Algebr Represent Theor (2007) 10:135–155
DOI 10.1007/s10468-006-9039-9

Applications of Duality to the Pure-injective Envelope

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Received: 1 October 2001/ Accepted: 1 February 2005 /
Published online: 19 December 2006

Abstract Given an R-T-bimodule ${}_RK_T$ and an R-S-bimodule ${}_RM_S$, we study how properties of ${}_RK_T$ affect the K-double dual $M^{**} = \mathrm{Hom}_T[\mathrm{Hom}_R(M, K), K]$, considered as a right S-module. If ${}_RK$ is a cogenerator, then for every R-S-bimodule, the natural morphism $\Phi_M : M \to M^{**}$ is a pure-monomorphism of right S-modules. If ${}_RK$ is the minimal (injective) cogenerator and K_T is quasi-injective, then M^{**} is a pure-injective right S-module. If ${}_RK$ is the minimal (injective) cogenerator, and $T = \mathrm{End}_R K$, it is shown that K_T is quasi-injective if and only if the K-topology on R is linearly compact. If the ${}_RK$-topology on R is of finite type, then the natural morphism $\Phi_R : R \to R^{**}$ is the pure-injective envelope of R_R as a right module over itself.

Key words bicommutator · M-topology · minimal injective cogenerator · pure-injective envelope · quasi-injective

Mathematics Subject Classifications (2000) 16D50 · 16D90 · 16S90

Let R and T be associative rings with identity, and ${}_RK_T$ an R-T-bimodule. The bimodule ${}_RK_T$ induces an additive endofunctor $(-)^{**}$: **R-Mod** $\to$ **R-Mod**, the K-double dual, of the category of left R-modules. It is defined by

$$M^{**} = \mathrm{Hom}_R[\mathrm{Hom}_T({}_RM, {}_RK_T), {}_RK_T].$$

There is a natural transformation $\Phi : 1_{\mathbf{R-Mod}} \to (-)^{**}$ from the identity functor to the K-double dual, given by the evaluation morphism $\Phi_M : M \to M^{**}$, $\Phi_M(m) : \xi \mapsto \xi(m)$. The functorial nature of $(-)^{**}$ implies that if M has an R-S-bimodule

Presented by Ken Goodearl.

The author is partially supported by NSF Grant DMS-02-00698.

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structure, then so does M^{**}. Furthermore, the naturality of Φ implies that $\Phi_M : M \to M^{**}$ is an R-S-morphism.

Warfield [18] noticed that if M is a right S-module, viewed as a $\mathbb{Z}$-S-bimodule, and K is the multiplicative group of complex numbers of Modulus 1, viewed as a $\mathbb{Z}$-$\mathbb{Z}$-bimodule (thus $R = T = \mathbb{Z}$), then the right S-module M_S^{**} is pure-injective, and the evaluation map $\Phi_M : M_S \to M_S^{**}$ is a pure-monomorphism of right S-modules. This argument may then be applied to prove the existence [6, 18, 21] of pure-injective envelopes in the category **Mod-S** of right S-modules.

This article is devoted to the study of how conditions on ${}_RK_T$ affect the nature of the evaluation morphism $\Phi_M : M_S \to M_S^{**}$. For example, we prove the following.

Theorem A *Let ${}_RK$ be a left R-module, and $T = End_R K$.*

(1) (*Proposition* 1) *The left R-module ${}_RK$ is a cogenerator if and only if for every R-S-bimodule M, the morphism $\Phi_M : M_S \to M_S^{**}$ is a pure-monomorphism as a morphism of right S-modules.*
(2) (*Theorem* 1) *If ${}_RK$ is the minimal (injective) cogenerator and K_T is quasi-injective, then for every R-S-bimodule M which is finitely generated as an R-module, the K-double dual M_S^{**} is pure-injective as a right S-module.*
(3) (*Corollary* 3) *Suppose that ${}_RK$ is the minimal (injective) cogenerator, K_T is quasi-injective and that the ${}_RK$-topology on ${}_RR$ is of finite type. If ${}_RM_S$ is an R-S-bimodule which is finitely presented as a left R-module, and $M \otimes_S M^* \in \sigma[{}_RK]$, then the evaluation morphism $\Phi_M : M_S \to M_S^{**}$ is the pure-injective envelope of the right S-module M_S.*

For example, if R is a right Noetherian left FBN ring, a left one-dimensional domain, a left Noetherian V-ring or an almost maximal uniserial commutative domain, then the minimal (injective) cogenerator ${}_RK$ has the property that K_T is quasi-injective and the ${}_RK$-topology on R is of finite type. While it may seem standard to prove such results only when ${}_RK$ is the minimal injective cogenerator, we follow Menini and Orsatti [9] by noting that a parallel theory may be developed under the weaker assumption that ${}_RK$ is the minimal cogenerator.

Theorem B (Theorem 5, Proposition 8) *Let ${}_RK_T$ be an R-T-bimodule and ${}_RK$ the minimal (injective) cogenerator. The right T-module K_T is quasi-injective if and only if the ${}_RK$-topology on R is linearly compact, that is, for every open left ideal I of R, the quotient module R/I is linearly compact. In that case, K_T is strongly quasi-injective.*

Recall that a basis of open left ideals of the ${}_RK$-topology on ${}_RR$ is given by the annihilators of finite subsets of K. The ${}_RK$-topology is said to be of finite type if it may be given by a basis of finitely generated left ideals. If ${}_RK$ is the minimal cogenerator, then the injective envelope $E(K)$ is the minimal injective cogenerator. As every open left ideal I in the K-topology is also open in the $E(K)$-topology, Theorem B indicates the advantage of working with the minimal cogenerator.

Müller [12] clarified the relationship between linearly compact modules and duality, while Sandomierski [16] realized the connection between duality and quasi-injective modules. In this paper, we apply results of Menini and Orsatti [9] and Zelmanowitz [20] developed further in this direction to study the bicommutator $B = R^{**}$ of the module ${}_RK$, when ${}_RK$ is a minimal (injective) cogenerator and K_T is quasi-injective. The bicommutator may also be viewed as the K-adic completion [1] of the ring R. Now clearly $R \otimes_R R^* = R \otimes_R K = K \in \sigma[{}_RK]$, so if the ${}_RK$-topology on ${}_RR$ is of finite type, then Theorem A.3 implies that B_R is the pure-injective envelope of R_R.

A left R-module M is called *K-generated* if there is an R-epimorphism $f : K^{(\alpha)} \to M$ from a direct sum of copies of ${}_RK$. Recall (cf. [19, Section 15]) that $\sigma[{}_RK]$ denotes the full subcategory of **R-Mod** of submodules of K-generated modules. As ${}_RK$ is a cogenerator [19, 15.8], restriction of scalars along the ring morphism $\Phi_R : R \to B$ induces an equivalence of categories $\mathrm{res}^B_R : \sigma[{}_BK] \to \sigma[{}_RK]$. Most of our results are consequences of the following.

Theorem C (Theorem 13) *Let ${}_RK$ be the minimal (injective) cogenerator and $T = \mathrm{End}_RK$. If K_T is quasi-injective and the K-topology on ${}_RR$ is of finite type, then the inverse equivalence of res^B_R is given by $B \otimes_R - : \sigma[{}_RK] \to \sigma[{}_BK]$.*

For example, it follows (cf. Lemma 18) that $B = \mathrm{End}_R\, B_R$. As B_R is pure-injective as a right R-module the ring B is the endomorphism ring of a pure-injective module. Thus $B/J(B)$ is von Neumann regular, where $J(B)$ denotes the Jacobson radical of B. The next result is a decomposition theorem which indicates that $B/J(B)$ is essential over its socle.

Theorem D (Corollary 19) (cf. [21, Thm. 6.1]) *Let ${}_RK$ be the minimal (injective) cogenerator and suppose that K_T is quasi-injective. If the ${}_RK$-topology on ${}_RR$ is of finite type, then B_R is the pure-injective envelope (as a right R-module) of a direct sum of pure-injective indecomposable right R-modules B_i :*

$$B_R = \mathrm{PE}(R_R) = \mathrm{PE}(\oplus_i B_i).$$

If B_R is a flat R-module, then so are the B_i.

In all of the examples of rings cited above, the pure injective envelope of R_R is a flat right R-module. Theorem D may thus be used to find examples of pure-injective indecomposable flat right R-modules.

1 Purity

Let R and T be associative rings with identity. The category of left R-modules is denoted by **R-Mod**; the category of right T-modules by **Mod-T**. Throughout the article, we fix an R-T-bimodule ${}_RK_T$, and denote the contravariant functor

$$\mathrm{Hom}_R(-, {}_RK_T) : \textbf{R-Mod} \to \textbf{Mod-T}$$

by $M \mapsto M^*$. When there is no danger of confusion, we use the same notation to denote the functor $\mathrm{Hom}_T(-, {}_RK_T) : \textbf{Mod-T} \to \textbf{R-Mod}$. The association $M \mapsto M^{**}$ is the composition of two contravariant functors, and is therefore a covariant functor

$$(-)^{**} : \textbf{R-Mod} \to \textbf{R-Mod}.$$

There is a natural transformation $\Phi : 1_{\textbf{R-Mod}} \to (-)^{**}$ from the identity functor, where given a left R-module ${}_RM$, the R-morphism $\Phi_M : M \to M^{**}$ is given by the *evaluation map*

$$[\Phi_M(m)](\xi) = \xi(m).$$

Since $(-)^{**}$ is a functor, an R-S-bimodule structure ${}_RM_S$ yields an R-S-bimodule structure on M^{**}. The naturality of Φ implies that $\Phi_M : M \to M^{**}$ is an R-S-bimodule homomorphism.

The module ${}_RK$ is a *cogenerator* if for every left R-module M and nonzero $m \in M$, there is a $\xi \in M^*$ such that $\xi(m) \neq 0$. In other words, ${}_RK$ is a cogenerator if and only if the evaluation map Φ_M is a monomorphism for every left R-module M. A left R-module ${}_RM$ is *K-reflexive* if the embedding $\Phi_M : M \to M^{**}$ is an isomorphism. We say the same for a right T-module V_T if $\Phi_V : V \to V^{**}$ is an isomorphism.

In this paper, we will be interested in two cases:

(1) The module ${}_RK$ is the minimal cogenerator $\oplus_{X\in\Omega} E(X)$, where Ω denotes the set of isomorphism classes of simple left R-modules.
(2) The module ${}_RK$ is the minimal injective cogenerator, the injective envelope of Item (1).

In both cases, we will take $T = \mathrm{End}_R K$. If ${}_RK$ is the minimal cogenerator and n a natural number, then every element $a \in K^n$ is contained in an injective direct summand of ${}_RK^n$. It follows that if $a \in K^m$, $b \in K^n$ are such that $\mathrm{ann}_R(a) \subseteq \mathrm{ann}_R(b)$, then there is an R-morphism $f : {}_RK^m \to {}_RK^n$ such that $f(a) = b$. This property of ${}_RK$ will be invoked without comment.

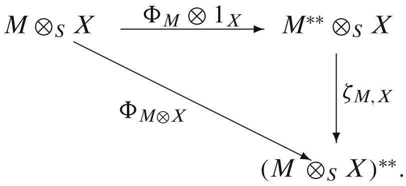
From several equivalent definitions of purity, we choose the following: a morphism of right S-modules $m : M_S \to N_S$ is a *pure monomorphism* if for every left S-module ${}_SX$, the morphism of Abelian groups $m \otimes 1_N : M \otimes_S X \to N \otimes_S X$ is a monomorphism. A right S-module M_S is called *pure-injective* if every pure-monomorphism of right S-modules $m : M_S \to N_S$ has a retraction.

Proposition 1 *Let ${}_RK$ be a cogenerator. If ${}_RM_S$ is an R-S-bimodule, then the evaluation morphism $\Phi_M : M \to M^{**}$ is a pure monomorphism of right S-modules.*

Proof Let ${}_SX$ be a left S-module. We may define a morphism of left R-modules

$$\zeta_{M,X} : M^{**} \otimes_S X \to (M \otimes_S X)^{**}$$

as follows: Let $\xi \otimes x$ be an elementary tensor in $M^{**} \otimes_S X$. If $g \in (M \otimes_S X)^*$, then $g : M \otimes X \to {}_RK$, so that $g(- \otimes x) : {}_RM \to {}_RK$ belongs to M^*. Thus $\xi[g(- \otimes x)]$ makes sense and belongs to K. Define $\zeta_{M,X}(\xi \otimes x) : g \mapsto \xi[g(- \otimes x)]$. It is straightforward to check that $\zeta_{M,X}$ is well-defined and S-bilinear. It therefore extends uniquely to the tensor product $M^{**} \otimes_S X$. It is a morphism of left R-modules and fits into a commutative diagram as follows:



Since the morphism $\Phi_{M\otimes X} : M \otimes_S X \to (M \otimes X)^{**}$ is a monomorphism, the commutativity of the diagram implies that $\Phi_M \otimes 1_X : M \otimes_S X \to M^{**} \otimes_S X$ is also a monomorphism. □

A right T-module U is *K-generated* if there is a T-epimorphism $g : K^{(\alpha)} \to U_T$ from a direct sum of copies of K. Denote by $\sigma[K_T]$ the full subcategory of **Mod-T** of submodules of K-generated modules. The category $\sigma[K_T]$ (cf. [19, Section 15]) is a Grothendieck category and so admits injective envelopes. The module K_T is *quasi-injective* if it is injective as an object of $\sigma[K_T]$. In this case, the functor $(-)^* :$ **Mod-T** $\to$ **R-Mod** is exact when restricted to $\sigma[K_T]$.

Theorem 2 *Suppose that ${}_RK$ is a cogenerator and that K_T is quasi-injective. If ${}_RM_S$ is finitely generated as a left R-module, then M^{**} is pure-injective as a right S-module.*

Proof As ${}_RM$ is finitely generated, there is an epimorphism $f : {}_RR^n \to {}_RM$. Applying the left exact functor $(-)^*$ gives a monomorphism $f^* : M^* \to (R^n)^* = (R^*)^n = (K)^n$ of right T-modules. As $\sigma[K_T]$ is closed under direct sums and submodules, $M^* \in \sigma[K_T]$.

If N_S is a right S-module, then the tensor product $N \otimes_S M^*_T$ belongs to $\sigma[K_T]$. To see this, consider an epimorphism $g : S^{(\alpha)} \to N_S$ from a free right S-module to N_S. The tensor functor $- \otimes_S M^*_T$ preserves epimorphisms, so that we obtain an epimorphism of right T-modules

$$g \otimes 1_{M^*} : (M^*)^{(\alpha)} \to N \otimes_S M^*.$$

Now $(M^*)^{(\alpha)} \in \sigma[K_T]$, since $\sigma[K_T]$ is closed under direct sums, and $N \otimes_S M^* \in \sigma[K_T]$ because $\sigma[K_T]$ is closed under factor modules.

Consider a pure-monomorphism $f : M^{**} \to N_S$. The morphism $f \otimes 1_{M^*} : M^{**} \otimes_S M^* \to N \otimes_S M^*$ is a monomorphism of right T-modules. Both T-modules belong to $\sigma[K_T]$, and, because K_T is injective in $\sigma[K_T]$, we obtain an epimorphism

$$(f \otimes 1, K) : \mathrm{Hom}_T(N \otimes_S M^*, K_T) \to \mathrm{Hom}_T(M^{**} \otimes_S M^*, K_T).$$

The domain is naturally isomorphic to $\mathrm{Hom}_S(N_S, \mathrm{Hom}_T(M^*_T, K_T)) \cong \mathrm{Hom}_S(N_S, M^{**})$; the codomain to $\mathrm{Hom}_T(M^{**}, M^{**}) = \mathrm{End}_T(M^{**})$. Any preimage of the identity in $\mathrm{End}_T(M^{**})$ along these isomorphisms is a retraction of f. □

2 The K-topology

Notation: Unless otherwise specified, ${}_RK$ will from now on denote the minimal cogenerator or the minimal injective cogenerator in R-Mod and $T = \mathbf{End}_R K$. The *K-topology* on R is the left linear topology whose *basic open left ideals* ${}_RI$ are those for which the quotient R/I admits an embedding into a direct sum K^n of finitely many copies of the module ${}_RK$. Equivalently, I is the annihilator of finitely many elements of K. The collection of such ideals is closed under finite intersections, and a left ideal ${}_RI$ is *open* in the K-topology provided it contains a basic open left ideal. The collection of open left ideals is a filter that satisfies the axioms [17, p.144] for a fundamental system of open neighborhoods of 0 for a left linear topology on R.

There is a bijective correspondence [17, Prop. VI.4.2] between left linear topologies on the ring R and hereditary pretorsion classes of **R-Mod**. Associated to the K-topology on R is $\sigma[{}_RK]$, which is also the smallest hereditary pretorsion class of **R-Mod** containing K. By Theorem IV.5.1 of [17], the K-topology on R is a left Gabriel topology if and only if the category $\sigma[{}_RK]$ is closed under extensions.

If ${}_RK$ is the minimal cogenerator, then a left ideal $I \subseteq R$ is a basic open left ideal if and only if it is *cofinite,* that is, the module R/I is finitely cogenerated. Evidently, every basic open left ideal in the K-topology is a basic open left ideal in the $E(K)$-topology. The topologies are the same if every cyclic submodule of $E(K)$ is finitely cogenerated. This occurs, for example, when the ring R is left Noetherian or if there are only finitely many simple left R-modules up to isomorphism.

Lemma 3 *If the left ideal $I \subseteq R$ is a basic open left ideal in the K-topology, then*

$$(R/I)^* \cong ann_K(I)$$

is a finitely generated T-submodule of K_T. In that case, the left R-module R/I is K-reflexive.

Proof First note that the isomorphism is given by the rule $\zeta \mapsto \zeta(1+I)$. If $I \subseteq R$ is a basic open left ideal, then R/I admits an embedding into a finite direct sum of copies of ${}_RK$, say $f: R/I \to {}_RK^n$, for some n. Write $(1+I)f = a = \Sigma_{i=1}^n a_i$, where a_i belongs to the ith copy of K. If $v \in \mathrm{ann}_K(I)$, then there is an R-morphism $g: K^n \to K$ such that

$$v = (a)g = \Sigma_{i=1}^n (a_i)g = \Sigma_{i=1}^n (a_i)g_i,$$

where g_i is the restriction of g to the ith copy of K. Thus $g_i \in T$ and $\mathrm{ann}_K(I) = \Sigma_i a_i T$.

Recall that a left R-module M is called *K-dense* (cf. [19, p. 430]) if given $\xi \in M^{**}$ and finitely many $f_1, \ldots, f_n \in M^*$, there is an $m \in M$ such that for every i, $1 \leq i \leq n$, $\xi(f_i) = \Phi_M(m)(f_i) = (m) f_i$. Since $_RK$ is a cogenerator, [19, 47.6(4)] implies that every left R-module is K-dense. So if $_RM$ is such that M^*_T is finitely generated, then $\Phi_M : M \to M^{**}$ is onto and the left R-module M is K-reflexive. If $I \subseteq R$ is a basic open left ideal of R, then the dual module $(R/I)^*$ is a finitely generated right T-module. Thus R/I is K-reflexive. □

Lemma 4 *Every finitely generated T-submodule V_T of K_T is of the form*

$$V_T = ann_K(I) = (R/I)^*$$

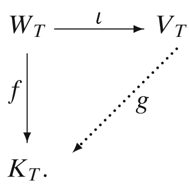
for some basic open left ideal $I \subseteq R$ in the K-topology. In that case, $ann_R(V) = I$.

Proof Let $V_T = \Sigma_{i=1}^n a_i T$ be a finitely generated submodule of K_T. Then $I = \mathrm{ann}_R(V) = \cap_{i=1}^n \mathrm{ann}_R(a_i)$ is a basic open left ideal of R and $V \subseteq \mathrm{ann}_K(I)$. Consider the element $a = \Sigma_{i=1}^n a_i \in K^n$, where a_i belongs to the ith copy of K. Then $I = \mathrm{ann}_R(a)$, so if $b \in \mathrm{ann}_K(I)$, then there is a morphism $f : K^n \to K$ such that $(a) f = b$. But then $b = \Sigma_{i=1}^n (a_i) f = \Sigma_{i=1}^n (a_i) f_i$, where f_i is the restriction of f to the ith copy of K. It follows that $b \in \Sigma_i a_i T = V$. □

The two lemmas imply that the rule $I \mapsto \mathrm{ann}_K(I)$ is an inclusion-reversing correspondence between the basic open left ideals I and the finitely generated submodules V_T of K_T; the inverse is given by $V_T \mapsto \mathrm{ann}_R(V)$.

3 Quasi-injectivity

Recall that the module K_T is *V-injective* (cf. [19, Section 16], [11, Section I.2]) if given any submodule $W_T \subseteq V_T$, every T-morphism $f : W_T \to K_T$ may be extended to a morphism $g : V_T \to K_T$;



By [19, 16.3] or [11, Prop. 1.4], K_T is quasi-injective if and only if it is V_T-injective for every finitely generated submodule $V_T \subseteq K_T$.

A left R-module $_RM$ is *linearly compact* if every family $\{a_i + N\}_{i \in I}$ of cosets of submodules that satisfies the *finite intersection property* has a non-empty intersection:

$$\bigcap_{i \in I} a_i + N \neq \emptyset.$$

The K-topology on R is said to be *linearly compact* if for every open left ideal $I \subseteq R$, the quotient module R/I is linearly compact.

Theorem 5 *The module K_T is quasi-injective if and only if the K-topology on R is linearly compact.*

Proof Suppose that K_T is quasi-injective. If $I \subseteq R$ is a basic left ideal, then by Lemma 3, R/I is K-reflexive and $(R/I)^*$ is a finitely generated submodule of K_T. As K_T is $(R/I)^*$-injective, [19, 47.8(2)] implies that R/I is linearly compact. By [19, 29.8(2)], any factor module of a linearly compact module is itself linearly compact. It follows that R/I is linearly compact for any open left ideal I of R.

Conversely, assume that the K-topology on R is linearly compact. In particular, R/I is linearly compact for every basic open left ideal I of R. By [19, 47.8(1.ii)], K_T is $(R/I)^*$-injective. By Lemma 4, every finitely generated submodule of K_T is of the form $(R/I)^*$ for some basic open left ideal $I \subseteq R$. Thus K_T is quasi-injective. □

If a short exact sequence of left R-modules is given

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0,$$

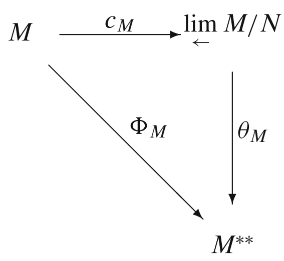
then by [19, 29.8(2)], the module ${}_RM$ is linearly compact if and only if N and M/N are linearly compact. So if the K-topology on R is linearly compact, then every cyclic submodule of ${}_RK$ is linearly compact, and therefore, every finitely generated submodule of ${}_RK$ is linearly compact. A module ${}_RM$ is termed *locally linearly compact,* if every finitely generated submodule is linearly compact. We conclude that the K-topology is linearly compact if and only if every module in the category $\sigma[{}_RK]$ is locally linearly compact.

Given a left R-module ${}_RM$, the K-topology on M is defined to be the linear topology given by the fundamental system of open neighborhoods of zero that consists of those ${}_RN \subseteq {}_RM$ for which M/N admits an embedding into a direct sum K^n of finitely many copies of ${}_RK$.

Proposition 6 *Suppose that K_T is quasi-injective and let ${}_RM$ be finitely generated. Given a submodule ${}_RN \subseteq {}_RM$, the quotient module M/N belongs to $\sigma[{}_RK]$ if and only if it admits an embedding into some direct sum K^n of finitely many copies of ${}_RK$.*

Proof Only one direction requires proof, so assume that $M/N \in \sigma[{}_RK]$. As M/N is a finitely generated left R-module, the foregoing comments imply that M/N is linearly compact. It is therefore essential over a socle of finite length and thus admits an embedding into some K^n. □

If M is a left R-module, then the *K-adic completion* of M with respect to the K-topology is the inverse limit $\varprojlim M/N$ indexed by the basic submodules of M in the K-topology. The universal property of the inverse limit ensures the existence of the obvious R-morphism $c_M : M \to \varprojlim M/N$. Since every left R-module is K-dense, Proposition 2.1 and Corollary 2.7 of [1] and Proposition 6 imply that if ${}_RM$ is finitely generated, then there is a natural isomorphism $\theta_M : \varprojlim M/N \to M^{**}$ of left R-modules such that the diagram



commutes.

Let us consider some examples of rings R for which the minimal (injective) cogenerator ${}_RK$ is quasi-injective when considered as a module K_T over its endomorphism ring $T = \mathrm{End}_R K$. In all of the examples, save perhaps Example (3), every cyclic submodule of the minimal injective cogenerator is finitely cogenerated, so that the K-topologies are the same, whether K is the minimal cogenerator or the minimal injective cogenerator.

Examples

(1) If R is a right Noetherian left FBN ring (cf. [4, Ch. 8]) and R/I is a cyclic module essential over its socle, then Theorem 8.11 of [4] implies that R/I is Artinian. Whether ${}_RK$ is the minimal cogenerator or the minimal injective cogenerator, the open left ideals I of the K-topology are those for which the quotient R/I is of finite length. This example includes all commutative Noetherian rings A; in that case, a basis of open ideals may be given by finite products of maximal ideals (cf. [18]).

(2) Let R be a *left one-dimensional domain* (cf. [2]). This means that R is a domain with the property that for every nonzero left ideal I of R, the quotient module R/I is Artinian. There are Noetherian examples of one-dimensional domains that do not belong to the class described in Example (1). For example, the first Weyl algebra $A_1(k)$ over an algebraically closed field k of characteristic zero is a Noetherian one-dimensional domain that is not FBN.

(3) A ring R is called a left *V-ring* if every simple left R-module is injective. If ${}_RK$ is the minimal cogenerator and $I \subseteq R$ a basic left ideal in the K-topology, then R/I is semisimple of finite length. If ${}_RM$ is a finitely generated left R-module, the foregoing discussion shows that ${}_RM^{**}$ is isomorphic to the projective limit of an inverse system of semisimple modules of finite length, with all the structural morphisms epimorphisms.

(4) A uniserial commutative domain D is called *almost maximal* [3, p. 78] if every collection $\{r_i + I_i\}_i$ of cosets of ideals I_i of R, that satisfies the finite intersection property and $\cap_i I_i \neq 0$, has nonempty intersection. Such a domain D is a local ring, with maximal ideal P, and the minimal (injective) cogenerator is the injective envelope ${}_RK = E(D/P)$ of the unique simple module over D. If

$I \subseteq D$ is a basic open ideal, then D/I is a uniform module, and therefore may be embedded into K. It follows that D/I has a simple submodule, and since D is a domain, $I \neq 0$. By the almost maximal property of D, the quotient module D/I is linearly compact.

(5) Let R be a local ring with maximal ideal J with $J^2 = 0$ (cf. [22]). Suppose further that ${}_RJ$ is infinitely generated, while J_R is simple. As in the case of the uniserial domain, the minimal (injective) cogenerator is the injective envelope ${}_RK = E(R/J)$ of the unique simple left R-module. If I is an open left ideal, then R/I embeds into a finite direct sum K^n, so that R/I is essential over a socle of finite length. It follows that R/I must be of finite length. Let us note that the K-topology on R is not a Gabriel topology or, equivalently, that the category $\sigma[{}_RK]$ is not closed under extensions. Indeed, ${}_RJ$ and R/J both belong to $\sigma[{}_RK]$, but ${}_RR$ does not (since it is not linearly compact).

All the examples above, save perhaps the almost maximal uniserial domain, have the property that the K-topology is *Artinian*, that is, for every basic left ideal $I \subseteq {}_RR$ the quotient module R/I is Artinian. This case was considered by Menini and Orsatti [10]. Recall that a module is said to be *locally Noetherian* if every finitely generated submodule is Noetherian.

Theorem 7 *The K-topology on R is Artinian if and only if the module K_T is locally Noetherian.*

Proof If the K-topology is Artinian, then K_T is locally Noetherian by [10, Prop. 2.3.a]. Suppose now that K_T is locally Noetherian, and let $I \subseteq R$ be a basic open left ideal in the K-topology. The T-module $V_T = \mathrm{ann}_K(I)$ is finitely generated, and therefore Noetherian. If

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I$$

were a properly descending chain of left ideals containing I, then, as ${}_RK$ is a cogenerator, the corresponding chain of annihilators would be a proper *ascending* chain

$$0 = \mathrm{ann}_K(I_0) \subseteq \mathrm{ann}_K(I_1) \subseteq \mathrm{ann}_K(I_2) \subseteq \cdots \subseteq \mathrm{ann}_K(I) = V_T,$$

contradicting the assumption that V_T is Noetherian. □

Under the conditions of Theorem 7, every module in $\sigma[{}_RK]$ is locally Artinian and every module in $\sigma[K_T]$ is locally Noetherian.

A quasi-injective right T-module V_T is called *strongly quasi-injective* if it is a self cogenerator, that is, if it is a cogenerator in the category $\sigma[V_T]$.

Proposition 8 If K_T is quasi-injective, then K_T is strongly quasi-injective.

Proof Since K_T is an injective object of $\sigma[K_T]$, it suffices to verify that every simple module $X_T \in \sigma[K_T]$ may be embedded into K_T. There is a finitely generated $V_T \subseteq K_T$ for which there is a nonzero morphism $\eta : V_T \to X_T$. By Lemma 4, there is a basic open left ideal $I \subseteq R$ for which $V = \mathrm{ann}_K(I)$. Consider the kernel $W_T = \mathrm{Ker}\,\eta$ and express it as the directed sum $W_T = \Sigma_i W_i$ of its finitely generated submodules. For each W_i, the left ideal $I_i = \mathrm{ann}_R(W_i) \subseteq R$ is a basic open left ideal and $I_i \supseteq I$. As all of these inclusion are proper, each of the quotient maps $p_i : R/I \to R/I_i$ has nonzero kernel. Because R/I is linearly compact, it is finitely cogenerated, so the intersection of the kernels is nonzero. Thus the inclusion $I' = \cap_i I_i \supseteq I$ is proper. Let $r \in I' \setminus I$ and consider the morphism $\lambda_r : V \to K_T$ defined by the action of r on K_T from the left. Since $r \notin I$, the morphism λ_r is nonzero. As $r \in I'$, the kernel of λ_r contains all the W_i, hence W_T. The image of the morphism is therefore isomorphic to the simple module $V/W \cong X_T$. □

4 The Bicommutator

Assumption: From now on, suppose that K_T is quasi-injective. The K-double dual, or, equivalently, the K-adic completion, of the module ${}_RR$ will be denoted $B := R^{**} = \mathrm{End}_T(K_T)$. Since T is the endomorphism ring of ${}_RK$, the ring B is the *bicommutator* of the left R-module ${}_RK$. The action of B gives K the structure of a left B-module. As the K-double dual of R, the bicommutator comes equipped with the structure of an R-ring induced by the evaluation morphism $\Phi_R : R \to B$, which is also a morphism of rings.

The B-T-bimodule ${}_BK_T$ is *balanced* in the sense that $\mathrm{End}_B({}_BK) = T$ and $\mathrm{End}_T(K_T) = B$. As in the case of the R-T-bimodule ${}_RK_T$, two contravariant functors $\mathrm{Hom}_B(-, {}_BK)$: **B-Mod** → **Mod-T** and $\mathrm{Hom}_T(-, {}_BK_T)$: **Mod-T** → **B-Mod** are induced. We denote both functors by $(-)^\dagger$ and a left B-module M is ${}_BK_T$*-reflexive* if the natural transformation Ψ (given by the evaluation map) from the identity functor on **B-Mod** to the functor $(-)^{\dagger\dagger}$ is a B-isomorphism at ${}_BM$.

The *restriction of scalars* functor res^B_R : **B-Mod** → **R-Mod** along the ring morphism $\Phi_R : R \to B$, associates to every left B-module ${}_BM$ a left R-module structure denoted ${}_RM$. To the B-module ${}_BK$ is associated the R-module ${}_RK$, so this functor induces a functor from $\sigma[{}_BK]$ to $\sigma[{}_RK]$.

Proposition 9 (cf. [1]) The restriction of scalars functor along the ring morphism $\Phi_R : R \to B$. induces an equivalence of categories,

$$\mathrm{res}^B_R : \sigma[{}_BK] \subseteq \sigma[{}_RK].$$

Proof Since ${}_RK$ is a cogenerator, $\Phi_R(R) \subseteq B$ is dense [19, 47.6(4)]. By [19, 15.8], the functor res^B_R is an equivalence of categories. □

The theorem implies that for ${}_BM \in \sigma[{}_BK]$ the restriction functor res^B_R commutes with the K-double dual. For, consider the diagram

$$\begin{array}{ccc} {}_BM & \xrightarrow{\mathrm{res}^B_R} & {}_RM \\ \Big\downarrow{\scriptstyle \Psi_M} & & \Big\downarrow{\scriptstyle \Phi_M} \\ M^{\dagger\dagger} & \xrightarrow{\mathrm{res}^B_R} & ({}_RM)^{**}. \end{array}$$

We claim that it is commutative. First note that $\mathrm{res}^B_R(M^{\dagger\dagger}) = M^{\dagger *}$. Then, as both ${}_BM$ and ${}_BK$ belong to $\sigma[{}_BK]$, the equivalence given by the theorem implies that $M^\dagger_T \cong M^*_T$. One consequence is that a left B-module M in $\sigma[{}_BK]$ is ${}_BK$-reflexive if and only if ${}_RM = \mathrm{res}^B_R(M)$ is ${}_RK$-reflexive.

In view of the fact that the bimodule ${}_BK_T$ is balanced, greater effort (cf. [9, 20]) has been devoted to the study of the K-duality induced between the categories **Mod-T** and **B-Mod**. If ${}_RK$ is the minimal injective cogenerator, then because it is locally linearly compact, the pair (T, K_T) is an example of what Menini [8] calls an *l-couple* for R. As K_T is strongly quasi-injective, the ${}_BK_T$-reflexive right T-modules have been characterized by Zelmanowitz [20, Thm. 3.3] as the modules copresented by K_T. For the class of submodules of K_T this characterization yields the following.

Theorem 10 (cf. [9],[20]) *There is an inclusion-reversing bijective correspondence between the submodules $V_T \subseteq K_T$ and left ideals ${}_BX \subseteq B$ which are closed in the ${}_BK$-topology on ${}_BB$. The correspondence is given by the rules*

$$V_T \mapsto \mathrm{ann}_B(V) \text{ and } {}_BX \mapsto \mathrm{ann}_K(X),$$

which are mutual inverses. The left ideal ${}_BX \subseteq B$ is open in the ${}_BK$-topology if and only if it is the anihilator of a finitely generated submodule $V_T \subseteq K_T$.

Proof Everything in the first sentence follows from [9, Thm. 4.7] and [20, Thm. 3.3], except for the assertion that $\mathrm{ann}_B(V)$ is closed in the ${}_BK$-topology. But that is immediate from the second statement and the observation that $\mathrm{ann}_B(V) = \cap_{V'} \mathrm{ann}_B(V')$, where the intersection is indexed by the finitely generated submodules of V_T. The second statement is proved in the same manner as were Lemmas 3 and 4; one appeals to the consequence of Proposition 9 that ${}_BK$ is the minimal (injective) cogenerator of $\sigma[{}_BK]$. □

Let us give some examples of left ideals of B that are closed in the ${}_BK$-topology:

(1) Let $I \subseteq R$ be a left ideal and $V_T = \mathrm{ann}_K(I)$, and denote by $\mathrm{cl}(I)$ the closure of I in B. Then $\mathrm{cl}(I) = \mathrm{ann}_B(V)$. If $I \subseteq R$ is a two-sided ideal, then $\mathrm{cl}(I)$ is a two-sided ideal of B. This is because $V_T = \mathrm{ann}_T(I)$ is an R-T-subbimodule of ${}_RK_T$ and is therefore, by Proposition 9 a B-T-subbimodule.

(2) Suppose that $Y_B \subseteq B$ is a right ideal of B. Because B is the bicommutator of ${}_RK$, the left annihilator of Y in B is $\mathrm{ann}_B(YK_T)$ and is therefore a closed left ideal. In particular, if $e \in B$ is idempotent, then the summand $Be = \mathrm{ann}_B[(1-e)K]$ is closed.

(3) By Lemma 4 and the fact that ${}_RK$ is a cogenerator, the simple submodules of K_T are of the form $\mathrm{ann}_K(I)$, where I is a maximal left ideal of R. It follows that $\mathrm{soc}_R(K) = \mathrm{soc}_T(K)$ is a B-T-bimodule of ${}_BK_T$. Since K_T is an injective object of $\sigma[K_T]$ and is essential over its socle, the Jacobson radical $J(B)$ of the endomorphism ring $B = \mathrm{End}_T(K_T)$ is given [19, 22.1(1)] by

$$J(B) = \mathrm{ann}_B[\mathrm{soc}(K)].$$

If R is a left Noetherian V-ring, then $K = \mathrm{soc}(K)$, so that $J(B) = 0$. Since B is the endomorphism ring of an injective object of $\sigma[{}_RK]$, [19, 22.1(1)] implies that $B = B/J(B)$ is von Neumann regular.

5 Topologies of Finite Type

All of the examples given in Section 3, except the last, have the property that the K-topology on R may be given by a fundamental system of open neighborhoods of 0 which are finitely generated left ideals. Such a linear topology is said to be of *finite type*. Examples of linear topologies of finite type are the finite matrix topologies considered in [5]; Gabriel topologies of finite type are treated in [17, Section XIII.1].

Consider the following commutative diagram.

$$\begin{array}{ccccccc}
M_1 & \xrightarrow{f} & M_2 & \xrightarrow{g} & M_3 & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Phi_{M_1}} & & \downarrow{\scriptstyle \Phi_{M_2}} & & \downarrow{\scriptstyle \Phi_{M_3}} & & \\
M_1^{**} & \xrightarrow{f^{**}} & M_2^{**} & \xrightarrow{g} & M_3^{**} & \longrightarrow & 0
\end{array}$$

If the top row is exact and consists of finitely generated left R-modules, then, as in the proof of Theorem 2, the right T-modules M_i^* will all belong to $\sigma[{}_RK]$. As K_T is quasi-injective, the functor $(-)^*$ is exact on $\sigma[{}_RK]$. So if the K-double dual functor is restricted to the finitely generated left R-modules, it is the composition of a contravariant left exact functor with a contravariant exact functor. Thus it is a right exact functor on the finitely generated modules. We infer that the bottom row is an exact sequence of left R-modules (resp., B-modules). In particular, if M_1 and M_2 are K-reflexive, then so is M_3.

Proposition 11 *The K-topology on R is of finite type if and only if every module $M \in \sigma[{}_RK]$ is a direct limit $M = \varinjlim M_i$ of finitely presented K-reflexive modules M_i that belong to $\sigma[{}_RK]$.*

Proof If the K-topology on R is of finite type, then the finitely presented modules of the form R/I, where I is a basic finitely generated open left ideal of R, form a generating set for the category $\sigma[{}_RK]$. Such a module R/I is finitely presented and K-reflexive. Every module M in $\sigma[{}_RK]$ may therefore be presented by coproducts of such modules

$$\oplus_{a\in A} R/I_a \xrightarrow{\quad f \quad} \oplus_{b\in B} R/I_b \longrightarrow {}_RM \longrightarrow 0.$$

An argument as in [7, Appendice] shows that M is the direct limit of finitely presented modules M_i each of which is given by a finite subpresentation of the above. More precisely, for each i, there are finite subsets $A_i \subseteq A$, $B_i \subseteq B$ such that a presentation of M_i is given by

$$\oplus_{a\in A_i} R/I_a \xrightarrow{\quad f_i \quad} \oplus_{b\in B_i} R/I_b \longrightarrow {}_R M_i \longrightarrow 0,$$

where f_i is the restriction of f to $\oplus_{a\in A_i} R/I_a$. Both direct sums are finite hence K-reflexive. The foregoing observations indicate that each M_i is also K-reflexive.

Conversely, suppose that every $M \in \sigma[{}_R K]$ is the direct limit of finitely presented modules in $\sigma[{}_R K]$. If $I \subseteq R$ is an open left ideal, the hypothesis implies that there is an epimorphism $g : {}_R N \to R/I$, where ${}_R N$ is a finitely presented module belonging to $\sigma[{}_R K]$. Now $R/I = \varinjlim R/I'$, where the direct limit is indexed by the finitely generated left ideals $I' \subseteq I$. As ${}_R N$ is finitely presented, the functor $\mathrm{Hom}_R(N, -)$ commutes with direct limits [17, Prop. V.3.4]. The morphism $g : N \to R/I$ thus factors through one of the quotient maps $p : R/I' \to R/I$, where $I' \subseteq I$ is finitely generated,

$$\begin{array}{ccc} N & & \\ {\scriptstyle f}\downarrow & \searrow^{g} & \\ R/I' & \xrightarrow{\ p\ } & R/I. \end{array}$$

Let $J_0 \subseteq R$ be a finitely generated left ideal with the property that $(J_0 + I')/I'$ is the image of f. As g is an epimorphism, $J_0 + I = R$. Thus there is a finitely generated left ideal $I_0 \subseteq I$ such that $J_0 + I_0 = R$. Now replace I' with $I' + I_0$. The diagram remains commutative, with the added feature that f is now an epimorphism. Thus $R/I' \in \sigma[{}_R K]$ and hence $I' \subseteq I$ is a left ideal of R, which is finitely generated and open in the K-topology. □

Recall the morphism $\zeta_{R,M} : {}_R B \otimes_R M \to {}_R M^{**}$ used in the proof of Proposition 1. It is natural in ${}_R M$, which means that

$$\zeta_{R,-} : {}_R B \otimes_R - \to (-)^{**}$$

is a natural transformation of endofunctors of **R-Mod**. The naturality implies that if M is an R-S-bimodule, then $\zeta_{R,M}$ is an R-S-morphism. The following proposition implies that if ${}_R M_S$ is an R-S-bimodule such that ${}_R M$ is finitely presented, then the right S-module $B \otimes_R M_S$ is pure-injective.

Proposition 12 *Restricted to the category* **R-mod** *of finitely presented left R-modules, the natural transformation* $\zeta_{R,-} : {}_R B \otimes_R - \to (-)^{**}$ *is a natural isomorphism.*

Proof We need to prove that for every finitely presented left R-module M, the R-morphism $\zeta_{R,M} : {}_R B \otimes_R M \to M^{**}$ is an R-isomorphism. The functors $B \otimes_R -$ and $(-)^{**}$ agree on the value at R, $B \otimes_R R = B = R^{**}$. Thus they agree on every finitely generated free left R-module. The functor $(-)^{**}$ is right exact when restricted to finitely generated left R-modules, so if we apply the two functors to a free presentation

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0$$

of the finitely presented left R-module ${}_RM$ (cf. the argument in [17, Proposition IV.10.1]), then the naturality of the morphism ζ ensures that $\zeta_{R,M} : B \otimes_R M \to M^{**}$ is an R-isomorphism,

$$\begin{array}{ccccccc} B \otimes R^m & \xrightarrow{f} & B \otimes R^n & \longrightarrow & B \otimes M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \zeta_{R,R^m}} & & \downarrow{\scriptstyle \zeta_{R,R^n}} & & \downarrow{\scriptstyle \zeta_{R,M}} & & \\ B^m & \xrightarrow{f^{**}} & B^n & \longrightarrow & M^{**} & \longrightarrow & 0. \end{array}$$

□

Theorem 13 *If the K-topology on ${}_RR$ is of finite type, then the functor $B \otimes_R - : \sigma[{}_RK] \to \sigma[{}_BK]$ is the equivalence inverse of res_R^B.*

Proof We need to prove that ${}_RB \otimes_R - : \sigma[{}_RK] \to \sigma[{}_RK]$ is naturally isomorphic to the identity functor on $\sigma[{}_RK]$, and that $B \otimes_R \mathrm{res}_R^B(-) : \sigma[{}_BK] \to \sigma[{}_BK]$ is naturally isomorphic to the identity functor on $\sigma[{}_BK]$.

Proposition 12 implies that $\zeta_{R,M} : {}_RB \otimes_R M \to M^{**}$ is a natural isomorphism on the category of finitely presented left R-modules. As $M \mapsto M^{**}$ is a natural isomorphism on the category of K-reflexive module, we see that the identity functor and the functor ${}_RB \otimes_R -$ are naturally isomorphic on the category of K-reflexive finitely presented modules. Now both the identity functor and ${}_RB \otimes_R -$ commute with direct limits, and Proposition 11 implies that every object of $\sigma[{}_RK]$ is a direct limit of K-reflexive finitely presented modules in $\sigma[{}_RK]$. The natural isomorphism from ${}_RB \otimes_R -$ to the identity functor therefore extends to all of $\sigma[{}_RK]$ via the direct limit.

To show that the endofunctor $B \otimes_R \mathrm{res}_R^B(-)$ of $\sigma[{}_BK]$ is isomorphic to the identity functor, we proceed similarly, by first showing that the two are naturally isomorphic on the subcategory of $\sigma[{}_BK]$ of ${}_BK$-reflexive modules ${}_BM$ for which ${}_RM$ is finitely presented. Theorem 9, the comments that follow the theorem, and Proposition 11 imply that every module in $\sigma[{}_BK]$ is a direct limit of modules in this subcategory. As both functors commute with direct limits, the isomorphism will extend to all of $\sigma[{}_BK]$.

Notice first that the definition of $\zeta_{R,-}$ may be used to define a natural transformation

$$\zeta'_{R,-} : {}_BB \otimes_R - \to (-)^{*\dagger},$$

of functors from **R-Mod** to **B-Mod**. If ${}_BM$ is K-reflexive and ${}_RM$ is finitely presented, then we get natural isomorphisms

$$\begin{aligned} {}_BB \otimes_R \mathrm{res}_R^B(M) &= {}_BB \otimes_R M \\ &\cong ({}_RM)^{*\dagger} \cong ({}_BM)^{\dagger\dagger} \cong {}_BM. \end{aligned}$$

The first natural isomorphism follows from Proposition 12; the second from the consequence $({}_RM)^* \cong ({}_BM)^\dagger$ of Theorem 9; and the third from the assumption on ${}_BM$ that it is ${}_BK$-reflexive. □

The main point of the theorem is that for a left R-module M in $\sigma[{}_RK]$ the tensor product ${}_BB \otimes_R M$ is a B-module in $\sigma[{}_BK]$ whose restriction to R is naturally isomorphic to M.

6 Applications

Assumption: Unless otherwise specified, we assume from now on that the K-topology on ${}_RR$ is of finite type. Let us describe some of the consequences of Theorem 13. In the following, the embedding $R_R \subseteq B_R$ is understood to be given by Φ_R.

Corollary 14 *For every left R-module $M \in \sigma[{}_RK]$, $B/R \otimes_R M = 0$.*

Proof Apply the functor $- \otimes_R M$ to the pure-exact sequence of right R-modules

$$0 \longrightarrow R_R \xrightarrow{\Phi_R} B_R \longrightarrow B/R \longrightarrow 0$$

to obtain the exact sequence of left R-modules

$$0 \longrightarrow {}_RM \xrightarrow{\Phi_R \otimes 1_M} B \otimes_R M \longrightarrow B/R \otimes_R M \longrightarrow 0.$$

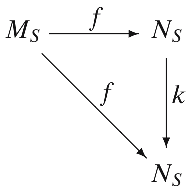
By Theorem 13, the morphism $\Phi_R \otimes 1_M$ is an isomorphism of left R-modules. □

The pure-injective envelope of a right S-module M_S is a pure-monomorphism $f : M_S \to N_S$ with N_S pure-injective such that:

(1) Any S-morphism $g : M_S \to N'_S$ with N'_S pure-injective factors through f,

$$\begin{array}{ccc} M_S & \xrightarrow{f} & N_S \\ & \searrow^{g} & \vdots\, h \\ & & N'_S. \end{array}$$

(2) Any endomorphism $k : N_S \to N_S$ for which the diagram



commutes is an automorphism.

The second condition ensures that the pure-injective envelope of a right S-module M_S is unique up to isomorphism over M_S. It will be denoted by $\mathrm{PE}_S(M)$.

Corollary 15 *The morphism $\Phi_R : R \to B$ of right R-modules is the pure-injective envelope of R_R.*

Proof Since $\Phi_R : R_R \to B_R$ is a pure-monomorphism, any pure-monomorphism $f : R_R \to N_R$, with N_R pure-injective, factors through Φ_R. We will prove that any endomorphism $h : B_R \to B_R$ that fixes R pointwise is the identity $1_B : B_R \to B_R$ by showing that $\mathrm{Hom}_R(B/R, B) = 0$. Then, if $h : B_R \to B_R$ fixed R pointwise, the induced morphism $1_B - h\ : B/R \to B$ would be zero, $h = 1_B$. Corollary 13 implies $B/R \otimes_R K = 0$, so that

$$\begin{aligned}\mathrm{Hom}_R(B/R, B) &= \mathrm{Hom}_R(B/R, \mathrm{Hom}_T({}_RK_T, K_T)) \\ &\cong \mathrm{Hom}_T(B/R \otimes_R K_T, K_T) = 0.\end{aligned}$$

□

Example (5) of Section 3 shows that when the K-topology is not of finite type, the bicommutator of ${}_RK$ need not be the pure-injective envelope of R_R. Indeed, the ring R is pure-injective as a right module over itself. It is therefore its own pure-injective envelope, while the bicommutator B_R of the minimal (injective) cogenerator ${}_RK$ is a proper extension of R_R.

Theorem 2 showed that if ${}_RM_S$ is finitely generated as a left R-module, then the evaluation morphism $\Phi_M : M_S \to M^{**}_S$ is a pure monomorphism into a pure-injective right S-module. The following result provides a sufficient condition for the S-morphism Φ_M to be the pure-injective envelope.

Corollary 16 *If the R-S-bimodule ${}_RM_S$ is finitely presented as an R-module and ${}_RM \otimes_S M^* \in \sigma[{}_RK]$, then the pure-injective envelope of M_S is given by $\Phi_M : M_S \to M^{**}_S$.*

Proof As in the previous proof, we will show that $\mathrm{Hom}_S(M^{**}/M, M^{**}) = 0$. An argument as in the proof of Corollary 14 together with the isomorphism $M^{**} \cong B \otimes_R M$ of Proposition 12 shows that M^{**}/M is isomorphic to $B/R \otimes_R M_S$. Now if

$M \otimes_S M^* \in \sigma[{}_R K]$, then Corollary 14 implies that $B/R \otimes_R M \otimes_S M^* = 0$. It follows that

$$\begin{aligned}\mathrm{Hom}_S(M^{**}/M, M^{**}) &\cong \mathrm{Hom}_S(B/R \otimes_R M, M^{**}) \\ &= \mathrm{Hom}_S(B/R \otimes_R M_S, \mathrm{Hom}_T({}_S M^*_T, K_T)) \\ &\cong \mathrm{Hom}_T(B/R \otimes_R M \otimes_S M^*_T, K_T)) = 0. \qquad \square\end{aligned}$$

Let S be a ring whose center $R = C(S)$ is a Noetherian ring and which is finitely generated as an R-module. If M_S is finitely generated, then it is also finitely generated over the center R. We can consider M as an R-S-bimodule that is finitely presented as a left R-module. As K is a faithful R-module and R is commutative, there is an obvious embedding $R \subseteq T$. As ${}_R M$ is finitely generated, the dual module $M^*_T \in \sigma[K_T]$ (cf. proof of Theorem 2). Thus M^* considered as an R-module belongs to $\sigma[{}_R K]$ and therefore $M \otimes_S M^*$ considered as an R-module belongs to $\sigma[{}_R K]$ (*ibid*). The corollary implies that the pure-injective envelope of M_S is given by the evaluation morphism $\Phi_M : M_S \to M^{**}_S$.

The pure-injective envelope $M_S \subseteq \mathrm{PE}_S(M)$ of a right S-module is an elementary extension (cf. [13, Thm. 4.21], [16]). If the ring R is left coherent, then the class of flat right R-modules is elementary (cf. [13, Thm. 14.18],[15]), so that the pure-injective envelope of R_R is flat. For example, suppose that R is left Noetherian. Then the K-topology on ${}_R R$ is of finite type, so the bicommutator B_R of ${}_R K$ is flat. If B_R is flat, Proposition 12 implies that K-adic completion is an exact functor when restricted to the finitely presented left R-modules.

Corollary 17 *Let R be a left one-dimensional domain. The bicommutator B_R of ${}_R K$ is flat.*

Proof A left one-dimensional domain is a left Ore domain, so its left field of fractions Q is flat when considered as a right R-module. We will prove that the right R-module *B/R* is a vector space over Q and is therefore itself flat. Considering the short exact sequence

$$0 \longrightarrow R_R \xrightarrow{\Phi_R} B_R \longrightarrow B/R \longrightarrow 0$$

immediately yields the result.

Let $r \in R$ be nonzero. Then R/Rr is Artinian and so belongs to $\sigma[{}_R K]$. By Corollary 14,

$$(B/R) \otimes_R R/Rr \cong (B/R)/(B/R)r = 0,$$

so that B/R is a divisible right R-module. To see that it is torsion-free, suppose there is an element $b \in B$ such that $br \in R$. As R_R is pure in B_R, there is a $b' \in R$ such that $br = b'r$. Thus $(b - b')r = 0$. But R_R is torsion-free and B_R is an elementary extension, so that B_R is also torsion-free, which implies that $b = b' \in R$. It follows that B/R is a torsion-free divisible right R-module. $\square$

Let $e \in B$ be an idempotent element, $e^2 = e$. By Theorem 13,

$$eB \otimes_B K = e(B \otimes_B K) = e(B \otimes_R K) = eB \otimes_R K.$$

Lemma 18 If $e \in B$ is idempotent, then $\mathrm{End}_R(eB_R) = \mathrm{End}_B(eB) = eBe$.

Proof Since $eB \cong \mathrm{Hom}_T(K, eK)$, we get that

$$\begin{aligned}\mathrm{Hom}_B(eB, eB) &\cong \mathrm{Hom}_B(eB, \mathrm{Hom}_T({}_BK_T, eK_T)) \\ &\cong \mathrm{Hom}_T(eB \otimes_B K_T, eK_T) \\ &\cong \mathrm{Hom}_T(eB \otimes_R K_T, eK_T) \\ &\cong \mathrm{Hom}_R(eB, \mathrm{Hom}_T({}_RK_T, eK_T)) \cong \mathrm{End}_R(eB).\end{aligned}$$

□

For example, if $e \in B$ is an irreducible idempotent, then eB is indecomposable as a right B-module. By the lemma, it is also indecomposable as a right R-module. Since eB_R is a direct summand of the pure-injective right R-module B_R, it is itself pure-injective, which implies that eBe is a local ring [23, Thm. 9]. If $e = 1$, the lemma yields the equation $\mathrm{End}_R(B_R) = \mathrm{End}_B(B_B)$, which indicates that every R-endomorphism of B_R is of the form $\lambda_b : x \mapsto bx$ for some element $b \in B$.

Corollary 19 (cf. [21, Thm. 6.1]) There exists a collection $\{e_a\}_{a\in\Theta}$ of mutually orthogonal irreducible idempotents in the bicommutator B_R such that B_R is the pure-injective envelope of a direct sum of pure-injective indecomposable R-modules

$$B_R = \mathrm{PE}_R(R_R) = \mathrm{PE}_R\left(\bigoplus_{a\in\Theta} e_a B\right).$$

Proof First, we will show that K_T is essential over its socle. It is clear that if $J \subseteq R$ is a maximal left ideal, then $\mathrm{ann}_K(J)$ is a simple T-module. But we know that every finitely generated T-submodule of K_T is of the form $V_T = \mathrm{ann}_K(I)$ for some basic open left ideal $I \subseteq R$. So take a maximal left ideal $J \supseteq I$; then $\mathrm{ann}_K(J) \subseteq V$ is contained in the socle of K_T.

Write $\mathrm{soc}(K_T) = \oplus_{a\in\Theta} W_a$ as a direct sum of simple T-modules. Let $E_\sigma(W_a)$ denote the injective envelope of W_a in the category $\sigma[K_T]$. Then

$$K_T = E_\sigma(\oplus_a E_\sigma(W_i)).$$

Let $\{e_a\}_{a\in\Theta}$ be the family of mutually orthogonal irreducible idempotents in B corresponding to this decomposition. By the comments following Lemma 18, each direct summand $e_a B$ is a pure-injective indecomposable right R-module.

Consider the pure-injective envelope of the direct sum $\mathrm{PE}_R(\oplus_{a\in\Theta} e_a B)$. Since the direct sum is a pure submodule of B_R, and B_R is pure-injective, the pure-injective envelope is a direct summand P_R of B_R. Write $B_R = P_R \oplus P'_R$ and let $e \in \mathrm{End}_R(B_R)$ be the idempotent that projects onto P_R with respect to this decomposition. Thus $P_R = eB_R$. By the comments following Lemma 18, $e \in \mathrm{End}_B(B_B) = B$. Now $e_a B \subseteq eB$ for every $a \in \Theta$, so the right B-module eB is a direct summand of B_B that contains all the e_a, $a \in A$. The idempotent $1 - e$ therefore annihilates the socle of K, and therefore $1 - e \in J(B)$. But that forces $1 - e = 0$, and the pure-injective envelope of the right R-module $\oplus_{a\in\Theta}\, e_a B$ is B_R as claimed. □

Suppose that R is a left Noetherian V-ring. We noted earlier that the ring B is von Neumann regular. The socle of B is generated as a right or left B-module by the irreducible idempotents. As the collection $\{e_a\}_{a\in\Theta}$ is a maximal set of mutually orthogonal irreducible idempotents, we have that

$$\mathrm{soc}(B) = \oplus_{a\in\Theta}\, e_a B = \oplus_{a\in\Theta}\, Be_a.$$

The corollary implies that, considered as a right R-module, the pure-injective envelope of $\mathrm{soc}(B)$ is the right R-module B_R (cf. [14, Example 5.2]).

By contrast, let us show that if $\mathrm{soc}(B)$ is considered as a left R-module, then it is a direct summand of ${}_RB$. It suffices to prove that ${}_B\mathrm{soc}(B) \in \sigma[{}_BK]$. For then, Proposition 9 implies that ${}_R\mathrm{soc}(B)$ is semisimple, hence injective. Consider an irreducible idempotent $e \in B$, and pick $k \in K$ such that $ek \neq 0$. There is a morphism from $\eta : Be \to K$ determined by $\eta : e \mapsto ek$; it is a nonzero morphism. As Be is simple, the morphism η is an embedding of Be into ${}_BK$. Thus $Be \in \sigma[{}_BK]$, and hence ${}_B\mathrm{soc}(B) \in \sigma[{}_BK]$.

Acknowledgements I am grateful to Vic Camillo who provoked my interest in this topic, and to the referee for helping bring the paper to its final form.

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