On L_p-affine surface areas

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Abstract

Let K be a convex body in $\mathbb{R}^n$ with centroid at 0 and B be the Euclidean unit ball in $\mathbb{R}^n$ centered at 0. We show that

$$\lim_{t \to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{O_p(K)}{O_p(B)},$$

where $|K|$ respectively $|B|$ denotes the volume of K respectively B, $O_p(K)$ respectively $O_p(B)$ is the p-affine surface area of K respectively B and $\{K_t\}_{t \geq 0}$, $\{B_t\}_{t \geq 0}$ are general families of convex bodies constructed from K, B satisfying certain conditions. As a corollary we get results obtained in [23, 25, 26, 31].

1 Introduction and notation

During the past two decades affine surface area, originally a basic invariant from the field of affine differential geometry, has become an important tool in convex geometry. For this to happen affine surface area had first to be extended so that it was defined on all convex bodies (see e.g. [8, 14, 22, 24, 29]). Work on the "extension problem" for affine surface area lead to the solution of the "uppersemicontinuity problem" for affine surface area [14] which has recently had an impact in the study of affine PDE's (see e.g. Trudinger and Wang [27] and Wang [28]). The flurry of activity surrounding affine surface area ultimately led to the Ludwig-Reitzner Characterization theorem [9] and various deep results in the area of e.g. combinatorics (see e.g. [1]), polytopal approximation (see e.g. [5, 13, 25]) and the theory of valuations (see e.g. [10, 11]).

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During the past decade it has come to be seen that the classical Brunn-Minkowski theory of convex bodies is a part of a more general L_p-Brunn-Minkwoski theory (see e.g. [2, 19, 20, 21]) and recent advances in this area have influenced the work in PDE's and ODE's of Chen, Chou, Hu, Ma, Shen and Wang [3, 4, 6] (among others).

Within the L_p-Brunn-Minkwoski theory, L_p-extensions have been found of the basic affine invariants of classical convex geometry. One of the most important affine invariants for which L_p-extensions have been discovered is affine surface area. For $1 < p < \infty$, L_p-affine surface area was defined by Lutwak [14, 16] for all convex bodies K in $\mathbb{R}^n$ and extended by Hug [7] to $0 \leq p < 1$:

$$O_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{< x, N_K(x) >^{\frac{n(p-1)}{n+p}}} d\mu(x).$$

Here $\kappa_K(x)$ is the (generalized) Gaussian curvature and $N_K(x)$ the outer unit normal in $x \in \partial K$, the boundary of K, and μ is the surface measure on ∂K.

The next step was taken by Meyer & Werner [22] who extended the definition of L_p-affine surface area to all $p \in (-n, \infty)$ and also gave a geometric definition of L_p-affine surface area in terms of L_p-*Santaló bodies* for $p \in (-n, \infty)$. Schütt & Werner went on to show how L_p-affine surface area, for all $p \in [-\infty, \infty]$ (except $p = -n$) has a natural definition in terms of *random polytopes* [25], and gave another definition in terms of the *surface bodies* [26]. Werner [31] provided yet another definition, for all p, in terms of L_p-*floatation bodies*.

It is easy to see that $O_p(K)$ is finite for all p with $0 \leq p \leq \infty$ (see [26]). This need not to be so for negative values of p [26]. Moreover, $O_0(K) = n \ \mathrm{vol}_n(K)$ and (for K with ∂K C^2 and a.e. strictly positive Gaussian curvature) $O_{\pm\infty} = n \ \mathrm{vol}_n(K^0)$, where K^0 is the polar body of K (see below). Note also that for all $p \neq -n$, $O_p(B) = \mathrm{vol}_{n-1}(\partial B)$.

As remarked above, in the L_p-extension process new classes of convex bodies were discovered through which L_p-affine surface area can be characterized geometrically. This new geometric characterizations have a common feature:
First a specific family $\{K_t\}_{t\geq 0}$ of convex bodies is constructed. This family is different in each of the extensions but of course related to the given convex body K. The L_p-affine surface area is then obtained by using expressions involving volume differences $|K| - |K_t|$. Examples for such families K_t are the L_p-Santaló bodies [23], the L_p-floatation bodies or weighted floating bodies [31] and the surface bodies [26] (see below for the definitions).

Therefore it seemed natural to ask whether there are completely general conditions on a family $\{K_t\}_{t\geq 0}$ of convex bodies in $\mathbb{R}^n$ that (in connection with volume difference expressions) will give L_p-affine surface area. A positive answer to this question was given in [30] in the case $p = 1$. We show here that this also holds in the general case.

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$B(a,r) = B_2^n(a,r)$ is the n-dimensional Euclidean ball with radius r centered at a. We put $B = B(0,1) = B_2^n(0,1)$ and $r\ B = B(0,r)$. By $||.||$ we denote the standard Euclidean norm on $\mathbb{R}^n$, by $<\cdot,\cdot>$ the standard inner product on $\mathbb{R}^n$. For two points x and y in $\mathbb{R}^n$ $[x,y] = \{\alpha x + (1-\alpha)y : 0 \le \alpha \le 1\}$ denotes the line segment from x to y.

For a set $A \subset \mathbb{R}^n$, $\mathrm{int}(A)$ is the interior of A. For a convex body $K \in \mathbb{R}^n$, ∂K denotes the boundary of K. μ is the usual surface area measure on ∂K. $\partial B = S^{n-1}$. For $x \in \partial K$, $N_K(x)$ is the outer unit normal vector to ∂K in x. We denote the n-dimensional volume of K by $|K|$. H^+ and H^- are the closed halfspaces determined by the hyperplane H.

We will assume from now on that a convex body K in $\mathbb{R}^n$ is positioned such that the centroid of K is at the origin. We choose the centroid to be at 0 but we could have chosen any interior point of K instead. We will then denote by $\mathcal{K}$ the set of convex bodies in $\mathbb{R}^n$ with centroid at the origin.

Let K be a convex body in $\mathbb{R}^n$. For $x \in \partial K$ let $r(x)$ be the radius of the biggest Euclidean ball contained in K that touches ∂K at x. It was shown in [24] that μ-a.e. on ∂K $r(x) > 0$ and that for $0 \le \alpha < 1$

$$\int_{\partial K} r(x)^{-\alpha} d\mu(x) < \infty \tag{1}$$

It was also noted in [24] that in general α cannot be chosen to be equal to 1.

Therefore, μ-a.e. on ∂K there exists a centered ellipsoid $E_c(x)$ such that

$$E_c(x) \cap \partial K = \{x\}, \qquad N_K(x) = N_{E_c(x)}(x)$$

$$|E_c(x)| = |<x, N_K(x)>|^{\frac{n+1}{2}} \left(\frac{r(x)}{2}\right)^{\frac{n-1}{2}} |B|.$$

Moreover there exists a hyperplane $H_c(x)$ such that

$$x \in \mathrm{int}\bigg(H_c(x)^-\bigg) \quad \text{and} \quad E_c(x) \cap H_c^-(x) \subseteq K \cap H_c^-(x). \tag{2}$$

We call $E_c(x)$ a *contained ellipsoid*. It should be noted that $E_c(x)$ is not unique. Likewise, the hyperplane $H_c(x)$ such that (2) holds, is not unique: if f.i. $H_c(x)$ is such that (2) holds, then for any hyperplane $H \subset \left(H_c(x)^-\right)$ parallel to $H_c(x)$ and with $x \in \text{int } H^-$, (2) holds as well.

Let $K \in \mathbb{R}^n$ and $x \in \partial K$ with unique outer unit normal vector $N_K(x)$ and strictly positive Gauss curvature $\kappa_K(x)$. Then (see f.i. [8, 25]), for $\varepsilon > 0$ given, there exist centered ellipsoids $E_a^i(x)$ and $E_a^c(x)$ such that

$$x \in \partial E_a^i(x), \quad x \in \partial E_a^c(x), \tag{3}$$

$$N_K(x) = N_{E_a^i(x)}(x) = N_{E_a^c(x)}(x), \tag{4}$$

$$\kappa_{E_a^i(x)}(x) = \frac{\kappa_K(x)}{(1-\varepsilon)^{n-1}}, \ \kappa_{E_a^c(x)}(x) = \frac{\kappa_K(x)}{(1+\varepsilon)^{n-1}} \tag{5}$$

and such that $E_a^i(x)$ and $E_a^c(x)$ have common axes: of length $< x, N(x) >$ once and of length $\left((1-\varepsilon)\frac{<x,N(x)>}{\kappa(x)^{\frac{1}{n-1}}}\right)^{\frac{1}{2}}$ respectively $\left((1+\varepsilon)\frac{<x,N(x)>}{\kappa(x)^{\frac{1}{n-1}}}\right)^{\frac{1}{2}}$ $(n-1)$-times. Thus

$$|E_a^i(x)| = (1-\varepsilon)^{\frac{n-1}{2}} |< x, N(x) >|^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}} |B| \tag{6}$$

$$|E_a^c(x)| = (1+\varepsilon)^{\frac{n-1}{2}} |< x, N(x) >|^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}} |B|. \tag{7}$$

Moreover there exists a hyperplane $H_a(x, \varepsilon)$ such that $x \in \text{int}\left(H_a(x,\varepsilon)^-\right)$ and such that

$$E_a^i(x) \cap H_a^-(x,\varepsilon) \subseteq K \cap H_a^-(x,\varepsilon) \subseteq E_a^c(x) \cap H_a^-(x,\varepsilon). \tag{8}$$

We call $E_a^i(x)$ and $E_a^c(x)$ the *approximating ellipsoids*. Again, the hyperplane $H_a(x,\varepsilon)$ is not unique. Similar comments as above for $H_c(x)$ apply.

If $x \in \partial K$ is such that $\kappa_K(x) = 0$, then it is well known that the indicatrix of Dupin at x is an elliptic cylinder (see [8, 24]). Hence there is a centered ellipsoid $E_0(x)$ that touches ∂K in x and which has at least one axis that is arbitrarily large: for all $\varepsilon > 0$ there exists a centered ellipsoid $E_0(x)$ with at least one axis $a = a(\varepsilon)$ with $a > \frac{1}{\varepsilon}$ (see [24]). Moreover there exists a hyperplane $H_0(x,\varepsilon))$ (not unique) such that

$$E_0(x) \cap H_0^-(x,\varepsilon) \subset K \cap H_0^-(x,\varepsilon). \tag{9}$$

We want to emphazise that above as well as in the following definition the hyperplanes have a mostly auxiliary function: the only information we need about K is the behavior near the boundary and the role of the hyperplanes is to express just that.

2 Definiton and Examples

Definition

Let $p \in \mathbb{R}$, $p \neq -n$ be fixed. For every $t \in \mathbb{R}$, $t \geq 0$, let $\mathcal{F}_t^p : \mathcal{K} \to \mathcal{K}, \quad K \longmapsto \mathcal{F}_t^p(K) = K_t^p$ be a map. We write in short $K_t = K_t^p$. We say that

(i) $\mathcal{F}_t^p$ is *p-limiting* if for all invertible linear maps T,

$$lim_{t\to 0}\frac{|(T(B)| - |(T(B))_t|}{|B| - |B_t|} \ = \ |\mathrm{det}T|^{\frac{(n-p)}{n+p}}$$

(ii) $\mathcal{F}_t^p$ is *C-inclusion preserving*, if there exists a constant $C = C(K) \geq 1$ such that μ-a.e. on ∂K we have: There exists t_1 such that, whenever $E_c(x)$ is a contained ellipsoid such that (2) holds for some hyperplane $H_c(x)$, then for all $t \leq t_1$

$$\Bigg(E_c(x)\Bigg)_{Ct} \cap H_c^-(x) \subseteq (K)_t \cap H_c^-(x).$$

(iii) $\mathcal{F}_t^p$ is *local* if it is 1-inclusion preserving for the approximating ellipsoids: Let $\varepsilon > 0$ be given and let $x \in \partial K$ with approximating ellipsoids $E_a^i(x)$ and $E_a^c(x)$ such that (8) holds for some $H_a(x,\varepsilon)$. Then there exits $t_2 = t_2(\varepsilon, x)$ such that for all $t \leq t_2$

$$\Bigg(E_a^i(x)\Bigg)_t \cap H_a^-(x,\varepsilon) \subseteq K_t \cap H_a^-(x,\varepsilon) \subseteq \Bigg(E_a^c(x)\Bigg)_t \cap H_a^-(x,\varepsilon).$$

(iv) $\mathcal{F}_t^p$ is *monotone* if for every centered ellipsoid E, E_t is again a centered ellipsoid homothetic to E and if the radius $r(t) = 1 - f(t)$ of the centered ball B_t is such that $f(0) = 0$, f is increasing and there is a constant d (independent of t) such that

$$f(Ct) \leq d \ f(t),$$

where C is the constant in (ii).

We would like to make some comments on the definition of the map $\mathcal{F}_t^p$.

Firstly, in the definition we restrict ourselves to considering the case $B_t \subseteq B$, $K_t \subseteq K$ for all $t \geq 0$. A similar construction can be given for $B \subseteq B_t$, $K \subseteq K_t$ for all $t \geq 0$ (compare [30]). Then theorems similar to Theorems 1 and 2 below hold.

The maps $\mathcal{F}_t^p$ are essentially determined by their behavior with respect to (affine images of) Euclidean balls.

The conditions (ii) and (iii) state that it is enough to know the behavior locally and close to the boundary of a convex body K. Globally we need only to know how the maps $\mathcal{F}_t^p$ behave for ellipsoids as stated in (i) and (iv). Again, we want to stress that the hyperplanes $H_c(x)$, $H_a(x,\varepsilon)$ and $H_0(x,\varepsilon)$ play mostly an auxiliary role: through them we express what counts which is the behavior near the boundary of K.

Let K and L be convex bodies in $\mathcal{K}$ such that $L \subset K$. Then we cannot expect that $L_t \subset K_t$. We do not even have necessarily, if $E_c(x)$ is the contained ellipsoid, that $\big(E_c(x)\big)_t \cap H_c^-(x) \subseteq (K)_t \cap H_c^-(x)$. This is why we require the condition *C-inclusion preserving.* Its geometric meaning is that, even if inclusion relation in general is not preserved passing to the "floating bodies", it is preserved -at least with a constant- for the contained ellipsoids.

Consider $B_\infty^n = \{x \in \mathbb{R}^n : \max_{1\le i\le n}|x_i| \le 1\}$ and for $0<\varepsilon<1$ let K_ε be B_∞^n with "rounded" vertices. We describe the boundary of K_ε in the first quadrant- the description in the other quadrants is accordingly:

For $0 \le x_i \le 1-\varepsilon$, $1 \le i \le n-1$, let $x_n = 1$.

For $1-\varepsilon \le x_i \le 1$, $1 \le i \le n-1$, let $\sum_{i=1}^n (x_i - (1-\varepsilon))^2 = \varepsilon^2$.

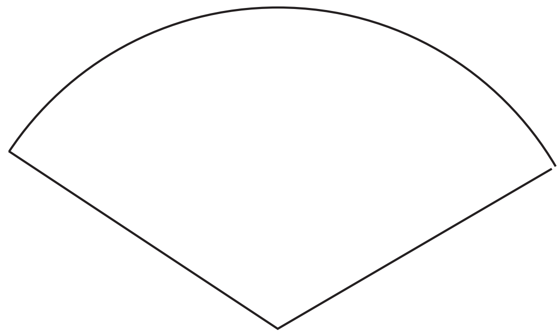
Then the Euclidean unit ball B is the contained ellipsoid of K_ε at $e_n = (0,\dots,1)$. For $f(x) = f_1(x) = \kappa(x)^{\frac{1}{n+1}}$, the surface body $B_{f,t}$ (see the examples below) of B is the Euclidean ball $(1 - s_n\, t^{\frac{2}{n-1}})\, B$, where s_n is a constant depending on n only.

For all $t > \frac{1}{2}\, \varepsilon^{n\frac{n-1}{n+1}}\, \mathrm{vol}_{n-1}(\partial B)$, $(K_\varepsilon)_{f,t} = \emptyset$, hence does not contain $B_{f,t}$.

For $t \le \frac{1}{2}\, \varepsilon^{n\frac{n-1}{n+1}}\, \mathrm{vol}_{n-1}(\partial B)$, $(K_\varepsilon)_{f,t}$ is contained in the half-space $\{x \in \mathbb{R}^n : x_n \le 1 - s_n\, t^{\frac{2}{n-1}}\, \varepsilon^{-\frac{n-1}{n+1}}\}$ and thus does not contain $(1 - s_n\, t^{\frac{2}{n-1}})\, B$ if ε is small enough.

Similarily, we cannot expect that the inclusion relations (8) and (9) pass to the "floating bodies". Therefore we require it as a condition - *locality* - which means geometrically that the inclusion relations (8) and (9) are preserved passing to the "floating bodies". Locality is necessary. To see that, consider the following example (see figure): Let K be the convex body in the plane which consists of a half circle and a triangle attached to it. We consider the surface body (see below) $K_{f_K,t}$ of K where the function $f_K = f_{1,K}$ is such that it is equal to 0 on the lines of the triangle and constant equal to 1 on the half circle.

Then, since $K_{f_K,0}$ does not contain the triangular part of K and since $|B| -$



$|B_{f_B,t}| = s_n \; t^{\frac{2}{n-1}}$ (see below)

$$\lim_{t\to 0} \frac{|K| - |K_{f_K,t}|}{|B| - |B_{f_B,t}|} = \infty$$

while

$$\frac{O_1(K)}{O_1(B)} = \frac{\int_{\partial K} \kappa^{\frac{1}{n+1}}}{\mathrm{vol}_{n-1}(\partial B)}$$

is clearly finite. $t \to K_{f_K,t}$ is not local. Let x be in the triangular part of the boundary of K. Then there is a centered ellipsoid $E_0(x)$ such that (9) holds. But $E_0(x)_{f_E,0} \cap H_0^-$ has non-empty intersection with the triangular part of K and thus is not contained in $K_{f_K,0} \cap H_0^-$.

Examples

1. The L_p-Santaló bodies [23]

Let $t \geq 0$ be in $\mathbb{R}$ and let $K \in \mathcal{K}$. Let $\beta > \frac{n+1}{2}$. Let

$$S_\beta(K,t) = \{x \in K : \int_{K^0} \frac{dy}{(1- < x, y >)^\beta} \leq \frac{1}{t}\}. \tag{10}$$

Here $K^0 = \{y \in \mathbb{R}^n : < x, y > \leq 1 \text{ for all } x \in K\}$ is the polar body of K.

We assume again that ∂K is C^2 with strictly positive Gaussian curvature everywhere. Then it follows from [23] that the family of L_p-Santaló bodies $\{K_t = S_\beta(K,t)\}_{t\geq 0}$, satisfiy the Definition.

Indeed, if K and L are convex bodies in $\mathcal{K}$ such that $L \subset K$, then $S_\beta(L,t) \subset S_\beta(K,t)$. Then (ii) and (iii) of the Definition hold, (ii) with $C = 1$. We refer to [23] for the details.

If B is the Euclidean unit ball in $\mathbb{R}^n$, then $S_\beta(B,t)$ is again a ball with radius $r(t) = 1 - (c_{n,\beta}\ t)^{\frac{1}{\beta - \frac{n+1}{2}}}$. $c_{n,\beta} = 2^{\frac{n-1}{2}}\ |B_2^{n-1}(0,1)|\ B(\frac{n+1}{2}, \beta - \frac{n+1}{2})$ is a constant depending on β and n only. For $x, y > 0$, $B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds$ is the Beta function. Thus for the L_p-Santaló bodies the function f of (iv) is $f(t) = (c_{n,\beta}\ t)^{\frac{1}{\beta - \frac{n+1}{2}}}$ which satisfies what is required in (iv).

By [23] for all linear invertible maps T: $S_\beta(T(K),t) = T(S_\beta(K, |\mathrm{det}T|\ t))$. This shows that for a centered ellipsoid E, E_t is again a centered ellipsoid homothetic to E. Moreover

$$\lim_{t\to 0}\frac{|T(B)| - |(T(B))_t|}{|B| - |B_t|} = |\mathrm{det}T|\ \ \lim_{t\to 0}\frac{1 - \left(1 - (c_{n,\beta}\ \frac{t}{|\mathrm{det}T|})^{\frac{1}{\beta - \frac{n+1}{2}}}\right)^n}{1 - \left(1 - (c_{n,\beta}\ t)^{\frac{1}{\beta - \frac{n+1}{2}}}\right)^n}$$

$$= |\mathrm{det}T|^{\frac{\beta - \frac{n+3}{2}}{\beta - \frac{n+1}{2}}} = |\mathrm{det}T|^{\frac{n-p}{n+p}},$$

for $p = \frac{n}{2\beta - n - 2}$.

2. The surface bodies [26]

Let $K \in \mathcal{K}$ and $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable function. Let $\mathbb{M}_f$ be the measure $\mathbb{M}_f = f\mu$ on ∂K. Let $t \geq 0$.

The surface body $K_{f,t}$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $\mathbb{M}_f$-measure less than or equal to t from ∂K. More precisely,

$$K_{f,t} = \bigcap_{\mathbb{M}_f(\partial K \cap H^-) \leq t} H^+ \tag{11}$$

For $-\infty \leq q \leq \infty$, $q \neq -n$ let the functions $f_q : \partial K \to \mathbb{R}$ be given as follows: For $q = \pm\infty$ put

$$f_{\pm\infty}(x) = f_{\pm\infty,K}(x) = \frac{\kappa_K(x)}{< x, N_K(x) >^n}$$

and for all other values of q

$$f_q(x) = f_{q,K}(x) = \frac{\kappa_K(x)^{\frac{q}{n+q}}}{< x, N_K(x) >^{\frac{n(q-1)}{n+q}}}.$$

We assume also that ∂K is C^2 with strictly positive Gaussian curvature everywhere. Then it follows from [26] that $K \to K_t = K_{f_q,t}$ satisfy the Definition.

Indeed, if ρB is the Euclidean ball with radius ρ , then for all $-\infty \leq q \leq \infty$, $q \neq -n$, for all $x \in \partial(\rho B)$: $f_q(x) = \rho^{-\frac{2nq-n-q}{n+q}}$. Then $(\rho B)_t$ is again a ball centered at 0 with radius $r(t) = \rho\left(1 - s_n \; (\frac{t}{\rho^{n\frac{n-q}{n+q}}})^{\frac{2}{n-1}}\right)$. $s_n = \frac{1}{2(|B_2^{n-1}|)^{\frac{2}{n-1}}}$ is a constant that depends on n only.

Thus for the surface bodies the function f of (iv) is $f(t) = s_n \; t^{\frac{2}{n-1}}$ which satisfies what is required in (iv). It follows from [26], Lemma 10 that the surface body of a centered ellipsoid E is again a centered ellipsoid homothetic to E.

The C-inclusion relation also holds. We give a sketch of the proof-under somewhat simplified conditions. The proof without any restrictions goes along the same lines, working with ellipsoids instead of balls and normals $N_K(x)$ that are not parallel to x. The calculations are just longer and more tedious to carry out but add no further insight. Let $x \in \partial K$ such that $N = N_K(x) = \frac{x}{\|x\|}$. Let $r = r(x)$ be the radius of the biggest Euclidean ball contained in K that touches ∂K in x. r can be taken as $r = \min_{1 \leq i \leq n} a_i$ where a_i is the length of the i-th axis of the approximating ellipsoid $E_a = E_a(x)$. Thus also $\kappa = \kappa(x) = \prod_{i=1}^{n-1} \frac{a_n}{a_i^2}$ (see e.g. [25]). Let $E_c = E_c(x)$ be the contained ellipsoid. Then, by construction of E_c, for $\varepsilon > 0$ given, there exists a hyperplane H such that

$$B\big(x - (1-\varepsilon) rN, (1-\varepsilon) r\big) \cap H^- \subset E_c \cap H^- \subset B(x - rN, r) \cap H^- \subset K \cap H^-.$$

And moreover

$$E_a^i \cap H^- \subset K \cap H^- \subset E_a^c \cap H^-$$

Then, as

$$\big((1-\varepsilon)\; r\big)^{n\frac{n-q}{n+q}} \; \mu\bigg(\partial B(x - (1-\varepsilon) rN, (1-\varepsilon) r) \cap H^-\bigg) \leq$$

$$\int_{\partial E_c \cap H^-} f_{q,E_c} \partial \mu_{E_c} \leq r^{n\frac{n-q}{n+q}} \; \mu\bigg(\partial B(x - rN, r) \cap H^-\bigg),$$

for t small enough,

$$(E_c)_{Ct} \cap H^- \subset \bigg(B\big(x - (1-\varepsilon)\; rN, (1-\varepsilon) r\big)\bigg)_{Ct} \cap H^-$$

$$= B\bigg(x - rN, (1 - r(1 - s_n(\frac{Ct}{r^{n\frac{n-q}{n+q}}})^{\frac{2}{n-1}}\bigg) \cap H^-.$$

To simplify the calculations we assume now in addition that

$$E_a^i \cap H^- = B\bigg(x - \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}} N, \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}}\bigg) \cap H^-.$$

Then

$$(E_a^i)_t \cap H^- = B\left(x - \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}}N, \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}}((1-s_n(\frac{t}{((1-\varepsilon)\kappa^{-1/(n-1)})^{n\frac{n-q}{n+q}}})^{\frac{2}{n-1}}\right) \cap H^-,$$

which is contained in $K_t \cap H^-$ as for H suitably chosen

$$\int_{\partial E_a^i \cap H^-} f_{q,E_a^i} \partial\mu_{E_a^i} \leq \int_{\partial K \cap H^-} f_{q,K} \partial\mu_K \leq \int_{\partial E_a^c \cap H^-} f_{q,E_a^c} \partial\mu_{E_a^c}$$

and for t small enough $\int_{\partial E_a^c \cap H^-} f_{q,E_a^c} \partial\mu_{E_a^c} \leq (1+\varepsilon) \int_{\partial E_a^i \cap H^-} f_{q,E_a^i} \partial\mu_{E_a^i}$. Hence it is enough to show that there exists C such that

$$B\left(x - rN, r(1 - s_n\left(\frac{Ct}{r^{n\frac{n-q}{n+q}}}\right)^{\frac{2}{n-1}}\right) \cap H^- \subset$$

$$B\left(x - \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}}N, \frac{(1-\varepsilon)}{\kappa^{\frac{1}{n-1}}}((1 - s_n(\frac{t}{((1-\varepsilon)\kappa^{-1/(n-1)})^{n\frac{n-q}{n+q}}})^{\frac{2}{n-1}}\right) \cap H^-,$$

which holds if we choose $C \geq 2\frac{a_{max}^2}{a_{min}^2}\left(\frac{a_{max}}{a_{min}}\right)^{n\frac{n-q}{n+q}}$ if $q \leq n$ and $C \geq 2\frac{a_{max}^2}{a_{min}^2}\left(\frac{a_{min}^2}{a_{max}^2}\right)^{n\frac{n-q}{n+q}}$ if $q \geq n$ where $a_{min} = \min_{x\in\partial K}\min_{1\leq i\leq n} a_i(x)$ and $a_{max} = \max_{x\in\partial K}\max_{1\leq i\leq n} a_i(x)$.

Similarly one shows the local-property.

Finally, it follows from Theorem 14 and Proposition 9 of [26] that

$$\lim_{t\to 0}\frac{|T(B)| - |(T(B))_t|}{|B| - |B_t|} = \frac{1}{n|B|s_n} \lim_{t\to 0}\frac{|T(B)| - |(T(B))_t|}{t^{\frac{2}{n-1}}} =$$

$$\frac{O_p(T(B))}{n|B|} = |\mathrm{det}T|^{\frac{n-p}{n+p}}\frac{O_p(B)}{n|B|} = |\mathrm{det}T|^{\frac{n-p}{n+p}}$$

3. The L_p-floatation bodies [31]

Let $K \in \mathcal{K}$ and $t \geq 0$. Let $f : K \to \mathbb{R}$ be an integrable function such that $f > 0$ m-a.e. where m is the Lebesgue measure on $\mathbb{R}^n$.

The L_p-floatation body $F(K, f, t)$ is the intersection of all the closed half-spaces H^+ whose defining hyperplanes H cut off a set of $(f\ m)$-measure less than or equal to t from K. More precisely,

$$F_{f,t} = F(K, f, t) = \bigcap_{\int_{K\cap H^-} f\ dm \leq t} H^+ \tag{12}$$

Let $-\infty \leq p \leq \infty$, $p \neq -n$ and let

$$f_p(x) = f_{p,K}(x) = \frac{< x, N_K(x) >^{\frac{n(n+1)(p-1)}{2(n+p)}}}{\kappa_K(x)^{\frac{n(p-1)}{2(n+p)}}}, \quad \text{for } x \in \partial K.$$

If K is such that f_p is continuous on ∂K, we extend f_p to all of K such that f_p is continuous on K. We assume also that ∂K is C^2 with μ-a.e. strictly positive Gaussian curvature. Then it follows from [31] that the family $\{K_t = F_{f_p,t}\}_{t\geq 0}$ satisfy the Definition.

This is shown similarly to the surface bodies.

3 The Theorems and their proofs

Theorem 1 *Let $0 \leq p < \frac{n}{n-2}$ be fixed. Let $t \in \mathbb{R}$, $t \geq 0$, let $\mathcal{F}_t^p : \mathcal{K} \to \mathcal{K}$, $K \longmapsto \mathcal{F}_t^p(K) = K_t$ be a p-limiting, C-inclusion preserving, monotone, local map.*

Then

$$lim_{t\to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{O_p(K)}{O_p(B)}.$$

Theorem 2 *Let $p \neq -n$ be fixed. Let $t \in \mathbb{R}$, $t \geq 0$, let $\mathcal{F}_t^p : \mathcal{K} \to \mathcal{K}$, $K \longmapsto \mathcal{F}_t^p(K) = K_t$, be a p-limiting, C-inclusion preserving, monotone, local map.*

Let $K \in \mathcal{K}$ be such that ∂K is C^2 and has strictly positive Gaussian curvature everywhere. Then

$$lim_{t\to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{O_p(K)}{O_p(B)}.$$

Remarks

1. As corollaries to Theorems 1 and 2 we obtain results of [23, 25, 26, 31]. Some of these results were obtained under weaker assumptions.

2. The restriction $p < \frac{n}{n-2}$ in Theorem 1 is due to (1) (see the proof of Theorem 1 below). This restriction can be removed if a modified version of (1) can be proved (see the Remark after the proof of Lemma 4) .

For the proof of Theorems 1 and 2 we need several Lemmas.

Lemma 3 *Let K and L be two convex bodies in $\mathbb{R}^n$ such that $0 \in int(L)$ and $L \subseteq K$. Then*

$$|K| - |L| = \frac{1}{n} \int_{\partial K} < x, N(x) > \left(1 - (\frac{||x_L||}{||x||})^n\right) d\mu(x),$$

where $x_L = [0, x] \cap \partial L$.

The proof of Lemma 3 is standard.

Lemma 4 *Let* $p \neq -n$ *be fixed. Let* $t \in \mathbb{R}$, $t \geq 0$, *let* $\mathcal{F}_t^p : \mathcal{K} \to \mathcal{K}$, $K \longmapsto \mathcal{F}_t^p(K) = K_t$ *be a p-limiting, C-inclusion preserving, monotone, local map.*

Let $K \in \mathcal{K}$. *Then* μ*-a.e. on* ∂K *there exists* t_0 *such that*

$$0 \leq < x, N_K(x) > \frac{1 - (\frac{\|x_{K_t}\|}{\|x\|})^n}{|B| - |B_t|} \leq max \left\{ \frac{\gamma}{|B|} \ \ (r(x))^{-\frac{p(n-1)}{n+p}}, \frac{1}{|B| - |B_{t_0}|} \right\}$$

where $x_{K_t} = [0, x] \cap \partial K_t$ *and* γ *is a constant.*

Lemma 5 *Let* $p \neq -n$ *be fixed. Let* $t \in \mathbb{R}$, $t \geq 0$, *let* $\mathcal{F}_t^p : \mathcal{K} \to \mathcal{K}$, $K \longmapsto \mathcal{F}_t^p(K) = K_t$ *be a p-limiting, C-inclusion preserving, monotone, local map.*

Let $K \in \mathcal{K}$ *and* $x \in \partial K$. *Then*

(i) if $\kappa_K(x) > 0$,

$$\lim_{t \to 0} \frac{< x, N_K(x) > \left(1 - (\frac{\|x_{K_t}\|}{\|x\|})^n\right)}{|B| - |B_t|} = \frac{\kappa_K(x)^{\frac{p}{n+p}}}{|B| \ \ < x, N_K(x) >^{\frac{n(p-1)}{n+p}}}.$$

(ii) if $\kappa_K(x) = 0$ *and* $p > 0$ *or* $p < -n$

$$\lim_{t \to 0} \frac{< x, N_K(x) > \left(1 - (\frac{\|x_{K_t}\|}{\|x\|})^n\right)}{|B| - |B_t|} = 0.$$

$x_{K_t} = [0, x] \cap \partial K_t$.

Proof of Theorems 1 and 2

By Lemma 3

$$\lim_{t \to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \lim_{t \to 0} \frac{1}{n} \int_{\partial K} \frac{< x, N_K(x) > (1 - (\frac{||x_{K_t}||}{||x||})^n)}{|B| - |B_t|} d\mu(x).$$

By Lemma 4 the functions under the integral sign are uniformly bounded in t by the function

$$g(x)=\max\left\{\frac{\gamma}{|B|}\ \left(r(x)\right)^{-\frac{p(n-1)}{n+p}},\frac{1}{|B|-|B_{t_0}|}\right\}.$$

As assumed in Theorem 1, $0\leq p<\frac{n}{n-2}$ and therefore g is integrable by (1). We apply Lebesgue's convergence theorem to interchange integration and limit. By Lemma 5 the functions under the integral are converging pointwise.

For the proof of Theorem 2 note that, under the assumptions of the theorem, the functions under the integral sign are uniformly bounded in t by the constant

$$\max\left\{\frac{\gamma}{|B|}\ r_0^{-\frac{p(n-1)}{n+p}},\frac{1}{|B|-|B_{t_0}|}\right\}$$

which is integrable for all p and where $r_0=\min\{r(x):x\in\partial K\}$.

Proof of Lemma 4

Let $x\in\partial K$ be such that $r=r(x)>0$ and let $E_c=E_c(x)$ be the corresponding contained ellipsoid such that (2) holds. By the C-inclusion property there exists C and t_1 such that for all $t\leq t_1$

$$\Big(E_c(x)\Big)_{Ct}\cap H_c^-(x)\subseteq (K)_t\cap H_c^-(x).$$

Hence for all $t\leq t_1$

$$\|x_{(E_c)_{Ct}}\|\leq\|x_{K_t}\|.$$

where $x_{(E_c)_{Ct}}=[0,x]\cap\partial(E_c)_{Ct}$ and thus

$$\frac{1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B|-|B_t|}\leq\frac{1-\left(\frac{\|x_{(E_c)_{Ct}}\|}{\|x\|}\right)^n}{|B|-|B_t|}$$

By (iv) $(E_c)_{Ct}$ is homothetic to E_c: $(E_c)_{Ct}=a_E(Ct)\ E_c$, hence $\frac{\|x_{(E_c)_{Ct}}\|}{\|x\|}=a_E(Ct)$. Thus

$$\frac{1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B|-|B_t|}\leq\frac{1-\big(a_E(Ct)\big)^n}{|B|-|B_t|}.$$

Let T be a linear invertible map such that $E_c=T(B)$.

As $|E_c| = |<x, N_K(x)>|^{\frac{n+1}{2}} \left(\frac{r(x)}{2}\right)^{\frac{n-1}{2}} |B|$,

$$|\mathrm{det} T| = |<x, N_K(x)>|^{\frac{n+1}{2}} \left(\frac{r(x)}{2}\right)^{\frac{n-1}{2}} . \tag{13}$$

It follows from property (i) that there is a constant c_1 such that for all $t \leq t_2 = t_2(x)$

$$\frac{|T(B)| - |(T(B))_{Ct}|}{|B| - |B_{Ct}|} = \frac{|E_c| - |(E_c)_{Ct}|}{|B| - |B_{Ct}|} = \frac{|E_c|\Big(1 - \big(a_E(Ct)\big)^n\Big)}{|B| - |B_{Ct}|} =$$

$$\frac{|\mathrm{det} T| \ |B| \ \Big(1 - \big(a_E(Ct)\big)^n\Big)}{|B| - |B_{Ct}|} \leq c_1 \ |\mathrm{det} T|^{\frac{n-p}{n+p}}.$$

Hence $1 - \big(a_E(Ct)\big)^n \leq c_1 \ \frac{B| - |B_{Ct}|}{|B|} \ |\mathrm{det} T|^{-\frac{2p}{n+p}}$ and thus for all $t \leq \min\{t_1, t_2\}$

$$\frac{1 - \left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B| - |B_t|} \leq c_1 \ \frac{|\mathrm{det} T|^{\frac{-2p}{n+p}}}{|B|} \ \frac{|B| - |B_{Ct}|}{|B| - |B_t|} \tag{14}$$

As $\mathcal{F}_t$ is monotone, by (iv) there exists t_3 such that for all $t \leq t_3$

$$\frac{|B| - |B_{Ct}|}{|B| - |B_t|} = \frac{1 - (1 - f(Ct))^n}{1 - (1 - f(t))^n} \leq 2d, \tag{15}$$

where d is the constant from (iv).

Let $t_0 = \min\{t_1, t_2, t_3\}$. Then by (13), (14) and (15), for all $t \leq t_0$ with a new constant c_2

$$\frac{1 - (\frac{\|x_{K_t}\|}{\|x\|})^n}{|B| - |B_t|} \leq \frac{c_2}{|B|} \ \left(\frac{r(x)}{2}\right)^{\frac{-p(n-1)}{n+p}} <x, N(x)>^{\frac{-p(n+1)}{n+p}} .$$

Now notice that there is an $0 < \beta \leq 1$ such that

$$B(0, \beta) \subseteq K \subseteq B(0, \frac{1}{\beta}).$$

and therefore

$$\beta^3 \leq <x, N(x)> \leq \frac{1}{\beta}.$$

Thus, with a new constant γ, we get for all $t \le t_0$

$$< x, N_K(x) > \ \frac{1-(\frac{\|x_{K_t}\|}{\|x\|})^n}{|B|-|B_t|} \le \frac{\gamma}{|B|} \ (r(x))^{-\frac{p(n-1)}{n+p}} \ . \tag{16}$$

Let now $t \ge t_0$. Then

$$\frac{1-(\frac{\|x_{K_t}\|}{\|x\|})^n}{|B|-|B_t|} \le \frac{1}{|B|-|B_t|} \le \frac{1}{|B|-|B_{t_0}|}.$$

Remark

For $x \in \partial K$ let H be a hyperplane through 0 with normal $N(x)$. H^- is such that $x \in H^-$. Let $E(x)$ be a centered ellipsoid such that $x \in E(x)$ and such that $E(x) \cap H^- \subset K$. Let $v(x)$ be the maximum volume of such ellipsoids.

Conjecture [12] *Let $\beta < 2$. Then we have for all K in $\mathcal{K}$*

$$\int_{\partial K} v(x)^{-\beta} d\mu(x) < \infty.$$

If the conjecture holds true, then (13) can be replaced by $|\det T| = \frac{v(x)}{|B|}$ and hence the right hand side of inequality (16) can be replaced by $\gamma' \ |B|^{\frac{n-p}{n+p}} \ v(x)^{-\frac{2p}{n+p}}$. By the conjecture this is integrable, if $\frac{2p}{n+p} < 2$, which holds for all $p > -n$.

Proof of Lemma 5

(i) Let $x \in \partial K$ be such that $\kappa(x) = \kappa_K(x) > 0$. Let $\varepsilon > 0$ be given. Let $E_a^i = E_a^i(x)$ and $E_a^c = E_a^c(x)$ be the corresponding approximating ellipsoids with properties (3) - (8).

By the locality property (iii) there exists t_1 such that for all $t \le t_1$

$$\left(E_a^i(x)\right)_t \cap H_a^-(x) \subseteq (K)_t \cap H_a^-(x) \subset \left(E_a^c(x)\right)_t \cap H_a^-(x)$$

Hence for all $t \le t_1$ as in the proof of Lemma 4

$$\frac{1-\left(\frac{\|x_{(E_a^c)_t}\|}{\|x\|}\right)^n}{|B|-|B_t|} \le \frac{1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B|-|B_t|} \le \frac{1-\left(\frac{\|x_{(E_a^i)_t}\|}{\|x\|}\right)^n}{|B|-|B_t|}$$

By (iv) $(E_a^i)_t$ is homothetic to E_a^i and $(E_a^c)_t$ is homothetic to E_a^c: $(E_a^i)_t = a_i(t)\ E_a^i$ and $(E_a^c)_t = a_c(t)\ E_a^c$ and hence $\frac{\|x_{(E_a^i)_t}\|}{\|x\|} = a_i(t)$ and $\frac{\|x_{(E_a^c)_t}\|}{\|x\|} = a_c(t)$. Thus

$$\frac{1-\big(a_c(t)\big)^n}{|B|-|B_t|} \le \frac{1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B|-|B_t|} \le \frac{1-\big(a_i(t)\big)^n}{|B|-|B_t|}.$$

Let T_i be a linear invertible map such that $E_a^i = T_i(B)$ and let T_c be a linear invertible map such that $E_a^c = T_c(B)$. Then, as

$$|E_a^i| = (1-\varepsilon)^{\frac{n-1}{2}} |< x, N(x) >|^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}} |B|$$

and

$$|E_a^c| = (1+\varepsilon)^{\frac{n-1}{2}} |< x, N(x) >|^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}} |B|$$

$$|\det T_i| = |(1-\varepsilon)^{\frac{n-1}{2}} | < x, N(x) > |^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}}, \tag{17}$$

and

$$|\det T_c| = |(1+\varepsilon)^{\frac{n-1}{2}} | < x, N(x) > |^{\frac{n+1}{2}} \kappa(x)^{-\frac{1}{2}}. \tag{18}$$

It then follows from property (i) that there is t_2 such that for all $t \le t_2 = t_2(x)$

$$\frac{|T_i(B)| - |(T_i(B))_t|}{|B|-|B_t|} = \frac{|E_a^i| - |(E_a^i)_t|}{|B|-|B_t|} = \frac{|\det T_i|\ |B| \left(1-\big(a_i(t)\big)^n\right)}{|B|-|B_t|} \le (1+\varepsilon)\ |\det T_i|^{\frac{n-p}{n+p}}$$

and

$$\frac{|T_c(B)| - |(T_c(B))_t|}{|B|-|B_t|} = \frac{|E_a^c| - |(E_a^c)_t|}{|B|-|B_t|} = \frac{|\det T_c|\ |B| \left(1-\big(a_c(t)\big)^n\right)}{|B|-|B_t|} \ge (1-\varepsilon)\ |\det T_c|^{\frac{n-p}{n+p}}.$$

Hence $1-\big(a_i(t)\big)^n \le (1+\varepsilon)\ \frac{B|-|B_t|}{|B|}\ |\det T_i|^{-\frac{2p}{n+p}}$ and $1-\big(a_c(t)\big)^n \ge (1-\varepsilon)\ \frac{B|-|B_t|}{|B|}\ |\det T_c|^{-\frac{2p}{n+p}}$ and thus for all $t \le t_0 = \min\{t_1, t_2\}$

$$(1-\varepsilon)\ \frac{|\det T_c|^{\frac{-2p}{n+p}}}{|B|} \le \frac{1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B|-|B_t|} \le (1+\varepsilon)\ \frac{|\det T_i|^{\frac{-2p}{n+p}}}{|B|} \tag{19}$$

Putting (17) and (18) in (19) we get with (new) constants c_1 and c_2 for all $t \le t_0$

$$\frac{(1-c_1\varepsilon)}{|B|}\ \frac{\kappa_K(x)^{\frac{p}{n+p}}}{< x, N(x) >^{\frac{n(p-1)}{n+p}}} \le \frac{< x, N(x) > \left(1-\left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n\right)}{|B|-|B_t|}$$

$$\leq \frac{(1+c_2\varepsilon)}{|B|} \ \frac{\kappa_K(x)^{\frac{p}{n+p}}}{<x, N(x)>^{\frac{n(p-1)}{n+p}}}$$

(ii) Now we consider the case that $x \in \partial K$ is such that $\kappa(x) = 0$. Then, by (9), then there is a centered ellipsoid $E_0(x)$ that touches ∂K in x and which has at least one axis that is arbitrarily large and there exists a hyperplane $H_0(x)$ such that

$$E_0(x) \cap H_0^-(x) \subset K \cap H_0^-(x).$$

Then we continue as in the proof of Lemma 4 and find that for all t small enough

$$0 \leq <x, N(x)> \ \frac{1 - \left(\frac{\|x_{K_t}\|}{\|x\|}\right)^n}{|B| - |B_t|} \leq c_1 \ \frac{|\mathrm{det} T|^{\frac{-2p}{n+p}}}{|B|} \ \frac{|B| - |B_{Ct}|}{|B| - |B_t|} \leq c_2 \ \frac{|\mathrm{det} T|^{\frac{-2p}{n+p}}}{|B|}$$

where c_1 and c_2 are constants and T is the linear invertible map such that $E_0(x) = T(B)$. Thus $|\mathrm{det}T| = \Pi_{i=1}^n a_i$, where a_i, $1 \leq i \leq n$ are the lengths of the principal axes of $E_0(x)$. As $|\mathrm{det}T|^{\frac{-2p}{n+p}} = \left(\frac{1}{\Pi_{i=1}^n a_i}\right)^{\frac{2p}{n+p}}$, this finishes the proof of the lemma as $p > 0$ or $p < -n$ and as one of the axes is arbitrarily large.

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