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Using Fractional Order Adjustment Rules and Fractional Order Reference Models in Model-Reference Adaptive Control

B. M. VINAGRE
Department of Electronics and Electromechanical Engineering, Industrial Engineering School, University of Extremadura, E-06071 Badajoz, Spain

I. PETRÁŠ and I. PODLUBNY
Department of Informatics and Process Control, BERG Faculty, Technical University of Košice, 04200 Košice, Slovak Republic

Y. Q. CHEN
Department of Electrical and Computer Engineering, CSOIS, Utah State University, Logan, UT 84322-4160, U.S.A.

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Abstract. This paper investigates the use of Fractional Order Calculus (FOC) in conventional Model Reference Adaptive Control (MRAC) systems. Two modifications to the conventional MRAC are presented, i.e., the use of fractional order parameter adjustment rule and the employment of fractional order reference model. Through examples, benefits from the use of FOC are illustrated together with some remarks for further research.

Keywords: Fractional order calculus, model reference adaptive control, fractional order parameter adjustment rule, fractional order reference model.

1. Introduction

Fractional calculus is a 300-years-old topic. The theory of fractional order derivative was developed mainly in the 19th century. Recent books [4–6] provide a good source of references on fractional calculus. However, applying fractional order calculus to dynamic systems control is just a recent focus of interest [9, 13, 14, 16]. For pioneering work on this regard, we cite [10, 14].

The model reference approach was developed by Whitaker and his colleagues around 1960 [11]. MRAC (Model Reference Adaptive Control) has become a standard part in textbooks on adaptive control [1, 3]. The well known MIT rule for MRAC is to adjust or update the unknown parameter using gradient information.

The major contribution of this paper is to introduce the fractional order calculus into the MRAC in two ways: the use of fractional order parameter adjustment rule and the employment of fractional order reference model. The objective of this paper is to show that FOC can be used to extend many existing conventional results. Mainly via simulation results, benefits from the use of FOC are illustrated together with some remarks for further research although detailed theoretical analysis is left out in this paper.

This paper is organized as follows: in Section 2, the MRAC is briefly reviewed followed by an introduction on fractional order operators and related dynamic systems in Section 3. In Section 4 the use of fractional order calculus into MRAC via fractional order adjustment

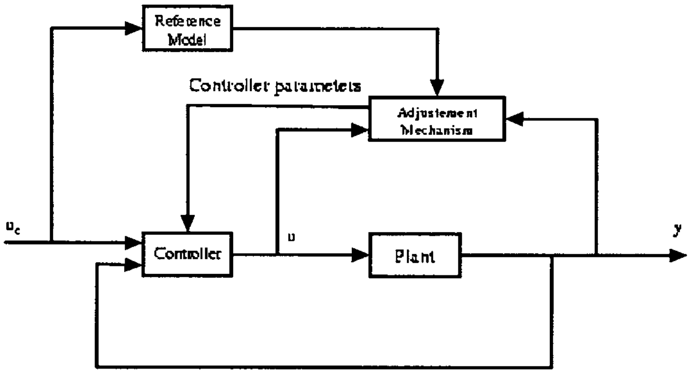

Figure 1. Basic principles of model reference adaptive system.

rule (Section 4.1) and fractional order reference model (Section 4.2) is presented together with some illustrative simulation results. Section 5 concludes this paper with some remarks on future research.

2. MRAC: A Brief Review

2.1. The MRAC Problem

The Model Reference Adaptive System (MRAS) is one of the main approaches to adaptive control, in which the desired performance is expressed in terms of a reference model (a model that describes the desired input-output properties of the closed-loop system) and the parameters of the controller are adjusted based on the error between the reference model output and the system output. These basic principles are illustrated in Figure 1. As can be seen from Figure 1, there are two loops: an inner loop which provides the ordinary control feedback, and an outer loop which adjusts the parameters in the inner loop.

2.2. The Gradient Approach

The gradient approach to model reference adaptive control is based on the assumption that the parameters change more slowly than the other variables in the system. This assumption, which admits a quasi-stationary treatment, is essential for the computation of the sensitivity derivatives that are needed in the adaptation.

Let e denote the error between the system output, y, and the reference output, y_m. Let θ denote the parameters to be updated. By using the criterion

$$J(\theta) = \frac{1}{2}e^2, \tag{1}$$

the adjustment rule for changing the parameters in the direction of the negative gradient of J is that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\gamma\frac{\partial J}{\partial\theta} = -\gamma e\frac{\partial e}{\partial\theta}. \tag{2}$$

If it is assumed that the parameters change much more slowly than the other variables in the system, the derivative $\partial e/\partial\theta$, that is, the sensitivity derivative of the system, can be evaluated under the assumption that θ is constant.

There are many variants about the MIT rules for the parameter adjustment. For example, the *sign-sign* algorithm is widely used in communication systems [1]; the PI-adjustment rule is used in [8]. In this paper, we will introduce a new variant of the MIT rules for the parameter adjustment by using the fractional order calculus. In addition, we shall extend the reference model to the fractional order. The fundamental of this paper is the fractional order calculus and the related notion about the fractional order dynamic systems, which will be very briefly introduced in the next section.

3. Fractional Order Operators and Fractional Order Control Systems

In this section, for the purpose of self-containing, a very brief introduction on fractional order operators and related dynamic systems is presented.

3.1. Fractional Order Operators

Fractional calculus is a generalization of integration and differentiation to non-integer (fractional) order fundamental operator ${}_a\mathrm{D}_t^\alpha$, where a and t are the limits and α, ($\alpha \in \mathbb{R}$) the order of the operation. The two definitions used for the general fractional integro-differential are the Grünwald–Letnikov (GL) definition and the Riemann–Liouville (RL) definition [5, 6]. The GL definition is that

$$ {}_a\mathrm{D}_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t-a/h]} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{3}$$

where $[\cdot]$ means the integer part while the RL definition

$$ {}_a\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \, \mathrm{d}\tau, \tag{4}$$

for $(n-1 < r < n)$ and where $\Gamma(\cdot)$ is the Euler's *gamma* function.

For convenience, Laplace domain notion is usually used to describe the fractional integro-differential operation [6]. The Laplace transform of the RL fractional derivative/integral (4) under zero initial conditions for order α, $(0 < \alpha < 1)$ is given by [5]

$$ \mathcal{L}\{{}_a\mathrm{D}_t^{\pm\alpha} f(t); s\} = s^{\pm\alpha} F(s). \tag{5}$$

3.2. Fractional Order Control Systems

In theory, the control systems can include both the fractional order dynamic system to be controlled and the fractional order controller. A fractional order plant to be controlled can be described by a typical n-term linear FODE in time domain

$$ a_n \mathrm{D}_t^{\beta_n} y(t) + \cdots + a_1 \mathrm{D}_t^{\beta_1} y(t) + a_0 \mathrm{D}_t^{\beta_0} y(t) = 0, \tag{6}$$

where a_k $(k = 0, 1, \ldots, n)$ are constant coefficients of the FODE; β_k $(k = 0, 1, 2, \ldots, n)$ are real numbers. Without loss of generality, assume that $\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0 \geq 0$. Consider a control function which acts on the FODE system (6) as follows:

$$ a_n \mathrm{D}_t^{\beta_n} y(t) + \cdots + a_1 \mathrm{D}_t^{\beta_1} y(t) + a_0 \mathrm{D}_t^{\beta_0} y(t) = u(t). \tag{7}$$

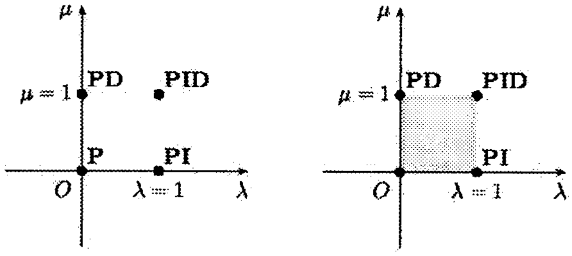

Figure 2. PID-controller: from *points* to *plane*.

By Laplace transform, we can get a fractional transfer function

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\beta_n} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}. \tag{8}$$

In general, a fractional order dynamic system can be represented by a transfer function of the form:

$$G_p(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\alpha_m} + \cdots + b_1 s^{\alpha_1} + b_0 s^{\alpha_0}}{a_n s^{\beta_n} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}. \tag{9}$$

However, in control practice, more common is to consider the fractional order controller. This is due to the fact that the plant model may have already been obtained as an integer order model in classical sense. In most cases, our objective is to apply the Fractional Order Control (FOC) to enhance the system control performance. Taking the conventional PID controller as an example, its fractional order version, $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller, was studied in time domain in [13] and in frequency domain in [12]. The time domain formula is that

$$u(t) = K_p e(t) + T_i \,\mathrm{D}_t^{-\lambda} e(t) + T_d \,\mathrm{D}_t^{\mu} e(t) \quad (\mathrm{D}_t^{(*)} \equiv {}_0\,\mathrm{D}_t^{(*)}). \tag{10}$$

It can be expected that $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller (10) may enhance the systems control performance due to more tuning knobs introduced, which is intuitively illustrated by Figure 2.

In what follows, we shall present a motivative example on a simple fractional order controller for a double integrator plant $H(s) = A/s^2$ where A is the open-loop plant gain. Suppose a fractional order controller of the form $D(s) = s^{\alpha}$, $0 < \alpha < 1$ is to be used. The open-loop transfer function of the overall controlled system will be of the form:

$$F_o(s) = D(s)G(s) = \frac{A}{s^{2-\alpha}},$$

which is in fact the form of *the Bode's ideal transfer function* [6]. It has the following characteristics:

(a) Open loop:

1. The Bode amplitude plot has constant slope of $-(2-\alpha)$.
2. The crossover frequency depends only on A.
3. The phase curve is a horizontal line at $-(2-\alpha)(\pi/2)$.
4. The Nyquist curve is a straight line through the origin with argument $-(2-\alpha)(\pi/2)$.

(b) Closed loop with unity feedback:

1. The transfer function has the form

$$F_c(s) = \frac{A}{s^{2-\alpha} + A}. \tag{11}$$

2. The gain margin is infinite.
3. The phase margin is constant,

$$\Phi_m = \pi\left(1 - \frac{2-\alpha}{2}\right).$$

4. The step response has the expression (see [6, 2]):

$$y(t) = At^{2-\alpha}\, \mathrm{E}_{2-\alpha,2-\alpha+1}\left(-At^{2-\alpha}\right),$$

where $\mathrm{E}_{2-\alpha,2-\alpha+1}(-At^{2-\alpha})$ is the Mittag–Leffler function in two parameters. Assuming $A \in \mathbb{R}^+$, such a step response exhibits an overshoot independent of parameter A and dependent only on the parameter α, the fractional order.

Clearly, from the above discussions, it is a very desirable property that the overshoot is independent of parameter A (related to load in vehicle suspension system) and dependent only on the fractional order α. This has been explored by Oustaloup in terms of *iso-damping* [15].

4. Using Fractional Order Calculus in MRAC Scheme

In this section, the fractional order calculus is introduced into MRAC scheme in two ways. One is the use of fractional derivatives for the MIT adjustment rules and the other one is the use of fractional order reference models. The modified MRAC schemes are explained with some simulation illustrations.

4.1. Fractional Order Adjustment Rule

4.1.1. *The New Adjustment Rule*

As can be observed in Equation (2), the rate of change of the parameters depends solely on the adaptation gain, γ. Taking into account the properties of the fractional differential operator, it is possible to make the rate of change depending on both the adaptation gain, γ, and the derivative order, α, by using the adjustment rule

$$\frac{\mathrm{d}^\alpha\theta}{\mathrm{d}^\alpha t} = -\gamma\frac{\partial J}{\partial\theta} = -\gamma e\frac{\partial e}{\partial\theta}, \tag{12}$$

where α is a real number denoting the fractional order derivative. In other words, the above parameter updating rule can be expressed as follows:

$$\theta = -\gamma I^\alpha\left[\frac{\partial J}{\partial\theta}\right] = -\gamma I^\alpha\left[e\frac{\partial e}{\partial\theta}\right]; \quad I^\alpha \equiv D^{-\alpha}. \tag{13}$$

For example, consider the first order SISO system to be controlled:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + ay = bu, \tag{14}$$

where y is the output, u is the input and the system parameters a and b are unknown constants or unknown slowly time-varying. Assume that the corresponding reference model is given by

$$\frac{\mathrm{d}y_m}{\mathrm{d}t} + a_m y_m = b_m u_c, \tag{15}$$

where u_c is the reference input signal for the reference model, y_m is the output of the reference model and a_m and b_m are known constants. Perfect model-following can be achieved with the controller defined by

$$u(t) = \theta_1 u_c(t) - \theta_2 y(t), \tag{16}$$

where

$$\theta_1 = \frac{b_m}{b}; \quad \theta_2 = \frac{a_m - a}{b}. \tag{17}$$

From Equations (14) and (16), assuming that $a + b\theta_1 \approx a_m$, and taking into account that b can be absorbed in γ, the equations for updating the controller parameters can be designed as (see, e.g., [1]),

$$\frac{\mathrm{d}^\alpha \theta_1}{\mathrm{d}t^\alpha} = -\gamma \left(\frac{1}{p + a_m}\right) u_c e, \tag{18}$$

$$\frac{\mathrm{d}^\alpha \theta_2}{\mathrm{d}t^\alpha} = \gamma \left(\frac{1}{p + a_m}\right) y e, \tag{19}$$

where $p = \mathrm{d}/\mathrm{d}t$, and γ is the adaptation gain, a small positive real number. Equivalently, in frequency domain, (18) and (19) can be written as

$$\theta_1 = -\frac{\gamma}{s^\alpha} \left(\frac{1}{s + a_m}\right) u_c e, \tag{20}$$

$$\theta_2 = \frac{\gamma}{s^\alpha} \left(\frac{1}{s + a_m}\right) y e. \tag{21}$$

Clearly, the conventional MRAC [1] is the case when $\alpha = 1$.

A block diagram for the above MRAC scheme for adjusting the unknown parameters θ_1 and θ_2 is shown in Figure 3.

In Figure 4, simulation results for $a = 1$, $b = 0.5$, $a_m = b_m = 2$, $\gamma = 3$ are presented. Two cases are considered for $\alpha = 1$ and $\alpha = 1.25$. As can be observed from Figure 4, under the same conditions, compared to the case when $\alpha = 1$, the updating of the unknown parameters is faster when $\alpha = 1.25$. The benefit due to the use of a slightly higher order of the derivatives is clearly demonstrated in Figure 4.

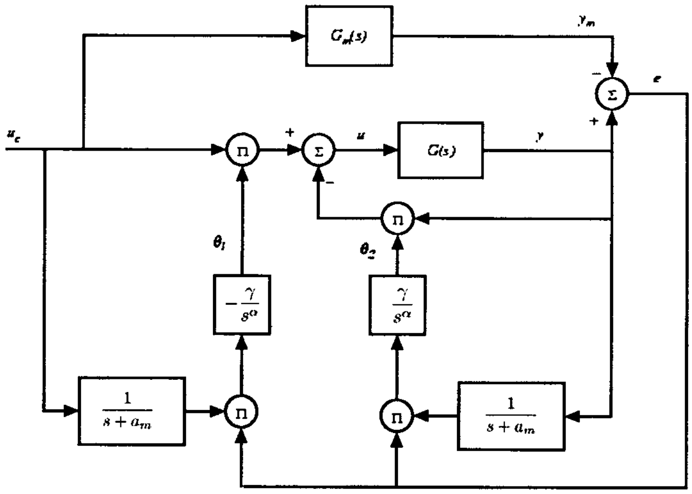

Figure 3. Block diagram of a simple MRAC scheme.

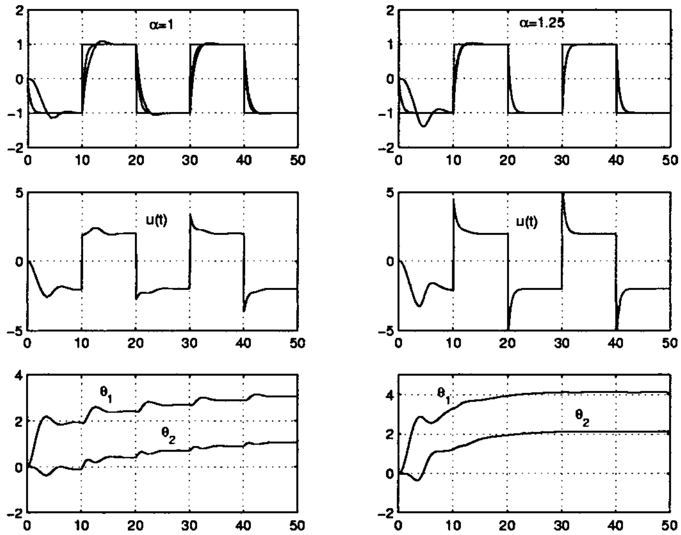

Figure 4. Simulation results for fractional order MRAC.

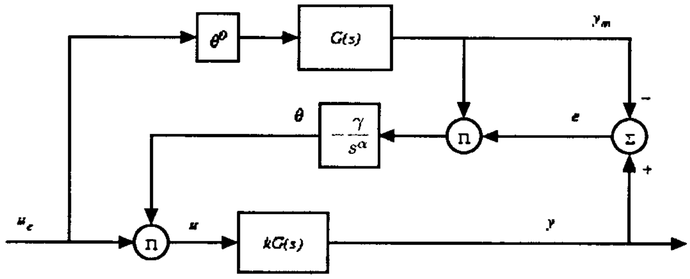

Figure 5. A simple MRAC scheme for feedforward gain adjustment.

4.1.2. *Stability Considerations*

Equation (2), usually known as MIT rule, performs well if the adaptation gain is small. The allowable value depends on both the magnitude of the reference signal, u_c, and the process gain. So, if not properly handled, the MIT rule may give an unstable closed-loop system.

As an example, consider the MRAS scheme in Figure 5 in which the problem is to adjust a feedforward gain, θ, to the value θ_0 [1]. Consider the transfer function of the system

$$G(s) = \frac{1}{s^2 + a_1 s + a_2}. \tag{22}$$

The MIT rule gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\gamma e y_m, \tag{23}$$

where

$$e = G(s)\,(\theta - \theta_0)\,u_c. \tag{24}$$

The governing differential equation for the overall adaptive system is

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} + a_1 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_2 \frac{\mathrm{d}y}{\mathrm{d}t} + \gamma u_c y_m y = \theta \frac{\mathrm{d}u_c}{\mathrm{d}t} + \gamma u_c y_m^2. \tag{25}$$

Some insight into the behavior of the system can be obtained by assuming that adaptation mechanism is connected when the equilibrium is reached. That is, when $u_c = u_c^0 = y_m = y_m^0$, the time-varying differential equation (25) is transformed into a differential equation with constant coefficients that describes an LTI system with its characteristic equation given by

$$s^3 + a_1 s^2 + a_2 s + \gamma u_c^0 y_m^0 = 0. \tag{26}$$

It is easy to test the stability condition by using the Routh test which yields

$$a_1 a_2 > \gamma (u_c^0)^2. \tag{27}$$

So, if the adaptation gain γ or the reference signal u_c are sufficiently large, the system may become unstable.

As an illustrative simulation example, let $a_1 = a_2 = \theta^0 = 1$ and $\gamma = 0.1$. For reference signal amplitude $u_c = 0.1$, 1 and 3.5, the results are shown in Figure 6. As can be observed from Figure 6, when $\left(u_c^0\right)^2 > 10$, the system becomes unstable.

Here we adopt an alternative adjustment rule using the FOC as follows:

$$\frac{\mathrm{d}^\alpha \theta}{\mathrm{d}t^\alpha} = -\gamma e y_m, \quad 0 < \alpha < 1. \tag{28}$$

With the flexibility in selecting both the derivative order and the adaptation gain, one can expect an enlarged range of reference signal magnitude with which the system is stable.

For example, with $\gamma = 0.1$ and $\alpha = 0.75$ the simulation results shown in Figure 7 demonstrate clearly that for $u_c = 0.1$ and $u_c = 1$, the behavior of the adaptive system is similar to the conventional case when $\alpha = 1$ shown in Figure 6. However, with $\alpha = 0.75$, the overall adaptive system is still stable even when $u_c = 5$.

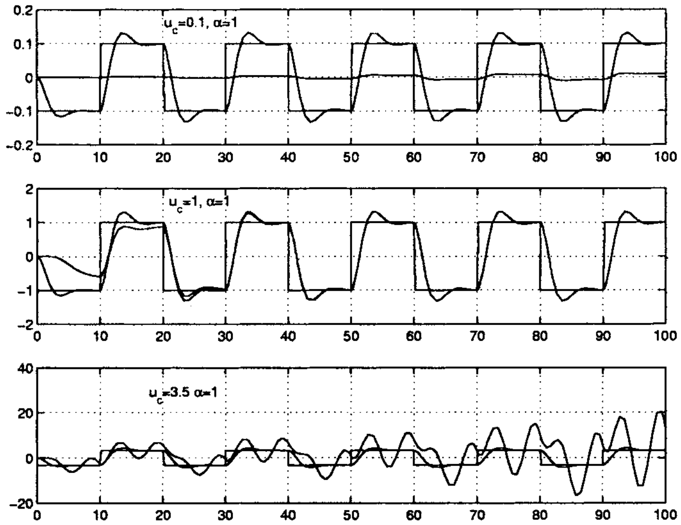

Figure 6. Unstable behavior in conventional MRAC with respect to the magnitude of reference input signal.

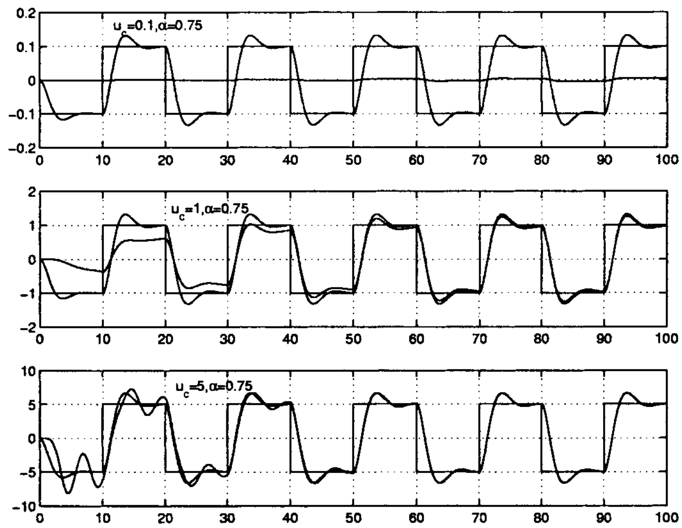

Figure 7. Improved stability behavior of MRAC with respect to the magnitude of reference input signal using FOC.

To obtain some insight into the above beneficial fact, it is noted that for different choices of the design parameter-pair (γ, α) in the operator γ/s^{α} in order to achieve similar transient performances on the adjustment rate, the induced phase response depends only on α. This could be a desired behavior that the conventional MRAC cannot have.

4.2. Fractional Reference Model

Now, we will introduce another modification to MRAC problem by introducing fractional order system as the reference model. In the simplest MRAC problems, the usual reference models are FIRs or the second order dynamic systems. Clearly, the set of candidates of the

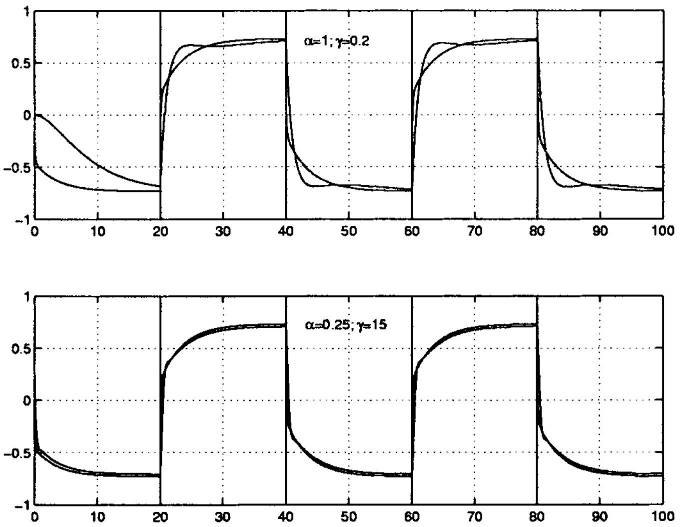

Figure 8. The effect of using fractional order reference model in MRAC.

reference models can be enlarged by using fractional order systems. In addition, transient response of MRAC systems can be improved. This is illustrated by a simulation example.

Consider a system described by the transfer function

$$G(s) = \frac{1}{s+1}. \tag{29}$$

The adaptive scheme shown in Figure 5 is used to adjust the feedforward gain in order to track the reference model output

$$y_m = \frac{1}{s^{0.25}+1} u_c. \tag{30}$$

With $\alpha = 1$ in the parameter adjusting rule (28), it would be very difficult, if not impossible, to track the reference output even after a significant time interval. However, when a fractional order reference model is used, it is an easy task if we choose $\alpha \in (0, 1)$.

Again, as an illustrative simulation example, in Figure 8 the results using the pairs $(\gamma_1, \alpha_1) = (0.2, 1)$ and $(\gamma_2, \alpha_2) = (15, 0.25)$ are shown. We can observe a quite large transient for $(\gamma_1, \alpha_1) = (0.2, 1)$ as shown in the top subplot of Figure 8. However, when we choose $(\gamma_2, \alpha_2) = (15, 0.25)$, i.e., a fractional order reference model is used, the tracking performance is almost perfect as shown in the bottom subplot of Figure 8. Note that in the latter case, the adjustment gain γ can be chosen as large as 15!

5. Concluding Remarks

In this paper, we have presented two ideas to extend the conventional Model Reference Adaptive Control (MRAC) by using fractional order parameter adjustment rule and the employment of fractional order reference model. Through examples, benefits from the use of FOC are illustrated.

Concluding this paper, we offer the following brief remarks regarding the further research:

– The stability bound for γ and α.
– The optimal design of the fractional order α.
– The stability analysis in frequency domain.

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