International Mathematical Forum, 5, 2010, no. 24, 1179 - 1183

Comaximal Submodules of Multiplication Modules

Saeed Rajaee

Department of Mathematics, Faculty of Science
Payame Noor University of Mashhad, Mashhad, Iran
saeed_rajaee@pnu.ac.ir

Abstract

Let M be a multiplication module over a commutative ring R. In this paper we investigate some results on prime comaximal submodules of a prime multiplication module M.

Mathematics Subject Classification: 13C05, 13C13, 13A15

Keywords: Multiplication module, Prime submodule, Prime module, Comaximal submodules

1 Introduction

Throughout this work, R denotes a commutative ring with identity and M denotes a unital R-module. For a submodule N of M, the set

$$(N :_R M) = \{r \in R : rM \subseteq N\}$$

is called colon of N and it is an ideal of R.
Let I be an ideal of R, the submodule $(N :_M I)$ of M is defined by

$$(N :_M I) = \{m \in M \mid im \in N, \forall i \in I\}$$

Similarly, for an element $s \in R$, the submodule $(N :_M s)$ is defined by:

$$(N :_M s) = \{m \in M \mid sm \in N\}$$

we investigate some properties of multiplication modules. It is clear that every cyclic module is multiplication, and that a multiplication module over a local ring is cyclic, see [2], [8], [11].
In this paper, we obtain some results on the prime comaximal submodules of a prime multiplication module M, which are concern to comaximal ideals.

2 Multiplication modules

Definition 2.1. An R-module M is called a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$. In this case we can take $I = (N :_R M)$ and I is called a presentation ideal of N.

Definition 2.2. A submodule N of R-module M is called prime submodule if $N \neq M$ and for $r \in R$ and $m \in M$, we have

$$rm \in N \Rightarrow r \in (N :_R M) \ or \ \ m \in N$$

Equivalently, if $rm \in N$ for some $r \in R$ and $m \in M \setminus N$, then $rM \subseteq N$.
In the special case in which $N = 0$, the ideal $(0 :_R M) = Ann_R(M)$ of R is called the annihilator of M.

Lemma 2.3. *Let N be a prime submodule of M, then*

$$p = (N : M) \in spec(R)$$

proof: Let $ab \in (N : M)$, and $a \notin (N : M)$, then $abM \subseteq N$, hence for any $m \in M$,

$$(ab)m = (ba)m = b(am) \in N \qquad (*)$$

Since $a \notin (N : M)$, hence $aM \nsubseteq N$, then there exists $s \in M$, where $as \notin N$. Since N is prime by $(*)$ we have

$$(ab)s = b(as) \in N \Rightarrow b \in (N : M) \ \ or \ \ as \in N$$

Since $as \notin N$, hence $b \in (N : M)$.

Definition 2.4. *Let N be a prime submodule of M, then $p = (N : M)$ is a prime ideal of R and N is said to be p- prime submodule.*

Theorem 2.5. [[10] Theorems 1.3, 1.5]*Let P be a proper submodule of a multiplication R-module M. Then the following statements are equivalent:*
(i) P is a prime submodule;
(ii) for every submodules $N, K \subseteq M$, we have

$$NK \subseteq P \Longrightarrow N \subseteq P \ or \ K \subseteq P$$

(iii) for every $m, n \in M$, if $mn \subseteq P$ then $m \in P$ or $n \in P$.
(iv) $Ann_R(M/P)$ is a prime ideal of M.
(v) $P = QM$ for some prime ideal Q of R with $Ann_R(M) \subseteq Q$.

Definition 2.6. *Let N be a submodule of R-module M and I an ideal of R, then*

$$(N :_M I) = \{m \in M \,|\, Im \subseteq N\} = \{m \in M \,|\, im \in N, \forall i \in I\}$$

is a submodule of M.
In the special case in which $N = 0$, the submodule $(0 :_M I) = Ann_M(I)$ of M is called the annihilator of I in M. Similarly, for an element $s \in R$, the submodule $(N :_M s)$ is equal to:

$$(N :_M s) = \{m \in M \,|\, sm \in N\}$$

In particular, if $I = (i_1, \dots, i_k)$ be a finitely generated ideal of R, then

$$(N :_M I) = \cap_{s=1}^{k}(N :_M i_s)$$

Theorem 2.7. *Let M_1 and M_2 be R-modules, then $M = M_1 \oplus M_2$ is a multiplication R-module if and only if M_1 and M_2 be multiplication modules.*
proof. Let $M = M_1 \oplus M_2$ be a multiplication module and N_1 be a submodule of M_1 and thereby a submodule of M. Therefore there exists an ideal I of R, where $N_1 = IM = I(M_1 \oplus M_2) = IM_1 \oplus IM_2$.
Since $IM_1 \cap IM_2 = 0$, hence $N_1 = IM_1$, which implies that M_1 is multiplication. Similarly, M_2 is a multiplication R-module.
Conversely, let $N \neq M$ be a submodule of M, then there exist submodules N_1 and N_2 respectively of M_1 and M_2 such that $N = N_1 + N_2$ and $N_1 \cap N_2 = 0$. There exist two ideals I and J of R, where $N_1 = IM_1$ and $N_2 = JM_2$. Therefore $N = N_1 + N_2 = IM + JM = (I + J)M$, where $I + J$ is an ideal of R, which implies that M is a multiplication module. In this case $I = (N_1 :_R M_1)$ and $J = (N_2 :_R M_2)$, then $r \in I \cap J$ if and only if $rM_1 \subseteq N_1$ and $rM_2 \subseteq N_2$ if and only if $rM = r(M_1 + M_2) = rM_1 + rM_2 \subseteq N_1 + N_2$ if and only if $r \in (N_1 + N_2 :_R M)$. Therefore $(N_1 :_R M_1) \cap (N_2 :_R M_2) = (N_1 + N_2 :_R M)$.

Corollary 2.8. *Let $\{M_i\}_{i \in \Lambda}$ be a finite collection of R-modules, then the direct sum $M = \bigoplus_{i \in \Lambda} M_i$ is a multiplication module if and only if each of M_i is multiplication module.*

Corollary 2.9. *Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a split exact sequence of R-modules, then M is a multiplication module if and only if M_1 and M_2 are multiplication modules.*

3 Comaximal submodules

Definition 3.1. *Let M be an R-module and M_1 and M_2 are submodules of M, then M_1 and M_2 are called comaximal, whenever $M = M_1 \oplus M_2$.*

Definition 3.2. *An R-module M is called a prime module if zero submodule of M is prime submodule of M. It is clear that M is a prime module if and only if $Ann_R(M) = Ann_R(m)$, for every nonzero $m \in M$.*

Definition 3.3. *An R-module M is a weak cancellation module, whenever $IM = JM$ for some ideals I and J of R, then $I + Ann_R(M) = J + Ann_R(M)$. In particular, if $Ann_R(M) = 0$, then M is called a cancellation module.*

Proposition 3.4. [[10] Proposition 3.5] *Every prime multiplication module is weak cancellation module and hence is finitely generated.*

Theorem 3.5. *Let M be a prime multiplication R-module, N_1 and N_2 be comaximal prime submodules of M, where $N_1 = IM$ and $N_2 = JM$ for some ideals I and J of R, then I and J are comaximal ideals of R.*
proof. Since $M = N_1 \oplus N_2$, hence $N_1 \cap N_2 = 0$ and $M = N_1 + N_2$.
We have $I = (N_1 : M)$ and $J = (N_2 : M)$, and hence
$I \cap J = (N_1 : M) \cap (N_2 : M) = (N_1 \cap N_2 : M) = (0 : M) = Ann_R(M)$.
Whereas M is a prime module, then $Ann_R(M) = 0$, hence $I \cap J = 0$. On the other side, $M = RM = N_1 + N_2 = IM + JM = (I + J)M$. Every prime multiplication module is weak cancellation module and thereby $R + Ann_R(M) = (I + J) + Ann_R(M)$ and since $Ann_R(M) = 0$, hence $R = I + J$. Therefore $R = I \oplus J$.

Definition 3.6. *A submodule Q of an R-module M is called primary submodule, if $Q \neq M$ and for $r \in R$ and $m \in M$, we have*

$$rm \in Q \Rightarrow r \in \sqrt{(Q : M)} \ \text{ or } \ m \in Q$$

Lemma 3.7. *Let Q be a primary submodule of an R-module M, then ideal $(Q : M)$ is a primary ideal of R.*
proof: Let $ab \in (Q : M)$, and $a \notin (Q : M)$, we must show that there exists a positive integer number n, where $b^n \notin (Q : M)$. We have $abM \subseteq Q$ and $aM \nsubseteq Q$, hence for at least one $s \in M$, $as \notin Q$.

$$(ab)s = (ba)s = b(as) \in (ab)M \subseteq Q \overset{as \notin Q}{\longrightarrow} b \in \sqrt{Q : M}$$

Therefore there exists $n \in \mathbb{N}$, where $b^n \notin (Q : M)$, hence $(Q : M)$ is a primary ideal of R.

If Q be a primary submodule of an R-module M, then $(Q : M)$ is a primary ideal of R, and we say that Q is a q- primary submodule, where $q = \sqrt{(Q : M)}$ is a prime ideal of R.
If Q be a q-primary submodule of M and $q = (Q : M)$, then Q is a prime submodule of M (see [4] Proposition 1).

Corollary 3.8. *Let Q_1 and Q_2 be comaximal primary submodules of M, where $q_1 = (Q_1 : M)$ and $q_2 = (Q_2 : M)$, then $R = q_1 \oplus q_2$.*

References

[1] Atiyah, M.F. and I.G. MacDonald, *Introduction to commutative Algebra,* New York, Addison-Wesley Publishing Company, 1969.

[2] A. Barnard,, *Multiplication modules,* J. Algebra **71** (1981), no.1, 174-178.

[3] Bourbaki, Nicolas, *Commutative Algebra,* Paris: Hermann, Publishers in Arts and Science, 1972.

[4] C.-P.Lu, *Prime submodules of modules*, Comm. Math. Univ. Sancti Pauli, **33** (1984) pp. 61-69.

[5] D.Eisenbud, Commutative algebra with a view toward algebraic geometry, Springer-Verlag, New York, 1995.

[6] J. Jenkins and P. F. Smith., *On the prime radical of a module over a commutative ring,* Comm. Algebra, **20** (1992), 3593-3602.

[7] Matsumura, Hideyuki., *Commutative ring theorey,* Cambridge: Cambridge University Press, 1980.

[8] P.F. Smith., *Some remarks on multiplication modules,* Arch. Math., **50** (1988), 223-235.

[9] R. Ameri, *On the prime submodules of multiplication modules,* International Journal of Mathematics and Mathematical Science (2003), no.27, 1715-1724.

[10] R.Jahani-Nezhad and M.H. Naderi., *On Prime and Semiprime Submodules of Multiplication Modules,* International Mathematical Forum, **4** (2009), no.26, 1257-1266.

[11] Z. A. El-Bast and P.F. Smith., *Multiplication modules,* Comm. Algebra, **16** (1988), no.4, 755-779.

Received: January, 2010