Diagonal Algebraic Space-Time Block Codes

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Abstract— **We construct a new family of linear space-time block codes by the combination of rotated constellations and the Hadamard transform, and we prove them to achieve the full transmit diversity over a quasi-static or fast fading channels. The proposed codes transmit at a normalized rate of 1 symbol/sec. When the number of transmit antennae $n = 1, 2$ or n is a multiple of 4 we spread a rotated version of the information symbol vector by the Hadamard transform and send it over n transmit antennae and n time periods; for other values of n, we construct the codes by sending the components of a rotated version of the information symbol vector over the diagonal of an $n \times n$ space-time code matrix. The codes maintain their rate, diversity and coding gains for all real and complex constellations carved from the complex integers ring $\mathbb{Z}[i]$, and they outperform the codes from orthogonal design when using complex constellations for $n > 2$. The maximum likelihood decoding of the proposed codes can be implemented by the sphere decoder at a moderate complexity. It is shown that using the proposed codes in a multi-antenna system yields good performances with high spectral efficiency and moderate decoding complexity.**

Index Terms— **Block codes, diversity methods, maximum likelihood decoding, MIMO systems.**

I. Introduction

RECENTLY, many works have been done on signal processing and modulation techniques for transmit diversity over fading channels [1]-[5]. The motivation is the severe attenuation of the wireless channel from one part (see [6] and references therein) and the large capacity of a multi-antenna system from another part [7], [8]. Since it is impossible to recover a severely attenuated signal, it is necessary to provide the receiver by less attenuated replicas of the transmitted signal, which is done by diversity techniques [6]. The first space-time (ST) code of a normalized rate of 1 symbol/sec was proposed by Alamouti over 2 transmit antennae and 2 time periods in [1]. In [2], Tarokh *et al.* gave the construction criteria, tradeoff between constellation size, data rate, diversity advantage and complexity, and some hand constructed trellis ST codes which satisfy the proposed criteria. The generalization of the Alamouti scheme to more than two transmit antennae by using the theory of orthogonal design was done by Tarokh *et al.* in [4]. The ST block codes proposed in [4] have a normalized rate of 1 symbol/sec over real constellations; over complex constellations, ST block codes of normalized rates of 1/2 and 3/4 symbol/sec have been proposed for $n = 3, 4$. Due to their orthogonal structure, the maximum likelihood (ML) decoding of these codes can be implemented by a linear decoder [4]. In [9], DaSilva and Sousa proposed a diagonal scheme in order to achieve diversity over $n = 2, \ldots, 5$ transmit antennae, where the components of rotated n-dimensional BPSK modulations were transmitted over the different transmit antennae. The scheme in [9] has a normalized rate of 1 symbol/sec, and the ML decoding was done by exhaustive search over all n-dimensional rotated BPSK vectors, for $n = 2, \ldots, 5$.

In this paper we construct a new family of linear ST block codes by the use of rotated constellations and the Hadamard transform; the so-called diagonal algebraic space-time (DAST) block codes. The DAST block codes have a normalized rate of 1 symbol/sec and achieve the full diversity over n transmit and m receive antennae. The DAST block codes maintain their diversity and coding gains over all real or complex constellations carved from the ring of complex integers $\mathbb{Z}[i]$, with $i = \sqrt{-1}$, such as pulse-amplitude modulation (PAM) or quadrature-amplitude modulation (QAM). Due to the lattice structure of the DAST block codes, the ML decoding can be implemented by the sphere decoder at a moderate complexity independent of the transmission rate [10], [11]. The DAST block codes outperform the ST codes from orthogonal design [4] for $n > 2$; at the same throughput and same SNR, the DAST block codes have smaller error rates. The difference in performance is further enhanced when the spectral efficiency of the used constellation or the number of receive antennae increases.

The paper is organized as follows: in Section II, we recall the construction criteria of ST codes and discuss the possibility of transmitting at high data rates along with maximum transmit diversity in the light of the existence Theorems in [2]. Section III gives a brief summary of the theory of rotated constellations and the properties of the Hadamard transform. The code construction, properties, and decoding schemes are given in Section IV. Simulation results and comments on the DAST block codes performances are given in Section V. In Section VI we discuss the obtained results.

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II. Transmit Diversity and Maximum Achievable Rates

A. System Model

Consider a system of n transmit and m receive antennae, where at each time we transmit n symbols simultaneously from all the transmit antennae. The radiated power by each transmit antenna is proportional to $1/n$ so that the total radiated power is independent[1] of n. The antennae are supposed to be sufficiently spaced such that the fadings are assumed uncorrelated. At each receive antenna the received signal is the superposition of the n transmitted symbols affected by independent fadings and disturbed by additive white Gaussian noise. The transmission is done by burst of length l over a quasi-static Rayleigh fading channel changing randomly every l symbol durations. Let $\boldsymbol{H}$ denote the $m \times n$ channel transfer matrix, with h_{kj} denoting the fading between transmit antenna j and receive antenna k. The fadings h_{kj} are modeled by independent complex Gaussian random variables of variance 0.5 per real dimension. Transmit diversity is obtained by ST coding which encodes the information symbols over n antennae and l symbol periods. A ST block code[2] associates with each information symbol vector $\boldsymbol{x} = (x_1, \ldots, x_d)$, for $d > 0$, an $n \times l$ matrix $\boldsymbol{B}(\boldsymbol{x})$ with entries b_{jt}, $j = 1 \ldots n$, $t = 1 \ldots l$, such that b_{jt} is sent over transmit antenna j at time t. We say that the ST code is linear if $\boldsymbol{B}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{B}(\boldsymbol{x}_1) + \boldsymbol{B}(\boldsymbol{x}_2)$ for any pair of information symbol vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. When there is no notational confusion, the ST code is denoted by $\boldsymbol{B}$. If the symbol period is normalized to 1 second then the normalized rate of the ST code $\boldsymbol{B}$ is d/l symbol/sec. Over m receive antennae and l time periods, the received signal can be written as

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{B} + \boldsymbol{\nu} \tag{1}$$

where $\boldsymbol{\nu}$ is the observation noise represented by an $m \times l$ complex matrix which entries are independent Gaussian distributed random variables of variance σ^2 per real dimension.

B. Design Criteria

Assuming a perfect channel state information (CSI) at the receiver, then over a quasi-static fading channel, the pairwise error probability (PEP) of decoding the codeword $\boldsymbol{e}$ given that the codeword $\boldsymbol{x} \neq \boldsymbol{e}$ was sent, is upper bounded by [2]

$$\Pr(\boldsymbol{x} \longrightarrow \boldsymbol{e}) \leq \left(\prod_{j=1}^{r} \lambda_j\right)^{-m} \left(\frac{\bar{E}_s}{8\sigma^2}\right)^{-mr} \tag{2}$$

[1]In this paper, we adopt the equivalent power normalization given in [5] by multiplying the noise variance by n at the considered SNR (see Section V).

[2]In [4], the term ST block codes is used only for codes from orthogonal design.

where $\bar{E}_s$ is the average energy per symbol, r is the minimum rank of the set of matrices $\boldsymbol{B}(\boldsymbol{x} - \boldsymbol{e})$ for all pairs of codewords $\boldsymbol{x} \neq \boldsymbol{e}$, and λ_j, $j = 1 \ldots r$, are the nonzero eigenvalues of $\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{e}) \triangleq \boldsymbol{B}(\boldsymbol{x}-\boldsymbol{e})\boldsymbol{B}^H(\boldsymbol{x}-\boldsymbol{e})$, with the superscript ^{H} denoting the transpose conjugate. Thus, minimizing the PEP is equivalent to the rank and determinant criteria defined as [2]

- *The rank criterion:* the minimum rank r of $\boldsymbol{B}(\boldsymbol{x} - \boldsymbol{e})$ taken over all distinct codewords pairs $(\boldsymbol{x}, \boldsymbol{e})$, is the diversity gain.
- *The determinant criterion:* the minimum of the geometric mean of the nonzero eigenvalues of $\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{e})$, $\left(\prod_{j=1}^{r} \lambda_j\right)^{1/r}$ in (2), taken over all distinct codewords pairs $(\boldsymbol{x}, \boldsymbol{e})$ is the coding gain.

Over a fast fading channel, the previous criteria become

- *The distance criterion:* In order to achieve the diversity vm in a rapid fading environment, for any two codewords $\boldsymbol{x} \neq \boldsymbol{e}$ the vectors $\boldsymbol{x}_t = (x_{1t}, x_{2t}, \cdots, x_{nt})^T$ and $\boldsymbol{e}_t = (e_{1t}, e_{2t}, \cdots, e_{nt})^T$ must be different at least for v values of $1 \leq t \leq l$.
- *The product criterion:* Let $\mathcal{V}(\boldsymbol{x}, \boldsymbol{e})$ denote the set of time instants $1 \leq t \leq l$, such that $\boldsymbol{x}_t \neq \boldsymbol{e}_t$. Then to achieve the maximum coding gain in a rapid fading environment, the minimum of the products $\prod_{t \in \mathcal{V}(\boldsymbol{x},\boldsymbol{e})} |\boldsymbol{x}_t - \boldsymbol{e}_t|^2$ taken over distinct codewords must be maximized.

ST codes satisfying the criteria for both quasi-static and fast fading channels were called 'smart and greedy' in [2].

C. Maximum Achievable Rates

We note that [2, Corollary 3.3.1] states that for a constellation $Q \subset \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, if the size of Q is 2^b elements, and if the diversity gain is nm, then the transmission rate is at most b bits per second per Hertz. Note that in this work, the authors supposed that the ST code matrix belonged to the constellation Q^{nl}, which is true for the ST codes given in [1], [2], [4]. When using rotated constellations in dimension $d = n$, the original constellation Q is extended to Q_1 defined by

$$Q_1 \triangleq \left\{x,\, x = \sum_{i=1}^{n} m_{ji} a_i,\, j = 1 \ldots d\right\} \tag{3}$$

where $a_i \in Q$, and m_{ji} are the elements of the jth row of the rotation matrix $\boldsymbol{M}_d$. Since the rotations used in this paper have the property that all their rows are equal to the first one up to a permutation and multiplications by ± 1, one obtains the size of Q_1 equal to $\left(2^b\right)^d = 2^{db}$ elements. This makes the maximum achievable rate db bits per second per Hertz according to [2, Corollary 3.3.1] when using the rotated constellation Q_1. Note that the increase in the size of the used constellation, caused by the rotation, induces no power neither bandwidth penalty. Also, there is no penalty in the decoding complexity caused by the increase of the constellation size if the sphere decoder is used at the receiver [10], [11]. We also note that the DAST block codes are not concerned by [4, Theorem 5.4.2], which states that the Alamouti scheme [1] is the unique linear processing full diversity complex system transmitting at a normalized rate of 1 symbol/sec. This is because the DAST block codes do not have linear processing. Nevertheless, their ML decoding has a moderate complexity as is shown in subsection IV-C.

III. Rotated Constellations and the Hadamard Transform

The idea of rotated constellations was first proposed by Boullé and Belfiore in [12]. Rotations were proposed based on the fact that given a constellation Q in dimension d, if any given vector $\boldsymbol{x} \in Q$ has its components $x_1, \ldots, x_d$ different from all the other components of the vectors in Q, then affecting $(x_1, \ldots, x_d)$ by independent fadings $\longrightarrow (\alpha_1 x_1, \ldots, \alpha_d x_d)$, allows the receiver to recover $(x_1, \ldots, x_d)$ unless all the components fall in deep fading, i.e., $|\alpha_j| \ll 1$, $\forall j$. The latter property is called full modulation diversity of the constellation Q, and it can be measured by the minimum product distance of Q [12].

Definition: *The minimum product distance of the constellation Q is given by*

$$d_{d,min} \triangleq \min_{\boldsymbol{y}=\boldsymbol{x}_1-\boldsymbol{x}_2, \boldsymbol{x}_1 \neq \boldsymbol{x}_2 \in Q} \prod_{j=1}^{d} |y_j|. \tag{4}$$

The optimal rotations in the sense of modulation diversity are those that have full modulation diversity and maximize the minimum product distance. Number field theory was used in [13], [14] in order to construct quasi-optimal rotations in certain dimensions. The use of rotated constellations was further extended to construct collision resistant signal sets [16], and to construct T-user codes over the QAM constellations [17]. Here we only report the quasi-optimal rotations with the best values of the minimum product distance found in [13], [14]. In [13], the construction of rotations $\boldsymbol{M}_d$ in dimension d was done in an iterative manner in a 'Hadamard' way as follows

$$\boldsymbol{M}_d = \begin{bmatrix} \boldsymbol{M}^1_{d/2} & -\boldsymbol{M}^2_{d/2} \\ \boldsymbol{M}^2_{d/2} & \boldsymbol{M}^1_{d/2} \end{bmatrix} \tag{5}$$

where $\boldsymbol{M}^1_{d/2}$ is the optimal real rotation in dimension $d/2$ and $\boldsymbol{M}^2_{d/2}$ is an orthogonal transformation in dimension $d/2$ depending only on one parameter [13]. Then, one varies this parameter in order to choose the rotation that maximizes the minimum product distance. This method works very well for the dimensions $d = 2, 3, 4$ and 6. It becomes less successful for $d \geq 8$, since one excludes too many parameters in the rotations in high dimensions. We report in Table I the first row of the optimal real rotations found in [13] along with $d_{d,min}$ for $d = 2, 4$ that we use to construct our DAST block codes (the rest of the rotation matrix can easily be obtained from (5)). For the dimensions $d = 2^q, q \geq 3$, we use the rotations given in [14] constructed on the real part of the cyclotomic number field of degree $4d$: $\mathbb{Q}\left(\cos\frac{2\pi}{8d}\right)$, which relatively give good values of the minimum product distance[3] [14]

$$d_{d,min} = \frac{\sqrt{2}}{(2d)^{d/2}}. \tag{6}$$

Table II shows the MATLAB program[4] which generates the rotation matrix $\boldsymbol{M}_d$ of any dimension $d = 2^q$. This method to generate full modulation diversity rotations is appealing, especially for large d. Note that the best rotations in [13] give better $d_{d,min}$ for $d = 2, 4$, while starting from $d = 8$, the rotations given in Table II are better. For example, take $d = 8$, then the rotation in [13] gives $d_{8,min} = 3.69\,10^{-6}$, whereas the rotation given in Table II yields $d_{8,min} = 2.1579\,10^{-5}$ (see [13, Table III] for comparisons). We find appropriate here to give some comments on the underlying principles of rotated constellations. It is well known that in order to protect the transmitted information symbols from noise one needs to add redundancy [18]. In the geometric representation of the transmitted signals, redundancy can be seen as an extension of the geometric dimension of the original space containing the information symbols. When using rotated constellations, instead of adding redundancy and increasing the geometric dimension of the transmitted signal, we increase the algebraic dimension [19] of the rotated constellation, which is translated by the increase of the constellation size. For example, in dimension two, let the information symbols (a_1, a_2) belong to the BPSK constellation, i.e., $a_1, a_2 \in Q = \{+1, -1\}$ which can be seen as a subset of the field of rational numbers $\mathbb{Q}$. When we rotate $(a_1, a_2)^T$ by the rotation $\boldsymbol{M}_2$ given in Table I, the resulted vector $(x_1, x_2)^T = \boldsymbol{M}_2(a_1, a_2)^T$ still has a geometric dimension of 2, but one has $x_1, x_2 \in Q_1 = \{\pm 0.5257 \pm 0.8507\}$ which does not belong any more to $\mathbb{Q}$ but is now a subset of the algebraic number field $\mathbb{Q}(\sqrt{5})$ that can be seen as a vector space of dimension 2 over $\mathbb{Q}$. The increase in the algebraic dimension can also be seen as the increase of the constellation's cardinal: one has $\#(Q) = 2$ and $\#(Q_1) = 4$. This can be seen as follows: each component of the rotated vector contains information about all the transmitted symbols. The increase of the algebraic dimension could increase the decoding complexity since one can not decode the different components separately. Nevertheless, the sphere decoder exploits the lattice structure of the rotated constellations in a similar way a decoder exploits the algebraic structure

[3] In [14] $d_{d,min}$ was proved to be $\geq 1/(2d)^{(d/2)}$ after normalization. The values of $d_{d,min}$ in (6) are computed over d-dimensional 4-QAM constellations with average energy per symbol $\bar{E}_s = 1$.

[4] The generated matrix $\boldsymbol{M}_d$ is a rotation with $\boldsymbol{M}_d \boldsymbol{M}_d^T = \boldsymbol{I}_d$ for any dimension d. But $\boldsymbol{M}_d$ is proved to have full modulation diversity only for $d = 2^q$ [14].

of an error correcting code in order to reduce the complexity. Even though the complexity of the lattice decoder is, in general, larger than that of an error correcting code decoder, the sphere decoder yields a very good tradeoff between the complexity and the allowed spectral efficiency [10], [11].

The Hadamard transform is a real unitary transformation that exists for 1, 2 and all the dimensions multiple of 4 [20]. In dimension n, the Hadamard transform, $\boldsymbol{\mathcal{H}}_n$, satisfies $\boldsymbol{\mathcal{H}}_n\boldsymbol{\mathcal{H}}_n^T = n\boldsymbol{I}_n$, with $\boldsymbol{I}_n$ the identity matrix in dimension n.

IV. Codes Construction, Properties, and Decoding

A. The Coding Algorithm

Let $\boldsymbol{M}_n$ be a rotation of dimensions $n \times n$ (with $n = 1, 2$ or n is a multiple of 4), which generates a full modulation diversity lattice, we construct the DAST block code in dimensions $n \times n$ as follows[5].

$$\boldsymbol{\Xi}_n \quad \triangleq \quad \boldsymbol{\mathcal{H}}_n \mathrm{diag}(x_1, \ldots, x_n), \tag{7}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^T = \boldsymbol{M}_n\boldsymbol{a}$, and $\boldsymbol{a} = (a_1, \ldots, a_n)^T$ is the information symbol vector. In the sequel, we denote the entries of the Hadamard matrix $\boldsymbol{\mathcal{H}}_n$ by $\hbar_{ij}$ in order to differentiate them from the entries of the channel transfer matrix h_{ij}.

Examples: For $n = 2$ the corresponding DAST block code is given by

$$\boldsymbol{\Xi}_2 \quad \triangleq \quad \begin{bmatrix} x_1 & x_2 \\ x_1 & -x_2 \end{bmatrix}, \tag{8}$$

where $\boldsymbol{x} = (x_1, x_2)^T = \boldsymbol{M}_2\boldsymbol{a}$, and $\boldsymbol{M}_2$ is the 2-dimensional rotation matrix given in Table I. For $n = 4$ the corresponding DAST block code is given by

$$\boldsymbol{\Xi}_4 \quad \triangleq \quad \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & -x_2 & x_3 & -x_4 \\ x_1 & x_2 & -x_3 & -x_4 \\ x_1 & -x_2 & -x_3 & x_4 \end{bmatrix}, \tag{9}$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T = \boldsymbol{M}_4\boldsymbol{a}$, and $\boldsymbol{M}_4$ is the 4-dimensional rotation matrix given in Table I.

B. Properties

Proposition 1: The DAST block code $\boldsymbol{\Xi}_n$ has a transmit diversity equal to n under quasi-static fading assumption. When n is a power of 2 and for the rotations given in Section III, the coding gain of the DAST block code[6], equals

$$\delta_n \quad = \quad \begin{cases} \frac{2^{\frac{2}{n}}}{\sqrt{5}}, & \text{for} \quad n = 2, 4 \\ \frac{1}{2^{\frac{n-1}{n}}}, & \text{for} \quad n \geq 8 \end{cases} \tag{10}$$

Proof. Let $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e} = \boldsymbol{M}_n(\boldsymbol{a} - \boldsymbol{b})$ such that $\boldsymbol{a} \neq \boldsymbol{b}$. We can write the DAST block code at the word $\boldsymbol{y}$ as (7)

$$\boldsymbol{\Xi}_n \quad = \quad \boldsymbol{\mathcal{H}}_n \mathrm{diag}(y_1, \ldots, y_n).$$

Since $\boldsymbol{M}_n$ generates a full modulation diversity lattice, one has $y_j \neq 0 \ \forall j = 1 \ldots n$ taken over all the vectors $\boldsymbol{a} \neq \boldsymbol{b}$ in the considered constellation. It follows that the matrix $\mathrm{diag}(y_1, \ldots, y_n)$ is full rank, and also $\boldsymbol{\Xi}_n$ is full rank over all the differences of codewords. For the coding gain one computes

$$\begin{aligned} \det(\boldsymbol{\Xi}_n\boldsymbol{\Xi}_n^H) &= \\ \det\left(\boldsymbol{\mathcal{H}}_n \mathrm{diag}(y_1, \ldots, y_n)\mathrm{diag}(y_1^*, \ldots, y_n^*)\boldsymbol{\mathcal{H}}_n^T\right) &= \\ \det\left(n\boldsymbol{I}_n \mathrm{diag}(|y_1|^2, \ldots, |y_n|^2)\right) = n^n \prod_{j=1}^{n} |y_j|^2. \end{aligned} \tag{11}$$

By taking the minimum over $\boldsymbol{y}$ of the determinant above and then taking the n-th root one obtains the coding gain of the DAST block code

$$\delta_n \quad = \quad n(d_{n,min})^{2/n}. \tag{12}$$

From Table I one has $d_{2,min} = \frac{1}{\sqrt{5}}$ which gives $\delta_2 = \frac{2}{\sqrt{5}}$, and $d_{4,min} = \frac{1}{40}$ which gives $\delta_4 = \frac{\sqrt{2}}{\sqrt{5}}$.
For $n \geq 8$, and for the rotations given in Section III, Replacing (6) in (12) one obtains $\delta_n = \frac{1}{2^{\frac{n-1}{n}}}$. □

Note that the coding gain given in (10) is greater than 0.5 and it approaches this value when n increases. For example $\delta_8 = 0.5453$ and $\delta_{32} = 0.5109$. We also note that one has the multiplicative factor n in the coding gain expression in (12) because in our model we normalize the radiated power at a given SNR by the number of transmit antennae n by multiplying the noise variance by n (see Section V). If the normalization is done at the transmitter side then the coding gain will be $\delta_n = (d_{n,min})^{2/n}$.

Proposition 2: The DAST block codes are also suitable for fast fading.

Proof. Let $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{e} = \boldsymbol{M}_n(\boldsymbol{a} - \boldsymbol{b})$ such that $\boldsymbol{a} \neq \boldsymbol{b}$. The DAST block code satisfies the distance criterion II-B, because $y_j \neq 0$, $\forall j$, thus the strings $\hbar_{1t}y_t, \hbar_{2t}y_t, \cdots, \hbar_{nt}y_t$ are $\neq 0$ in $v = n$ values for $1 \leq t \leq n$, thanks to the Hadamard transform, which also helps maximizing the product crite-

[5] In a sense, the DAST block codes can be considered as a generalization of the scheme proposed in [9] for BPSK and $n = 2, \ldots, 5$, where the rotations used in [9] were optimized either by exhaustive search or by the gradient method over the BPSK constellation in order to obtain modulation diversity. Here, we use rotations constructed on algebraic number fields from [12]-[15] that guarantee the maximum transmit diversity and coding gains for a large set of n, over all PAM and QAM constellations.

[6] The coding gain is defined as the minimum of $\det\left(\boldsymbol{\Xi}_n\boldsymbol{\Xi}_n^H\right)^{1/n}$, computed over all the differences between distinct codeword pairs $\boldsymbol{x} - \boldsymbol{e}$, (2).

rion given by

$$\min_{\boldsymbol{a}\neq\boldsymbol{b}}\left(\prod_{t=1}^{n}\sum_{j=1}^{n}|\hbar_{jt}y_t|^2\right) = \min_{\boldsymbol{a}\neq\boldsymbol{b}}\left(\prod_{t=1}^{n}n|y_t|^2\right) = n^n d_{n,min}^2. \quad (13)$$

Thus the DAST block codes realize the maximum coding advantage in a fast fading environment. □

The proposed DAST block codes spread each component x_j of the rotated vector $\boldsymbol{x}$ by the Hadamard sequence $\boldsymbol{\mathcal{H}}_{n,j}$ which corresponds to the jth column of the Hadamard transform $\boldsymbol{\mathcal{H}}_n$. The spread component is then sent over n antennae at the time j. The following proposition shows that the DAST block codes are quasi-optimum in the sense of maximizing the coding gains in (11) and (13).

Proposition 3: For a given rotated constellation in dimension n with $d_{n,min} > 0$, the DAST block code $\boldsymbol{\Xi}_n$ has a maximum coding gain over the ST codes formed by the conjunction of the rotated constellation and linear transformations with entries from $\{+1, -1\}$.

Proof. Let $\boldsymbol{S}$ denote $n \times n$ matrix with entries $s_{ij} \in \{+1, -1\}$. Suppose that we use the columns of $\boldsymbol{S}$ to spread the rotated vector $\boldsymbol{x}$ of dimension n. Then the proposed code matrix is given by (7)

$$\boldsymbol{S}\text{diag}(x_1, \ldots, x_n). \quad (14)$$

To maximize the coding gain over a quasi-static (11) or fast fading (13) environment one should maximize the quantity

$$\Delta \triangleq \det\left(\boldsymbol{S}\boldsymbol{S}^T\right). \quad (15)$$

Let $\boldsymbol{A} \triangleq \boldsymbol{S}\boldsymbol{S}^T$, then the diagonal elements of $\boldsymbol{A}$ $a_{ii} = n$ for $i = 1 \ldots n$. Let $\lambda_1 \ldots \lambda_n$ denote the eigenvalues of $\boldsymbol{A}$, then it follows

$$\sum_{j=1}^{n}\lambda_j = \text{trace}(\boldsymbol{A}) = n^2. \quad (16)$$

We want to maximize

$$\Delta = \prod_{j=1}^{n}\lambda_j. \quad (17)$$

It is well known that the solution of this maximization[7] yields

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n. \quad (18)$$

Combining (18) with (16) gives

$$\boldsymbol{S}\boldsymbol{S}^T = n\boldsymbol{I}_n, \quad (19)$$

which is satisfied by the Hadamard transform. On the other hand, it is proved that for $n = 2^q$ there is only n orthogonal sequences with entries from $\{+1, -1\}$ [22], which makes the Hadamard matrix the unique transform which maximizes the coding gain of the DAST block codes. □

Remarks:

1. In general, if we do not restrain the entries of $\boldsymbol{S}$ to belong to $\{+1, -1\}$, i.e., we allow the antennae to have different transmit powers (like in beamforming) when keeping the total power fixed to n, the solution of (19) is given by $\boldsymbol{S} = \sqrt{n}\boldsymbol{U}$, where $\boldsymbol{U}$ is an $n \times n$ unitary matrix. In this context, $\boldsymbol{U} = \boldsymbol{I}_n$ is also helpful to maximize the coding gain over a quasi-static fading channel, and allows for constructing the DAST block codes over any n transmit antennae provided that a rotation matrix with full modulation diversity exists [24]. Nevertheless, the Hadamard transform is useful for reducing the high peak-to-average power ratio over different transmit antennae which makes the power amplifiers operate in their nonlinear region. Under fast fading, the Hadamard transform is useful to satisfy the distance criterion, i.e., making the ST code column vectors different in n positions over distinct codewords pairs; however, maximizing the product criterion in eqn.(13) can be done by any scaled unitary matrix, and in particular the scaled identity matrix. Since the rotated components are transmitted over the diagonal in space and time, affecting them by independent fadings is guaranteed over quasi-static or fast fading. Note that the "pure" quasi-static or fast fading model is not realistic: first, sufficiently spaced antennae is limited by the small space available in mobile handsets, and second, the use of very long interleavers in order to generate independent fading coefficients at each time instant is not practical in systems with delay constraints. Hence, the combination of spatial and temporal variations of the fading coefficients in a realistic fading environment is profitable for the DAST block codes.
2. Note that the DAST block codes keep their transmit diversity advantage with real constellations carved from the cubic lattice like the PAM, and also for the complex constellations like QAM. Because in (11) one has the product of the absolute values of the entries y_j, $j = 1 \ldots n$. Which, for a normalized constellation, has the same value whether $y_1, \ldots y_n$ are real or complex.
3. During n periods of time the received signal is given by an $m \times n$ matrix (1)

$$\boldsymbol{r} = \boldsymbol{H}\left(\boldsymbol{\mathcal{H}}_n\text{diag}(x_1, \ldots, x_n)\right) + \boldsymbol{\nu} \quad (20)$$

where the $m \times n$ complex matrix $\boldsymbol{\nu}$ has independent Gaussian distributed random variables of variance σ^2 per real dimension as entries. Equivalently, one has

$$\begin{aligned}\boldsymbol{r} &= (\boldsymbol{H}\boldsymbol{\mathcal{H}}_n)\,\text{diag}(x_1, \ldots, x_n) + \boldsymbol{\nu} \\ \text{vec}(\boldsymbol{r}^T) &= \begin{bmatrix}\text{diag}(\boldsymbol{H}_1\boldsymbol{\mathcal{H}}_n) \\ \vdots \\ \text{diag}(\boldsymbol{H}_m\boldsymbol{\mathcal{H}}_n)\end{bmatrix}\boldsymbol{x} + \text{vec}(\boldsymbol{\nu}^T) \quad (21)\end{aligned}$$

[7] The solution can be obtained by the use of Lagrange multipliers for example.

where $\boldsymbol{H}_j$ denotes the jth row of $\boldsymbol{H}$ representing the jth receive antenna, and $\text{vec}(\boldsymbol{r})$ arranges the matrix $\boldsymbol{r}$ in a one column vector by putting its columns one after the other. If at each receive antenna $1 \leq j \leq m$ we write $\text{diag}(\boldsymbol{H}_j \boldsymbol{\mathcal{H}}_n) = \text{diag}(\alpha_{j1}, \ldots, \alpha_{jn})$, then the received signal is given by

$$\begin{aligned} \boldsymbol{r}_1 \quad &\triangleq \quad \text{vec}(\boldsymbol{r}^T) = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ 0 & \alpha_{12} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{1n} \\ & & \vdots & \\ \alpha_{m1} & 0 & \cdots & 0 \\ 0 & \alpha_{m2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{mn} \end{bmatrix} \boldsymbol{M}_n \boldsymbol{a} \\ &+ \quad \text{vec}(\boldsymbol{\nu}^T) \triangleq \boldsymbol{\mathcal{A}} \boldsymbol{M}_n \boldsymbol{a} + \boldsymbol{\nu}_1. \end{aligned} \tag{22}$$

Since the Hadamard transform is an orthogonal transformation, the variables α_{ij}, $i = 1 \ldots m$, $j = 1 \ldots n$ are independent identically distributed (i.i.d.) complex Gaussian variables with variance $\frac{n}{2}$ per real dimension. Writing the received signal as in (22) allows one to understand how the DAST block codes exploit the transmit diversity. Using a DAST block code over n transmit antennae and m receive antennae is equivalent to sending the word $(x_1, \ldots, x_n)$ over one transmit antenna and m receive antennae during n periods of time, where the channel changes randomly every time instant (since the fadings between each transmit-receive antenna pair are independent). The latter scheme has a diversity of mn since the lattice from which we transmit the words has a full modulation diversity [13].

C. Decoding

A perfect CSI is assumed available at the receiver. First we perform maximum ratio combining of (22). This yields

$$\begin{aligned} \boldsymbol{r}_2 &= \boldsymbol{\mathcal{A}}^H \boldsymbol{r}_1 \qquad (23) \\ &= \text{diag}\left(\sum_{j=1}^m |\alpha_{j1}|^2, \ldots, \sum_{j=1}^m |\alpha_{jn}|^2 \right) \boldsymbol{M}_n \boldsymbol{a} + \boldsymbol{\nu}_2 \end{aligned}$$

where $\boldsymbol{\nu}_2$ is an colored Gaussian noise with covariance matrix $E[\boldsymbol{\nu}_2 \boldsymbol{\nu}_2^H] = 2\sigma^2 \boldsymbol{\mathcal{A}}^H \boldsymbol{\mathcal{A}}$. In order to whiten the noise, we multiply the received signal in (23) by $(\boldsymbol{\mathcal{A}}^H \boldsymbol{\mathcal{A}})^{-1/2}$, giving

$$\begin{aligned} \boldsymbol{r}_3 &= (\boldsymbol{\mathcal{A}}^H \boldsymbol{\mathcal{A}})^{-1/2} \boldsymbol{r}_2 \qquad (24) \\ &= \text{diag}\left(\sqrt{\sum_{j=1}^m |\alpha_{j1}|^2}, \ldots, \sqrt{\sum_{j=1}^m |\alpha_{jn}|^2} \right) \boldsymbol{M}_n \boldsymbol{a} + \boldsymbol{\nu}_3 \end{aligned}$$

with $\boldsymbol{\nu}_3$ an $n \times 1$ additive white Gaussian noise. Then we apply the sphere decoder [10], [11] on the real and imaginary parts of (24). The sphere decoder takes advantage of the lattice structure of the received signals and proceeds as follows. It searches the closest lattice points to the received signal which are enclosed in a sphere of radius C_0 centered at the received signal. At each time a lattice point of a norm less than C_0 is found, we reduce the sphere radius accordingly and restart the search until an empty sphere is reached. The choice of C_0 depends on the considered lattice, which is generated by $\text{diag}\left(\sqrt{\sum_{j=1}^m |\alpha_{j1}|^2}, \ldots, \sqrt{\sum_{j=1}^m |\alpha_{jn}|^2} \right) \boldsymbol{M}_n$ in (24), as well as on the additive noise level [25]. The complexity of the sphere decoder is polynomial in the lattice dimension n, and independent of the transmission rate [11]. Recent results show that an efficient implementation of the sphere decoder yields low complexity (roughly cubic in n) at moderate and large SNR [25]. We also note that other sub-optimal detection schemes based on successive interference cancellation such as [26], [27] can be useful to decode (24), especially when $m > 1$ [30].

Note that the use of rotations $\boldsymbol{M}_n$ with real entries [13], [14] in the proposed codes design simplifies the decoding process. In (24), the received signal is written as a combination of the information symbols $\boldsymbol{a}$ by the real-valued combining matrix $\text{diag}\left(\sqrt{\sum_{j=1}^m |\alpha_{j1}|^2}, \ldots, \sqrt{\sum_{j=1}^m |\alpha_{jn}|^2} \right) \boldsymbol{M}_n$, and disturbed by additive white Gaussian noise. Hence, when the information symbols are complex, QAM for example, one can decode separately the real and imaginary parts of (24) using the sphere decoder or other sub-optimal schemes. However, if we choose $\boldsymbol{M}_n$ with complex entries [15, VI-C] then one can not separate the real and imaginary parts of (24) because the combining matrix is complex. In this case, one should represent the complex received signal in real dimensions in $\mathbb{R}^{2n}$ where $\mathbb{R}$ is the field of real numbers, which considerably increases the complexity of the used decoder when n increases [11]. Complex rotations were proposed in [15] to construct constellations matched for both the Gaussian and the Rayleigh fading channels. In a multi-antenna system where the channel is assumed Rayleigh or Rice, the use of complex rotations could enhance the coding gain since one doubles the degrees of freedom when using complex rotations compared to real rotations. For example, the complex rotations constructed over the cyclotomic number fields in [15, VI-C] give the following minimum product distances: $d_{2,min} = 1/2$ and $d_{4,min} = 1/16$ which yield a common coding gain of 1 for $n = 2$ and $n = 4$ transmit antennae when used in the DAST block codes. Comparing with (10), one expects a gain of approximately 0.5 dB for $n = 2$ and a gain about 2 dB for $n = 4$ when using complex instead of real rotations in the DAST block codes; however, in simulations we noticed similar performances for $n = 2$ and a less than 1 dB gain for

$n = 4$ at large SNR. The explanation can be given by the involvement of other parameters in the performances of the DAST block codes, such as the *kissing number* [?], which is under investigation. We noticed in simulations that the obtained gain in performances when using complex rotations is not worth the increase in complexity[8].

V. Simulation Results

In simulations, we use normalized QAM constellations with average energy per symbol $\bar{E}_s = 1$. The transfer matrix $\boldsymbol{H}$ is modeled as in subsection II-A, and the additive white Gaussian noise has a variance $\sigma^2 = n/(2\text{SNR})$ per real dimension. Error probability curves are plotted as a function of SNR in dB. We simulate the DAST block codes for the dimensions $n = 2, 3, 4, 8, 16, 32$, for different values of m and different size of constellations. Comparisons with the ST block codes from orthogonal design [4] (denoted by $\boldsymbol{\mathcal{G}}_n$) are done for $n = 2, 4$, at the same spectral efficiency. Fig. 1 shows the performances of the codes Ξ_n, for $n = 2, 3, 4, 8, 16, 32$ transmit antennae and $m = 1$ receive antenna. The modulation used is the 4-QAM with a spectral efficiency of 2 bits/sec/HZ. We also plotted the performances of the uncoded 4-QAM over a Rayleigh fading channel and over a Gaussian channel [28]. At a symbol error rate (SER) of 10^{-5}, Ξ_4 has a gain of about 7.5 dB over Ξ_2. The gain increases with n to attain 16 dB for $n = 32$. Comparing the performances of the code Ξ_{32} and the uncoded 4-QAM over the Gaussian channel shows only about 1 dB of difference. The latter comparison confirms the results of Foschini and Gans [8] concerning the capacity of a multi-antenna system with n transmit antennae and $m = 1$ receive antenna over a quasi-static fading channel

$$C = \log_2\left(1 + \frac{\rho}{n}\chi^2_{2n}\right), \tag{25}$$

where χ^2_{2n} is a chi-squared random variable that has $2n$ degrees of freedom, formed by summing the squares of $2n$ independent Gaussian normalized and centered random variables, and ρ is the signal to noise ratio. When n is large, the capacity in (25) tends in distribution by the strong law of large numbers to the capacity of the Gaussian channel [29]. Fig. 2 presents the performances of the codes Ξ_n, for $n = 2, 3, 4, 8, 16, 32$ transmit antennae and $m = 2$ receive antennae with 4-QAM. It is noticed that the gain obtained by increasing n is less important than the case with one receive antenna. For example at a SER of 10^{-5}, Ξ_4 has slightly more than 3.5 dB of gain over Ξ_2, whereas Ξ_{32} shows a gain around 7.5 dB over Ξ_2. This is because much of the diversity gain is already achieved using two transmit antennae and two receive antennae [5]. In Fig. 3 we compare the Alamouti code $\boldsymbol{\mathcal{G}}_2$ [1] with the code Ξ_2 for one and two receive antennae with the 4-QAM modulation. At the same spectral efficiency of 2 bits/sec/HZ, the Alamouti scheme shows almost 1 dB of gain over the code Ξ_2. For $n = 2$ transmit antennae it seems difficult to outperform the Alamouti scheme since it is the unique complex orthogonal design transmitting at a normalized rate of 1 symbol/sec [4]. Nevertheless, when n increases, the DAST block codes give better performances. Fig. 4 shows the performances of Ξ_4 of normalized rate 1 symbol/sec and $\boldsymbol{\mathcal{G}}_4$ of normalized rate 1/2 symbol/sec, with one receive antenna and different spectral efficiencies. The two codes are compared at the same spectral efficiency. For example, at a spectral efficiency of 2 bits/sec/HZ, the code Ξ_4 uses the normalized 4-QAM modulation, and the code $\boldsymbol{\mathcal{G}}_4$ uses the normalized 16-QAM modulation. The code Ξ_4 has a gain of almost 1 dB over $\boldsymbol{\mathcal{G}}_4$ at 2 bits/sec/HZ. This gain is enhanced when increasing the size of the constellation. For example it reaches almost 5 dB at 4 bits/sec/HZ, and almost 16 dB at 8 bits/sec/HZ. The latter property is explained by the following: for a normalized constellation, the ST codes Ξ_4 and $\boldsymbol{\mathcal{G}}_4$ keep their diversity advantage of $4m$ and their coding gains of $\sqrt{2/5}$ and 2 respectively. So the ratio of their coding gain given in dB equals[9] -5 dB. On the other hand, when increasing the size of the constellation, one looses approximately 3 dB per each added bit[10]. Since at the same SNR Ξ_4 transmits two times the number of bits transmitted by $\boldsymbol{\mathcal{G}}_4$, the difference in performance is exacerbated when the size of the constellation increases. Fig. 5 compares the performances of Ξ_4 and $\boldsymbol{\mathcal{G}}_4$ for $m = 2$ receive antennae and different spectral efficiencies. We notice that the gain of the DAST block code is even enhanced when m increases. It attains more than 2 dB at 2 bits/sec/HZ, and almost 19 dB at 8 bits/sec/HZ. Recent results by Hassibi and Hochwald in [31], and Sandhu and Paulraj in [32], offer more insight about the performances of the linear ST block codes, and especially those from orthogonal design when the number of receive antennae increases. In summary, the ST codes from orthogonal design do not exploit all the degrees of freedom offered by the multi-antenna system due to the restrictions of the orthogonal structure. On the other hand, the DAST block codes have higher data rates (for $n > 2$) than the ST codes from orthogonal design; thus, it can be proved that the information loss (compared to the multi-antenna capacity) is smaller for the DAST block codes, which implies a quantified gain proportional to the information loss incurred; the latter gain increases when the number of receive antennae m or the spectral efficiency increases (see [31] and [32] for more details). Fig. 6 compares the performances of Ξ_4 and $\boldsymbol{\mathcal{G}}_4$ for $m = 4$ receive antennae. It confirms the remark made above on Fig. 5. For example, Ξ_4 outperforms $\boldsymbol{\mathcal{G}}_4$ by almost 4 dB at 2 bits/sec/HZ. At the spectral efficiency of 8 bits/sec/HZ, Ξ_4 is more than 20 dB better than $\boldsymbol{\mathcal{G}}_4$.

[8] Note that in our scheme, the use of complex rotations doubles the complexity of the encoder, but multiplies the complexity of the decoder by a factor ≥ 8, since if the complexity of the decoder is $O(n^3)$ with real rotations, it becomes $O(2^3 n^3)$ with complex rotations.

[9] The simulation results slightly differ from the coding gain expression since the latter is only an approximation for large SNR.

[10] This is also an approximation resulted directly from the capacity formula [7].

TABLE I

First row of the optimal real rotation matrices in dimensions 2 and 4.

TABLE II

Full modulation diversity rotations in dimension $d = 2^q$, constructed on the number fields $\mathbb{Q}\left(\cos \frac{2\pi}{8d}\right)$.

VI. Conclusions

A new class of bandwidth efficient linear space-time block codes has been constructed and studied in this paper. The so-called diagonal algebraic space-time (DAST) block codes are constructed by the conjunction of the rotated constellations having full modulation diversity and good minimum product distance values from one part, and the Hadamard transform from another part. These codes satisfy the construction criteria of space-time codes design for the PAM and QAM constellations of any size, under quasi-static and fast fading. The ML decoding can be implemented by the sphere decoder at a moderate complexity. At the same spectral efficiency, the DAST block codes improve over the space-time codes from orthogonal design for $n > 2$ and for complex constellations. This improvement becomes larger when the size of the constellation or the number of transmit and receive antennae increases. The DAST block codes are proved to be quasi-optimal in the sense of maximizing the coding gain over both fast and quasi-static fading environment in the absence of multipath.

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Fig. 1. Average SER as a function of SNR, one receive antenna.

Fig. 2. Average SER as a function of SNR, two receive antennae.

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Fig. 3. Average SER as a function of SNR, two transmit, one and two receive antennae.

Fig. 4. Average BER as a function of SNR, four transmit and one receive antennae.

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Fig. 5. Average BER as a function of SNR, four transmit and two receive antennae.

Fig. 6. Average BER as a function of SNR, four transmit and four receive antennae.

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TABLE I

dimension	column					$d_{d,min}$
2	1-2	0.5257	0.8507			$\frac{1}{\sqrt{5}}$
4	1-4	0.2012	0.3255	-0.4857	-0.7859	$\frac{1}{40}$

TABLE II

$$M = \mathrm{sqrt}(2/d) * \cos\left(pi/(4*d)*(4*[1:d]'-1)*(2*[1:d]-1)\right);$$

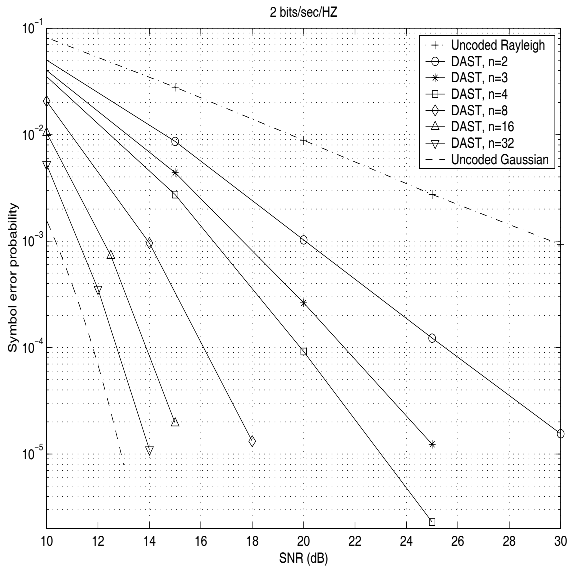
2 bits/sec/HZ
Uncoded Rayleigh
DAST, n=2
DAST, n=3
DAST, n=4
DAST, n=8
DAST, n=16
DAST, n=32
Uncoded Gaussian
Symbol error probability
SNR (dB)

Fig. 1.

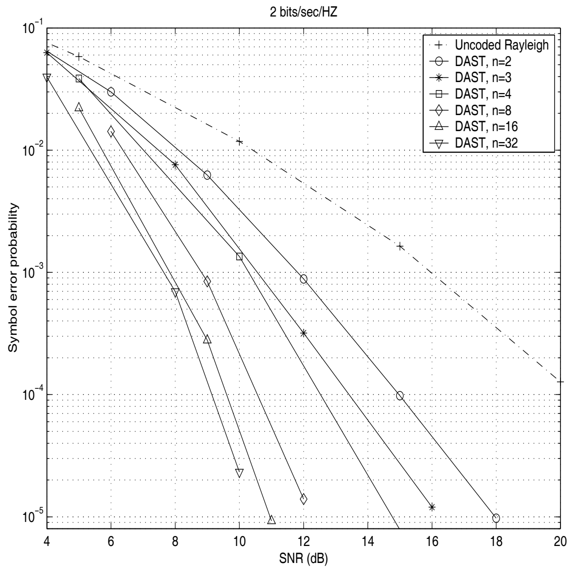
2 bits/sec/HZ
Uncoded Rayleigh
DAST, n=2
DAST, n=3
DAST, n=4
DAST, n=8
DAST, n=16
DAST, n=32
Symbol error probability
SNR (dB)

Fig. 2.

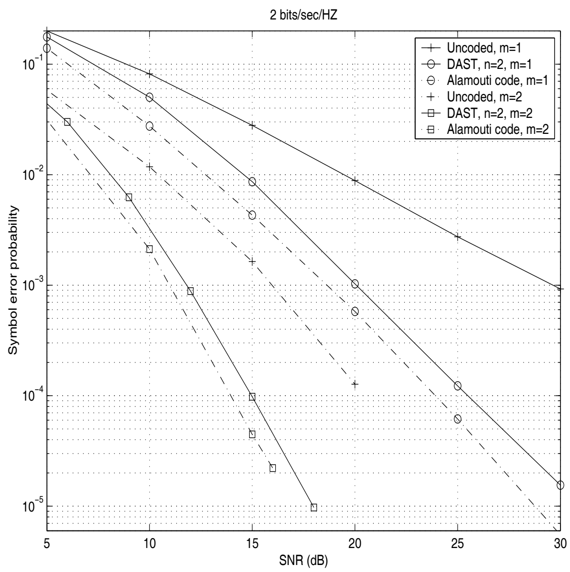
2 bits/sec/HZ
Uncoded, m=1
DAST, n=2, m=1
Alamouti code, m=1
Uncoded, m=2
DAST, n=2, m=2
Alamouti code, m=2
Symbol error probability
SNR (dB)
10^-1
10^-2
10^-3
10^-4
10^-5
5
10
15
20
25
30

Fig. 3.

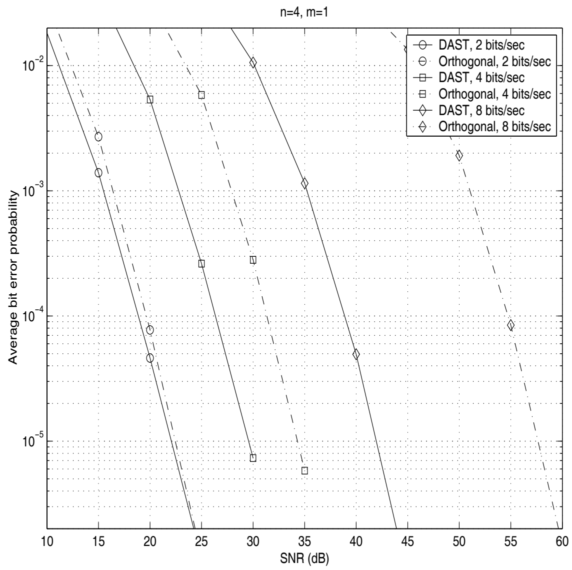
n=4, m=1
DAST, 2 bits/sec
Orthogonal, 2 bits/sec
DAST, 4 bits/sec
Orthogonal, 4 bits/sec
DAST, 8 bits/sec
Orthogonal, 8 bits/sec
Average bit error probability
SNR (dB)
10
15
20
25
30
35
40
45
50
55
60

Fig. 4.

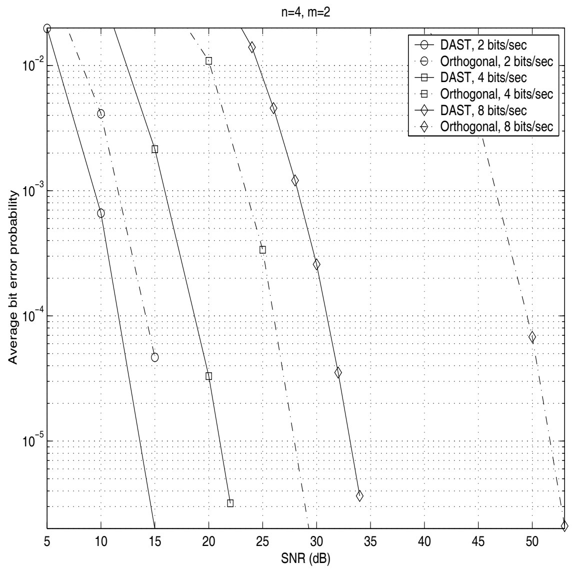
n=4, m=2
DAST, 2 bits/sec
Orthogonal, 2 bits/sec
DAST, 4 bits/sec
Orthogonal, 4 bits/sec
DAST, 8 bits/sec
Orthogonal, 8 bits/sec
Average bit error probability
SNR (dB)

Fig. 5.

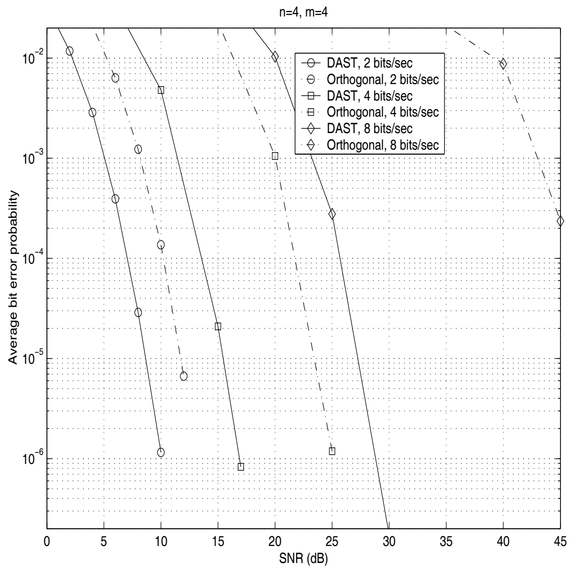
n=4, m=4
DAST, 2 bits/sec
Orthogonal, 2 bits/sec
DAST, 4 bits/sec
Orthogonal, 4 bits/sec
DAST, 8 bits/sec
Orthogonal, 8 bits/sec
Average bit error probability
SNR (dB)
10^-2
10^-3
10^-4
10^-5
10^-6
0
5
10
15
20
25
30
35
40
45

Fig. 6.