Broadcast Channel with Degraded Source Random Variables And Receiver Side Information

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***Abstract*— The problem of sending a pair of correlated sources through a broadcast channel with correlated side information at the receivers is studied from a joint source-channel coding perspective. Sufficient and necessary conditions are provided for reliable transmission. The two conditions are identical except for the left-hand side of one of three inequalities. For two special cases the problem is solved completely: when one side information is a function of the sources, and when a certain Markov property is satisfied on the sources and side information.**

I. Introduction

Consider the system shown in Fig. 1, which consists of a discrete memoryless broadcast channel with transition probability $p(y_1, y_2|x)$, a pair of discrete memoryless correlated sources (U, V), and a pair of side information random variables $(\tilde{U}, \tilde{V})$. The sources and side information have the joint distribution $p(u, v, \tilde{u}, \tilde{v})$. There is further an encoder and decoders (f, g_1, g_2) such that

$$X^n = f(U^n, V^n) \tag{1}$$

$$\hat{U}^n(1) = g_1(Y_1^n, \tilde{U}^n) \tag{2}$$

$$(\hat{U}^n(2), \hat{V}^n) = g_2(Y_2^n, \tilde{V}^n). \tag{3}$$

We seek the set of pairs $(p(y_1, y_2|x),\ p(u, v, \tilde{u}, \tilde{v}))$ for which there exist (f_1, g_1, g_2) such that for any $\epsilon > 0$ we can make

$$Pr((\hat{U}^n(1) \neq U^n) \cup (\hat{U}^n(2) \neq U^n) \cup (\hat{V}^n \neq V^n)) \leq \epsilon \tag{4}$$

for arbitrarily small ϵ. We call this problem the broadcast channel with degraded source random variables and receiver side information.

This problem has been studied in various settings in recent years. In [1], Tuncel considered the problem of sending a single source to both receivers through a broadcast channel with different side information at the receivers. In [2]–[5], the problem of sending two independent messages to two receivers through a broadcast channel with the undesired message as the side information at the receiver is investigated. This is a channel coding problem because it begins with messages. However, it indeed can be interpreted as a special case of Tuncel's joint source-channel coding problem. An extension of this problem is given by Kramer and Shamai in [4], where degraded messages are sent through the broadcast channel with parts of the messages provided as side information at the receivers. Our work in this paper is to extend the results of [4] to a joint source-channel coding scenario.

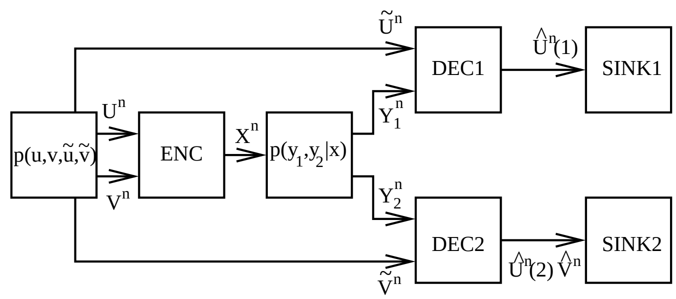

Fig. 1. System model.

II. Main Results

Theorem 1 *A sufficient condition for the reliable transmission in the above system is*

$$I(U, W; U, V) < I(U, W; Y_1, \tilde{U}) \tag{5}$$

$$H(U, V) < I(U, W; Y_1, \tilde{U}) + I(V, X; Y_2, \tilde{V}|U, W) \tag{6}$$

$$H(U, V) < I(U, V, X; Y_2, \tilde{V}) \tag{7}$$

and a necessary condition for the reliable transmission is

$$H(U) < I(U, W; Y_1, \tilde{U}) \tag{8}$$

$$H(U, V) < I(U, W; Y_1, \tilde{U}) + I(V, X; Y_2, \tilde{V}|U, W) \tag{9}$$

$$H(U, V) < I(U, V, X; Y_2, \tilde{V}) \tag{10}$$

with

$$p(u, v, \tilde{u}, \tilde{v}, w, x, y_1, y_2) = p(u, v, \tilde{u}, \tilde{v})p(w, x|u, v)p(y_1, y_2|x) \tag{11}$$

Remark:

1) The sufficient condition and the necessary condition are identical except the left-hand side of the first inequality, which is $I(U, W; U, V)$ in the sufficient condition and $H(U) = I(U; U, V) \leq I(U, W; U, V)$ in the necessary condition.
2) The auxiliary random variable W represents a code to be decoded by both receivers. Supposedly, the only

information that is decoded by both receivers is U with entropy $H(U)$ as in (8). However, to construct a code with the joint distribution in (11), we pay the price of increasing the amount of information decoded by both receivers from $H(U)$ to $I(U,W;U,V)$.

3) Theorem 1 implies that the broadcast channel with side information can be viewed as a parallel broadcast channel. That is, in addition to the broadcast channel $p(y_1,y_2|x)$, there is a virtual broadcast channel with input (U,V) and two outputs $\tilde{U}$ and $\tilde{V}$. We note that the inputs of the two broadcast channels may be correlated.

In the following two special cases, the sufficient and necessary conditions meet.

Theorem 2 *If $\tilde{U}$ is a deterministic function of (U,V), then the sufficient and necessary condition for reliable transmission is*

$$H(U,\tilde{U}) < I(U,\tilde{U};\tilde{U}) + I(S;Y_1) \tag{12}$$

$$H(U,V) < I(U,\tilde{U};\tilde{U}) + I(S;Y_1) + I(V;\tilde{V}|U,\tilde{U}) + \\ + I(X;Y_2|S) \tag{13}$$

$$H(U,V) < I(U,V;\tilde{V}) + I(X;Y_2) \tag{14}$$

with

$$p(u,v,\tilde{u},\tilde{v},s,x,y_1,y_2) = p(u,v,\tilde{u},\tilde{v})p(s)p(x|s)p(y_1,y_2|x) \tag{15}$$

Remark: In this case, $(\tilde{U},S)$ plays the role of W in the general problem.

Remark: Suppose we choose U and V to be independent with entropy R_1 and R_2, respectively, and side information $\tilde{U}$ and $\tilde{V}$ as functions of V and U, respectively, with entropies $H(\tilde{U}) = R_2'$ and $\tilde{V} = R_1'$. Then Theorem 2 simplifies to

$$R_1 \le I(S;Y_1) \tag{16}$$

$$R_1 + R_2 \le R_2' + I(S;Y_1) + I(X;Y_2|S) \tag{17}$$

$$R_1 + R_2 \le R_1' + I(X;Y_2) \tag{18}$$

with

$$p(s,x,y_1,y_2) = p(s)p(x|s)p(y_1,y_2|x) \tag{19}$$

which is equivalent to Theorem 3 in [4]. This means that Theorem 2 includes Kramer-Shamai's result as a special case.

Theorem 3 *If $\tilde{U} \longrightarrow U \longrightarrow V$, then the sufficient and necessary condition of the reliable transmission is*

$$H(U) < I(U;\tilde{U}) + I(W;Y_1) \tag{20}$$

$$H(U,V) < I(U;\tilde{U}) + I(W;Y_1) + I(V;\tilde{V}|U) + \\ + I(X;Y_2|W) \tag{21}$$

$$H(U,V) < I(U,V;\tilde{V}) + I(X;Y) \tag{22}$$

with

$$p(u,v,\tilde{u},\tilde{v},w,x,y_1,y_2) = p(u,v,\tilde{u},\tilde{v})p(w)p(x|w)p(y_1,y_2|x) \tag{23}$$

III. Proof of Theorem 1

Sufficient Condition: Consider a given joint distribution

$$p(u,v,\tilde{u},\tilde{v},w,x,y_1,y_2) = p(u,v,\tilde{u},\tilde{v})p(w,x|u,v)p(y_1,y_2|x) \tag{24}$$

We will show that (5) and (6) are equivalent to the following

$$I(U,W;U,V) + R < I(U,W;Y_1,\tilde{U}) \tag{25}$$

$$H(V|U,W) - R < I(V,X;Y_2,\tilde{V}|U,W) \tag{26}$$

for some $0 \le R \le H(V|U,W)$. It is straightforward that (25) and (26) implies (5) and (6). Conversely, (6) implies that there exists $\delta > 0$, such that

$$H(U,V) + \delta \le I(U,W;Y_1,\tilde{U}) + I(V,X;Y_2,\tilde{V}|U,W) \tag{27}$$

Define $R \triangleq I(U,W;Y_1,\tilde{U}) - I(U,W;U,V) - \frac{\delta}{2}$, where we assume that $R \ge 0$. Then (25) is valid. We note that

$$\begin{aligned} R &\le I(U,W;Y_1,\tilde{U}) - I(U,W;U,V) \\ &\le H(U,V) - I(U,W;U,V) = H(V|U,W) \end{aligned} \tag{28}$$

From (27), we have

$$\begin{aligned} &H(U,V) - I(U,W;Y_1,\tilde{U}) + \delta \\ &= H(U,V) - I(U,W;U,V) - R - \frac{\delta}{2} + \delta \\ &= H(V|U,W) - R + \frac{\delta}{2} \le I(V,X;Y_2,\tilde{V}|U,W) \end{aligned} \tag{29}$$

which complete the converse direction. Thus, in the sequal, we will show that (25), (26), (7) and (11) are sufficient condition for the reliable transmission.

To achieve the proposed sufficient condition, we will construct a superpostion code. The inner code (U^n,W^n) is according to $p(u,w)$ with rate $I(U,W;U,V)+R$ and decoded by both decoders. The outer code (U^n,V^n,X^n) is according to $p(u,v,x|u,w)$ with rate $H(V|U,W) - R$ and decoded by decoder 2. We note that part of the code is provided by the sources (U^n,V^n). Thus, the task is to generated the rest of the code joint typical with the sources.

1. *Random Codebook Generation*: Independently generate L sequences uniformly distributed in the strong typical set $\mathcal{T}^n_{[W]}$ as defined in [6, Definition 1.2.8], say $W^n(1),\ldots,W^n(L)$, where $\frac{1}{n}\ln L > I(U,V;W)$. For every pair of sequences $(u^n,v^n) \in \mathcal{U}^n \times \mathcal{V}^n$, define $\mathcal{W}(u^n,v^n)$ as follows.

$$\mathcal{W}(u^n,v^n) \triangleq \left\{ w^n : \begin{array}{c} w^n \in \{W^n(0),\ldots,W^n(L)\} \\ (u^n,v^n,w^n) \in \mathcal{T}^n_{[UVW]} \end{array} \right\} \tag{30}$$

If $\mathcal{W}(u^n,v^n) \neq \emptyset$, then uniformly select an element from $\mathcal{W}(u^n,v^n)$ and denote it as $w^n(u^n,v^n)$, otherwise, let $w^n(u^n,v^n)$ be an arbitrary sequence in $\{W^n(1),\ldots,W^n(L)\}$. For each $(u^n,v^n) \in \mathcal{U}^n \times \mathcal{V}^n$ generate one sequence $x^n(u^n,v^n,w^n)$ according to $\prod_{i=1}^n p(x_i|u_i,v_i,w_i)$ where $w^n = w^n(u^n,v^n)$.

2. *Encoding*: Define $f(u^n,v^n) = x^n(u^n,v^n,w^n(u^n,v^n))$.

3. *Decoding*: Define the set $\mathcal{C}$ as

$$\mathcal{C} \triangleq \mathcal{T}^n_{[UW]} \cap (\mathcal{U}^n \times \{W^n(1),\ldots,W^n(L)\}) \tag{31}$$

For any $(y_1^n, \tilde{u}^n) \in \mathcal{Y}_1^n \times \tilde{\mathcal{U}}^n$, if (u^n, w^n) is the only sequence pair in $\mathcal{C}$ such that $(u^n, w^n, y_1^n, \tilde{u}^n) \in \mathcal{T}^n_{[UWY_1\tilde{U}]}$, then decoder g_1 is defined as $g_1(y_1^n, \tilde{u}^n) = u^n$, otherwise, let $g_1(y_1^n, \tilde{u}^n)$ be an arbitrary sequence in $\mathcal{T}^n_{[U]}$. For any $(y_2^n, \tilde{v}^n) \in \mathcal{Y}_2^n \times \tilde{\mathcal{V}}^n$, if (u^n, v^n) is the only sequence pair such that $(u^n, v^n, w^n(u^n, v^n), x^n(u^n, v^n, w^n(u^n, v^n)), y_2^n, \tilde{v}^n) \in \mathcal{T}^n_{[UVXWY_2\tilde{V}]}$, then decoder $g_2(y_1^n, \tilde{v}^n) = (u^n, v^n)$, otherwise, let $g_2(y_2^n, \tilde{v}^n)$ be an arbitrary pair of sequences in $\mathcal{T}^n_{[UV]}$.

4. *Probability of Error*: Let $(U^n, V^n, \tilde{U}^n, \tilde{V}^n)$ be the sources and side information, (W^n, X^n) be the code, and (Y_1^n, Y_2^n) be the channel outputs. The average probability error

$$\begin{aligned} P_e &\le Pr(E_0 \cup E_1 \cup E_2) \\ &\le Pr(E_0) + Pr(E_1 \backslash E_0) + Pr(E_2 \backslash E_0) \end{aligned} \tag{32}$$

where

$$E_0 \triangleq (U^n, V^n, \tilde{U}^n, \tilde{V}^n, W^n, X^n, Y_1^n, Y_2^n) \notin \mathcal{T}^n_{[UV\tilde{U}\tilde{V}WXY_1Y_2]} \tag{33}$$

$$E_1 \triangleq \bigcup_{(\bar{U}^n, \bar{W}^n) \neq (U^n, W^n)} (\bar{U}^n, \bar{W}^n, Y_1^n, \tilde{U}^n) \in \mathcal{T}^n_{[UWY_1\tilde{U}]} \tag{34}$$

$$E_2 \triangleq \bigcup_{(\bar{U}^n, \bar{V}^n) \neq (U^n, V^n)} (\bar{U}^n, \bar{V}^n, \bar{W}^n, \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} \tag{35}$$

$$\bar{W}^n \triangleq w^n(\bar{U}^n, \bar{V}^n) \tag{36}$$

$$\bar{X}^n \triangleq x^n(\bar{U}^n, \bar{V}^n, w^n(\bar{U}^n, \bar{V}^n)). \tag{37}$$

Furthermore, we have

$$\begin{aligned} Pr(E_0) = {} & Pr((U^n, V^n, W^n) \notin \mathcal{T}^n_{[UVW]}) + \\ & + Pr((U^n, V^n, \tilde{U}^n, \tilde{V}^n, W^n, X^n, Y_1^n, Y_2^n) \notin \\ & \mathcal{T}^n_{[UV\tilde{U}\tilde{V}WXY_1Y_2]} | (U^n, V^n, W^n) \in \mathcal{T}^n_{[UVW]}) \end{aligned} \tag{38}$$

With $\frac{1}{n} \ln L > I(U, V; W)$ and sufficiently large n , we have

$$Pr((U^n, V^n, W^n) \notin \mathcal{T}_\delta(UVW)) \le \epsilon \tag{39}$$

and also because of the codebook generation, we have

$$\begin{aligned} & Pr((\tilde{U}^n, \tilde{V}^n, X^n, Y_1^n, Y_2^n) \notin \\ & \mathcal{T}^n_{[\tilde{U}\tilde{V}XY_1Y_2]}(U^n, V^n, W^n) | (U^n, V^n, W^n) \in \mathcal{T}^n_{[UVW]}) \le \epsilon \end{aligned} \tag{40}$$

Thus, we have [6, Lemma 1.2.10]

$$\begin{aligned} Pr((U^n, V^n, \tilde{U}^n, \tilde{V}^n, W^n, X^n, Y_1^n, Y_2^n) \notin \mathcal{T}^n_{[UV\tilde{U}\tilde{V}WXY_1Y_2]} \\ \le 2\epsilon \end{aligned} \tag{41}$$

For the second term in (32), we have

$$\begin{aligned} & Pr(E_1 \backslash E_0) = Pr(E_1 \cap E_0^c) \\ & = \sum_{(u^n, w^n) \in \mathcal{C}} p(u^n, w^n) \sum_{\substack{(\bar{U}^n, \bar{W}^n) \neq (u^n, w^n) \\ (\bar{U}^n, \bar{W}^n) \in \mathcal{C}}} \\ & \quad Pr((\bar{U}^n, \bar{W}^n, Y_1^n, \tilde{U}^n) \in \mathcal{T}^n_{[UWY_1\tilde{U}]} | (u^n, w^n) \text{ sent } \cap E_0^c) \\ & \le \max_{(u^n, w^n) \in \mathcal{C}} \sum_{\substack{(\bar{U}^n, \bar{W}^n) \neq (u^n, w^n) \\ (\bar{U}^n, \bar{W}^n) \in \mathcal{C}}} \\ & \quad Pr((\bar{U}^n, \bar{W}^n, Y_1^n, \tilde{U}^n) \in \mathcal{T}^n_{[UWY_1\tilde{U}]} | (u^n, w^n) \text{ sent } \cap E_0^c) \\ & \le L \times \exp(n(H(U|W) + 2\delta)) \times \frac{\exp(n(H(U, W|\tilde{U}, Y_1) + 2\delta))}{(1-\epsilon)\exp(n(H(U, W) - 2\delta))} \\ & \le L \times \exp(n(H(U|W) - I(U, W; \tilde{U}, Y_1) + 7\delta)) \end{aligned} \tag{42}$$

Thus, $Pr(E_1 \backslash E_0) \le \epsilon$ with sufficiently large n if

$$R + I(U, V; U, W) < I(U, W; \tilde{U}, Y_1) \tag{43}$$

where $R \triangleq \frac{1}{n} \ln L - I(U, V; W)$.

For the third term in (32), we have

$$\begin{aligned} Pr(E_2 \backslash E_0) &= Pr(E_2 \cap E_0^c) = Pr((E_{21} \cup E_{22}) \cap E_0^c) \\ &\le Pr(E_{21} \cap E_0^c) + Pr(E_{22} \cap E_0^c) \end{aligned} \tag{44}$$

where

$$E_{21} \triangleq \bigcup_{\substack{(\bar{U}^n, \bar{W}^n) = (U^n, W^n) \\ (\bar{U}^n, \bar{V}^n) \neq (U^n, V^n) \\ (\bar{U}^n, \bar{V}^n) \in \mathcal{T}^n_{[UV]}}} (\bar{U}^n, \bar{V}^n, \bar{W}^n, \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} \tag{45}$$

$$E_{22} \triangleq \bigcup_{\substack{(\bar{U}^n, \bar{W}^n) \neq (U^n, W^n) \\ (\bar{U}^n, \bar{V}^n) \neq (U^n, V^n) \\ (\bar{U}^n, \bar{V}^n) \in \mathcal{T}^n_{[UV]}}} (\bar{U}^n, \bar{V}^n, \bar{W}^n, \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} \tag{46}$$

While the error event E_2 can be decomposed differently, the current decomposition yields the expressions most close to the necessary condition. For the first term in (44), we have

$$\begin{aligned} & Pr(E_{21} \cap E_0^c) \\ & = \sum_{(u^n, v^n) \in \mathcal{T}^n_{[UV]}} p(u^n, v^n) \sum_{\substack{(\bar{U}^n, \bar{W}^n) = (u^n, w^n) \\ (\bar{U}^n, \bar{V}^n) \neq (u^n, v^n) \\ (\bar{U}^n, \bar{V}^n) \in \mathcal{T}^n_{[UV]}}} \\ & \quad Pr((\bar{U}^n, \bar{V}^n, \bar{W}^n, \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} | \\ & \qquad ((u^n, v^n) \text{ sent } \cap E_0^c) \\ & = \max_{(u^n, v^n) \in \mathcal{T}^n_{[UV]}} \sum_{\substack{(\bar{U}^n, \bar{W}^n) = (u^n, w^n) \\ (\bar{U}^n, \bar{V}^n) \neq (u^n, v^n) \\ (\bar{U}^n, \bar{V}^n) \in \mathcal{T}^n_{[UV]}}} \\ & \quad Pr((\bar{U}^n, \bar{V}^n, \bar{W}^n, \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} | \\ & \qquad (u^n, v^n) \text{ sent } \cap E_0^c) \\ & \le \frac{\exp(n(H(V|U, W) + 2\delta)) \exp(n(H(W) + \delta))}{(1-\epsilon)\exp(n(H(W|U, V) - 2\delta)) L} \times \\ & \quad \times \frac{\exp(n(H(V, X|U, W, Y_2, \tilde{V}) + 2\delta))}{(1-\epsilon)\exp(n(H(U, V, X) - 2\delta))} \\ & \le \exp(n(H(V|U, W) - R - I(V, X; Y_2, \tilde{V}|U, W) + 11\delta)) \end{aligned} \tag{47}$$

Thus, $Pr(E_(21) \cap E_0^c) \leq \epsilon$ with sufficiently large n if

$$H(V|U,W) - R < I(V,X;Y_2,\tilde{V}|U,W) \tag{48}$$

For the second term in (44), we have

$$\begin{aligned}
Pr(E_{22} \cap E_0^c) &= \sum_{(u^n,v^n)\in\mathcal{T}^n_{[UV]}} p(u^n,v^n) \sum_{\substack{(\bar{U}^n,\bar{W}^n)\neq(u^n,w^n)\\(\bar{U}^n,\bar{V}^n)\neq(u^n,v^n)\\(\bar{U}^n,\bar{V}^n)\in\mathcal{T}^n_{[UV]}}} Pr((\bar{U}^n,\bar{V}^n,\bar{W}^n,\\
&\quad \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} | (u^n,v^n) \text{ sent } \cap E_0^c)\\
&\leq \max_{(u^n,v^n)\in\mathcal{T}^n_{[UV]}} \sum_{\substack{(\bar{U}^n,\bar{W}^n)\neq(u^n,w^n)\\(\bar{U}^n,\bar{V}^n)\neq(u^n,v^n)\\(\bar{U}^n,\bar{V}^n)\in\mathcal{T}^n_{[UV]}}} Pr((\bar{U}^n,\bar{V}^n,\bar{W}^n,\\
&\quad \bar{X}^n, Y_2^n, \tilde{V}^n) \in \mathcal{T}^n_{[UVWXY_2\tilde{V}]} | (u^n,v^n) \text{ sent } \cap E_0^c)\\
&\leq \exp(n(H(U,V)+\epsilon)) \frac{\exp(n(H(U,V,X|Y_2,\tilde{V})+2\delta))}{(1-\epsilon)\exp(n(H(U,V,X)-2\delta))}\\
&\leq \exp(n(H(U,V) - I(U,V,X;Y_2,\tilde{V}) + 6\delta))
\end{aligned} \tag{49}$$

Thus, $Pr(E_{22} \cap E_0^c) \leq \epsilon$ with sufficiently large n if

$$H(U,V) < I(U,V,X;Y_2,\tilde{V}) \tag{50}$$

The average probability of error with sufficiently large n

$$\begin{aligned}
P_e &\leq Pr(E_0) + Pr(E_1 \backslash E_0) + Pr(E_{21} \backslash E_0) + Pr(E_{22} \backslash E_0)\\
&\leq 2\epsilon + \epsilon + \epsilon + \epsilon = 5\epsilon
\end{aligned} \tag{51}$$

if the following condition is satisfied

$$I(U,W;U,V) + R < I(U,W;Y_1,\tilde{U}) \tag{52}$$

$$H(V|U,W) - R < I(V,X;Y_2,\tilde{V}|U,W) \tag{53}$$

$$H(U,V) < I(U,V,X;Y_2,\tilde{V}) \tag{54}$$

Necessary Condition: From Fano's inequality, we have

$$\begin{aligned}
nH(U) &\leq I(U^n;Y_1^n,\tilde{U}^n) + n\epsilon\\
&= \sum_{i=1}^n I(U^n;Y_{1i},\tilde{U}_i|Y_1^{i-1},\tilde{U}^{i-1}) + n\epsilon\\
&\leq \sum_{i=1}^n I(U^n,Y_1^{i-1},\tilde{U}^{i-1};Y_{1i},\tilde{U}_i) + n\epsilon\\
&= \sum_{i=1}^n I(U^n,Y_1^{i-1},\tilde{U}^{i-1},Y^n_{2(i+1)},\tilde{V}^n_{i+1};Y_{1i},\tilde{U}_i) -\\
&\quad - I(Y^n_{2(i+1)},\tilde{V}^n_{i+1};Y_{1i},\tilde{U}_i|Y_1^{i-1},\tilde{U}^{i-1},U^n) + n\epsilon\\
&= \sum_{i=1}^n I(U_i,Z_i;Y_{1i},\tilde{U}_i) -\\
&\quad - I(Y^n_{2(i+1)},\tilde{V}^n_{i+1};Y_{1i},\tilde{U}_i|Y_1^{i-1},\tilde{U}^{i-1},U^n) + n\epsilon \qquad (55)\\
&\leq \sum_{i=1}^n I(U_i,Z_i;Y_{1i},\tilde{U}_i) + n\epsilon \qquad (56)
\end{aligned}$$

where $Z_i \triangleq (U^{i-1},U^n_{i+1},Y_1^{i-1},\tilde{U}^{i-1},Y^n_{2(i+1)},\tilde{V}^n_{i+1})$. From Fano's inequality, we further have

$$\begin{aligned}
nH(V|U) &\leq I(V^n;Y_2^n,\tilde{V}^n|U^n) + n\epsilon\\
&= \sum_{i=1}^n I(V^n,X^n;Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1}) + n\epsilon\\
&= \sum_{i=1}^n I(V_i,X_i;Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1}) + n\epsilon\\
&\leq \sum_{i=1}^n I(V_i,X_i,Y_1^{i-1},\tilde{U}^{i-1};Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1}) + n\epsilon\\
&= \sum_{i=1}^n I(V_i,X_i;Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1},Y_1^{i-1},\tilde{U}^{i-1}) +\\
&\quad + I(Y_1^{i-1},\tilde{U}^{i-1};Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1}) + n\epsilon\\
&= \sum_{i=1}^n I(V_i,X_i;Y_{2i},\tilde{V}_i|U_i,Z_i) +\\
&\quad + I(Y_1^{i-1},\tilde{U}^{i-1};Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1}) + n\epsilon
\end{aligned} \tag{57}$$

Because of the following equality [6],

$$\begin{aligned}
&\sum_{i=1}^n I(Y^n_{2(i+1)},\tilde{V}^n_{i+1};Y_{1i},\tilde{U}_i|Y_1^{i-1},\tilde{U}^{i-1},U^n)\\
&\quad = \sum_{i=1}^n I(Y_1^{i-1},\tilde{U}^{i-1};Y_{2i},\tilde{V}_i|U^n,Y^n_{2(i+1)},\tilde{V}^n_{i+1})
\end{aligned} \tag{58}$$

(55) and (57) yields

$$\begin{aligned}
nH(U,V) &\leq \sum_{i=1}^n I(U_i,Z_i;Y_{1i},\tilde{U}_i) +\\
&\quad + I(V_i,X_i;Y_{2i},\tilde{V}_i|U_i,Z_i) + 2n\epsilon
\end{aligned} \tag{59}$$

Finally, from Fano's inequality, we have

$$\begin{aligned}
nH(U,V) &\leq I(U^n,V^n,X^n;Y_2^n,\tilde{V}^n) + n\epsilon\\
&= \sum_{i=1^n} I(U^n,V^n,X^n;Y_{2i},\tilde{V}_i|Y_2^{i-1},\tilde{V}^{i-1}) + n\epsilon\\
&\leq \sum_{i=1^n} I(U_i,V_i,X_i;Y_{2i},\tilde{V}_i) + n\epsilon
\end{aligned} \tag{60}$$

Let Q be a random variable uniformly distributed on $\{1,\ldots,n\}$, and random variabls $(Z,U,V,\tilde{U},\tilde{V},X,Y_1,Y_2,Q)$ such that

$$\begin{aligned}
&Pr\left(\begin{array}{c} Z=z, U=u, V=v, \tilde{U}=\tilde{u}\\ \tilde{V}=\tilde{v}, X=x, Y_1=y_1, Y_2=y_2 \end{array} \middle| \, Q=i \right)\\
&\quad = Pr\left(\begin{array}{c} Z_i=z, U_i=u, V_i=v, \tilde{U}_i=\tilde{u}\\ \tilde{V}_i=\tilde{v}, X_i=x, Y_{1i}=y_1, Y_{2i}=y_2 \end{array} \right)
\end{aligned} \tag{61}$$

Also define $W=(Z,Q)$, then from (56) we have

$$\begin{aligned}
nH(U) &\leq \sum_{i=1}^n I(U_i,Z_i;Y_{1i},\tilde{U}_i) + n\epsilon\\
&= nI(U,Z;Y_1,\tilde{U}|Q) + n\epsilon\\
&\leq I(U,Z,Q;Y_1,\tilde{U}) + n\epsilon\\
&= I(U,W;Y_1,\tilde{U}) + n\epsilon
\end{aligned} \tag{62}$$

and from (59) we have

$$\begin{aligned}nH(U,V) &\le \sum_{i=1}^{n} I(U_i,Z_i;Y_{1i},\tilde{U}_i)+\\ &\quad + I(V_i,X_i;Y_{2i},\tilde{V}_i|U_i,Z_i)+2n\epsilon\\ &= nI(U,Z;Y_1,\tilde{U}|Q)+nI(V,X;Y_2,\tilde{V}|U,Z,Q)+2n\epsilon\\ &\le nI(U,W;Y_1,\tilde{U})+nI(V,X;Y_2\tilde{V}|U,W)+2n\epsilon \qquad (63)\end{aligned}$$

and from (60) we have

$$\begin{aligned}nH(U,V) &\le \sum_{i=1^n} I(U_i,V_i,X_i;Y_{2i},\tilde{V}_i)+n\epsilon\\ &= nI(U,V,X;Y_2,\tilde{V}|Q)+n\epsilon\\ &= nH(Y_2,\tilde{V}|Q)-nH(Y_2,\tilde{V}|U,V,X)+n\epsilon\\ &\le nH(Y_2,\tilde{V})-nH(Y_2,\tilde{V}|U,V,X)+n\epsilon\\ &= nI(U,V,X;Y_2,\tilde{V})+n\epsilon \qquad (64)\end{aligned}$$

which completes the proof for the necessary condition.

IV. Proof of Theorem 2

The sufficient condition part follows immediately from the sufficient condition in the general case with $W=(\tilde{U},S)$ and the joint distribution satisfying (15). We prove the necessary condition part as follows. We define a different decoding function g_1' as

$$g_1'(Y_1^n,\tilde{U}^n)=(\hat{U}^n(1),\hat{\tilde{U}}^n)=(g_1(Y_1^n,\tilde{U}^n),\tilde{U}^n) \qquad (65)$$

We note that decoder g_1' generates an error *iff* decoder g generates an error. Thus,

$$Pr(\hat{U}^n(1)\ne U^n)=Pr((\hat{U}^n(1),\hat{\tilde{U}}^n)\ne(U^n,\tilde{U}^n)) \qquad (66)$$

By applying Fano's inequality on $Pr((\hat{U}^n(1),\hat{\tilde{U}}^n)\ne(U^n,\tilde{U}^n))$, we have

$$\begin{aligned}nH(U,\tilde{U}) &\le I(U^n,\tilde{U}^n;Y_1^n,\tilde{U^n})+n\epsilon\\ &\le I(U^n,\tilde{U}^n;\tilde{U}^n)+I(U^n,\tilde{U}^n;Y_1^n|\tilde{U^n})+n\epsilon\\ &\le nI(U,\tilde{U};\tilde{U})+I(U^n,\tilde{U}^n,\tilde{V}^n;Y_1^n)+n\epsilon \qquad (67)\end{aligned}$$

We also have

$$\begin{aligned}nH(V|U,\tilde{U}) &\le I(V^n,X^n;Y_2^n,\tilde{V}^n|U^n,\tilde{U}^n)+n\epsilon\\ &= I(V^n,X^n;\tilde{V}^n|U^n,\tilde{U}^n)+\\ &\quad + I(V^n,X^n;Y_2^n|U^n,\tilde{U}^n,\tilde{V}^n)+n\epsilon\\ &= nI(V;\tilde{V}|U,\tilde{U})+I(X^n;Y_2^n|U^n,\tilde{U}^n,\tilde{V}^n)+n\epsilon \qquad (68)\end{aligned}$$

and

$$\begin{aligned}nH(U,V) &\le I(U^n,V^n,X^n;Y_2^n,\tilde{V}^n)+n\epsilon\\ &= I(U^n,V^n,X^n;\tilde{V}^n)+I(U^n,V^n,X^n;Y_2^n|\tilde{V}^n)+n\epsilon\\ &\le nI(U,V;\tilde{V})+I(X^n;Y_2^n)+n\epsilon \qquad (69)\end{aligned}$$

Then, we follow similar steps as in the proof of the necessary condition of Theorem 1, and the necessary condition part of this Theorem can be proven.

V. Proof of Theorem 3

The sufficient part is straightfarward. For the necessary condition part, we have

$$\begin{aligned}nH(U^n) &\le I(U^n;\tilde{U}^n,Y_1^n)+n\epsilon\\ &= I(U^n;\tilde{U}^n)+I(U^n;Y_1^n|\tilde{U^n})+n\epsilon\\ &\overset{(a)}{\le} nI(U;\tilde{U})+I(U^n;Y_1^n)+n\epsilon\\ &\le nI(U;\tilde{U})+I(U^n,\tilde{V}^n;Y_1^n)+n\epsilon \qquad (70)\end{aligned}$$

where (a) is because

$$\begin{aligned}p(\tilde{u}^n,u^n,y_1^n) &= \sum_{v^n} p(\tilde{u}^n,u^n,v^n,y_1^n)\\ &= \sum_{v^n} p(\tilde{u}^n,u^n)p(v^n|u^n)p(y_1^n|u^n,v^n)\\ &= p(\tilde{u}^n,u^n)\sum_{v^n} p(v^n|u^n)p(y_1^n|u^n,v^n)\\ &= p(\tilde{u}^n,u^n)p(y_1^n|u^n) \qquad (71)\end{aligned}$$

And we also have

$$\begin{aligned}nH(V|U) &\le I(V^n,X^n;Y_2^n,\tilde{V}^n|U^n)+n\epsilon\\ &= I(V^n,X^n,;\tilde{V}^n|U^n)+\\ &\quad + I(V^n,X^n;Y_2^n|U^n,\tilde{V}^n)+n\epsilon\\ &= nI(V;\tilde{V}|U)+I(V^n,X^n;Y_2^n|U^n,\tilde{V}^n)+n\epsilon \qquad (72)\end{aligned}$$

Then, we follow the same steps as in the proof of Theorem 2, and this theorem can be proven.

VI. Conclusion

In this paper, we studied the problem of sending a pair of correlated sources through a broadcast channel with correlated side information at the receivers from a joint source-channel coding perspective. We provided a sufficient condition and a necessary condition for the reliable transmission. These two conditions are identical except for the left-hand side of one of three inequalities. We also studied two special cases of this problem. In one special case, one side information is a function of the sources, and in the other special case, a Markov condition is assumed. We obtained the sufficient and necessary conditions for the reliable transmission in both special cases.

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