = 2$ in Theorem 3.3, we get:

Corollary 4.20. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, satisfies

$$\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + \frac{(1-\gamma)z}{\gamma(1-z)^2}, \ \gamma > 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\gamma(1-z)}{\gamma - z}, \ z \in \mathbb{E}.$$

For $\alpha = 1$ in Theorem 3.4, we obtain:

Corollary 4.21. If $f \in \mathcal{A}$, $f'(z) \neq 0$, satisfies

$$\frac{zf''(z)}{f'(z)} \prec \frac{(1-\gamma)z}{(1-z)(\gamma - z)}, \ \gamma > 1,$$

then

$$f'(z) \prec \frac{\gamma(1-z)}{\gamma - z}, \ z \in \mathbb{E}.$$

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Creation of a summation formula involving hypergeometric function

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Abstract The main aim of present paper is the development of a summation formula involving Contiguous relation and Hypergeometric function.

Keywords Contiguous relation, recurrence relation, Legendre's duplication formula.
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§1. Introduction

Generalized Gaussian Hypergeometric function of one variable is defined by :

$$ {}_AF_B\left[\begin{array}{lr} a_1, a_2, \cdots, a_A \;; & \\ & z \\ b_1, b_2, \cdots, b_B \;; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1} $$

or

$$ {}_AF_B\left[\begin{array}{lr} (a_A) \;; & \\ & z \\ (b_B) \;; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lr} (a_j)_{j=1}^{A} \;; & \\ & z \\ (b_j)_{j=1}^{B} \;; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}. \tag{2} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers. The series converges for all finite z if $A \leq B$, converges for $\mid z \mid< 1$ if $A = B + 1$, diverges for all z, $z \neq 0$ if $A > B + 1$.

Contiguous Relation is defined by :

[Abramowitz p.558(15.2.19)]

$$ \begin{aligned} &(a-b)\;(1-z)\;{}_2F_1\left[\begin{array}{lr} a,\; b\;; & \\ c\;\;; & z \end{array}\right] \\ = \;\; &(c-b)\;{}_2F_1\left[\begin{array}{lr} a,\; b-1\;\;; & \\ c\;\;; & z \end{array}\right] + (a-c)\;{}_2F_1\left[\begin{array}{lr} a-1,\; b\;\;; & \\ c\;; & z \end{array}\right]. \end{aligned} \tag{3} $$

Recurrence relation is defined by :

$$\Gamma(z+1) = z\ \Gamma(z). \tag{4}$$

Legendre's duplication formula is defined by:

$$\sqrt{\pi}\ \Gamma(2z) = 2^{(2z-1)}\ \Gamma(z)\ \Gamma\left(z+\frac{1}{2}\right) \tag{5}$$

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)} \\ &= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)}.\end{aligned} \tag{6}$$

In the monograph of Prudnikov et al., a summation formula is given in the form [Prudnikov, 491, equation(7.3.7.8)].

$${}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & & ; & \end{matrix}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\Gamma(b)}\right]. \tag{7}$$

Now using Legendre's duplication formula and Recurrence relation for Gamma function, the above formula can be written in the form

$${}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & & ; & \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right]. \tag{8}$$

It is noted that the above formula [Prudnikov, 491, equation(7.3.7.8)], i.e., equation (7) or (8) is not correct. The correct form of equation (7) or (8) is obtained by [Asish et. al(2008), p.337(10)]

$${}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-1}{2} & & ; & \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})}\left\{\frac{(b+a-1)}{(a-1)}\right\} + \frac{2\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\right]. \tag{9}$$

Involving the formula obtained by [Asish et. al(2008), p.337(10)], the main formula is developed.

§2. Main summation formula

For the main formula $a \neq b$,

$$\begin{aligned}&{}_2F_1\left[\begin{matrix} a, & b & ; & \frac{1}{2} \\ \frac{a+b-27}{2} & & ; & \end{matrix}\right] \\ &= \frac{2^{(b-1)}\ \Gamma(\frac{a+b-27}{2})}{(a-b)\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-27}{2})}\left\{\frac{(213458046676875a - 491250187505700a^2)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\right.\right. \\ &\quad + \frac{(435512515705695a^3 - 209814739262856a^4 + 63324503917311a^5 - 12906154537276a^6)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\end{aligned}$$

$$+\frac{(1854829867891a^7-192666441968a^8+14632679633a^9-812840028a^{10}+32645613a^{11})}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-922376a^{12}+17381a^{13}-196a^{14}+a^{15}-213458046676875b+703059560256555a^2b)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-764206606631664a^3b+424783145160213a^4b)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-123961558508816a^5b+28028453942867a^6b)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-3633191343712a^7b+425612752519a^8b-27340124448a^9b+1778577801a^{10}b)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-54933424a^{11}b+1944943a^{12}b-22736a^{13}b+377a^{14}b+491250187505700b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-703059560256555ab^2+410654159060166a^3b^2-252495431002668a^4b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(98834442709527a^5b^2-18037624016160a^6b^2+3187069911108a^7b^2-256860290484a^8b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(24819343731a^9b^2-905704800a^{10}b^2+50019606a^{11}b^2-665028a^{12}b^2+20097a^{13}b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-435512515705695b^3+764206606631664ab^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-410654159060166a^2b^3+75757901167095a^4b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-27839681691360a^5b^3+8189570751180a^6b^3-914370528960a^7b^3+130095417375a^8b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-5944083600a^9b^3+479849370a^{10}b^3-7600320a^{11}b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(356265a^{12}b^3+209814739262856b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-424783145160213ab^4+252495431002668a^2b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-75757901167095a^3b^4+5499323707710a^5b^4)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-1198578429720a^6b^4+276636429090a^7b^4-17287439400a^8b^4+2025762375a^9b^4)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-40060020a^{10}b^4+2731365a^{11}b^4-63324503917311b^5+123961558508816ab^5)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-98834442709527a^2b^5+27839681691360a^3b^5)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-5499323707710a^4b^5+170755274970a^6b^5)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-20378504160a^7b^5+3806472285a^8b^5-101970960a^9b^5+10015005a^{10}b^5)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(12906154537276b^6-28028453942867ab^6)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(18037624016160a^2b^6-8189570751180a^3b^6)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(1198578429720a^4b^6-170755274970a^5b^6+2219549220a^7b^6-111775860a^8b^6)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(17298645a^9b^6-1854829867891b^7+3633191343712ab^7-3187069911108a^2b^7)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(914370528960a^3b^7-276636429090a^4b^7+20378504160a^5b^7)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-2219549220a^6b^7+9694845a^8b^7)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(192666441968b^8-425612752519ab^8+256860290484a^2b^8-130095417375a^3b^8)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(17287439400a^4b^8-3806472285a^5b^8+111775860a^6b^8)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-9694845a^7b^8-14632679633b^9)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(27340124448ab^9-24819343731a^2b^9+5944083600a^3b^9)}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$

$$+\frac{(-2025762375a^4b^9+101970960a^5b^9)}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$

$$+\frac{(-17298645a^6b^9+8128400028b^{10}-1778577801ab^{10}+905704800a^2b^{10}-479849370a^3b^{10})}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$

$$+\frac{(40060020a^4b^{10}-10015005a^5b^{10}-32645613b^{11}+54933424ab^{11}-50019606a^2b^{11})}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$

$$+\frac{(7600320a^3b^{11}-2731365a^4b^{11}+922376b^{12}-1944943ab^{12}+665028a^2b^{12}-356265a^3b^{12})}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$

$$\left.+\frac{(-17381b^{13}+22736ab^{13}-20097a^2b^{13}+196b^{14}-377ab^{14}-b^{15})}{\prod\limits_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}\right\}$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-26}{2})}\left\{\frac{(-327685276755900a+556774391637180a^2-352109148087096a^3)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}\right.$$

$$+\frac{(150650472413496a^4-33670631171300a^5+6664432280548a^6-695586859408a^7)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(75024207248a^8-3912675780a^9+240770244a^{10}-5959096a^{11}+201656a^{12}-1820a^{13})}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(28a^{14}+327685276755900b-468705291631704a^2b+466260662646672a^3b)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(-172939780772236a^4b+53680058062448a^5b-7631694182672a^6b+1191972696736a^7b)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(-78351365148a^8b+6951650784a^9b-208643864a^{10}b+10742992a^{11}b-116116a^{12}b)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(3248a^{13}b-556774391637180b^2+468705291631704ab^2-149918371031784a^3b^2)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(100819750068084a^4b^2-22806421978320a^5b^2)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(5419326852576a^6b^2-476374180752a^7b^2)}{\prod\limits_{\Xi=1}^{13}\left\{a-2\Xi\right\}}$$

$$+\frac{(60318126252a^8b^2-2265212040a^9b^2+167605152a^{10}b^2-2198664a^{11}b^2+95004a^{12}b^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(352109148087096b^3-466260662646672ab^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(149918371031784a^2b^3-16250835339600a^4b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(8006146283040a^5b^3-1088289382320a^6b^3+205431554880a^7b^3-10180305000a^8b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(1066306800a^9b^3-17417400a^{10}b^3+1085760a^{11}b^3-150650472413496b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(172939780772236ab^4-100819750068084a^2b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(16250835339600a^3b^4-692464059000a^5b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(263201054760a^6b^4-19769079600a^7b^4+3045922200a^8b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-65032500a^9b^4+5722860a^{10}b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(33670631171300b^5-53680058062448ab^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(22806421978320a^2b^5-8006146283040a^3b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(692464059000a^4b^5-11695284720a^6b^5+3556657440a^7b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-113456700a^8b^5+14567280a^9b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-6664432280548b^6+7631694182672ab^6-5419326852576a^2b^6+1088289382320a^3b^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$
\begin{aligned}
&+\frac{(-263201054760a^4b^6+11695284720a^5b^6-63871920a^7b^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(15967980a^8b^6+695586859408b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-1191972696736ab^7+476374180752a^2b^7-205431554880a^3b^7+19769079600a^4b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-3556657440a^5b^7+63871920a^6b^7-750242207248b^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(78351365148ab^8-60318126252a^2b^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(10180305000a^3b^8-3045922200a^4b^8+113456700a^5b^8-15967980a^6b^8+3912675780b^9)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-6951650784ab^9+2265212040a^2b^9-1066306800a^3b^9+65032500a^4b^9-14567280a^5b^9)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-240770244b^{10}+208643864ab^{10}-167605152a^2b^{10}+17417400a^3b^{10}-5722860a^4b^{10})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(5959096b^{11}-10742992ab^{11}+2198664a^2b^{11}-1085760a^3b^{11}-201656b^{12}+116116ab^{12})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\left.\left.\frac{(-95004a^2b^{12}+1820b^{13}-3248ab^{13}-28b^{14})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\right\}\right] \qquad (10)
\end{aligned}
$$

§3. Derivation of summation formula

Substituting $c=\frac{a+b-27}{2}$ and $z=\frac{1}{2}$ in equation (3), we get

$$
\begin{aligned}
&(a-b)\ {}_2F_1\left[\begin{matrix} a,\ b & ; & \frac{1}{2} \\ \frac{a+b-27}{2} & ; & \end{matrix}\right]\\
= \ &(a-b-27)\ {}_2F_1\left[\begin{matrix} a,\ b-1 & ; & \frac{1}{2} \\ \frac{a+b-27}{2} & ; & \end{matrix}\right]+(a-b+27)\ {}_2F_1\left[\begin{matrix} a-1,\ b & ; & \frac{1}{2} \\ \frac{a+b-27}{2} & ; & \end{matrix}\right].
\end{aligned}
$$

Now involving (9), we get

$$
\begin{aligned}
&L.H.S\\
=\ &\frac{2^{(b-1)}\ \Gamma(\frac{a+b-27}{2})}{\Gamma(b)}\Bigg[\frac{(a-b-27)(b-1)}{(a-b+1)}\ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-27}{2})}\Bigg\{\frac{(-213458046676875)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(269886287248200a+65733460216605a^2)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-223263203842224a^3+138221227795833a^4)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-45299044646856a^5+9373582275057a^6-1314993712032a^7+129330291519a^8)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-9029828808a^9+445756311a^{10}-15213744a^{11}+341523a^{12}-4536a^{13}+27a^{14})}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(491250187505700b-830156182620750ab)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(328340483265960a^2b+96134123718324a^3b)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-118937687443812a^4b+45891929444238a^5b)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-9895540551120a^6b+1424016147672a^7b)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-136339704036a^8b+9405623214a^9b-427953240a^{10}b+13924404a^{11}b-252252a^{12}b)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(2898a^{13}b-435512515705695b^2+819657938551680ab^2-478607127705810a^2b^2)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(83560532622240a^3b^2+28156349135595a^4b^2)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-16884008933760a^5b^2+4303580247540a^6b^2)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}\\
&+\frac{(-605753547840a^7b^2+61026213735a^8b^2-3711302400a^9b^2)}{\prod\limits_{\Lambda=1}^{14}\big\{a-(2\Lambda-1)\big\}}
\end{aligned}
$$

$$+\frac{(173929470a^{10}b^2-4099680a^{11}b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(77805a^{12}b^2+209814739262856b^3-415325620994652ab^3+277758623059272a^2b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-80019646553100a^3b^3+6701096653200a^4b^3+2705571036360a^5b^3-860921031600a^6b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(149712360840a^7b^3-12822283800a^8b^3+893860500a^9b^3-27232920a^{10}b^3+807300a^{11}b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-63324503917311b^4+128105658877704ab^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-91327401815049a^2b^4+30673432773600a^3b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-4930737871350a^4b^4+187346078640a^5b^4+100930660110a^6b^4-16610176800a^7b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(1997482725a^8b^4-81627000a^9b^4+3798795a^10b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(12906154537276b^5-26573613236450ab^5)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(19022311763280a^2b^5-6930110869080a^3b^5+1305574560360a^4b^5-120750589260a^5b^5)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(1557153360a^6b^5+1483370280a^7b^5-100763460a^8b^5)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(8351070a^9b^5-1854829867891b^6)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(3764670584640ab^6-2815452865260a^2b^6+976958962560a^3b^6-207520925850a^4b^6)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(21782295360a^5b^6-1106195580a^6b^6+7020405a^8b^6)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(192666441968b^7-397852161736ab^7)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(273134198736a^2b^7-104886323640a^3b^7+18640515600a^4b^7-2434957560a^5b^7)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(116731440a^6b^7-2674440a^7b^7-14632679633b^8+28551488472ab^8-21460692357a^2b^8)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(6458821200a^3b^8-1454821875a^4b^8+110761560a^5b^8-8947575a^6b^8+8128400028b^9)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-1648153650ab^9+977450760a^2b^9-377991900a^3b^9+44401500a^4b^9-6216210a^5b^9)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-32645613b^{10}+57794880ab^{10}-42767010a^2b^{10}+8442720a^3b^{10}-1924065a^4b^{10})}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(922376b^{11}-1795612ab^{11}+727272a^2b^{11}-278460a^3b^{11}-17381b^{12}+24024ab^{12})}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$\left.+\frac{(-17199a^2b^{12}+196b^{13}-350ab^{13}-b^{14})}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\right\}$$

$$+\frac{(a-b-27)}{(a-b+1)}\,\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-26}{2})}\left\{\frac{(-335591130336525+190559508787980a+177398458571439a^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\right.$$

$$+\frac{(-218789371836456a^3+98760262908247a^4-26171769096172a^5+4452817513683a^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-531476681648a^7+43749464681a^8-2623611276a^9+106326077a^{10}-3062696a^{11})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(51597a^{12}-532a^{13}+a^{14}+874463879976480b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-829169202307410ab+563916462625444a^2b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(215998698359292a^3b-124722423077872a^4b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(35004566087330a^5b-6191872009568a^6b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(726212553736a^7b-60054910656a^8b+3394286610a^9b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-135168176a^{10}b+3411772a^{11}b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-53872a^{12}b+350a^{13}b-906448967123661b^{2}+1042318056685464ab^{2})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-347811359059998a^{2}b^{2}-29649579132744a^{3}b^{2}+54918061714089a^{4}b^{2})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-18173122364880a^{5}b^{2}+3378803927436a^{6}b^{2}-398114860752a^{7}b^{2}+32182394517a^{8}b^{2})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-1725764040a^{9}b^{2}+63585522a^{10}b^{2}-1356264a^{11}b^{2})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(17199a^{12}b^{2}+511028628050304b^{3})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-642114557715732ab^{3}+287766279345840a^{2}b^{3})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-42785053168500a^{3}b^{3}-7242740577600a^{4}b^{3})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(4698245189400a^{5}b^{3}-934192637280a^{6}b^{3}+118103363640a^{7}b^{3}-8685144000a^{8}b^{3})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(471685500a^{9}b^{3}-13075920a^{10}b^{3}+278460a^{11}b^{3})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-179201798851617b^{4}+236171571722196ab^{4})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-117758430558795a^{2}b^{4}+27376521006000a^{3}b^{4})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-1756255126650a^{4}b^{4}-452341922760a^{5}b^{4})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(157977877050a^{6}b^{4}-18003618000a^{7}b^{4}+1550083275a^{8}b^{4})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-56241900a^{9}b^{4}+1924065a^{10}b^{4})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(420128058461766b^5-57073865773518ab^5+29703933609120a^2b^5-7827388198920a^3b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(1031971540320a^4b^5-17314303860a^5b^5-9768114720a^6b^5+2099716920a^7b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-108501120a^8b^5+6216210a^9b^5-70128494260665b^6+9352627138512ab^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-5105678113620a^2b^6+1363410120240a^3b^6-209509446510a^4b^6+15463612080a^5b^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(211988700a^6b^6-63871920a^7b^6+8947575a^8b^6+819314537472b^7-1142446945176ab^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(576748533600a^2b^7-168750399240a^3b^7+23969937600a^4b^7-2196973800a^5b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(74884320a^6b^7+2674440a^7b^7-74146872231b^8+92637235188ab^8-52057083735a^2b^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(12232251000a^3b^8-2128762125a^4b^8+133800660a^5b^8-7020405a^6b^8+4487884128b^9)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-6317241294ab^9+2695602000a^2b^9-815642100a^3b^9+77859600a^4b^9-8351070a^5b^9)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-230219847b^{10}+243421464ab^{10}-138614190a^2b^{10}+20918040a^3b^{10}-3798795a^4b^{10})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(6724224b^{11}-9532692ab^{11}+2611440a^2b^{11}-807300a^3b^{11}-190827b^{12}+134316ab^{12})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-77805a^2b^{12}+2016b^{13}-2898ab^{13}-27b^{14})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\Bigg\}\Bigg]+\frac{2^{(b-1)}\ \Gamma(\frac{a+b-27}{2})}{\Gamma(b)}\Bigg[\frac{(a-b+27)}{(a-b-1)}$$

$$\times\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-26}{2})}\Bigg\{\frac{(335591130336525-874463879976480a+906448967123661a^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-819314537472a^7+74146872231a^8-4487884128a^9+230219847a^{10}-6724224a^{11})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(190827a^{12}-2016a^{13}+27a^{14}-190559508787980b+829169202307410ab)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-1042318056685464a^2b+642114557715732a^3b-236171571722196a^4b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(57073865773518a^5b-9352627138512a^6b+1142446945176a^7b-92637235188a^8b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(6317241294a^9b-243421464a^{10}b+9532692a^{11}b-134316a^{12}b+2898a^{13}b)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-1773984585714439b^2-56391646262544ab^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(347811359059998a^2b^2-287766279345840a^3b^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(117758430558795a^4b^2-29703933609120a^5b^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(5105678113620a^6b^2-576748533600a^7b^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(52057083735a^8b^2-2695602000a^9b^2+138614190a^{10}b^2-2611440a^{11}b^2+77805a^{12}b^2)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(218789371836456b^3-215998698359292ab^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(29649579132744a^2b^3+42785053168500a^3b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-27376521006000a^4b^3+7827388198920a^5b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-1363410120240a^6b^3+168750399240a^7b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(-12232251000a^8b^3+815642100a^9b^3-20918040a^{10}b^3)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$
$$+\frac{(807300a^{11}b^3-98760262908247b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(124722423077872ab^4-54918061714089a^2b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(7242740577600a^3b^4+1756255126650a^4b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-1031971540320a^5b^4+209509446510a^6b^4-23969937600a^7b^4+2128762125a^8b^4)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-77859600a^9b^4+3798795a^10b^4+26171769096172b^5-35004566087330ab^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(18173122364880a^2b^5-4698245189400a^3b^5+452341922760a^4b^5+17314303860a^5b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-15463612080a^6b^5+2196973800a^7b^5-133800660a^8b^5)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(8351070a^9b^5-4452817513683b^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(6191872009568ab^6-3378803927436a^2b^6+934192637280a^3b^6-157977877050a^4b^6)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(9768114720a^5b^6-211988700a^6b^6-74884320a^7b^6+7020405a^8b^6+531476681648b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-726212553736ab^7+398114860752a^2b^7-118103363640a^3b^7+18003618000a^4b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-2099716920a^5b^7+63871920a^6b^7-2674440a^7b^7)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-43749464681b^8+60054910656ab^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(-32182394517a^2b^8+8685144000a^3b^8-1550083275a^4b^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$+\frac{(108501120a^5b^8-8947575a^6b^8)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}$$

$$
\begin{aligned}
&+\frac{(26236111276b^9-3394286610ab^9+1725764040a^2b^9-471685500a^3b^9+56241900a^4b^9)}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-6216210a^5b^9-106326077b^{10}+135168176ab^{10}-63585522a^2b^{10}+13075920a^3b^{10})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\frac{(-1924065a^4b^{10}+3062696b^{11}-3411772ab^{11}+1356264a^2b^{11}-278460a^3b^{11}-51597b^{12})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\\
&+\left.\frac{(53872ab^{12}-17199a^2b^{12}+532b^{13}-350ab^{13}-b^{14})}{\prod\limits_{\Xi=1}^{13}\{a-2\Xi\}}\right\}+\frac{(a-b+27)}{(a-b-1)}\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-27}{2})}\\
&\times\left\{\frac{(213458046676875-491250187505700a+435512515705695a^2-209814739262856a^3)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\right.\\
&+\frac{(63324503917311a^4-12906154537276a^5+1854829867891a^6-192666441968a^7)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(14632679633a^8-812840028a^9+32645613a^{10}-922376a^{11}+17381a^{12}-196a^{13}+a^{14})}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(-269886287248200b+830156182620750ab)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(819657938551680a^2b+415325620994652a^3b)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(-128105658877704a^4b+26573613236450a^5b-3764670584640a^6b+397852161736a^7b)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(-28551488472a^8b+1648153650a^9b-57794880a^{10}b)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(1795612a^{11}b-24024a^{12}b+350a^{13}b)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(-657334602166050b^2-328340483265960ab^2)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}\\
&+\frac{(478607127705810a^2b^2-277758623059272a^3b^2)}{\prod\limits_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}
\end{aligned}
$$

$$+\frac{(91327401815049a^4b^2 - 19022311763280a^5b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(2815452865260a^6b^2 - 273134198736a^7b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(21460692357a^8b^2 - 977450760a^9b^2 + 42767010a^{10}b^2 - 727272a^{11}b^2 + 17199a^{12}b^2)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(223263203842224b^3 - 96134123718324ab^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-83560532622240a^2b^3 + 80019646553100a^3b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-30673432773600a^4b^3 + 6930110869080a^5b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-976958962560a^6b^3 + 104886323640a^7b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-6458821200a^8b^3 + 377991900a^9b^3 - 8442720a^{10}b^3)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(278460a^{11}b^3 - 138221227795833b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(118937687443812ab^4 - 28156349135595a^2b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-6701096653200a^3b^4 + 4930737871350a^4b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-1305574560360a^5b^4 + 207520925850a^6b^4 - 18640515600a^7b^4 + 1454821875a^8b^4)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(-44401500a^9b^4 + 1924065a^{10}b^4 + 45299044646856b^5 - 45891929444238ab^5)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$
$$+\frac{(16884008933760a^2b^5 - 2705571036360a^3b^5 - 187346078640a^4b^5 + 120750589260a^5b^5)}{\prod_{\Lambda=1}^{14}\{a-(2\Lambda-1)\}}$$

$$+\frac{(-21782295360a^6b^5+2434957560a^7b^5-110761560a^8b^5)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(6216210a^9b^5-9373582275057b^6)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(9895540551120ab^6-4303580247540a^2b^6+860921031600a^3b^6-100930660110a^4b^6)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-1557153360a^5b^6+1106195580a^6b^6-116731440a^7b^6)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(8947575a^8b^6+1314993712032b^7)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-1424016147672ab^7+605753547840a^2b^7-149712360840a^3b^7+16610176800a^4b^7)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-1483370280a^5b^7+2674440a^7b^7-129330291519b^8)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(136339704036ab^8-61026213735a^2b^8)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(12822283800a^3b^8-1997482725a^4b^8+100763460a^5b^8)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-7020405a^6b^8+9029828808b^9)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-9405623214ab^9+3711302400a^2b^9-893860500a^3b^9+81627000a^4b^9-8351070a^5b^9)}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(-445756311b^{10}+427953240ab^{10}-173929470a^2b^{10}+27232920a^3b^{10}-3798795a^4b^{10})}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\frac{(152137744b^{11}-13924404ab^{11}+4099680a^2b^{11}-807300a^3b^{11}-341523b^{12}+252252ab^{12})}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}$$
$$+\left.\left.\frac{(-77805a^2b^{12}+4536b^{13}-2898ab^{13}-27b^{14})}{\prod_{\Lambda=1}^{14}\left\{a-(2\Lambda-1)\right\}}\right\}\right].$$

On simplification, we get the main result.

§4. Conclusion

In this paper we have created a summation formula with the help of contiguous relation and hypergeometric function. However, the formula presented herein may be further developed to extend this result. Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions. Just as the mathematical properties of the Gauss hypergeometric function are already of immense and significant utility in mathematical sciences and numerous other areas of pure and applied mathematics, the elucidation and discovery of the formula of hypergeometric functions considered herein should certainly eventually prove useful to further developments in the broad areas alluded to above.

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A short interval result for the function $A(n)$[1]

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Abstract let $n > 1$ be an integer, $gcd(a, b)$ denote the greatest commom divisor of a and b. $A(n)$ denotes the arithmetic mean of $gcd(1, n), \cdots, gcd(n, n)$. In this paper, we shall establish a short interval result for the function $A(n)$.

Keywords *Gcd*-Sum function, short interval, convolution method.

§1. Introduction

The *gcd*-sum function (pillai's function) is defined by

$$P(n) = \sum_{k=1}^{n} gcd(k, n), \tag{1}$$

where $gcd(a, b)$ denotes the greatest commom divisor of a and b. Pillai [2] proved that

$$P(n) = \sum_{d|n} d\varphi(\frac{n}{d})$$

and

$$\sum_{d|n} P(d) = nd(n) = \sum_{d|n} \sigma(d)\varphi(\frac{n}{d}),$$

where φ is Euler's function, $d(n)$ and $\varphi(n)$ denote the number of divisors of n and the sum of the divisors of n respectively. This function is multiplicative and $P(p^a) = (a+1)p^a - ap^{a-1}$ for every prime power $p^a (a \geq 1)$.

Chidambaraswamy and Sitaramachandrarao [4] showed that, given an arbitrary $\epsilon > 0$,

$$\sum_{n \leq x} P(n) = C_1 x^2 \log x + C_2 x^2 + O(x^{1+\theta+\epsilon}), \tag{2}$$

where $C_1,\ C_2$ are computable constant and $0 < \theta < \frac{1}{2}$ is some exponent contained in

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{\theta+\epsilon}). \tag{3}$$

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The asymptotic formula (3) is the well-known Dirichlet divisor problem. The latest value of θ is $\theta = \frac{131}{416}$ proved by Huxley [5].

The arithmetic mean of $gcd(1, n), \cdots, gcd(n, n)$ is given by

$$A(n) = \frac{P(n)}{n} = \sum_{d|n} \frac{\varphi(d)}{d}. \tag{4}$$

The harmonic mean of $gcd(1, n), \cdots, gcd(n, n)$ is

$$H(n) = n \left(\sum_{k=1}^{n} \frac{1}{gcd(k, n)} \right)^{-1} = n^2 \left(\sum_{d|n} d\varphi(d) \right)^{-1}. \tag{5}$$

In this paper, we shall prove the following short interval result.

Theorem 1.1. If $x^{\frac{131}{416}+\epsilon} \le y \le x$, then

$$\sum_{x<n\le x+y} A(n) = C_1 y \log x + C_2 y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{131}{416}+\frac{\epsilon}{2}}), \tag{6}$$

where $C_1 = \frac{1}{\zeta(2)}$, $C_2 = \frac{2\gamma}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}$.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Proof of the Theorem

Lemma 2.1. Suppose s is a complex number ($\Re s > 1$), then

$$\sum_{n=1}^{\infty} \frac{A(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(s+1)}. \tag{7}$$

Proof. Here $A(n)$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{A(n)}{n^s} &= \prod_p \left(1 + \frac{A(p)}{p^s} + \frac{A(p^2)}{p^{2s}} + \frac{A(p^3)}{p^{3s}} + \frac{A(p^4)}{p^{4s}} + \cdots \right) \\
&= \prod_p \left(1 + \frac{\frac{2p-1}{p}}{p^s} + \frac{\frac{3p^2-2p}{p^2}}{p^{2s}} + \frac{\frac{4p^3-3p^2}{p^3}}{p^{3s}} + \cdots \right) \\
&= \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \frac{4}{p^{3s}} + \cdots - \frac{1}{p^{s+1}} - \frac{2}{p^{2s+1}} - \frac{3}{p^{3s+1}} - \cdots \right) \\
&= \zeta^2(s) \prod_p \left(1 - \frac{1}{p^{s+1}}\right) \\
&= \frac{\zeta^2(s)}{\zeta(s+1)}.
\end{aligned}$$

Lemma 2.2. Let $1 < y \le x$ be real number and

$$B(x, y; k, \epsilon) := \sum_{\substack{x < mn \le x+y \\ m > x^{\epsilon}}} \frac{1}{m},$$

Then we have

$$B(x,y;k,\epsilon) \ll yx^{-\epsilon}. \tag{8}$$

Proof.

$$\begin{aligned} B(x,y;k,\epsilon) &= \sum_{x^{\epsilon}<m\le x+y}\frac{1}{m}\sum_{\frac{x}{m}<n\le\frac{x+y}{m}}1 \\ &\ll \sum_{x^{\epsilon}<m\le x+y}\frac{y}{m^2} \\ &\ll yx^{-\epsilon}. \end{aligned}$$

Next we prove our Theorem 1.1. From Lemma 2.1, we get

$$A(n)=\sum_{n=mk}\frac{\mu(m)}{m}d(k).$$

So we have

$$\begin{aligned} \sum_{x<n\le x+y}A(n) &= \sum_{x<mk\le x+y}\frac{\mu(m)}{m}d(k) \\ &= \sum_1 + O(\sum_2), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \sum_1 &= \sum_{m\le x^{\varepsilon}}\frac{\mu(m)}{m}\sum_{\frac{x}{m}<k\le\frac{x+y}{m}}d(k), \\ \sum_2 &= \sum_{\substack{x\,<\,km\,\le\,x\,+\,y \\ m\,>\,x^{\epsilon}}}\left|\frac{\mu(m)d(k)}{m}\right|. \end{aligned}$$

In view of

$$\sum_{n\le x}d(n)=x\log x+(2\gamma-1)x+O\big(x^{\frac{131}{416}+\frac{\epsilon}{2}}\big),$$

then

$$\sum_1=\sum_{m\le x^{\epsilon}}\frac{\mu(m)}{m}\left(\frac{x+y}{m}\log(\frac{x+y}{m})-\frac{x}{m}\log\frac{x}{m}+(2\gamma-1)\frac{y}{m}+O\big((\frac{x}{m})^{\frac{131}{416}+\frac{\epsilon}{2}}\big)\right), \tag{10}$$

Let

$$x_0=\frac{x}{m},\ y_0=\frac{y}{m},$$

then

$$\begin{aligned} (x_0+y_0)\log(x_0+y_0)-x_0\log x_0 &= (x_0+y_0)\left(\log x_0+\log(1+\frac{y_0}{x_0})\right)-x_0\log x_0 \\ &= (x_0+y_0)\left(\log x_0+\frac{y_0}{x_0}+O((\frac{y_0}{x_0})^2\right)-x_0\log x_0 \\ &= y_0\log x_0+y_0+O\left(\frac{y_0^2}{x_0}\right). \end{aligned}$$

Then we have

$$\sum_{1} = \sum_{m\le x^{\epsilon}} \frac{\mu(m)y\log x}{m^2} - \sum_{m\le x^{\epsilon}} \frac{\mu(m)y\log m}{m^2} + 2\gamma \sum_{m\le x^{\epsilon}} \frac{\mu(m)}{m^2} + O(x^{\frac{131}{416}+\frac{\epsilon}{2}} + yx^{-\frac{\epsilon}{2}}), \quad (11)$$

Now we evaluate $\sum_{m\le x^{\epsilon}} \frac{\mu(m)}{m^2}$, $\sum_{m\le x^{\epsilon}} \frac{\mu(m)\log m}{m^2}$,

$$\sum_{m\le x^{\epsilon}} \frac{\mu(m)}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} - \sum_{m>x^{\epsilon}} \frac{\mu(m)}{m^2} = \frac{1}{\zeta(2)} + O(x^{-\epsilon}), \quad (12)$$

$$\sum_{m\le x^{\epsilon}} \frac{\mu(m)\log m}{m^2} = \sum_{m=1}^{\infty} \frac{\mu(m)\log m}{m^2} - \sum_{m\ge x^{\epsilon}} \frac{\mu(m)\log m}{m^2} = \frac{\zeta'(2)}{\zeta^2(2)} + O(x^{-\frac{\epsilon}{2}}). \quad (13)$$

Then we obtain

$$\sum_{1} = \frac{y\log x}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}y + 2\gamma\frac{y}{\zeta(2)} + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{131}{416}+\frac{\epsilon}{2}}). \quad (14)$$

In view of Lemma 2.2,

$$\begin{aligned}\sum_{2} &\ll \sum_{x^{\epsilon}<m\le x+y} \frac{1}{m} \sum_{\frac{x}{m}<n\le\frac{x+y}{m}} d(n) \\ &\ll x^{\frac{\epsilon}{2}} \sum_{x^{\epsilon}<m\le x+y} \frac{1}{m} \sum_{\frac{x}{m}<n\le\frac{x+y}{m}} 1 \\ &\ll yx^{-\frac{\epsilon}{2}}. \end{aligned} \quad (15)$$

Now our theorem follows from (14) and (15).

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A Note on Power Mean and Generalized Contra-Harmonic Mean

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Abstract In this note, the log convexity and a simple proof for monotonicity of power mean and generalized contra-harmonic mean are presented.

Keywords Power mean, generalized contra-harmonic mean, monotonicity and sequence.

§1. Introduction

It is well known that in Pythagorean school on the basis of proportions the ten Greek means are defined of which six means are named and four means are unnamed. The definitions and some distinguished results were discussed in [4]. The six named Greek means studied in terms of contra harmonic mean and it has some interesting properties and studied by several researchers. Further, it is generalized, the generalized version is called Lehmer mean. The power mean is the mean includes arithmetic mean, geometric, harmonic mean and etc. Some remarkable results on these means were found in [1-3,5].

For $a, b > 0$, then

$$C(a,b) = \frac{a^2+b^2}{a+b} \tag{1}$$

$$P_n(a,b) = \begin{cases} \left(\frac{a^n+b^n}{2}\right)^{\frac{1}{n}}, & n \neq 0 \\ \sqrt{ab}, & n = 0 \end{cases} \tag{2}$$

and

$$C_n(a,b) = \frac{a^n+b^n}{a^{n-1}+b^{n-1}} \tag{3}$$

are respectively called contra-harmonic mean, power mean and generalized contra-harmonic mean (or Lehmer mean).

The sequence g_n is said to be log convex (See $[1,2]$), if

$$g_n^2 \leq g_{n+1}c_{n-1} \tag{4}$$

and the sequence g_n is said to be log concave, if

$$g_n^2 \geq g_{n+1}c_{n-1}. \tag{5}$$

§2. Results

In this section, the monotonicity and log convexity results for power mean and generalized contra-harmonic mean were discussed.

Lemma 2.1. For $a,\ b > 0$, then the sequence

$$g_n = \sum_{n=0}^{\infty}(a^n + b^n) \tag{6}$$

is log convex.

Proof. Let $g_n = (a^n + b^n)$, consider

$$\begin{aligned} g_n^2 - g_{n+1}g_{n-1} &= (a^n + b^n)^2 - (a^{n+1} + b^{n+1})(a^{n-1} + b^{n-1}) \\ &= a^{n-1}b^{n-1}[2ab - a^2 - b^2] \\ &= -a^{n-1}b^{n-1}(a-b)^2 \leq 0. \end{aligned}$$

This proves that $g_n^2 \leq g_{n+1}g_{n-1}$.

Lemma 2.2. For $a,\ b > 0$, then the generalized contra-harmonic mean

$$C_n(a,b) = \frac{a^n + b^n}{a^{n-1} + b^{n-1}} \tag{7}$$

is increasing with respect to the parameter n, that is $C_{n+1}(a,b) > C_n(a,b)$ for all real n.

Proof. The proof is explored in [4].

Theorem 2.1. The generalized contra-harmonic mean is monotonically increasing with respect to the parameter n if and only if the sequence g_n of Lemma 2.1 is log-convex.

Proof. Consider, $g_n^2 \leq g_{n+1}g_{n-1}$. Substitute $g_n = a^2 + b^2$. Then,

$$\frac{a^n + b^n}{a^{n-1} + b^{n-1}} \leq \frac{a^{n+1} + b^{n+1}}{a^n + b^n}.$$

This implies that,

$$C_{n+1}(a,b) > C_n(a,b). \tag{8}$$

Again consider,

$$C_{n+1}(a,b) - C_n(a,b) = \frac{a^n + b^n}{a^{n-1} + b^{n-1}} - \frac{a^{n+1} + b^{n+1}}{a^n + b^n}$$

on simplifying gives,

$$= \frac{1}{(a^n + b^n)(a^{n-1} + b^{n-1})}[g_{n+1}g_{n-1} - g_n^2],$$

the eqn (8) is holds, if $g_{n+1}g_{n-1} - g_n^2 \geq 0$. This implies that,

$$g_n^2 \leq g_{n+1}g_{n-1}. \tag{9}$$

The proof is follows from eqs (8) and (9).

Remark 2.1. Theorem 2.3 can also be proved by considering the decreasing sequence $g_n = \sum_{n=0}^{\infty} \frac{1}{(a^n+b^n)}$.

Theorem 2.2. For $a,\ b > 0$, then the generalized contra-harmonic mean $C_n(a,b)$ is log convex.

Proof. Consider,

$$C_{n+1}^2(a,b) - C_{n+1}(a,b)C_{n-1}(a,b) = \left(\frac{a^n+b^n}{a^{n-1}+b^{n-1}}\right)^2 - \left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)\left(\frac{a^{n-1}+b^{n-1}}{a^{n-2}+b^{n-2}}\right)$$

on simplifying this,

$$= \frac{1}{(a^n+b^n)(a^{n-1}+b^{n-1})^2(a^{n-2}+b^{n-2})}[\Delta],$$

where

$$\Delta = (a^{n-2}+b^{n-2})(a^n+b^n)^3 - (a^{n+1}+b^{n+1})(a^{n-1}+b^{n-1})^3$$

$$\Delta = a^{n-2}b^{3n-3}[b^3-a^3] + a^{3n-3}b^{n-2}[a^3-b^3] + 3[a^{3n-2}b^{n-1}(b-a) + a^{n-1}b^{3n-2}(a-b)]$$

is equivalently,

$$\Delta = (a^{n-2}b^{n-2})(a^{2n-1}-b^{2n-1})(a-b)^3 > 0.$$

Theorem 2.3. For $a,\ b > 0$, then the power mean $P_n(a,b)$ satisfies the inequality

$$[P_n^n(a,b)]^2 \leq P_{n+1}^{n+1}(a,b)P_{n-1}^{n-1}(a,b).$$

Proof. From the definition of power mean $P_n(a,b) = \left(\frac{a^n+b^n}{2}\right)^{\frac{1}{n}}$, for $n \neq 0$, which is equivalently written as;

$$P_n^n(a,b) = \left(\frac{a^n+b^n}{2}\right),$$

consider,

$$\begin{aligned}[P_n^n(a,b)]^2 - P_{n+1}^{n+1}(a,b)P_{n-1}^{n-1}(a,b) &= \left(\frac{a^n+b^n}{2}\right)^2 - \left(\frac{a^{n+1}+b^{n+1}}{2}\right)\left(\frac{a^{n-1}+b^{n-1}}{2}\right) \\ &= \frac{1}{4}\left[-(a-b)^2a^{n-1}b^{n-1}\right] \leq 0.\end{aligned}$$

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A short interval result for the e-squarefree e-divisor function[1]

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Abstract Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. The aim of this paper is to establish a short interval result for the function $(t^{(e)}(n))^r$. This enriches the properties of the e-squarefree e-divisor function.

Keywords The e-squarefree e-divisor function, the generalized divisor function, short interval.

§1. Introduction and preliminaries

Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a_i}$. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of n if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$.

The integer $n > 1$ is called e-squarefree, if all exponents $a_1, \cdots, a_s$ are squarefree. The integer 1 is also considered to be e-squarefree. Consider now the exponential squarefree exponential divisor (e-squarefree e-divisor) of n. Here $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an e-squarefree e-divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$, if $b_1|a_1, \cdots, b_s|a_s$, $b_1, \cdots, b_s$ are squarefree. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$.

Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. The function $t^{(e)}(n)$ is called the e-squarefree e-divisor function, which is multiplicative and if $n = \prod_{i=1}^{s} p_i^{\alpha_i} > 1$, then (see [1])

$$t^{(e)}(n) = 2^{\omega(\alpha_1)} \cdots 2^{\omega(\alpha_s)},$$

where $\omega(\alpha) = s$ denotes the number of distinct prime factors of α. The properties of the function $t^{(e)}(n)$ were investigated by many authors; see example [6]. Let

$$A(x) := \sum_{n \leq x} (t^{(e)}(n))^r,$$

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Recently László Tóth proved that the estimate

$$\sum_{n\le x} t^{(e)}(n) = c_1 x + c_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\epsilon})$$

holds for every $\varepsilon > 0$, where

$$c_1 := \prod_p \left(1 + \sum_{\alpha=6}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)}}{p^\alpha}\right), \tag{$*$}$$

$$c_2 := \zeta(\frac{1}{2}) \prod_p \left(1 + \sum_{\alpha=4}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)} - 2^{\omega(\alpha-2)} + 2^{\omega(\alpha-4)}}{p^{\frac{\alpha}{2}}}\right).$$

The aim of this paper is to study the short interval case of $(t^{(e)}(n))^r$ and prove the following

Theorem 1.1. If $x^{\frac{1}{5}+2\varepsilon} < y \le x$, then

$$\sum_{x<n\le x+y} (t^{(e)}(n))^r = c_1 y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}),$$

where c_1 is given by $(*)$.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant. We assume that $1 \le a \le b$ are fixed integers, and we denote by $d(a,b;k)$ the number of representations of k as $k = n_1^a n_2^b$, where $n_1,\ n_2$ are natural numbers, that is,

$$d(a,b;k) = \sum_{k=n_1^a n_2^b} 1,$$

and $d(a,b;k) \ll n^{\epsilon^2}$ will be used freely.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. Suppose s is a complex number ($\Re s > 1$), then

$$F(s) := \sum_{n=1}^{\infty} \frac{(t^e(n))^r}{n^s} = \frac{\zeta(s)\zeta^{2^r-1}(2s)}{\zeta^{C_r}(4s)} G(s),$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{6}$, and $C_r = 2^{2r-1} - 2^{r-1}$.

Proof. Here $(t^{(e)}(n))^r$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(t^{(e)}(n))^r}{n^s} &= \prod_p\left(1+\frac{(t^{(e)}(p))^r}{p^s}+\frac{(t^{(e)}(p^2))^r}{p^{2s}}+\frac{(t^{(e)}(p^3))^r}{p^{3s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^s}+\frac{2^r}{p^{2s}}+\frac{2^r}{p^{3s}}+\frac{2^r}{p^{4s}}+\frac{2^r}{p^{5s}}+\frac{4^r}{p^{6s}}+\cdots\right)\\
&= \prod_p(1-\frac{1}{p^s})^{-1}\prod_p(1-\frac{1}{p^s})\left(1+\frac{1}{p^s}+\frac{2^r}{p^{2s}}+\frac{2^r}{p^{3s}}+\cdots\right)\\
&= \zeta(s)\zeta^{2^r-1}(2s)\prod_p(1-\frac{1}{p^{2s}})^{2^r-1}\left(1+\frac{2^r-1}{p^{2s}}+\frac{4^r-2^r}{p^{6s}}+\cdots\right)\\
&= \frac{\zeta(s)\zeta^{2^r-1}(2s)}{\zeta^{C_r}(4s)}G(s). \qquad (1)
\end{aligned}$$

Now we write $C_r = 2^{2r-1} - 2^{r-1}$ and $G(s) := \sum_{n=1}^{\infty}\frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $\Re s > \frac{1}{6}$.

Lemma 2.2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and

$$B(x, y; k, \epsilon) := \sum_{\substack{x < nm^k \leq x+y \\ m > x^{\epsilon}}} 1.$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}}\log x. \qquad (2)$$

Proof. This lemma is very important when studying the short interval distribution of 1-free number; see for example, [4].

Let $a(n)$, $b(n)$ and $c(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$):

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \zeta(s)G(s). \qquad (3)$$

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \zeta^{2^r-1}(2s). \qquad (4)$$

$$\sum_{n=1}^{\infty}\frac{c(n)}{n^s} = \zeta^{-C_r}(4s). \qquad (5)$$

Lemma 2.3. Let $a(n)$ be an arithmetic function defined by (3), then we have

$$\sum_{n\leq x} a(n) = Cx + O(x^{\frac{1}{6}+\epsilon}), \qquad (6)$$

where $C = \mathrm{Res}_{s=1}\zeta(s)G(s)$.

Proof. Using Lemma 2.1, it is easy to see that

$$\sum_{n\leq x}|g(n)| \ll x^{\frac{1}{6}+\epsilon}.$$

Therefore from the definition of $g(n)$ and (3), it follows that

$$\begin{aligned}\sum_{n\le x} a(n) &= \sum_{mn\le x} g(n)\\ &= \sum_{n\le x} g(n)\sum_{m\le \frac{x}{n}} 1\\ &= \sum_{n\le x} g(n)(\frac{x}{n}+O(1))\\ &= Cx+O(x^{\frac{1}{6}+\epsilon}),\end{aligned}$$

and $C=\mathrm{Res}_{s=1}\zeta(s)G(s)$.

Next we prove our Theorem 1.1. From Lemma 2.3 and the definition of $a(n)$, $b(n)$and $c(n)$, we get

$$(t^{(e)}(n))^r=\sum_{n=n_1n_2^2n_3^4} a(n_1)b(n_2)c(n_3),$$

and

$$a(n)\ll n^{\epsilon^2}, b(n)\ll n^{\epsilon^2}, c(n)\ll n^{\epsilon^2}. \tag{7}$$

So we have

$$\begin{aligned}A(x+y)-A(x) &= \sum_{x<n_1n_2^2n_3^4\le x+y} a(n_1)b(n_2)c(n_3)\\ &= \sum_1 +O(\sum_2+\sum_3),\end{aligned} \tag{8}$$

where

$$\begin{aligned}\sum_1 &= \sum_{\substack{n_2\le x^\epsilon\\ n_3\le x^\epsilon}} b(n_2)c(n_3)\sum_{\frac{x}{n_2^2n_3^4}<n_1\le\frac{x+y}{n_2^2n_3^4}} a(n_1),\\ \sum_2 &= \sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^\epsilon}} |a(n_1)b(n_2)c(n_3)|,\\ \sum_3 &= \sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_3>x^\epsilon}} |a(n_1)b(n_2)c(n_3)|.\end{aligned} \tag{9}$$

In view of Lemma 2.3,

$$\begin{aligned}\sum_1 &= \sum_{\substack{n_2\le x^\epsilon\\ n_3\le x^\epsilon}} b(n_2)c(n_3)\Big(\frac{Cy}{n_2^2n_3^4}+O\big((\frac{x}{n_2^2n_3^4})^{\frac{1}{6}+\epsilon}\big)\Big)\\ &= c_1y+O(yx^{-\frac{\epsilon}{2}}+x^{\frac{1}{6}+\frac{3}{2}\epsilon}),\end{aligned} \tag{10}$$

where $c_1=\mathrm{Res}_{s=1}F(s)$.

$$
\begin{aligned}
\sum_{2} \quad &\ll \sum_{\substack{x < n_1 n_2^2 n_3^4 \leq x+y \\ n_2 > x^{\epsilon}}} (n_1)^{\epsilon^2} \\
&\ll x^{\epsilon^2} \sum_{\substack{x < n_1 n_2^2 n_3^4 \leq x+y \\ n_2 > x^{\epsilon}}} 1 \\
&\ll x^{2\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\
&\ll yx^{2\epsilon^2-\epsilon} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}\log x \\
&\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \qquad (11)
\end{aligned}
$$

Similarly we have

$$\sum_{3} \ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \qquad (12)$$

Now our theorem follows from (8)-(12).

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On the modifications of the Pell-Jacobsthal numbers

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Abstract In this brief note, we formulate some modifications of the Pell-Jacobsthal numbers, which belong to a more general class, the Lucas sequences.

Keywords integer sequence, Lucas number, Pell-Jacobsthal number.
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§1. Introduction

Pell-Jacobsthal sequences have been used extensively in the field of enumerative combinatorics and cryptography. For $n \in \mathbb{N}$, the nth Pell-Jacobsthal number (see e.g.[1,2,3,4]) is defined by

$$j_n = 2^n + (-1)^n.$$

The first ten members of the sequence $\{j_n\}$ are given in Tab. 1.

j_0	j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8	j_9
2	1	5	7	17	31	65	127	257	511

Table 1: The first ten members of $\{j_n\}$.

We first consider the following modification of j_n:

$$j_n^s = s^n + (-1)^n,$$

where $n \in \mathbb{N}$ and $s \geq 0$ is a real number. When $s = 2$ we recover the standard Pell-Jacobsthal numbers.

The first five members of the sequence $\{j_n^s\}$ with respect to n are given in Tab. 2.

j_0^s	j_1^s	j_2^s	j_3^s	j_4^s
2	$s-1$	s^2+1	s^3-1	s^4+1

Table 2: The first five members of $\{j_n^s\}$ for $s \geq 0$.

In the case $s = 0$ we have

$$j_n^0 = (-1)^n.$$

In the case $s = 1$ we have

$$j_n^1 = 1 + (-1)^n.$$

In the case $s = 2$ we have

$$j_n^2 = 2^n + (-1)^n.$$

In Tab. 3 we show the first fifty members of $\{j_n^s\}$ with respect to n and s.

j_n^s	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
$s = 0$	1	-1	1	-1	1	-1	1	-1	1	-1
$s = 1$	2	0	2	0	2	0	2	0	2	0
$s = 2$	2	1	5	7	17	31	65	127	257	511
$s = 3$	2	2	10	26	82	242	730	2186	6562	19682
$s = 4$	2	3	17	63	257	1023	4097	16383	65537	262143

Table 3: The first fifty members of sequence $\{j_n^s\}$.

§2. Theorems

Theorem 2.1. For every $n \in \mathbb{N}$ and $s \geq 0$,

$$j_{n+1}^s = s j_n^s - (s+1)(-1)^n.$$

Proof. The justification is straightforward. We have

$$\begin{aligned} j_{n+1}^s &= s^{n+1} + (-1)^{n+1} \\ &= s s^n - (-1)^n \\ &= s(s^n + (-1)^n) - s(-1)^n - (-1)^n \\ &= s j_n^s - (s+1)(-1)^n. \end{aligned}$$

Next, we further extend the Pell-Jacobsthal numbers to the following form

$$j_n^{s,t} = s^n + (-t)^n,$$

where $n \in \mathbb{N}$, s and t are arbitrary real numbers. When $s = 2$ and $t = 1$ we recover the original Pell-Jacobsthal numbers.

Likewise, we have the following recursive equation.

Theorem 2.2. For every $n \in \mathbb{N}$ and real numbers $s,\ t$,

$$j_{n+1}^{s,t} = s j_n^{s,t} - (s+t)(-t)^n.$$

Proof. For every $n \geq 0$, we have

$$\begin{aligned} j_{n+1}^{s,t} &= s^{n+1} + (-t)^{n+1} \\ &= ss^n - t(-t)^n \\ &= s(s^n + (-t)^n) - s(-t)^n - t(-t)^n \\ &= sj_n^{s,t} - (s+t)(-t)^n. \end{aligned}$$

Finally, we mention the following equations regarding $j_n^{s,t}$:

- $j_n^{0,0} = 0$,
- $j_n^{s,0} = s^n$,
- $j_n^{0,t} = (-t)^n$,
- $j_n^{-t,t} = 2(-t)^n$,
- $j_n^{s,s} = s^n(1 + (-1)^n)$.

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The semi normed space defined by entire rate sequences

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Abstract In this paper we introduce the sequence spaces $\Gamma_\pi(p,\sigma,q,s)$, $\Lambda_\pi(p,\sigma,q,s)$ and define a semi normed space (X,q), semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Keywords Entire rate sequence, analytic sequence, invariant mean, semi norm.

§1. Introduction

A complex sequence, whose kth term is x_k, is denoted by $\{x_k\}$ or simply x. Let φ be the set of all finite sequences. A sequence $x=\{x_k\}$ is said to be analytic rate if $\sup_k \left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ_π. A sequence x is called entire rate sequence if $\lim_{k\to\infty} \left|\frac{x_k}{\pi_k}\right|^{\frac{1}{k}} = 0$. The vector space of all entire rate sequences will be denoted by Γ_π. Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n)=\sigma(\sigma^{m-1}(n))$, $m=1,2,3,\cdots$.

A continuous linear functional φ on Λ_π is said to be an invariant mean or a σ-mean if and only if

(1) $\varphi(x)\geq 0$ when the sequence $x=(x_n)$ has $x_n\geq 0$ for all n.

(2) $\varphi(e)=1$ where $e=(1,1,1,\cdots)$ and

(3) $\varphi(\{x_\sigma(n)\})=\varphi(\{x_n\})$ for all $x\in\Lambda_\pi$.

For certain kinds of mappings σ, every invariant mean φ extends the limit functional on the space C of all real convergent sequences in the sense that $\varphi(x)=\lim x$ for all $x\in C$. Consequently $C\subset V_\sigma$, where V_σ is the set of analytic sequences all of those σ-means are equal.

If $x=(x_n)$, set $Tx=(Tx)^{1/n}=(x_\sigma(n))$. It can be shown that

$$V_\sigma=\{x=(x_n): \lim_{m\to\infty} t_{mn}(x_n)^{1/n}=L \text{ uniformly in } n,\ L=\sigma-\lim_{n\to\infty}(x_n)^{1/n}\},$$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{1/n}}{m+1}. \tag{1}$$

Given a sequence $x = \{x_k\}$ its nth section is the sequence $x^{(n)} = \{x_1, x_2, \cdots, x_n, 0, 0, \cdots\}$, $\delta^{(n)} = (0, 0, \cdots, 1, 0, 0, \cdots)$, 1 in the nth place and zeros elsewhere. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1, 2, \cdots)$ are continuous.

§2. Definitions and preliminaries

Definition 2.1. The space consisting of all those sequences x in w such that $\left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right) \to 0$ as $k \to \infty$ is denoted by Γ_π. In other words $\left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)$ is a null sequence. Γ_π is called the space of entire rate sequences. The space Γ_π is a metric space with the metric

$$d(x, y) = \left\{\sup_k \left(\left|\frac{x_k - y_k}{\pi_k}\right|^{1/k}\right) : \ k = 1, 2, 3, \cdots\right\}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in Γ_π.

Definition 2.2. The space consisting of all those sequences x in w such that

$$\left\{\sup_k \left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)\right\} < \infty$$

is denoted by Λ_π. In other words $\left\{\sup_k \left(\left|\frac{x_k}{\pi_k}\right|^{1/k}\right)\right\}$ is a bounded sequence.

Definition 2.3. Let $p,\ q$ be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, then p and q are said to be equivalent.

Lemma 2.1. Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \le Mp(x)$ for all $x \in X$.

Definition 2.4. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$, for all $k \in N$.

Definition 2.5. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.1. From the above two definitions, it is clear that a sequence space E is solid implies that E is monotone.

Definition 2.6. A sequence E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup_k p_k = G$. Let $D = \max(1, 2^{G-1})$. Then for $a_k,\ b_k \in C$, the set of complex numbers for all $k \in N$ we have

$$|a_k + b_k|^{1/k} \le D\{|a_k|^{1/k} + |b_k|^{1/k}\}. \tag{2}$$

Let (X, q) be a semi normed space over the field C of complex numbers with the semi norm q. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over X. We define the following sequence spaces:

$$\Lambda_\pi(p,\sigma,q,s) = \Big\{x \in \Lambda(X) : \sup_{n,k} k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right)^{p_k} < \infty \quad \text{uniformly in } n \geq 0, s \geq 0\Big\},$$

$$\Gamma_\pi(p,\sigma,q,s) = \Big\{x \in \Gamma_\pi(X) : k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right)^{p_k} \to 0, \text{ as } k \to \infty \quad \text{uniformly in } n \geq 0, s \geq 0\Big\}.$$

§3. Main results

Theorem 3.1. $\Gamma_\pi(p,\sigma,q,s)$ is a linear space over the set of complex numbers.

Proof. The proof is easy, so omitted.

Theorem 3.2. $\Gamma_\pi(p,\sigma,q,s)$ is a paranormed space with

$$g(x) = \left\{\sup_{k\geq 1} k^{-s}\left(q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right), \text{ uniformly in } n > 0\right\},$$

where $H = \max\left(1, \sup_k p_k\right)$.

Proof. Clearly $g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence. It can be easily verified that $g(x+y) \leq g(x) + g(y)$. Next $x \to \theta$, λ fixed implies $g(\lambda x) \to 0$. Also $x \to \theta$ and $\lambda \to 0$ implies $g(\lambda x) \to 0$. The case $\lambda \to 0$ and x fixed implies that $g(\lambda x) \to 0$ follows from the following expressions.

$$g(\lambda x) = \left\{\sup_{k\geq 1} k^{-s} q\left(\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right), \text{ uniformly in } n,\ m \in N\right\},$$

$$g(\lambda x) = \left\{(|\lambda|\, r)^{pm/H} : \sup_{k\geq 1} k^{-s} q\left(\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right) r > 0, \text{ uniformly in } n,\ m \in N\right\},$$

where $r = 1/|\lambda|$. Hence $\Gamma_\pi(p,\sigma,q,s)$ is a paranormed space. This completes the proof.

Theorem 3.3. $\Gamma_\pi(p,\sigma,q,s) \cap \Lambda_\pi(p,\sigma,q,s) \subseteq \Gamma_\pi(p,\sigma,q,s)$.

Proof. The proof is easy, so omitted.

Theorem 3.4. $\Gamma_\pi(p,\sigma,q,s) \subset \Lambda_\pi(p,\sigma,q,s)$.

Proof. The proof is easy, so omitted.

Remark 3.1. Let q_1 and q_2 be two semi norms on X, we have

(i) $\Gamma_\pi(p,\sigma,q_1,s) \cap \Gamma_\pi(p,\sigma,q_2,s) \subseteq \Gamma_\pi(p,\sigma,q_1+q_2,s)$.

(ii) If q_1 is stronger than q_2, then $\Gamma_\pi(p,\sigma,q_1,s) \subseteq \Gamma_\pi(p,\sigma,q_2,s)$.

(iii) If q_1 is equivalent to q_2, then $\Gamma_\pi(p,\sigma,q_1,s) = \Gamma_\pi(p,\sigma,q_2,s)$.

Theorem 3.5. (i) Let $0 \leq p_k \leq r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\Gamma_\pi(r,\sigma,q,s) \subset \Gamma_\pi(p,\sigma,q,s)$.

(ii) $s_1 \leq s_2$ implies $\Gamma_\pi(p, \sigma, q, s_1) \subset \Gamma_\pi(p, \sigma, q, s_2)$.

Proof. (i)

$$\text{Let } x \in \Gamma_\pi(r, \sigma, q, s), \tag{3}$$

$$k^{-s}\left\{q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k} \to 0 \text{ as } k \to \infty. \tag{4}$$

Let $t_k = k^{-s}\left\{q\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda > \lambda_k$. Define $u_k = t_k$ $(t_k \geq 1)$; $u_k = 0$ $(t_k < 1)$ and $v_k = 0$ $(t_k \geq 1)$; $v_k = t_k$ $(t_k < 1)$; $t_k = u_k + v_k, t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$u_k^{\lambda_k} \leq t_k \quad \text{and} \quad v_k^{\lambda_k} \leq v_k^{\lambda}. \tag{5}$$

(i.e.) $t_k^{\lambda_k} \leq t_k + v_k^{\lambda}$ by (5),

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}\right)^{\lambda_k} \leq k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k}$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}^{r_k}\right)^{p_k/r_k} \leq k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k}$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \leq k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k}.$$

$$\text{But } k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{r_k} \to 0 \text{ as } k \to \infty \text{ by (4)}$$

$$k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \to 0 \text{ as } k \to \infty.$$

$$\text{Hence } x \in \Gamma_\pi(p, \sigma, q, s) \tag{6}$$

From (3) and (6) we get $\Gamma_\pi(r, \sigma, q, s) \subset \Gamma_\pi(p, \sigma, q, s)$. This completes the proof.

(ii) The proof is easy, so omitted.

Theorem 3.6. The space $\Gamma_\pi(p, \sigma, q, s)$ is solid and as such is monotone.

Proof. Let $\left(\frac{x_k}{\pi_k}\right) \in \Gamma_\pi(p, \sigma, q, s)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then

$$k^{-s}\left(q\left\{\left|\frac{\alpha_k x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \leq k^{-s}\left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \text{ for all } k \in N.$$

$$\left(q\left\{\left|\frac{\alpha_k x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \leq \left(q\left\{\left|\frac{x_{\sigma^k(n)}}{\pi_{\sigma^k(n)}}\right|^{1/k}\right\}\right)^{p_k} \text{ for all } k \in N.$$

This completes the proof.

Theorem 3.7. The space $\Gamma_\pi(p, \sigma, q, s)$ are not convergence free in general.

Proof. The proof follows from the following example.

Example 3.1. Let $s = 0$; $p_k = 1$ for k even and $p_k = 2$ for k odd. Let $X = C$, $q(x) = |x|$ and $\sigma(n) = n+1$ for all $n \in N$. Then we have $\sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2$ and $\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3$. Therefore, $\sigma^k(n) = (n+k)$ for all $n, k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = (1/k)^k \pi_k$ and $y_k = k^k \pi_k$ for all $k \in N$. (i.e.) $\left|\frac{x_k}{\pi_k}\right|^{1/k} = 1/k$ and $\left|\frac{y_k}{\pi_k}\right|^{1/k} = k$, for all $k \in N$.

Hence, $\left|\left(\frac{1}{n+k}\right)^{n+k}\right|^{p_k} \to 0$ as $k \to \infty$. Therefore $\left(\frac{x_k}{\pi_k}\right) \in \Gamma_\pi(p, \sigma)$. But $\left|\left(\frac{1}{n+k}\right)^{n+k}\right|^{p_k} r \not\to 0$ as $k \to \infty$. Hence $\left(\frac{y_k}{\pi_k}\right) \notin \Gamma_\pi(p, \sigma)$. Hence the space $\Gamma_\pi(p, \sigma, q, s)$ are not convergence free in general. This completes the proof.

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On the mean value of $|\mu^{(e)}(n)|$ over square-full number[1]

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Abstract In this paper, we shall study the mean value of $|\mu^{(e)}(n)|$ by the convolution method, where n is square-full number.

Keywords Square-full number, mean value, convolution method.

§1. Introduction

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. The integer n is called a square-full number if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $a_1 \geq 2,\ a_2 \geq 2, \cdots, a_r \geq 2$. Let $f_2(n)$ is the characteristic function of square-full integers.

The integer d is called an exponential divisor (e-divisor) of n if $d = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$, where $b_1|a_1,\ b_2|a_2, \cdots, b_r|a_r$, notion: $d|_e n$. By convention $1|_e 1$. The integer $n > 1$ is called exponentially square-full (e-square-full) if all the exponents $a_1,\ a_2, \cdots, a_r$ are square-full.

The exponential convolution (e-convolution) of arithmetic functions is defined by

$$(f \bigodot g)(n) = \sum_{b_1 c_1 = a_1} \sum_{b_2 c_2 = a_2} \cdots \sum_{b_r c_r = a_r} f(p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}) g(p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}),$$

where $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$.

These notions were introduced by M. V. Subbarao [1]. The e-convolution $\bigodot$ is commutative, associative and has the identity element μ^2, where μ is the Möbius function.

The inverse with respect to $\bigodot$ of the constant 1 function is called the exponential analogue of the Möbius function and it is denoted by $\mu^{(e)}$. Hence for every $n > 1$,

$$\sum_{d|_e n} \mu^{(e)}(d) = \mu^2(n).$$

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Here $\mu^{(e)}(1) = 1$ and for $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} > 1$,

$$\mu^{(e)}(n) = \mu(a_1)\mu(a_2)\cdots\mu(a_r).$$

The function $\mu^{(e)}$ is multiplicative and $\mu^{(e)}(p^a) = \mu(a)$ for every prime power p^a. Hence $\mu^{(e)}(n) \in \{-1, 0, 1\}$ for every $n > 1$ and for every prime p, $\mu^{(e)}(p) = 1$, $\mu^{(e)}(p^2) = -1$, $\mu^{(e)}(p^3) = -1$, $\mu^{(e)}(p^4) = 0, \ldots$

Let

$$S(x) := \sum_{\substack{n\le x \\ n \text{ is square-full}}} |\mu^{(e)}(n)| = \sum_{n\le x} |\mu^{(e)}(n)| f_2(n),$$

where $f_2(n) = \begin{cases} 1, & n \text{ is square-full;} \\ 0, & \text{otherwise.} \end{cases}$

In this paper, we shall prove a result about the mean value of $|\mu^{(e)}(n)|$ over square-full integers. Our main result is the following:

Theorem 1.1. For some $D > 0$,

$$\sum_{\substack{n\le x \\ n \text{ is square-full}}} |\mu^{(e)}(n)| = \frac{\zeta(\frac{3}{2})G(\frac{1}{2})}{\zeta(2)} x^{1/2} + \frac{\zeta(\frac{2}{3})G(\frac{1}{3})}{\zeta(\frac{4}{3})} x^{1/3} + O(x^{1/4} \exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})).$$

§2. Proof of the theorem

In order to prove our theorem, we need the following two lemmas.

Lemma 2.1. For $1 \le a < b$, then

$$\triangle(a, b; x) \ll \begin{cases} x^{\frac{2}{3a+3b}}, & \text{if } b < 2a; \\ x^{\frac{2}{9a}} \log x, & \text{if } b = 2a; \\ x^{\frac{2}{5a+2b}}, & \text{if } b > 2a. \end{cases}$$

Proof. For the proof of Lemma 2.1, see the Theorem 14.4 of Ivić [2].

Lemma 2.2. Let $f(n)$ be an arithmetical function for which

$$\sum_{n\le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a), \quad \sum_{n\le x} |f(n)| = O(x^{a_1} \log^r x),$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > 1/c > a \ge 0$, $r \ge 0$, $P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r, and $c \ge 1$ and $b \ge 1$ are fixed integers. Suppose for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}.$$

If $h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c)$, then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x),$$

where $R_1(t), \cdots, R_l(t)$ are polynomials in t of degrees not exceeding r, and for some $D > 0$

$$E_c(x) \ll x^{1/c} \exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})).$$

Proof. If $b = 1$, Lemma 2.2 is Theorem 14.2 of Ivić [2]. When $b \geq 2$, Lemma 2.2 can be proved by the same approach.

Next we prove our Theorem. Let

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is square-full}}}^{\infty} \frac{|\mu^{(e)}(n)|}{n^s} = \sum_{n=1}^{\infty} \frac{|\mu^{(e)}(n)| f_2(n)}{n^s}, \ (\Re s > 1)$$

where $f_2(n) = \begin{cases} 1, & n \text{ is square-full}; \\ 0, & \text{otherwise}. \end{cases}$

By the Euler product formula and $\mu^{(e)}(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = \mu(a_1)\mu(a_2)\cdots\mu(a_r)$, we get for $\Re s > 1$ that

$$F(s) = \frac{\zeta(2s)\zeta(3s)}{\zeta(4s)} G(s),$$

It is easy to prove that the Dirichlet series $G(s) := \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{7}$.

Let

$$\zeta(2s)\zeta(3s)G(s) = \sum_{n=1}^{\infty} \frac{\sum\limits_{n=ml} d(2,3;m)g(l)}{n^s} := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \ (\Re s > 1)$$

where $h(n) = \sum\limits_{n=ml} d(2,3;m)g(l)$.

By the Residue theorem and Lemma 2.1 we can get

$$\begin{aligned}\sum_{n\leq x} d(2,3;n) &= \zeta(\frac{3}{2})x^{1/2} + \zeta(\frac{2}{3})x^{1/3} + \Delta(2,3;x) \\ &= \zeta(\frac{3}{2})x^{1/2} + \zeta(\frac{2}{3})x^{1/3} + O(x^{2/15}), \qquad (1)\end{aligned}$$

Then from (1) and Abel integration formula we have the relation

$$\begin{aligned}\sum_{n\leq x} h(n) &= \sum_{ml\leq x} d(2,3;m)g(l) \\ &= \sum_{l\leq x} g(l) \sum_{m\leq x/l} d(2,3;m) \\ &= \sum_{l\leq x} g(l)\big[\zeta(\frac{3}{2})(\frac{x}{l})^{1/2} + \zeta(\frac{2}{3})(\frac{x}{l})^{1/3} + O((\frac{x}{l})^{2/15})\big] \\ &= \zeta(\frac{3}{2})x^{1/2}\sum_{l\leq x}\frac{g(l)}{l^{1/2}} + \zeta(\frac{2}{3})x^{1/3}\sum_{l\leq x}\frac{g(l)}{l^{1/3}} + O(x^{2/15}\sum_{l\leq x}\frac{|g(l)|}{l^{2/15}}) \\ &= \zeta(\frac{3}{2})x^{1/2}\sum_{n=1}^{\infty}\frac{g(l)}{l^{1/2}} + \zeta(\frac{2}{3})x^{1/3}\sum_{n=1}^{\infty}\frac{g(l)}{l^{1/3}} + O(x^{1/2}\sum_{l>x}\frac{|g(l)|}{l^{1/2}}) \\ &\quad + O(x^{1/3}\sum_{l>x}\frac{|g(l)|}{l^{1/3}}) + O(x^{2/15}\sum_{l\leq x}\frac{|g(l)|}{l^{2/15}}) \\ &= \zeta(\frac{3}{2})G(\frac{1}{2})x^{1/2} + \zeta(\frac{2}{3})G(\frac{1}{3})x^{1/3} + O(x^{1/7}).\end{aligned}$$

By Lemma 2.2 and Perron's formula we can get

$$\sum_{\substack{n\leq x \\ n \text{ is square-full}}} |\mu^{(e)}(n)| = \frac{\zeta(\frac{3}{2})G(\frac{1}{2})}{\zeta(2)}x^{1/2}+\frac{\zeta(\frac{2}{3})G(\frac{1}{3})}{\zeta(\frac{4}{3})}x^{1/3}+O(x^{1/4}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})),$$

where $D > 0$.

This completes the proof of Theorem 1.1.

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A short interval result for the function $\phi^{(e)}(n)$ [1]

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Abstract Let $n > 1$ be an integer, $\phi^{(e)}(n)$ is a multiplicative function. In this paper, we shall establish a short interval result for the function $\phi^{(e)}(n)$.

Keywords Convolution method, short interval.

§1. Introduction

Let $n > 1$ be an integer of canonical form $n = \prod_{i=1}^{r} p_i^{a_i}$. The integer d is called an exponential divisor of n if $d = \prod_{i=1}^{r} p_i^{c_i}$, where $c_i|a_i$ for every $1 \le i \le r$, notation: $d|_e n$. By convention $1_e 1$. This notion was introduced by M. V. Subbarao [2]. Note that 1 is not an exponential divisor of $n > 1$, the smallest exponential divisor of $n > 1$ is its squarefree kernel $\kappa(n) = \prod_{i=1}^{r} p_i$.

Let $\tau^{(e)}(n) = \sum_{d|_e n} 1$ and $\sigma^{(e)}(n) = \sum_{d|_e n} d$ denote the number and the sum of exponential divisors of n, respectively. The integer $n = \prod_{i=1}^{r} p_i^{a_i}$ is called exponentially squarefree if all the exponents a_i $(1 \le i \le r)$ are squarefree. Let $q^{(e)}$ denote the characterietic function of exponentially squarefree integers. Properties of these functions were investigated by several authors.

Two integers $n, m > 1$ have common exponential divisors if they have the same prime factors and in this case, i.e., for $n = \prod_{i=1}^{r} p_i^{a_i}$, $m = \prod_{i=1}^{r} p_i^{b_i}$, $a_i, b_i \ge 1 (1 \le i \le r)$, the greatest common exponential divisor of n and m is

$$(n, m)_e := \prod_{i=1}^{r} p_i^{(a_i, b_i)},$$

here $(1,1)_e = 1$ by convention and $(1,m)_e$ does not exist for $m > 1$.

The integers $n, m > 1$ are called exponentially coprime, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \le i \le r$, with the notation of above. In this case $(n,m)_e = \kappa(n) = \kappa(m)$. 1 and 1 are considered to be exponentially coprime. 1 and $m > 1$ are not exponentially coprime.

For $n = \prod_{i=1}^{r} p_i^{a_i}$, $a_i \ge 1$ $(1 \le i \le r)$, denote by $\phi^{(e)}(n)$ the number of integers $\prod_{i=1}^{r} p_i^{c_i}$ such that $1 \le c_i \le a_i$ and $(c_i, a_i) = 1$ for $1 \le i \le r$, and let $\phi^{(e)}(1) = 1$. Thus $\phi^{(e)}(n)$ counts the number of divisors d of n such that d and n are exponentially coprime .

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It is immediately , that $\phi^{(e)}$ is a prime independent multiplicative function and for $n > 1$,

$$\phi^{(e)}(n) = \prod_{i=1}^{r} \phi(a_i),$$

where ϕ is the Euler-function.

László Tóth [1] proved the following result:

$$\sum_{n \le x} \phi^e(n) = C_1 x + C_2 x^{\frac{1}{3}} + O(x^{\frac{1}{5}+\epsilon})$$

for every $\epsilon > 0$, where $C_1,\ C_2$ are constants given by

$$C_1 = \prod_p (1 + \sum_{a=3}^{\infty} \frac{\phi(a) - \phi(a-1)}{p^a}),$$

$$C_2 = \zeta(\frac{1}{3}) \prod_p (1 + \sum_{a=5}^{\infty} \frac{\phi(a) - \phi(a-1) - \phi(a-3) - \phi(a-4)}{p^{\frac{a}{3}}}),$$

In this paper, we shall prove the following short interval result.

Theorem 1.1. If $n > 1$ be an integer,

$$\sum_{x < n \le x+y} \phi^{(e)}(n) = Cy + O(yx^{\frac{-\epsilon}{3}} + x^{2\epsilon + \frac{1}{7}}), \tag{1}$$

where $C = G(1)\zeta(3)\zeta^2(5)$ is a constant.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

We assume that $1 \le a \le b$ are fixed integers, and we denote by $d(a, b; k)$ the number of representations of k as $k = n_1^a n_2^b$, where $n_1,\ n_2$ are natural numbers, that is,

$$d(a, b; k) = \sum_{k = n_1^a n_2^b} 1,$$

and $d(a, b; k) \ll n^{\epsilon^2}$ will be used freely.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. The Dirichlet series of $\phi^{(e)}(n)$ is absolutely convergent for $\Re s > 1$,

$$\sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} = \zeta(s)\zeta(3s)\zeta^2(5s)G(s), \tag{2}$$

where the Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

is absolutely convergent for $\Re s > \frac{1}{7}$.

Proof. Here $\phi^{(e)}(n)$ is multiplicative and by Euler product formula we have for $\Re s > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} &= \prod_p (1 + \frac{\phi^{(e)}(p)}{p^s} + \frac{\phi^{(e)}(p^2)}{p^{2s}} + \ldots + \frac{\phi^{(e)}(p^m)}{p^{ms}} + \ldots) \\
&= \prod_p (1 + \frac{1}{p^s} + \frac{\varphi(2)}{p^{2s}} + \frac{\varphi(3)}{p^{3s}} + \ldots + \frac{\varphi(m)}{p^{ms}} + \ldots) \\
&= \prod_p \frac{1}{1-p^{-s}} (1-p^{-s})(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \ldots + \frac{\varphi(m)}{p^{ms}} + \ldots) \\
&= \zeta(s) \prod_p (1 + \frac{1}{p^{3s}} + \frac{2}{p^{5s}} + \ldots) \\
&= \zeta(s) \prod_p \frac{1}{1-p^{-3s}} (1-p^{-3s})(1 + \frac{1}{p^{3s}} + \frac{2}{p^{5s}} + \ldots) \\
&= \zeta(s)\zeta(3s) \prod_p (1 + \frac{2}{p^{5s}} + \ldots) \\
&= \zeta(s)\zeta(3s) \prod_p \frac{1}{1-p^{-5s}} (1-p^{-5s})(1 + \frac{2}{p^{5s}} + \ldots) \\
&= \zeta(s)\zeta(3s)\zeta(5s) \prod_p (1 + \frac{1}{p^{5s}} + \ldots) \\
&= \zeta(s)\zeta(3s)\zeta(5s) \prod_p \frac{1}{1-p^{-5s}} (1-p^{-5s})(1 + \frac{1}{p^{5s}} + \ldots) \\
&= \zeta(s)\zeta(3s)\zeta^2(5s)G(s)
\end{aligned}$$

So we get

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

and by the properties of Dirichlet series, it is absolutely convergent for $\Re s > \frac{1}{7}$.

Lemma 2.2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and

$$B(x, y; k, \epsilon) := \sum_{\substack{x < nm^k \leq x+y \\ m > x^{\epsilon}}} 1,$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}} \log x. \tag{3}$$

Proof. This Lemma is very important when studying the short interval distribution of l-free numbers.

Next we prove our Theorem.

From Lemma 2.1, we have

$$\textstyle\sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} = \zeta(s)\zeta(3s)\zeta^2(5s)G(s).$$

Define

$$G_0(s) = \zeta(s)G(S),\ G_0(s) = \sum_{n=1}^{\infty} \frac{f_0(n)}{n^s},$$

Then

$$\sum_{n\le x} f_0(n) = G(1)x + O(x^{\frac{1}{7}+\epsilon}), \tag{4}$$

So

$$\phi^{(e)}(n) = \sum_{n=n_0 n_3^3 n_5^5} f_0(n_0)d_1(n_3)d_2(n_5), \tag{5}$$

Therefore

$$\begin{aligned}\sum_{x<n\le x+y} \phi^{(e)}(n) &= \sum_{x<n_0 n_3^3 n_5^5\le x+y} f_0(n)d_1(n)d_2(n)\\ &= \sum_{\substack{x\,<\,n_0 n_3^3 n_5^5\,\le\, x\,+\,y\\ n_j\,\le\, x^{\epsilon}}} f_0(n_0)d_1(n_3)d_2(n_5)\\ &\quad +O(\sum_{j=2}^{5}\sum_{x<n_0 n_3^3 n_5^5\le x+y} |f_0(n_0)d_1(n_3)d_2(n_5)|).\end{aligned} \tag{6}$$

Let

$$\sum_1 = \sum_{\substack{x\,<\,n_0 n_3^3 n_5^5\,\le\, x\,+\,y\\ n_j\,\le\, x^{\epsilon}}} f_0(n_0)d_1(n_3)d_2(n_5). \tag{7}$$

$$\sum_2 = O(\sum_{j=2}^{5}\sum_{x<n_0 n_3^3 n_5^5\le x+y} |f_0(n_0)d_1(n_3)d_2(n_5)|). \tag{8}$$

From (4), we can get

$$\sum_1 = G(1)\zeta(3)\zeta^2(5) + o(yx^{-\epsilon}) + O(x^{\frac{1}{7}+\epsilon}), \tag{9}$$

For $\sum_2$, we have

$$\sum_2 \ll \sum_{j=2}^{5}\sum_{x<n_0 n_3^3 n_5^5\le x+y} |f_0(n_0)d_1(n_3)d_2(n_5)| \ll x^{\epsilon^2}\sum_{j=2}^{5}\sum_{\substack{x\,<\,n_0 n_3^3 n_5^5\,\le\, x\,+\,y\\ n_j\,>\, x^{7\epsilon}}} 1,$$

By Lemma 2.2, we get

$$\sum_2 \ll yx^{\frac{-\epsilon}{2}+2\epsilon^2} + x^{\frac{1}{7}+\epsilon+2\epsilon^2}, \tag{10}$$

So

$$\sum_{x<n\le x+y} \phi^{(e)}(n) = \sum_1 + O(\sum_2) = Cy + O(yx^{\frac{-\epsilon}{3}} + x^{\frac{1}{7}+2\epsilon}). \tag{11}$$

Where $C = G(1)\zeta(3)\zeta^2(5)$.

Now our theorem follows from (6) and (11).

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Estimate of Second Hankel Determinant for certain classes of Analytic functions

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Abstract We introduce some classes of analytic-univalent functions and for any real μ, determine the sharp upper bounds of the functional $|a_2a_4 - a_3^2|$ for the functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to such classes of functions regular in the unit disc $E = \{z :| z |< 1\}$

Keywords Analytic functions, starlike functions, convex functions, alpha logarithmically convex functions, starlike functions with respect to symmetric points, convex functions with respect to symmetric points, Hankel determinant.

§1. Introduction and preliminaries

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in the unit disc $E = \{z :| z |< 1\}$. Let S be the class of functions $f(z) \in A$ and univalent in E. Let $M^{\alpha}(0 \leq \alpha \leq 1)$ be the class of functions which satisfy the condition

$$Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha}\right] > 0. \tag{2}$$

This class was studied by Darus and Thomas [1] and functions of this class are called α-logarithmically convex functions. Obviously $M^0 \equiv S^*$, the class of starlike functions and $M^1 \equiv K$, the class of convex functions.
In the sequel, we assume that $(0 \leq \alpha \leq 1)$ and $z \in E$.

$C_s^{*(\alpha)}$ denote the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left[\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{1-\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{\alpha}\right] > 0. \tag{3}$$

The following observations are obvious:

(i) $C_s^{*(0)} \equiv S_s^*$, the class of starlike functions with respect to symmetric points introduced by Sakaguchi [14].

(ii) $C_s^{*(1)} \equiv K_s$, the class of convex functions with respect to symmetric points introduced by Das and Singh [2].

C_s^{α} be the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left[\left(\frac{2zf'(z)}{g(z)-g(-z)}\right)^{1-\alpha}\left(\frac{2(zf'(z))'}{(g(z)-g(-z))'}\right)^{\alpha}\right] > 0, \tag{4}$$

where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S_s^*. \tag{5}$$

In particular

(i) $C_s^0 \equiv C_s$, the class of close-to-convex functions with respect to symmetric points introduced by Das and Singh [2].

(ii) $C_s^1 \equiv C_s^{'}$.

Let $C_{1(s)}^{\alpha}$ be the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left[\left(\frac{2zf'(z)}{h(z)-h(-z)}\right)^{1-\alpha}\left(\frac{2(zf'(z))'}{(h(z)-h(-z))'}\right)^{\alpha}\right] > 0, \tag{6}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in K_s. \tag{7}$$

We have the following observations:

(i) $C_{1(s)}^0 \equiv C_{1(s)}$.

(ii) $C_{1(s)}^1 \equiv C_{1(s)}^{'}$.

In 1976, Noonan and Thomas [11] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor [12] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by *Eq.* (1) with bounded boundary. Ehrenborg [3] studied the Hankel determinant of exponential polynomials and the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman [8]. Also Hankel determinant was studied by various authors including Hayman [5] and Pommerenke [13]. Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete and Szegö [4] then further generalised the estimate of $|a_3 - \mu a_2^2|$ where μ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$,

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

In this paper, we seek upper bound of the functional $|a_2a_4 - a_3^2|$ for functions belonging to the above defined classes.

§2. Main result

Let P be the family of all functions p analytic in E for which $Re(p(z)) > 0$ and

$$p(z) = 1 + p_1z + p_2z^2 + \cdots \tag{8}$$

for $z \in E$.

Lemma 2.1. If $p \in P$, then $|p_k| \leq 2$ $(k = 1, 2, 3, ...)$.

This result is due to Pommerenke [13].

Lemma 2.2. If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [9,10].

Theorem 2.1. If $f \in M^\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}\left[\frac{\alpha(11 + 36\alpha + 38\alpha^2 + 12\alpha^3 - \alpha^4)}{(1+3\alpha)(-4 + 263\alpha + 603\alpha^2 + 253\alpha^3 + 37\alpha^4)(1+\alpha)^4} + 1\right]. \tag{9}$$

Proof. As $f \in M^\alpha$, so from (2)

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha} = p(z). \tag{10}$$

On taking logarithm on both sides of (10), we get,

$$(1-\alpha)\log\left(\frac{zf'(z)}{f(z)}\right) + \alpha\log\left(\frac{(zf'(z))'}{f'(z)}\right) = \log p(z). \tag{11}$$

An easy calculation yields,

$$\log\left(\frac{zf'(z)}{f(z)}\right) = a_2z + \left(2a_3 - \frac{3}{2}a_2^2\right)z^2 + \left(3a_4 - 5a_2a_3 + \frac{7}{3}a_2^3\right)z^3 + \cdots, \tag{12}$$

$$\log\left(\frac{(zf'(z))'}{f'(z)}\right) = 2a_2z + \left(6a_3 - 6a_2^2\right)z^2 + \left(12a_4 - 30a_2a_3 + \frac{56}{3}a_2^3\right)z^3 + \cdots \tag{13}$$

and

$$\log p(z) = p_1z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1p_2 + \frac{p_1^3}{3}\right)z^3 + \cdots. \tag{14}$$

On substituting (12), (13) and (14) in (11), we get

$$\begin{aligned}
& (1-\alpha)\left[a_2 z+\left(2a_3-\frac{3}{2}a_2^2\right)z^2+\left(3a_4-5a_2a_3+\frac{7}{3}a_2^3\right)z^3+\cdots\right] \\
& +\alpha\left[2a_2 z+\left(6a_3-6a_2^2\right)z^2+\left(12a_4-30a_2a_3+\frac{56}{3}a_2^3\right)z^3+\cdots\right] \\
= \; & p_1 z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3-p_1p_2+\frac{p_1^3}{3}\right)z^3+\cdots. \qquad (15)
\end{aligned}$$

On equating the coefficients of z, z^2 and z^3 in (15), we obtain

$$a_2=\frac{p_1}{1+\alpha}, \qquad (16)$$

$$a_3=\frac{p_2}{2(1+2\alpha)}+\frac{(2+7\alpha-\alpha^2)p_1^2}{4(1+2\alpha)(1+\alpha)^2} \qquad (17)$$

and

$$a_4=\frac{p_3}{3(1+3\alpha)}+\frac{(3+19\alpha-4\alpha^2)p_1p_2}{6(1+\alpha)(1+2\alpha)(1+3\alpha)}+\frac{(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)p_1^3}{36(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \qquad (18)$$

Using (16), (17) and (18), it yields

$$a_2a_4-a_3^2=\frac{1}{C(\alpha)}\left[\begin{array}{r} 48(1+2\alpha)^2(1+\alpha)^3p_1p_3 \\ +(24(1+2\alpha)(1+\alpha)^2-36(1+3\alpha)(1+\alpha)^2(2+7\alpha-\alpha^2))p_1^2p_2 \\ +(4(1+2\alpha)(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4) \\ -9(1+3\alpha)(2+7\alpha-\alpha^2)^2)p_1^4-36(1+3\alpha)(1+\alpha)^4p_2^2 \end{array}\right], \qquad (19)$$

where $C(\alpha)=144(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4$.
Using Lemma 2.1 and Lemma 2.2 in (19), we obtain

$$|a_2a_4-a_3^2|=\frac{1}{C(\alpha)}\left|\begin{array}{r} -[81(1+3\alpha)^3-12(1+2\alpha)^2(1+\alpha)^3-12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -4(1+2\alpha)(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)]p_1^4 \\ +[24(1+2\alpha)^2(1+\alpha)^3+12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -54(1+\alpha)^2(1+3\alpha)^2)]p_1^2(4-p_1^2)x \\ -3(1+\alpha)^3[(1+4\alpha+7\alpha^2)p_1^2+12(1+4\alpha+3\alpha^2)](4-p_1^2)x^2 \\ +24(1+\alpha)^3(1+2\alpha)^2p_1(4-p_1^2)(1-|x|^2)z \end{array}\right|.$$

Assume that $p_1=p$ and $p\in[0,2]$, using triangular inequality and $|z|\le 1$, we have

$$|a_2a_4-a_3^2|\le\frac{1}{C(\alpha)}\left[\begin{array}{r} [81(1+3\alpha)^3-12(1+2\alpha)^2(1+\alpha)^3-12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -4(1+2\alpha)(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)]p^4 \\ +[24(1+2\alpha)^2(1+\alpha)^3+12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -54(1+\alpha)^2(1+3\alpha)^2]p^2(4-p^2)|x| \\ +3(1+\alpha)^3[(1+4\alpha+7\alpha^2)p^2+12(1+4\alpha+3\alpha^2)](4-p^2)|x|^2 \\ +24(1+\alpha)^3(1+2\alpha)^2p(4-p^2)(1-|x|^2) \end{array}\right]$$

or

$$|a_2a_4-a_3^2| \leq \frac{1}{C(\alpha)}\left[\begin{array}{r} [81(1+3\alpha)^3-12(1+2\alpha)^2(1+\alpha)^3-12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -4(1+2\alpha)(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)]p^4 \\ +24(1+\alpha)^3(1+2\alpha)^2p(4-p^2) \\ +[24(1+2\alpha)^2(1+\alpha)^3+12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ -54(1+\alpha)^2(1+3\alpha)^2]p^2(4-p^2)\delta \\ +3(1+\alpha)^3[(1+4\alpha+7\alpha^2)p^2-8(1+2\alpha)^2p+12(1+4\alpha+3\alpha^2)](4-p^2)\delta^2 \end{array}\right].$$

Therefore

$$|a_2a_4-a_3^2| \leq \frac{1}{C(\alpha)}F(\delta),$$

where $\delta = |x| \leq 1$ and

$$\begin{aligned} F(\delta) \ = \ & [81(1+3\alpha)^3-12(1+2\alpha)^2(1+\alpha)^3-12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ & -4(1+2\alpha)(6+23\alpha+154\alpha^2-47\alpha^3+8\alpha^4)]p^4 \\ & +24(1+\alpha)^3(1+2\alpha)^2p(4-p^2) \\ & +[24(1+2\alpha)^2(1+\alpha)^3+12(3+19\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ & -54(1+\alpha)^2(1+3\alpha)^2]p^2(4-p^2)\delta \\ & +3(1+\alpha)^3[(1+4\alpha+7\alpha^2)p^2-8(1+2\alpha)^2p+12(1+4\alpha+3\alpha^2)](4-p^2)\delta^2 \end{aligned}$$

is an increasing function. Therefore Max$F(\delta) = F(1)$.
Consequently

$$|a_2a_4-a_3^2| \leq \frac{1}{C(\alpha)}G(p), \tag{20}$$

where $G(p) = F(1)$.
So

$$G(p) = -A(\alpha)p^4 + B(\alpha)p^2 + 144(1+3\alpha)(1+\alpha)^4,$$

where

$$A(\alpha) = \alpha(-4+263\alpha+603\alpha^2+253\alpha^3+37\alpha^4)$$

and

$$B(\alpha) = 24\alpha(11+36\alpha+38\alpha^2+12\alpha^3-\alpha^4).$$

Now

$$G^{'}(p) = -4A(\alpha)p^3 + 2B(\alpha)p$$

and

$$G^{''}(p) = -12A(\alpha)p^2 + 2B(\alpha).$$

$G^{'}(p) = 0$ gives

$$p[2A(\alpha)p^2 - B(\alpha)] = 0.$$

$G^{''}(p)$ is negative at $p = \sqrt{\frac{12(11+36\alpha+38\alpha^2+12\alpha^3-\alpha^4)}{(-4+263\alpha+603\alpha^2+253\alpha^3+37\alpha^4)}} = p^{'}$.
So Max$G(p) = G(p^{'})$. Hence from (20), we obtain (9).

The result is sharp for $p_1 = p^{'}$, $p_2 = p_1^2 - 2$ and $p_3 = p_1(p_1^2 - 3)$.

For $\alpha = 0$ and $\alpha = 1$ respectively, we obtain the following results due to Janteng et al. [6].

Corollary 2.1. If $f(z) \in S^*$, then

$$|a_2 a_4 - a_3^2| \leq 1.$$

Corollary 2.2. If $f(z) \in K$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

Theorem 2.2. If $f \in C_s^{*(\alpha)}$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{(1+2\alpha)^2}. \tag{21}$$

Proof. Since $f \in C_s^{*(\alpha)}$, so from (3)

$$\left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{1-\alpha} \left(\frac{2(zf'(z))^{'}}{(f(z) - f(-z))^{'}}\right)^{\alpha} = p(z). \tag{22}$$

On taking logarithm on both sides of (22), we get,

$$(1-\alpha)\log\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) + \alpha \log\left(\frac{2(zf'(z))^{'}}{(f(z) - f(-z))^{'}}\right) = \log p(z). \tag{23}$$

After an easy calculation, we obtain,

$$\log\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) = 2a_2 z + 2(a_3 - a_2^2)z^2 + 2\left(2a_4 - 3a_2 a_3 + \frac{4}{3}a_2^3\right) z^3 + \cdots, \tag{24}$$

$$\log\left(\frac{2(zf'(z))^{'}}{(f(z) - f(-z))^{'}}\right) = 4a_2 z + 2\left(3a_3 - 4a_2^2\right) z^2 + 4\left(4a_4 - 9a_2 a_3 + \frac{16}{3}a_2^3\right) z^3 + \cdots. \tag{25}$$

On substituting (24), (25) and (14) in (23), we get

$$\begin{aligned} &(1-\alpha)\left[2a_2 z + 2(a_3 - a_2^2)z^2 + \left(2a_4 - 3a_2 a_3 + \frac{4}{3}a_2^3\right) z^3 + ...\right] \\ &+\alpha\left[4a_2 z + 2\left(3a_3 - 4a_2^2\right) z^2 + 4\left(4a_4 - 9a_2 a_3 + \frac{16}{3}a_2^3\right) z^3 + ...\right] \\ = \ & p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{3}\right) z^3 + \cdots. \end{aligned} \tag{26}$$

On equating coefficients of z, z^2 and z^3 in (26), we obtain

$$a_2 = \frac{p_1}{2(1+\alpha)}, \tag{27}$$

$$a_3 = \frac{p_2}{2(1+2\alpha)} + \frac{\alpha(1-\alpha)p_1^2}{4(1+2\alpha)(1+\alpha)^2} \tag{28}$$

and

$$a_4 = \frac{p_3}{4(1+3\alpha)} + \frac{(1+9\alpha-4\alpha^2)p_1 p_2}{8(1+\alpha)(1+2\alpha)(1+3\alpha)} + \frac{(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4)p_1^3}{48(1+2\alpha)(1+3\alpha)(1+\alpha)^3}. \tag{29}$$

Using (27), (28) and (29), it yields

$$a_2a_4 - a_3^2 = \frac{1}{C(\alpha)} \begin{bmatrix} 12(1+2\alpha)^2(1+\alpha)^3 p_1 p_3 \\ +(6(1+2\alpha)(1+\alpha)^2 - 24(1+3\alpha)(1+\alpha)^2(\alpha-\alpha^2))p_1^2 p_2 \\ +((1+2\alpha)(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4) \\ -6(1+3\alpha)(\alpha-\alpha^2)^2)p_1^4 - 24(1+3\alpha)(1+\alpha)^4 p_2^2 \end{bmatrix}, \tag{30}$$

where $C(\alpha) = 96(1+3\alpha)(1+2\alpha)^2(1+\alpha)^4$.
Using Lemma 2.1 and Lemma 2.2 in (30), we obtain

$$|a_2a_4 - a_3^2| = \frac{1}{C(\alpha)} \begin{vmatrix} [3(1+2\alpha)^2(1+\alpha)^3 + 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +(1+2\alpha)(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4) - 6(\alpha-\alpha^2)^2(1+3\alpha) \\ -12(\alpha-\alpha^2)(1+3\alpha)(1+\alpha)^2 - 6(1+3\alpha)(1+\alpha)^4]p_1^4 \\ -[-6(1+2\alpha)^2(1+\alpha)^3 - 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +12(\alpha-\alpha^2)(1+\alpha)^2(1+3\alpha) + 12(1+3\alpha)(1+\alpha)^4]p_1^2(4-p_1^2)x \\ -3(1+\alpha)^3[(1+2\alpha)^2p_1^2 + 2(1+3\alpha)(1+\alpha)(4-p_1^2)](4-p_1^2)x^2 \\ +6(1+\alpha)^3(1+2\alpha)^2 p_1(4-p_1^2)(1-|x|^2)z \end{vmatrix}.$$

Assume that $p_1 = p$ and $p \in [0,2]$, using triangular inequality and $|z| \le 1$, we have

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)} \begin{bmatrix} [3(1+2\alpha)^2(1+\alpha)^3 + 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +(1+2\alpha)(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4) - 6(\alpha-\alpha^2)^2(1+3\alpha) \\ -12(\alpha-\alpha^2)(1+3\alpha)(1+\alpha)^2 - 6(1+3\alpha)(1+\alpha)^4]p^4 \\ +[-6(1+2\alpha)^2(1+\alpha)^3 - 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +12(\alpha-\alpha^2)(1+\alpha)^2(1+3\alpha) + 12(1+3\alpha)(1+\alpha)^4]p^2(4-p^2)|x| \\ +3(1+\alpha)^3[(1+2\alpha)^2p^2 + 2(1+3\alpha)(1+\alpha)(4-p^2)](4-p^2)|x|^2 \\ +6(1+\alpha)^3(1+2\alpha)^2 p(4-p^2)(1-|x|^2) \end{bmatrix}$$

or

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)} \begin{bmatrix} [3(1+2\alpha)^2(1+\alpha)^3 + 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +(1+2\alpha)(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4) - 6(\alpha-\alpha^2)^2(1+3\alpha) \\ -12(\alpha-\alpha^2)(1+3\alpha)(1+\alpha)^2 - 6(1+3\alpha)(1+\alpha)^4]p^4 \\ +6(1+\alpha)^3(1+2\alpha)^2 p(4-p^2) \\ +[-6(1+2\alpha)^2(1+\alpha)^3 - 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ +12(\alpha-\alpha^2)(1+\alpha)^2(1+3\alpha) + 12(1+3\alpha)(1+\alpha)^4]p^2(4-p^2)\delta \\ +3(1+\alpha)^3[8(1+3\alpha)(1+\alpha) - 2(1+2\alpha)^2p - (1+4\alpha+2\alpha^2)p^2](4-p^2)\delta^2 \end{bmatrix}.$$

Therefore

$$|a_2a_4 - a_3^2| \le \frac{1}{C(\alpha)} F(\delta),$$

where $\delta = |x| \leq 1$ and

$$\begin{aligned} F(\delta) &= [3(1+2\alpha)^2(1+\alpha)^3 + 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ &+(1+2\alpha)(-7\alpha+16\alpha^2-17\alpha^3+8\alpha^4) - 6(\alpha-\alpha^2)^2(1+3\alpha) \\ &-12(\alpha-\alpha^2)(1+3\alpha)(1+\alpha)^2 - 6(1+3\alpha)(1+\alpha)^4]p^4 \\ &+6(1+\alpha)^3(1+2\alpha)^2 p(4-p^2) \\ &+[-6(1+2\alpha)^2(1+\alpha)^3 - 3(1+9\alpha-4\alpha^2)(1+2\alpha)(1+\alpha)^2 \\ &+12(\alpha-\alpha^2)(1+\alpha)^2(1+3\alpha) + 12(1+3\alpha)(1+\alpha)^4]p^2(4-p^2)\delta \\ &+3(1+\alpha)^3[8(1+3\alpha)(1+\alpha) - 2(1+2\alpha)^2 p - (1+4\alpha+2\alpha^2)p^2](4-p^2)\delta^2 \end{aligned}$$

is an increasing function. Therefore Max$F(\delta) = F(1)$.
Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{C(\alpha)} G(p), \tag{31}$$

where $G(p) = F(1)$.
So

$$G(p) = A(\alpha)p^4 - B(\alpha)p^2 + 96(1+3\alpha)(1+\alpha)^4,$$

where

$$A(\alpha) = \alpha(5+20\alpha+33\alpha^2+28\alpha^3+10\alpha^4)$$

and

$$B(\alpha) = 24(1+\alpha)^2(1+6\alpha+7\alpha^2+4\alpha^3).$$

Now

$$G^{'}(p) = 4A(\alpha)p^3 - 2B(\alpha)p$$

and

$$G^{''}(p) = 12A(\alpha)p^2 - 2B(\alpha).$$

$G^{'}(p) = 0$ gives

$$2p[2A(\alpha)p^2 - B(\alpha)] = 0.$$

Clearly $G(p)$ attains its maximum value at $p = 0$. So Max$G(p) = G(0) = 96(1+3\alpha)(1+\alpha)^4$. Hence from (31), we obtain (21).

The result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

For $\alpha = 0$ and $\alpha = 1$ respectively, we obtain the following results due to Janteng et al. [7].

Corollary 2.3. If $f(z) \in S_s^*$, then

$$|a_2a_4 - a_3^2| \leq 1.$$

Corollary 2.4. If $f(z) \in K_s$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}.$$

On the same lines, we can easily prove the following theorems:

Theorem 2.3. If $f \in C_s^{\alpha}$, then

$$|a_2a_4 - a_3^2| \le \frac{(3+2\alpha)^2}{9(1+2\alpha)^2}.$$

The result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.
For $\alpha = 0$ and $\alpha = 1$ respectively, we obtain the following results:

Corollary 2.5. If $f(z) \in C_s$, then

$$|a_2a_4 - a_3^2| \le 1.$$

Corollary 2.6. If $f(z) \in C_s^{'}$, then

$$|a_2a_4 - a_3^2| \le \frac{25}{81}.$$

Theorem 2.4. If $f \in C_{1(s)}^{\alpha}$, then

$$|a_2a_4 - a_3^2| \le \frac{(7+2\alpha)^2}{81(1+2\alpha)^2}.$$

The result is sharp for $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$. For $\alpha = 0$ and $\alpha = 1$ respectively, we obtain the following results:

Corollary 2.7. If $f(z) \in C_{1(s)}$, then

$$|a_2a_4 - a_3^2| \le \frac{49}{81}.$$

Corollary 2.8. If $f(z) \in C_{1(s)}^{'}$, then

$$|a_2a_4 - a_3^2| \le \frac{1}{9}.$$

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A short interval result for the function $a^2(n)$[1]

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Abstract Let $a(n)$ denote the number of nonisomorphic Abelian groups with n elements. In this paper, we shall establish a short internal result of $a^2(n)$.

Keywords Nonisomorphic Abelian groups, convolution method, short interval.

§1. Introduction

We define $a(n)$ to be the number of nonisomorphic Abelian groups with n elements. The properties of $a(n)$ were investigated by many authors.

P. Erdös and G. Szekeres [1] first proved that

$$\sum_{n\le x} a(n) = c_1x + O(x^{\frac{1}{2}}), \tag{1}$$

Kendall and Rankin [2] proved that

$$\sum_{n\le x} a(n) = c_1x + c_2x^{\frac{1}{2}} + O(x^{\frac{1}{3}}\log x), \tag{2}$$

H. -E. Richert [3] proved

$$\sum_{n\le x} a(n) = c_1x + c_2x^{\frac{1}{2}} + c_3x^{\frac{1}{3}} + O(x^{\frac{3}{10}}\log^{\frac{9}{10}} x), \tag{3}$$

Suppose $A(x) := \sum_{n\le x} a^2(n)$. Recently Lulu Zhang [4] proved that

$$A(x) = c_4x + c_5x^{\frac{1}{2}}\log^2 x + c_6x^{\frac{1}{2}}\log x + c_7x^{\frac{1}{2}} + O(x^{\frac{96}{245}+\epsilon}), \tag{4}$$

where $c_j (j = 4, 5, 6, 7)$ are computable constants.

In this short paper, we shall prove the following short interval result.

Theorem 1.1. If $x^{\frac{1}{5}+2\epsilon} \le y \le x$, then

$$\sum_{x<n\le x+y} a^2(n) = Cy + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3\epsilon}{2}}), \tag{5}$$

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Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant. If $1 \leq a \leq b \leq c$ are fixed integers, we define

$$d(a,b,c;n) = \sum_{n=n_1^a n_2^b n_3^c} 1.$$

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. Suppose s is a complex number ($\Re s > 1$), then

$$F(s) := \sum_{n=1}^{\infty} \frac{a^2(n)}{n^s} = \zeta(s)\zeta^3(2s)\zeta^5(3s)\zeta^{10}(4s)G(4s), \tag{6}$$

where $G(s)$ can be written as a Dirichlet series $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, which is absolutely convergent for $\Re s \geq \frac{1}{5}$.

Proof. The function $a^2(n)$ is multiplicative, So by Euler product formula, we have

$$\begin{aligned}
F(s) = \prod_P (1 + \sum_{\alpha=1}^{\infty} \frac{a^2(p^\alpha)}{p^{\alpha s}}) &= \prod_p (1 + \frac{1}{p^s} + \frac{4}{p^{2s}} + \frac{9}{p^{3s}} + \ldots) \\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{4}{p^{2s}} + \frac{9}{p^{3s}} + \ldots) \\
&= \zeta(s)\zeta^3(2s) \prod_p (1 - \frac{1}{p^{2s}})^3 (1 + \frac{3}{p^{2s}} + \frac{5}{p^{3s}} + \frac{16}{p^{4s}} + \cdots) \\
&= \zeta(s)\zeta^3(2s)\zeta^5(3s) \prod_p (1 - \frac{1}{p^{3s}})^5 (1 + \frac{5}{p^{3s}} + \frac{10}{p^{4s}} + \cdots) \\
&= \zeta(s)\zeta^3(2s)\zeta^5(3s)\zeta^{10}(4s)G(s).
\end{aligned}$$

Now we write $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $\Re s \geq \frac{1}{5}$.

Lemma 2.2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers. Then

$$\sum_{x < nm^k \leq x+y} 1 \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}} \log x.$$

Proof. This lemma is often used when studying the short internal distribution of 1-free numbers; see for example, [5].

Lemma 2.3. Let $G_0(s) = \zeta(s)G(s)$, $f_0(n)$ be the arithmetic function defined by

$$\sum_{n=1}^{\infty} \frac{f_0(n)}{n^s} = \zeta(s)G(s),$$

then we have

$$\sum_{n \leq x} f_0(n) = Ax + O(x^{\frac{1}{5}+\epsilon}), \tag{7}$$

where $A = Res_{s=1}\zeta(s)G(s)$.

Proof. From Lemma 2.1 the infinite series $\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ converges absolutely for $\sigma > \frac{1}{5}$, it follows that

$$\sum_{n\le x}|g(n)| \ll x^{\frac{1}{5}+\epsilon}.$$

Therefore from the definition of $g(n)$ and $f_0(n)$, it follows that

$$\sum_{n\le x} f_0(n) = \sum_{mn\le x} g(n) = \sum_{n\le x} g(n)[\frac{x}{n}] = Ax + O(x^{\frac{1}{5}+\epsilon}),$$

where $A = Res_{s=1}\zeta(s)G(s)$.

Now we prove our theorem. From Lemma 2.3 and convolution method, we obtain

$$a^2(n) = \sum_{n=kl^2u^3m^4} f_0(k)d_3(l)d_5(u)d_{10}(m)$$

and

$$f_0(n) \ll n^{\epsilon^2}, d_3(n) \ll n^{\epsilon^2}, d_5(n) \ll n^{\epsilon^2}, d_{10}(n) \ll n^{\epsilon^2}.$$

So we have

$$\begin{aligned} A(x+y) - A(x) &= \sum_{x<kl^2u^3m^4\le x+y} f_0(k)d_3(l)d_5(u)d_{10}(m) \\ &= \sum_1 + O(\sum_2 + \sum_3 + \sum_4), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \sum_1 &= \sum_{l,u,m\le x^{\varepsilon}} d_3(l)d_5(u)d_{10}(m) \sum_{\frac{x}{l^2u^3m^4}<k\le\frac{x+y}{l^2u^3m^4}} f_0(k), \\ \sum_2 &= \sum_{\substack{x\,<\,kl^2u^3m^4\,\le\,x\,+\,y \\ l\,>\,x^{\epsilon}}} |f_0(k)d_3(l)d_5(u)d_{10}(m)|, \\ \sum_3 &= \sum_{\substack{x\,<\,kl^2u^3m^4\,\le\,x\,+\,y \\ u\,>\,x^{\epsilon}}} |f_0(k)d_3(l)d_5(u)d_{10}(m)|, \\ \sum_4 &= \sum_{\substack{x\,<\,kl^2u^3m^4\,\le\,x\,+\,y \\ m\,>\,x^{\epsilon}}} |f_0(k)d_3(l)d_5(u)d_{10}(m)|. \end{aligned}$$

In view of Lemma 2.3,

$$\begin{aligned} \sum_1 &= \sum_{l,u,m\le x^{\varepsilon}} d_3(l)d_5(u)d_{10}(m) \sum_{\frac{x}{l^2u^3m^4}<k\le\frac{x+y}{l^2u^3m^4}} f_0(k) \\ &= \sum_{l,u,m\le x^{\varepsilon}} d_3(l)d_5(u)d_{10}(m)[\frac{Ay}{l^2u^3m^4} + O((\frac{x}{l^2u^3m^4})^{\frac{1}{5}+\varepsilon})] \\ &= Cy + O(yx^{-\frac{\epsilon}{2}} + O(x^{\frac{1}{5}+4\epsilon}), \end{aligned} \tag{9}$$

where $C = Res_{s=1}F(s)$.

By Lemma 2.2 with $k = 2$ we have

$$\begin{aligned}
\sum_{2} &\ll x^{\epsilon^2} \sum_{\substack{x < kl^2u^3m^4 \leq x+y \\ l > x^{\epsilon}}} 1 \\
&\ll x^{\epsilon^2} \sum_{\substack{x < kl^2 \leq x+y \\ l > x^{\epsilon}}} 1 \\
&\ll x^{\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\
&\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}, \qquad (10)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\sum_{3} &\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}, \\
\sum_{4} &\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \qquad (11)
\end{aligned}$$

Now our theorem follows from (8)-(11).

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Einstein-Hessian Manifolds and Einstein Curvature

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Abstract A Riemannian manifold with flat affine connection D and a non-degenerate metric expressed as the Hessian matrix of a function with respect to an affine coordinate system is called a Hessian manifold. Geometry of Hessian manifold is deeply related to Kaehlerian geometry. Taking into account of this fact we use second Kozsul form of a Hessian manifold as a Ricci tensor of a Kaehlerian manifold and obtain new results on Einstein-Hessian and Einstein-Kaehlerian manifolds in terms of it. Furthermore we define Einstein curvature tensor for Hessian manifolds and obtain Ricci-flatness condition for manifolds mentioned above. Also using a special type of a Hessian manifold we investigate new conditions for Euclidean space and affine homogeneous convex cones.

Keywords Hessian manifold, Ricci curvature, scalar curvature, Einstein manifold.
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§1. Introduction

In differential geometry, the Einstein curvature tensor, named after Albert Einstein, is used to express the curvature of a Riemannian manifold. Also in general relativity, the Einstein tensor oocurs in the Einstein field equation for gravitation describing space time curvature in a manner consistent with energy consideration [1].

According to George Hammond "Scientific evidence that God is a curvature in psychometry space". In his interesting and work he has discovered that God is caused by the Einstein curvature tensor and that Einstein's celebrated field equation is actually the mathematical equation of God. To him, Einstein's theory in other words is the explicit mathematical proof of God [2].

Passing from the personal to the mathematical, the Einstein curvature tensor is a trace-reversed version of the well-known Ricci tensor and is physically identified with the physical stress energy tensor.

On the other hand the pseudo-Riemannian Hessian manifold is the most suitable to define the fundamental properties of thermo dynamics systems based on function f and the metric g since it re-inforces the information obtained from $(U \subset R^n; h = Hess_\delta f)$ and build the first order nonconstant state potentials h_{ij} as components of a nonconstant metric. This new metric determines a space which is more appropriate for the thermodynamics theories [3].

From the mathematical point of view, Hessian manifolds structure have various applications. Let M be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M is said to be Hessian metric if g is locally expressed by $g = D^2u$ where u is a local smooth function. We call such a pair (D , g) a Hessian structure on M and a triple (M , D , g) a Hessian manifold [5−9]. Geometry of Hessian manifolds is deeply related to the following geometries:

i) Kaehlerian geometry and symplectic geometry. It will be suggested by the facts that the tangent bundle of a Hessian manifold naturally admits a Kaehlerian structure and that a Hessian structure is formally analogous to a Kaehlerian structure because a Kaehlerian metric is locally expressed as a complex Hessian.

ii) Affine differential geometry. Level surfaces of u are non-degenerate in the sense of affine differential geometry. For the study of the level surfaces affine differential geometric methods are quite useful.

iii) Information geometry. It is well known that many important smooth families of probability distributions (*e.g.* exponential families) admit Hessian structures. Thus on Hessian manifolds many interesting geometric areas intersect [6].

Ricci-flat manifolds are special Riemannian manifolds whose Ricci tensor vanishes. In physics, they represent vacuum solutions to analogous of Einstein's equations for Riemannian manifold of any dimension with vanishing cosmological constant. Ricci-flat manifolds are special type manifolds where the cosmological constant need not vanish. They are also related to holonomy groups. Physicians use this term with a different point of view. It is surely related to brane studies and also black holes. As is well-known Ricci-flat Kaehlerian manifolds have nice applications in superstring theory in appropriate dimensions [10].

Geometry of Hessian manifold finds connection with pure mathematical fields such as affine differential geometry, homogeneous spaces, cohomology on one hand, physics and applied science on the other. However inspite of it is importance there is not any work on Einstein curvature and Ricci-flatness of it.

The motivation of creating the article based on this fact. In this paper firstly we give basic concepts of Hessian manifolds and Einstein curvature tensor. Encouraged by the information above, we introduce Einstein curvature tensor of a Hessian manifold in terms of Hessian curvatures. Then using second Kozsul form β, we imply the relation between Kaehlerian and Hessian manifold. According to the sign of β we prove the Ricci-flatness of a Hessian manifold and obtain Einstein-Hessian manifolds with positive Einstein curvature. Due to these fact we prove that a Hessian manifold is Ricci-flat if and only if its universal covering $E_x = T_xM$ and it is a Euclidean space with respect to the induced metric. Also we show that a Hessian manifold is a Einstein-Hessian manifold with positive Einstein curvature if and only if E_x is an affine homogeneous convex domain not containing any full straight line.

§2. Preliminaries

Let $M^{\,m}$ be a Hessian manifold with Hessian structure $(D\,,\,g\,)$. We express various geometric concepts for the Hessian structure $(D\,,\,g)$ interms of affine coordinate system $\left\{\,x^1\,,\,\ldots\,,\,x^m\right\}$ with respect to D , i.e $D,\,dx^i = 0$.

i) The Hessian metric:

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j},$$

where u is a local smooth function.

ii) Let γ be a tensor field of type $(1,2)$ defined by

$$\gamma\,(X,Y) = \bigtriangledown_X Y - D_X Y,$$

where ∇ is the Riemannian connection for g. Then we have

$$\gamma^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k},$$

$$\gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k},$$

$$\gamma_{ijk} = \gamma_{jik} = \gamma_{kji},$$

where Γ^i_{jk} are the Christoffel's symbols of ∇ [6].

Definition 1.1. A Hermitian metric g on a complex manifold (M, J) is said to be a Kaehlerian metric if g can be locally expressed by the complex Hessian of a function φ,

$$g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j},$$

where $\{z_1, ..., z_n\}$ is a holomorphic coordinate system. The pair (J, g) is called a Kaehlerian structure on M. A complex manifold M with a Kaehlerian structure on M. A complex manifold M with a Kaehlerian structure (J, g) is said to be a Kaehlerian manifold and is denoted by (M, J, g) [9].

For a Hermitian metric g we set

$$\rho\,(X,Y) = g\,(JX,Y)\,.$$

Then the skew symmetric bilinear form ρ is called a Kaehlerian form for (J, g), and using a holomorphic coordinate system, we have

$$\rho = \sqrt{-1}\sum_{i,j} g_{i\bar{j}}\,dz_i \wedge d\overline{z}_j$$ [9].

Proposition 1.1. Let g be a Hermitian metric on a complex manifold M. Then the following conditions are equivalent.

(1) g is a Kaehlerian metric.

(2) The Kaehlerian form ρ is closed; $d\rho = 0$.

Let (M, D) be a flat manifold and let TM be the tangent bundle over M with projection $\pi : TM \to M$. For an affine coordinate system $\left\{x^1, ..., x^n\right\}$ on M, we set

$$z^j = \xi^j + \sqrt{-1}\xi^{n+j},$$

where $\xi^i = x_i \circ \pi$ and $\xi^{n+i} = dx^i$. Then n-tuples of functions given by $\{z_1, ..., z_n\}$ yield holomorphic coordinate systems on TM. We denote by J_D the complex structure tensor of the complex manifold TM. For a Riemannian metric g on M we put

$$g^T = \sum_{i,j=1}^{n} (g_{ij} \circ \pi)\, dz_i d\overline{z}_j.$$

Then g^T is a Hermitian metric on the complex manifold (TM, J_D) [9].

Proposition 1.2. Let (M, D) be a flat manifold and g a Riemannian metric on M. Then the following conditions are equivalent.

(1) g is a Hessian metric on (M, D).

(2) g^T is a Kaehlerian metric on (TM, J_D) [9].

§3. Einstein Curvature and Hessian manifolds

Definition 2.1. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ and let R and Ric denote its Riemannian curvature $(0,4)-$ tensor and Ricci $(0,2)-$ tensor, respectively .

Einstein tensor denoted by E is a combination of the metric tensor g and the Ricci tensor as follows

$$E = \frac{1}{2} sg - Ric,$$

where s denotes the scalar curvature function [4].

Recall that the Ricci curvature r is the function defined on the unit tangent bundle UM of (M, g) by

$$r(v) = Ric(v, v).$$

Similarly we define the Einstein curvature e to be the function defined on UM by

$$e(v) = 2E(v, v) = s - 2r(v),$$

where we multiplied by the constant 2 to make it equal $p-$curvature with $p = 1$, that is, the Einstein curvature precisely coincide with the average of the sectional curvature in the directions orthogonal to v

$$e(v) = \sum_{i,j \in I} R(e_i, e_j, e_i, e_j),$$

where $\{e_i, i \in I\}$ is any orthonormal basis for the orthogonal supplement $(\Re v)^{\perp}$ of the vector v in the tangent space $T_m M$ at m [3].

In order to show the relationship of Hessian and Riemannian structure we need the following definition and theorems.

Definition 2.2. Let (D, g) be a Hessian structure and let $\gamma = \nabla - D$ be the difference tensor between the Levi-Civita connection ∇ for g and D. A tensor field Q of type $(1,3)$ defined by the covariant differential

$$Q = D_{\gamma}$$

of γ is said to be the Hessian curvature tensor for (D,g). The components Q^i_{jkl} of Q with respect to an affine coordinate system $\left\{x^1,...,x^n\right\}$ are given by

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}.$$

Proposition 2.1. Let $g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}$. Then we have

$$\begin{aligned}(1)\ Q_{ijkl} &= \frac{1}{2}\frac{\partial^4 \varphi}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 \varphi}{\partial x^i \partial x^k \partial x^r}\frac{\partial^3 \varphi}{\partial x^j \partial x^l \partial x^s} \\ (2)\, Q_{ijkl} &= Q_{kjil} = Q_{klij} = Q_{ilkj} = Q_{jilk}.\end{aligned}$$

Proposition 2.2. Let R be the Riemannian curvature tensor for g. Then

$$R_{ijkl} = \frac{1}{2}\left(Q_{ijkl} - Q_{jikl}\right).$$

According to above Propositions we conclude that the Hessian curvature tensor Q carries more detailed information than the Riemannian curvature tensor R.

In [11] the authors obtained scalar and Ricci tensor for Hessian manifolds. In order to calculate Einstein tensor of Hessian manifolds we need following results. For the proof of theorems we refer to [11].

Theorem 2.1. Let (M,D,g) be a Hessian manifold with Hessian structure (D,g). The Ricci curvature tensor of (M,D,g) is

$$\sum_i Q_{ijli} = R^H_{jl} = \sum Q_{jjli} - 2R_{jl},$$

where Q_{jili} and R_{jl} are the components of Hessian curvature tensor and Ricci curvature tensor, respectively.

Theorem 2.2. Let (M,D,g) be a Hessian manifold with Hessian structure (D,g). The scalar curvature of (M,D,g) is

$$\sum_{jk} Q_{kjjk} = r^H = \sum_{jk} Q_{jkjk} - 2r,$$

where r is the scalar curvature of Riemannian manifold (M,g).

In the light of the theorems above we define Einstein tensor of (M,D,g) as follows.

Definition 2.3. Let (M,D,g) be a Hessian manifold with Hessian structure (D,g). Einstein tensor of M denoted by E^H expressed as follows

$$\begin{aligned}E^H &= \frac{1}{2}\sum_{jk} Q_{kjjk} - \sum_i Q_{ijli} \\ &= \frac{1}{2}\left(\sum_{jk} Q_{jkjk} - 2\sum_i Q_{jili}\right) - 2R_{jl} - r,\end{aligned}$$

where r is the scalar curvature and R_{jl} is the Ricci curvature of Riemannian manifold (M,g), respectively.

The Ricci curvature r is the function defined on the unit tangent bundle UM of (M, g) by $r(v) = Ric(v, v)$. Considering Proposition 1.2, a Hessian structure (D, g) on M induces a Kaehlerian structure $\left(J, g^T\right)$ on the tangent bundle TM so $r(v) = Ric(v, v)$ corresponds Ricci curvature tensor on the Kaehlerian manifold.

Similarly we define the Einstein curvature to be the functions on TM by

$$\begin{aligned} e(v) &= 2E(v, v) = R^T_{i\bar{j}l\bar{i}} - 2R^T_{k\bar{j}j\bar{k}} \\ &= \frac{1}{2} Q_{ijli} \circ \pi - Q_{kjjk} \circ \pi. \end{aligned}$$

The Einstein curvature tensor precisely coincides with the average of the holomorphic sectional curvature in the direction orthogonal to v

$$e(v) = \sum_{i,j} R^T\left(e_i, e_{\bar{j}}, e_i, e_{\bar{j}}\right) = \sum \frac{1}{2} Q_{ijij} \circ \pi,$$

where $\{e_i, i \in I\}$ is any orthonormal basis for the orthogonal supplement $(\Re v)^{\perp}$ of a vector v in the tangent space $T_m M$ at m.

Now let us comment on the Ricci tensor inequality in terms of Einstein-Hessian tensor.

Theorem 2.3. Suppose M is a complete, connected Hessian n-manifold whose Ricci tensor satisfies the following inequality for all $V \in TM$:

$$\frac{E^H}{2} + \frac{r}{2} + \frac{1}{2} \sum Q_{jjli} - \frac{1}{4} \left(\sum Q_{jkjk} - 2 \sum Q_{jili} \right) \geq \frac{n-1}{R^2} |V|^2.$$

Then M is compact, with a finite fundamental group, and diameter at most πR.

Proof. We use Myers's Theorem [12] for the proof. As is well known for a complete, connected Riemannian n-manifold whose Ricci tensor satisfies

$$R_{jl}(V, V) \geq \frac{n-1}{R^2} |V|^2.$$

Taking into account Theorem 2.1 we may write

$$\sum_i Q_{ijli} = R^H_{jl} = \sum Q_{jjli} - 2R_{jl}$$

for the Hessian type Ricci tensor. On the other hand it is not difficult to compute The Einstein tensor by using Definition 2.3. Considering these two facts together we obtain the Ricci tensor inequality for a Hessian manifold as follows

$$\frac{1}{2} \sum Q_{jjli} - \frac{1}{4} \left(\sum Q_{jkjk} - 2 \sum Q_{jili} \right) + \frac{r}{2} + \frac{E^H}{2} = R^H_{jl}.$$

Then using Myers's Theorem we complete the proof.

§4. Einstein-Hessian manifolds and Ricci-flatness

In this part of the study we define Einstein manifolds in general type on one hand and introduce Einstein-Hessian manifolds by a correspondence with Kaehlerian structure on the other.

As is well-known a Riemannian manifold (M, g) is an Einstein manifold if and only if there exists a real number λ such that $E = \lambda g$. This is also equivalent to say that the Einstein curvature e is a constant function on UM.

Kozsul forms are playing great important role in Hessian manifolds studies. The following definitions and its relation with curvature tensors are given by H. Shima [9].

Definition 3.1. Let w be the volume element of g we define a closed 1-form α and a symmetric bilinear form β by

$$\begin{aligned} D_X w &= \alpha(X) w \\ \beta &= D_\alpha. \end{aligned}$$

The forms α and β are called the first Kozsul form and the second Kozsul form for a Hessian structure (D, g) respectively.

Proposition 3.1.

$$\beta_{ij} = \frac{\partial \alpha_i}{\partial x^j} = \frac{1}{2}\frac{\partial^2 \log \det [g_{kl}]}{\partial x^i \partial x^j} = Q^r_{rij} = Q^r_{ijr}, [9].$$

Proposition 3.2. Let $R^T_{i\bar{j}}$ be the Ricci tensor on the Kaehlerian manifold $\left(TM, J, g^T\right)$. Then we have

$$R^T_{i\bar{j}} = \frac{1}{2}\beta_{ij} \circ \pi.$$

Definition 3.2. If a Hessian structure (D, g) satisfies the condition

$$\beta = \lambda g, \quad \lambda = \frac{\beta^i_i}{n},$$

then the Hessian structure is said to be Einstein-Hessian [9].

Theorem 3.1. Let (D, g) be a Hessian structure on M and let $\left(J, g^T\right)$ the Kaehlerian structure on the tangent bundle TM induced by (D, g). Then the following conditions (1) and (2) are equivalent:

(1) (D, g) is Einstein-Hessian.

(2) $\left(J, g^T\right)$ is Einstein-Kaehlerian [9].

Also accoding to these explanations one may also conclude that the second Kozsul form β plays a similar role to that of the Ricci tensor in Kaehlerian geometry.

Due to M. L. Labbi [4] a Kaehlerian manifold with positive Ricci curvature has positive Einstein curvature because for a Kaehlerian manifold each eigenvalue of Ricci has multiplicity at least 2.

Let us comment on the positivity of a Kaehlerian manifold by using second Kozsul form of the Hessian manifold.

Theorem 3.2. Let (M, D, g) be a compact oriented Hessian manifold. The second Kozsul form β_w for any volume element w can not be negative definite [9].

According to the theorem above we may express the following results.

Corollary 3.1. Let (M, D, g) be a compact oriented Hessian manifold and let $\left(J, g^T\right)$ the Kaehlerian structure on the tangent bundle TM induced by (D, g) then the Ricci curvature of a Kaehlerian manifold can not be negative definite.

Corollary 3.2. Let (M, D, g) be a compact oriented Hessian manifold and let $\left(J, g^T\right)$ the Kaehlerian structure on the tangent bundle TM induced by (D, g) then the Kaehlerian manifold satisfies one of the following conditions

(1) Kaehlerian manifold is Ricci-flat.

(2) Kaehlerian manifold has positive Einstein curvature.

Theorem 3.3. Let (M, D, g) be a compact oriented Hessian manifold. Then one of the following condition is satisfied

(1) M is Ricci-flat.

(2) M is a Einstein-Hessian manifold with positive Einstein curvature.

Proof. From the Corollary 3.1 it may be seen that the second Kozsul form of a compact oriented Hessian manifold can not be negative. This means that $\beta = 0$ or $\beta > 0$.

Suppose that $\beta = 0$. β fills the role of Ricci tensor of Hessian manifold M hence we conclude that M is also Ricci flat. On the other hand taking into account of $\beta > 0$ we conclude from [6] that M has positive Einstein curvature.

From the theorem above we may conclude the following corollary.

Corollary 3.3. Let (M, D, g) be a compact oriented Hessian manifold then it is divergence free.

Proof. As is well-known if M is compact, the Einstein tensor is gradient of the total scalar curvature Riemannian functional $\int_M s\,(g)\; dvol$ defined on the space of all Riemannian metrics on M. Since a Hessian metric is a special type of a Riemannian metric, it is clear that (M, D, g) also admits this condition. Consequently it is divergence free.

From now on we focus on the positivity of Hessian sectional curvature and its relation with Einstein curvature.

According to the definition of Einstein curvature we also have the following nice property

$$\begin{aligned} \text{positive sectional curvature} &\Rightarrow \text{positive Einstein curvature} \\ &\Rightarrow \text{positive scalar curvature.} \end{aligned}$$

On the other hand Shima [9] obtained the following proposition for Hessian manifolds with constant positive Hessian sectional curvature.

Proposition 3.2. Let c be positive real number and let

$$\Omega = \left\{ \left(x^1, ..., x^n\right) \in \mathbb{R}^n \,\middle|\, x^n > \frac{c}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2 \right\},$$

and let φ be a smooth function on Ω defined by

$$\varphi = -\frac{1}{c}\log\left\{ x^n - \frac{1}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2 \right\}.$$

Then $\left(\Omega, D,\ g = D^2\varphi\right)$ is a simply connected Hessian manifold of positive constant Hessian sectional curvature c.

Hence the following theorem can be proved as a consequence of the properties above.

Theorem 3.4. Let c be a positive real number and let

$$\Omega = \left\{ \left(x^1, ..., x^n\right) \in \mathbb{R}^n \left| x^n > \frac{c}{2} \sum_{i=1}^{n-1} \left(x^i\right)^2 \right. \right\},$$

and let φ be a smooth function on Ω defined by

$$\varphi = -\frac{1}{c} \log \left\{ x^n - \frac{1}{2} \sum_{i=1}^{n-1} \left(x^i\right)^2 \right\}.$$

Then $\left(\Omega, D,\ g = D^2 \varphi\right)$ is a simply connected Hessian manifold with positive Einstein curvature.

It is really surprising that (Ω, g) is isometric to hyperbolic space form $\left(H\left(-\frac{c}{4}\right), g\right)$ of constant sectional curvature $-c/4$;

$$\begin{aligned} H &= \left\{ \left(\xi^1, ..., \xi^{n-1}, \xi^n\right) \in \mathbb{R}^n \left| \xi^n > 0 \right. \right\}, \\ g &= \frac{1}{\left(\xi^n\right)^2} \left\{ \sum_{i=1}^{n} \left(d\xi^i\right)^2 + \frac{4}{c} \left(d\xi^n\right)^2 \right\}. \end{aligned}$$

For detailed information we refer to [9].

In [7] Shima introduced the homogeneous Hessian manifold concept by a close analogy with Kaehlerian structures. From now and sequel we deal with this type of Hessian manifold and second Kozsul form on it.

Definition 3.3. Let M be a Hessian manifold. A diffeomorphism of M on to itself is called an automorphism of M if it preserves both the flat affine structure and the Hessian metric. The set of all automorphisms of M, denoted by $Aut\,(M)$, forms a Lie group. A Hessian manifold M is said to be homogeneous if the group $Aut\,(M)$ acts transitively on M [7].

Theorem 3.5. Let M be a connected homogeneous Hessian manifold. Then we have

1) The domain of definition E_x for the exponential mapping $\exp_x$ at $x \in M$ given by the flat affine structure is a convex domain. Moreover E_x is the universal covering manifold of M with affine projection $\exp_x : E_x \to M$.

2) The universal covering manifold E_x of M has a decomposition $E_x = E_x^0 + E_x^+$ where E_x^0 is a uniquely determined vector subspace of the tangent space T_xM of M at x and E_x^+ is an affine homogeneous convex domain not containing any full straight line. Thus E_x admits a unique fibering with the following properties:

i) The base space is E_x^+.

ii) The projection $p : E_x \to E_x^+$ is given by the canonical projection from $E_x = E_x^0 + E_x^+$ onto E_x^+.

iii) The fiber $E_x^0 + v$ through $v \in E_x$ is characterized as the set of all points which can be joined with v by full straight lines contained in E_x. Moreover each fiber is an affine subspace of T_xM and is a Euclidean space with respect to the induced metric.

iv) Every automorphism of E_x is fiber preserving.

v) The group of automorphism of E_x which preserve every fiber, acts transitively on the fibers [7].

Corollary 3.4. Let β denote the canonical bilinear form on a connected homogeneous Hessian manifold M,

$$\beta_{ij} = \frac{\partial^2 \log F}{\partial x_i \partial x_j},$$

where $F=\sqrt{\det [g_{ij}]}$.

Then we have

i) β is positive semi-definite.

ii) The null space of β at $x \in M$ coincides with E_x^0. In particular,

iii) $\beta = 0$ if and only if $E_x = T_x M$ and its a Euclidean space with respect to the induced metric.

iv) β is positive definite if and only if E_x is an affine homogeneous convex domain not containing any full straight line [7].

Using this property we may prove the following theorem.

Theorem 3.6. Let (M, D, g) be a compact oriented homogeneous Hessian manifold. Then one of the following condition is satisfied.

1) M is Ricci-flat if and only if $E_x = T_x M$ and it is a Euclidean space with respect to the induced metric.

2) M is a Einstein-Hessian manifold with positive Einstein curvature if and only if E_x is an affine homogeneous convex domain not containing any full straight line.

Proof. By Theorem 3.3 compact oriented homogeneous Hessian manifold (M, D, g) is either Ricci-flat or Einstein-Hessian manifold with positive Einstein curvature. In virtue of Corollary 3.4 if M is Ricci-flat, the second Kozsul form β vanishes hence $E_x = T_x M$ and it is a Euclidean space with respect to the induced metric.

On the other hand if M is a Einstein-Hessian manifold with positive Einstein curvature we conclude that β is positive definite. Then as a consequence of Corollary 3.4 iv), the theorem is proved.

§5. Conclusion

Of the general theory of relativity you will be convinced, once you have studied it. Therefore I am not going to define it with a single word. Albert Einstein, In a postcard to A. Sommefeld.

The practicality of Einstein's obsevation is difficult to argue against. In honour of him, we have reinterpreted Einstein curvature in terms of Hessian curvatures. Upon examination of Kozsul form β, we prove the Ricci-flatness of a Hessian manifold and obtain Einstein-Hessian manifolds with positive Einstein curvature. Next we prove that a Hessian manifold is Ricci-flat if and only if its universal covering $E_x = T_x M$ and it is a Euclidean space with respect to the induced metric.

As mentioned in introduction, Einstein curvature and Hessian manifold theory have numerous applications from affine geometry to general relativity. In addition, Einstein manifolds usually relates to vacuum solutions to General relativity equations with a non-zero cosmological constant. We have clearly touched upon a new field for researchers working different area of science. We hope that the concepts in this paper draw new possible directions to the well-known facts of physics and mathematics.

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Mean value for the function $t^{(e)}(n)$ over square-full numbers [1]

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Abstract Let $n > 1$ be an integer, $n = \prod p_i^{a_i}$, $a_i \geq 2$, $i = 1, \cdots, r$. $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n, $\delta_2(n)$denote the characteristic function of square-full numbers. In this paper, we shall establish mean value for the function $\delta_2(n)t^{(e)}(n)$.

Keywords Square-full numbers, convolution method, mean value.

§1. Introduction

Let $n > 1$ be an integer, $n = \prod p_i^{a_i}$, $d = \prod p_i^{b_i}$, if $b_i|a_i$, $i = 1, 2, \cdots, r$, such that d is the e-divisor of n, notation: $d|_e(n)$. By convention $1|_e(n)$. The integer $n > 1$ is called e-squarefree if all exponents $a_1, \cdots, a_r$ are squarefree. Consider now the exponential squarefree exponential divisor of n, here $d = \prod p_i^{b_i}$ is an e-squarefree e-divisor of $n = \prod p_i^{a_i} > 1$, if $b_i|a_i$, $i = 1, 2, \cdots, r$ and $b_1, \cdots, b_r$ are squarefree. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$.

Let $t^{(e)}(n)$denote the number of e-squarefree e-divisor of n, which is multiplicative and if $n = \prod p_i^{a_i} > 1$, $i = 1, \cdots, r$, then

$$t^{(e)}(n) = 2^{\omega(a_1)} \cdots 2^{\omega(a_r)},$$

$\omega(n)$ denote the number of distinct prime factors of n, $\omega(1) = 0$; $\omega(n) = s$, $n = \prod p_i^{\alpha_i}$, $i = 1, \cdots, s$. Specially, for every prime p,

$$t^{(e)}(p) = 1,\ t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = 2,\ t^{(e)}(p^6) = 4, \cdots$$

The Dirichlet series of $t^{(e)}(n)$ is of form

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s),\ \Re s > 1,$$

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where $V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > 1/4$. László Tóth [1] proved that the estimate

$$\sum_{n\le x} t^{(e)}(n) = C_1 x + C_2 x^{1/2} + R(x),$$

where $R(x) = O(x^{1/4+\epsilon})$, holds for every $\epsilon > 0$, where

$$C_1 := \prod_p (1 + \sum_{\alpha=6}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)}}{p^{\alpha}}),$$

$$C_2 := \zeta(1/2) \prod_p (1 + \sum_{\alpha=4}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)} - 2^{\omega(\alpha-2)} + 2^{\omega(\alpha-4)}}{p^{\alpha/2}}).$$

Suppose RH is true, this was improved into $R(x) = O(x^{7/29+\epsilon})$ in [2].

In this paper, we shall prove a result about the mean value of $\delta_2(n)t^{(e)}(n)$, $\delta_2(n)$denote the characteristic function of square-full numbers, $\delta_2(n) = 1$, if n is squarefull numbers; otherwise, $\delta_2(n) = 0$. Our main result is the following:

Theorem 1.1. We have the asymptotic formula

$$\sum_{n\le x} \delta_2(n)t^{(e)}(n) = x^{1/2}R_{1,1}(\log x) + x^{1/3}R_{1,2}(\log x) + O(x^{1/4}exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})),$$

where $R_{1,k}(\log x)$, $k = 1, 2$ are polynomials of degree 1 in $\log x$, $D > 0$ is an absolute constant.

Notations 1.1. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Proof of the theorem

To prove the theorem, the following lemmas are needed.

Lemma 2.1. Let

$$d(2, 2, 3, 3; k) := \sum_{k=n^2m^3} d(n)d(m),$$

$$D(2, 2, 3, 3; x) := \sum_{1\le k\le x} d(2, 2, 3, 3; k),$$

such that

$$D(2, 2, 3, 3; x) = x^{1/2}P_{1,1}(\log x) + x^{1/3}P_{1,2}(\log x) + O(x^{19/80+\epsilon}),$$

$P_{1,1}(t)$, $P_{1,2}(t)$ are polynomials of degree 1 in t.

Proof. This is lemma 6 of Deyu Zhang [4].

Lemma 2.2. Let $f(m)$, $g(n)$ are arithmetical functions such that

$$\sum_{m\le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}),$$

$$\sum_{n\le x} |g(n)| = O(x^{\beta}),$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_J > \alpha > \beta > 0$, where$P_j(t)$ are polynomials in t. If $h(n) = \sum_{n=md} f(m)g(d)$ then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)$ are polynomials in t, $(j = 1, \cdots, J)$.

Proof. This is theorem 14.1 of Ivić [3].

Lemma 2.3. Let $f(n)$ be an arithmetical function for which

$$\sum_{n\leq x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^{a}),$$

$$\sum_{n\leq x} |f(n)| = O(x^{a_1}(\log x)^r),$$

where $a_1 \geq a_2 \geq \cdots \geq a_l > 1/c > a \geq 0$, $r \geq 0$, $P_j(t)$ are polynomials in t of degrees not exceeding r, $(j = 1, \cdots, l)$, and $c \geq 1$ and $b \geq 1$ are fixed integers. Suppose for $\Re s > 1$ that $\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}$, if $h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c)$, then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x),$$

where $R_j(t)$ are polynomials in t of degrees not exceeding r, $(j = 1, \cdots, l)$, and for some $D > 0$,

$$E_c(x) \ll x^{1/c} exp(-D(\log x)^{3/5}(\log\log x)^{-1/5}).$$

Proof. If $b = 1$, this is theorem 14.2 of Ivić [3]. When $b \geq 2$, Lemma 2.3 can be proved by the same approach.

Now we prove the Theorem. Let $\delta_2(n) = 1$, if n is squarefull numbers; otherwise, $\delta_2(n) = 0$. Let $\sum_{n=1}^{\infty} \frac{\delta_2(n)t^{(e)}(n)}{n^s} = F(s)$, $(\Re s > 1)$. By the Euler product formula we get for $\Re s > 1$ that

$$\begin{aligned}
F(s) &= \prod_p (1 + \frac{\delta_2(p)t^{(e)}(p)}{p^s} + \frac{\delta_2(p^2)t^{(e)}(p^2)}{p^{2s}} + \cdots) \\
&= \prod_p (1 + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \frac{2}{p^{5s}} + \frac{4}{p^{6s}} + \cdots) \\
&= \zeta(2s) \prod_p (1 + \frac{1}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{6s}} + \cdots) \\
&= \zeta^2(2s)\zeta(3s) \prod_p (1 + \frac{1}{p^{3s}} - \frac{1}{p^{4s}} - \frac{2}{p^{5s}} + \cdots) \\
&= \zeta^2(2s)\zeta^2(3s) \prod_p 1 - \frac{1}{p^{4s}} - \frac{2}{p^{5s}} - \frac{1}{p^{6s}} + \cdots) \\
&= \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)} \prod_p (1 - \frac{2}{p^{5s}} - \frac{2}{p^{9s}} - \frac{1}{p^{12s}} \cdots) \\
&= \frac{\zeta^2(2s)\zeta^2(3s)G(s)}{\zeta(4s)},
\end{aligned} \quad (1)$$

where $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, $G(s)$ is absolutely convergent for $\Re s > 1/5$, and

$$\sum_{n \le x} |g(n)| \ll x^{1/5+\epsilon}.$$

Let

$$\zeta^2(2s)\zeta^2(3s)G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \ \Re s > 1/2,$$

$$\zeta^2(2s)\zeta^2(3s) = \sum_{n=1}^{\infty} \frac{d(2,2,3,3;n)}{n^s},$$

such that

$$f(n) = \sum_{n=md} d(2,2,3,3;m)g(d). \tag{2}$$

From Lemma 2.1 and the definition of $d(2,2,3,3;m)$ we get

$$\sum_{m \le x} d(2,2,3,3;m) = x^{1/2}P_{1,1}(\log x) + x^{1/3}P_{1,2}(\log x) + O(x^{19/80+\epsilon}), \tag{3}$$

where $P_{1,k}(\log x)$ are polynomials of degrees 1 in $\log x$, $k = 1, 2$.

In addition we have

$$\sum_{n \le x} |g(n)| = O(x^{1/5+\epsilon}). \tag{4}$$

From (2), (3) and (4), in view of Lemma 2.2,

$$\sum_{n \le x} f(n) = x^{1/2}Q_{1,1}(\log x) + x^{1/3}Q_{1,2}(\log x) + O(x^{19/80+\epsilon}), \tag{5}$$

where $Q_{1,1}(t)$, $Q_{1,2}(t)$ are polynomials of degrees 1 in t, then we can get

$$\sum_{n \le x} |f(n)| \ll x^{1/2} \log x. \tag{6}$$

In view of $\frac{1}{\zeta(4s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{4s}}$, $\Re s > 1/4$, from (1) and (2) we have the relation

$$\delta(n)t^{(e)}(n) = \sum_{n=md^4} f(n)\mu(d). \tag{7}$$

From (5), (6) and (7), in view of Lemma 2.3, we can get

$$\sum_{n \le x} \delta_2(n)t^{(e)}(n) = x^{1/2}R_{1,1}(\log x) + x^{1/3}R_{1,2}(\log x) + O(x^{1/4}exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})). \tag{8}$$

Finally, the theorem is completely proved.

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Left Duo Seminear-rings

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Abstract In this paper we introduce the concept of left duo seminear-rings and discuss the properties of a seminear-ring R in which every left ideal is a right ideal. We also obtain some equivalent conditions for this seminear-ring.

Keywords Left duo seminear-ring, idempotents, nilpotents, mate functions and ideals.

§1. Introduction and preliminaries

Seminear-rings are a common generalization of near-rings and semi rings. A seminear-ring is an algebraic system $(R, +, \cdot)$, such that

(i) $(R, +)$ is a semigroup,

(ii) $(R, \cdot)$ is a semigroup, and

(iii)$(a + b)c = ac + bc$ for all $a, b, c \in R$.

If we replace (iii) by (iii)' $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a,\ b,\ c \in R$, then R is called a left seminear-ring. We write xy for $x \cdot y$ for all $x,\ y$ in R. Throughout this paper, R stands for a right seminear-ring $(R, +, \cdot)$ with an absorbing zero [2].

A non-empty subset I of a seminear-ring R is called a left ideal (right ideal) if,

(i) for all $x,\ y \in I$, $x + y \in I$, and

(ii) for all $x \in I$ and $r \in R$, $rx\ (xr) \in I$.

I is said to be an ideal of R, it is both a left ideal and a right ideal.

An ideal I of R is called a completely semiprime ideal if $x \in I$ whenever $x^2 \in I$. A seminear-ring R is said to have the

(i) Insertion of Factors Property-IFP for short, if for x, y in R, $xy = 0 \Rightarrow xry = 0$ for all r in R, if in addition, $xy = 0 \Rightarrow yx = 0$ for x, y in R we say R has $(*, IFP)$.

(ii) strong IFP if and only if for all ideals I of R, $xy \in I \Rightarrow xry \in I$ for all r in R and

(iii) property P_4 if for all ideals I of R $xy \in I \Rightarrow yx \in I$.

A left ideal B of a seminear-ring R is called essential, if $B \cap K = \{0\}$, where K is any left ideal of R, implies $K = \{0\}$. A seminear-ring R is said to be an integral if R has no non-zero zero-divisors.

A map $f : R \to R$ is called a mate function if $x = xf(x)x$ for all x in R. $f(x)$ is called a mate of x. This concept has been introduced in [2] with a view to handling the regularity structure in a seminear-ring with considerable ease and to discuss in detail the properties of the exact "mate" of each element.

Notations 1.1. We furnish below the notations that we make use of throughout this paper.

(1) $E = \{e \in R/e^2 = e\}$ - set of all idempotents of R.

(2) $L = \{x \in R/x^k = 0$ for some positive integer $k\}$ - set of all nilpotent elements of R.

(3) $C(R) = \{r \in R/rx = xr$ for all $x \in R\}$ - centre of R.

(4) $l(S) = \{x \in R/xs = 0$ for all $s \in S\}$ - left annihilator of a non-empty subset S in R.

2. Preliminary results

We freely make use of the following results from [2], [3] and [4] and designate them as (K1), (K2) etc.

(K1) A seminear-ring R has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in R.

(K2) If R has a mate function f, then R is a left (right) normal seminear-ring.

(K3) Let f be a mate function for R. Then every left ideal A of R is idempotent.

(K4) If R has a mate function f, then for every x in R, $xf(x)$, $f(x)x \in E$, $Rx = Rf(x)x$ and $xR = xf(x)R$.

(K5) Let r, m be two positive integers. We say that R is a $P(r, m)$ seminear-ring if $x^r R = Rx^m$ for all x in R .

(K6) A seminear-ring R is called a P_k seminear-ring (P'_k seminear-ring) if there exists a positive integer "k" such that $x^k R = xRx$ ($Rx^k = xRx$) for all x in R.

(K7) A seminear-ring R is called left (right) normal if $a \in Ra$ (aR) for each $a \in R$. R is normal if it is both left normal and right normal.

(K8) Let "r" be a positive integer. We say that R is a left-r-normal (right-r-normal) seminear-ring if $a \in Ra^r$ ($a^r R$) for each $a \in R$.

§3. Left Duo Seminear-rings

In this section we define the concept of left duo seminear-rings, furnish a few examples and prove some of its properties.

Definition 3.1. We say that a seminear-ring R is a left duo seminear-ring if every left ideal of R is a right ideal.

Example 3.1.

(i) A natural example of a left duo seminear-ring is the direct product of any two seminear-fields.

(ii) Let $R = \{0, a, b, c, d\}$. We define the semigroup operations "$+$" and "$\cdot$" in R as follows:

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	c
d	d	a	b	c	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
d	0	a	b	c	d

Obviously $(R, +, \cdot)$ is a left duo seminear-ring.

(iii) We consider the seminear-ring $(R, +, \cdot)$ where $R = \{0, a, b, c\}$ and the semigroup operations "$+$" and "$\cdot$" are defined as follows:

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	a	a

This is not a left duo seminear-ring. Since the left ideal $\{0, a\}$ is not a right ideal of R. It is worth noting that this is, in fact a near-ring. Thus, even in near-ring theory, a left ideal need not be a right ideal.

Proposition 3.1. Any $P(1, m)$ seminear-ring is an left duo seminear-ring.

Proof. Since R is a $P(1, m)$ seminear-ring. We have for all "x" in R, $xR = Rx^m$. Let "A" be any left ideal of R. Therefore $RA \subseteq A$. Let $a \in A$. For any $r \in R$, since $ar \in aR = Ra^m$ there exists $b \in R$ such that $ar = ba^m \in RA \subseteq A$. Thus $AR \subseteq A$ and the desired result now follows.

Proposition 3.2. Let R be a seminear-ring with a mate function f. If $E \subseteq C(R)$, then R is left duo seminear-ring.

Proof. Let A be any left ideal of R. Clearly then $RA \subseteq A$. For any "x" in A and "r" in R, $xr = xf(x)xr = xrf(x)x$ (since $f(x)x \in E$) $= xyx$ where $y = rf(x)$. Thus $xr = xyx = (xy)x \in RA \subseteq A$. This guarantees that $AR \subseteq A$ and hence R is left duo seminear-ring.

We furnish below a necessary and sufficient condition for a left duo seminear-ring to admit a mate function.

Theorem 3.1. Let R be a left duo right cancellative seminear-ring. Then R has a mate function if and only if every left ideal of R is idempotent and R is a left normal seminear-ring.

Proof. For the only if part, let us assume that R has a mate function f. For $x \in R$, $x = xf(x)x \in Rx$ and hence R is left normal seminear-ring. Also (K3) demands that, every left ideal of R is idempotent.

For the if part, let $x \in R$. We observe that R is a left normal seminear-ring $x \in Rx$ for every $x \in R$. Since R is a left duo seminear-ring and Rx is a left ideal of R, it is idempotent.

Clearly then $Rx = (Rx)^2 = (Rx)(Rx) = (RxR)x \subseteq (Rx)x = Rx^2$. Therefore $x \in Rx \subseteq Rx^2$. Hence there exists $y \in R$ such that $x = yx^2$. Now $x^2 = x(yx^2) = xyx \Rightarrow x = xyx$ (as R is right cancellative). By setting $y = g(x)$ we see that $x = xg(x)x$. Hence "g" is a mate function for R.

We shall obtain a complete characterisation of left duo seminear-rings in the following Theorem:

Theorem 3.2. Let R admit a mate function f. Then R is a left duo seminear-ring if and only if R is a P_1 seminear-ring.

Proof. Suppose R is a left duo seminear-ring. Clearly Rx being a left ideal for every "x" in R, is a right ideal of R. Therefore $(Rx)R \subseteq Rx$. Hence for any $r \in R$ there exists some $r' \in R$ such that $xr = (xf(x)x)r = x(f(x)xr) = xr'x \in xRx$. Thus $xR \subseteq xRx$. Obviously the reverse inclusion $xRx \subseteq xR$ always holds. Consequently we have $xR = xRx$ for all "x" in R. Thus R is P_1 seminear-ring.

For the converse, let A be any left ideal of R. Clearly then $RA \subseteq A$. For every $a \in R$, we have $aR = aRa = (aR)a \subseteq RA \subseteq A$. Consequently $AR \subseteq A$ and the result follows.

Theorem 3.3. Every left ideal of a left duo seminear-ring with mate functions is a left duo seminear-ring in its own right.

Proof. Let f be a mate function for R and let M be any left ideal of R. We observe that for all "x" in M, $f(x)xf(x) \in RMR \subseteq M$ (since R is left duo seminear-ring). We therefore define a map $g : M \to M$ such that, $g(x) = f(x)xf(x)$ for all "x" in R.

As $xg(x)x = x(f(x)xf(x))x = xf(x)(xf(x)x) = xf(x)x = x$, it is clear that g serves as a mate function for M. Now for all "x" in M, $Rx \subseteq M$. Also it is clear that $xM \subseteq xR = xRx = x(Rx) \subseteq xM$ (using Theorem left duo [1]) and therefore $xM = xR = xRx$ and $xRx = (xR)x = xMx$. Hence $xM = xMx$ for all "x" in M and Theorem 3.2 demands that M is also a left duo seminear-ring.

Remark 3.1. It is worth noting that the existence of a mate function and the property $xR = xRx$ for all x in R (P_1 seminear-ring) are preserved under homomorphisms. Consequently, if R admits mate functions and has the left duo property, any homomorphic image of R also does so.

In the following Theorem we prove some important properties of a left duo seminear-ring.

Theorem 3.4. Let R be a left duo seminear-ring admit a mate function "f". Then we have the following:

(i) For all ideals M_1 and M_2 of R, $M_1 \cap M_2 = M_1M_2$.

(ii) $Rx \cap Ry = Rxy$ for all x, y in R.

(iii) R has no non-zero nilpotent elements.

(iv) R has $(*, IFP)$.

(v) Every ideal of R is a completely semi prime ideal.

(vi) R has property P_4.

(vii) R has strong IFP.

(viii) Every left ideal of R is essential if R is integral.

Proof. Since R has a mate function f, then (K2) demands that R is left normal seminear-ring.

(i) Let $x \in M_1 \cap M_2$. Since f is a mate function, $x = xf(x)x = x(f(x)x) \in M_1M_2$. This implies

$$M_1 \cap M_2 \subseteq M_1M_2. \tag{1}$$

Let $r = yz \in M_1M_2$ with $y \in M_1$ and $z \in M_2$. Clearly $r \in M_1$, since R is left duo we have $r = yz \in RM_2 \subseteq M_2$ and so $r \in M_2$. Thus

$$M_1M_2 \subseteq M_1 \cap M_2. \tag{2}$$

From (1) and (2), we get $M_1 \cap M_2 = M_1M_2$.

(ii) For $x,\ y$ in R, $Rx \cap Ry = RxRy$-taking $M_1 = Rx$ and $M_2 = Ry$ in (i). We have, obviously, $Rx = Rx \cap R = RxR$ and this yields that $Rxy = RxRy$ and the result follows.

(iii) Since R is left normal seminear-ring, $a \in R \Rightarrow a \in Ra = Ra \cap Ra = Raa = Ra^2$ and therefore "$a^2 = 0 \Rightarrow a = 0$". Then (K1) demands that $L = \{0\}$. Hence R has no non-zero nilpotent elements.

(iv) Let $a,\ b \in R$. Suppose $ab = 0$. Then $a \in l(b)$. Clearly $l(b)$ is a left ideal of R. Since R is left duo seminear-ring, $l(b)$ is a right ideal of R. Hence $ar \in l(b)$ for any $r \in R$. Therefore $arb = 0$. Hence R is IFP seminear-ring.

Also, if $ab = 0$, then $(ba)^2 = ba(ba) = b(ab)a = b0a = b0 = 0$. Now by (iii) $ba = 0$ and (iv) follows.

(v) Let $x^2 \in M$ where M is any ideal of R. Since R is left normal seminear-ring. Therefore $x \in Rx = Rx \cap Rx = Rx^2 = RM \subseteq M$ and (v) follows.

(vi) Let $xy \in I$ where I is any ideal of R. Now $(yx)^2 = yxyx = y(xy)x \in RIR \subseteq I$ and (v) implies $yx \in I$. Hence the result follows.

(vii) Let I be any ideal of R and let $xy \in I$. Now (vi) implies $yx \in I$ and therefore, $yxr \in IR \subseteq I$ for all "r" in R. Thus we have $y(xr) \in I$ and again (vi) guarantees that $(xr)y = xry \in I$. Hence R has strong IFP.

(viii) Let P be a non-zero left ideal of R. Suppose there exists a left ideal Q of R such that $P \cap Q = \{0\}$. Then by (i), $PQ = \{0\}$ and since R is an integral seminear-ring, $Q = \{0\}$. This completes the proof of (viii).

We conclude our discussion with the following characterisation of left duo seminear-rings.

Theorem 3.5. Let R be a seminear-ring admit a mate function "f". Then the following statements are equivalent:

(i) R is left duo seminear-ring.

(ii) R is P_1 seminear-ring.

(iii) For all left ideals L_1 and L_2 of R, $L_1 \cap L_2 = L_1L_2$.

Proof. (i)$\Rightarrow$(ii) follows from Theorem 3.2.

(ii)$\Rightarrow$(iii) Let $x \in L_1 \cap L_2$. Clearly then $x = xf(x)x = x(f(x)x) \in L_1L_2$. Thus $L_1 \cap L_2 \subseteq L_1L_2$. Now let $r = yz \in L_1L_2$ with $y \in L_1$ and $z \in L_2$. We have $r = yz \in yR = yRy$ and therefore $r = ysy$ (for some s in R) $= y(sy) \in yL_1$. Since $yL_1 \subseteq L_1$ we have $r \in L_1$. This guarantees that $L_1L_2 \subseteq L_1$. Again as $r = yz \in RL_2 \subseteq L_2$ we see that $L_1L_2 \subseteq L_2$. Collecting all these pieces we get $L_1L_2 \subseteq L_1 \cap L_2$ and (iii) follows.

(iii)$\Rightarrow$(i) Let L_1 be any left ideal of R, and let $L_2 = R$. Then the assumption of (iii) implies that $L_1R = L_1 \cap R = L_1$, that is, L_1 is a right ideal of R and (i) follows.

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