Determining Closed Model Category Structures

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1 Introduction

Closed model categories are a general framework introduced by Quillen [15] in which one can do homotopy theory. An alternative framework has been developed by Baues [2]. Putting a closed model structure on a category not only allows one to use techniques of homotopy theory to study it but also allows one to better understand the category through the concepts and constructions that come with the structure. The usefulness of closed model categories in homotopy theory has been firmly established. It seems likely that they could also be useful in other areas of mathematics.

Our main theorem 7.1 and its variant 7.2 for the first time give general conditions for the existence of a closed model category structure on a category. All previous proofs of closed model category structures have depended on either ad hoc arguments or the existence of a closed model category structure on some other category. Our conditions have been designed to make giving a category a closed model structure a simple task. Only a few hypotheses need be verified. Hopefully our construction will prove useful as a tool for other researchers. As an illustrative example we give a model category structure for $CDGA(\mathbf{R})$.

In the next few paragraphs we give an overview of the contents and main arguments of the paper. For convenience we refer the reader to the relevant definitions within the paper. A closed model category (Definition 3.4) is a category with three distinguished classes of morphisms: cofibrations; fibrations; weak equivalences. We use the word acyclic as the adjective corresponding to the noun weak equivalence. Each map is required to have two factorizations. The first into a cofibration followed by an acyclic fibration and the second into an acyclic cofibration followed by a fibration. There are two corresponding lifting conditions. We explicitly construct the factorizations by a generalized infinite gluing construction [15] [10] and show the lifting conditions are satisfied. To define the gluing construction (Definition 4.5) we first need to specify a class of maps $\mathcal{G} \subset Hom_{\mathbf{C}}$. Take any map $f \in Hom_{\mathbf{C}}(X, Y)$. Then the gluing construction applied to f is the colimit over all possible dashed extensions of solid arrow diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \overset{h}{\nearrow} & \\ X(g) & & \end{array}$$

such that $g \in \mathcal{G}$. It is defined if the class of such extensions form a set. There are maps $X \xrightarrow{i(f)} colim X(g) \xrightarrow{p(f)} Y.$ The infinite gluing construction results from applying the gluing construction to $p(f)$ some cardinal number of times.

A cell category $(\mathbf{C}, I)$ (Definition 4.1) is a cocomplete category $\mathbf{C}$ together with a set of maps I, which we call cells. A relative cell complex (Definition 4.2) is a map that can be built from I using pushouts and direct limits, we take such complexes to be our cofibrations.

If for the domain A, which one can think of as the boundary, of any of our cells the functor $Hom(A, \_)$ commutes with direct limits of relative cell complexes of length κ then we call $(\mathbf{C}, I)$ κ small (Definition 4.4). Assuming this condition is enough to give us one of the liftings and factorizations. In this case we use the infinite gluing construction with the class of cofibrations with one cell and the cardinal κ. The lifting is first constructed for cofibrations with one cell by factoring the range of any cell through an non-final stage of the construction. The lifting for all cofibrations follows from their description as colimits. This technique is an extension of the small object argument of Quillen [15] [3] [9] [11].

Let us be given a class of weak equivalences. To get the second factorization and lifting we need all of our acyclic cofibrations to factor as acyclic cofibrations with at most κ cells where κ is some globally defined cardinal. We then call our class of acyclic cofibrations κ factorable (Definition 5.5). In this case we use the infinite gluing construction with the class of all acyclic cofibrations with fewer than κ cells and the cardinal κ. We construct the lifting for acyclic cofibrations with less than κ cells. There are some subtleties involved in the proof of this lifting and it also relies on the fact that $(\mathbf{C}, I)$ is κ small. We get the lifting for all acyclic cofibrations (Lemma 6.3) by κ factorability and a colimit argument. Bousfield also constructed this second factorization along with the lifting for various classes of weak equivalences on the category of simplicial sets [5].

We put our two factorizations together in Theorem 7.1 to get a closed model category structure. We state a special case of the theorem which can be understood with the terminology we have given in the introduction.

Theorem 1.1 *7.1 A κ small cell category $(\mathbf{C}, I)$ and a class of weak equivalences in $\mathbf{C}$ determine a closed model category if the acyclic cofibrations are κ factorable, closed under pushout and direct limits and all of the second maps in the first factorization are weak equivalences.*

An important step is our choice of fibrations as the closure under composition of the second map in our two factorizations of any map. In opposition is the usual method of defining fibrations as determined by a lifting property. Our choice makes it easier to see that our fibrations and acyclic fibrations have the needed lifting properties. In the special case that the acyclic cofibrations are generated by a set J, that is that they are the cofibrations in the cell category $(\mathbf{C}, J)$, we have a cofibrantly generated closed model category [11] [9]. The acyclic cofibrations are then automatically κ factorable and closed under pushouts and direct limits (Lemma 5.7). In comparison Hirschorn [11] and Dwyer, Kan and Hirschorn [9] add an additional smallness condition on the generating cofibrations, which in effect says that $(\mathbf{C}, J)$ is κ small, in order to get their closed model category structures. Of course in our situation we need not specify a set of generating acyclic cofibrations. It is sometimes more convenient just to give a class of weak equivalences as we have, for example, in Sections 8 and 9. In [17] we establish conditions under which the cells determine the weak equivalences and therefore the entire closed model structure.

At the end of the paper we give two examples of applications of our theory. The example of $DGM(\mathbf{R})$ is well known [15] [10] and has been included to illustrate the concepts. In the category of commutative differential graded algebras over a commutative ring $\mathbf{R}$, $CDGA(\mathbf{R})$, we let the cofibrations be the semi-free extensions and the weak equivalences be the homology equivalences. If $\mathbf{R}$ is the rationals this category models rational homotopy theory and it is well known that these definitions determine a closed model category structure. However the proof for characteristic p breaks down since in this case the fibrations cannot be the same as the surjections. We demonstrate for the first time that these definitions also determine a closed model category structure for a field of characteristic p.

Although the presentation is somewhat abstract, only some basic set theory and category theory are required to understand the facts herein presented. Perhaps the easiest way to read the paper is to first look at the simple example in Section 8. Going over its straightforward proofs is a good way to familiarize oneself with the new definitions and Theorem 7.1.

This paper started as a way of putting some earlier results [17] in the language of model categories. I would like to thank Rick Jardine for suggesting that I do that. I would also like to thank my Ph.D. supervisor Paul Selick especially for helping me with my many notational and other problems. Finally I would like to thank Bill Dwyer for reading over an earlier version of the manuscript and for drawing my attention to a mistake.

2 Notation

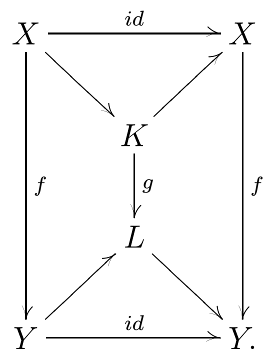
In a category $\mathbf{C}$, $Hom_{\mathbf{C}}(X, Y)$ denotes the set of morphisms from X to Y and $Hom_{\mathbf{C}}$ denotes the class of all morphisms. Top_* denotes the category of pointed topological spaces and pointed continuous maps. We let $|\ | :$ Sets $\rightarrow$ Cardinals be the function which assigns to each set its cardinality. For a cardinal κ we let κ^+ denote its successor cardinal. κ is called a regular cardinal if for every set of sets $\{A_\alpha\}_{\alpha\in\Gamma}$ such that $|A_\alpha| < \kappa$ and $|\bigcup A_\alpha| = \kappa$ we have $|\Gamma| \geq \kappa$. Not all cardinals are regular but the following lemma is true.

Lemma 2.1 *All successor cardinals are regular.*

3 Closed Model Categories

Here we give the definition of closed model category of Blanc [3]. He proves it is equivalent to the usual definition [16] [10].

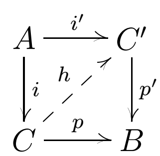
Definition 3.1 *Let the following be a diagram of maps in C*



Then we say that f is a retract of g.

If $\mathcal{B}$ is a class of morphisms then we say that a $\mathcal{B}'$ is a retract of $\mathcal{B}$ if $\mathcal{B} \subset \mathcal{B}'$ and every morphism in $\mathcal{B}'$ is a retract of a morphism in $\mathcal{B}$.

Definition 3.2 *Given the following commuting solid arrow diagram*



if a map h exists making both triangles commute then we say that the diagram has the **extension lifting property** *and call h an* **extension lift**. *Given i and p' if for any i' and p there exists such a h then we say that i has the* **extension property** *or* **left lifting property** *(LLP) with respect to p' or equivalently that p' has the* **lifting property** *or* **right lifting property** *(RLP) with respect to i.*

Lemma 3.3 *If f has the LLP with respect to g then any retract of f has the LLP with respect to any retract of g.*

Proof: Easy. □

Definition 3.4 *Let* **C** *be a category with three distinguished classes of morphisms: $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ all closed under composition. Assume that $\mathcal{W}$ is closed under retracts and contains all isomorphisms. Also assume that $\mathcal{W}$ satisfies the two out of three condition. That is for any two maps f, g in* **C** *such that $f \circ g$ is defined if two of f, g and $f \circ g$ are in $\mathcal{W}$ then the third is also. We call*

$\mathcal{W}$ *the* **weak equivalences**, $\mathcal{C}$ *the* **strong cofibrations** *and* $\mathcal{F}$ *the* **strong fibrations**.
Property 1:

Assume that for any $f : A \to B$ *a morphism in* $\mathbf{C}$ *there is a factorization* $f = p \circ i$ *where* $p \in \mathcal{F} \cap \mathcal{W}$ *and* $i \in \mathcal{C}$. *Also if there exist another such factorization* $f = p' \circ i'$ *then the following diagram has the extension lifting property*

$$\begin{array}{ccc} A & \xrightarrow{i'} & C' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p'} \\ C & \xrightarrow{p} & B \end{array} .$$

Now we consider the property dual to property 1
Property 2:

Assume that for any $f : A \to B$ *a morphism in* $\mathbf{C}$ *there is a factorization* $f = p \circ i$ *where* $p \in \mathcal{F}$ *and* $i \in \mathcal{C} \cap \mathcal{W}$. *Also if there exists another such factorization* $f = p' \circ i'$ *then the above diagram has the extension lifting property.*

If $\mathbf{C}$ *is closed under finite limits and satisfies property 2 it is called a* **right model category***(RMC).*

If $\mathbf{C}$ *is closed under finite colimits and satisfies property 1 it is called a* **left model category***(LMC).*

If $\mathbf{C}$ *is closed under both finite colimits and finite limits and both properties hold it is called a* **(closed) model category**.

Note that an initial object is the empty colimit and a terminal object if the empty limit. We call $\mathcal{C}$ the strong cofibrations since it is in fact their closure under retracts that would be the cofibrations in the usual definition of closed model category. A similar comment applies to $\mathcal{F}$.

4 Cell Categories

Definition 4.1 *Let* $\mathbf{C}$ *be a cocomplete category and* $\{\alpha : A(\alpha) \to B(\alpha)\}_{\alpha \in I}$ *a set of maps in* $\mathbf{C}$. *Then we call* $(\mathbf{C}, I)$ *a* **cell category** *and* I *the* **set of cells**.

An example is (Top_*, S) where $S = \bigcup_{n \in Z^+} \{S^n \to D^{n+1}\} \cup \{* \to S^n\}$. So we can think of the $A(\alpha)$ as the boundaries of the cells. In a closed model category our cells are sometimes called test cofibrations. We call them cells since we with to emphasize the analogy with standard cells in Top_*. This analogy motivates the following definition.

Definition 4.2 *Let* $(\mathbf{C}, I)$ *be a cell category. We define a* **(relative) cell complex**

$$(\kappa, \{X(n)\}_{n\leq\kappa}, \{K(n)\}_{n<\kappa}, \alpha, \{f(r)\}_{r\in\cup K(n)})$$

to be an ordinal κ*, objects* $X(n) \in \mathbf{C}$*, sets* $K(n)$*, a map* $\alpha : \bigcup K(n) \to I$ *and maps for every* $r \in K(n)$ $f(r) : A(\alpha(r)) \to X(n)$ *such that the following diagram is a pushout*

$$\begin{array}{ccc} \bigvee_{r\in K(n)} A(\alpha(r)) & \xrightarrow{\bigvee_{r\in K(n)} f(r)} & X(n) \\ \Big\downarrow{\scriptstyle \alpha(r)} & & \Big\downarrow \\ \bigvee_{r\in K(n)} B(\alpha(r)) & \longrightarrow & X(n+1) \end{array}$$

and if $n \leq \kappa$ *is a limit ordinal then*

$$X(n) = colim_{m<n} X(m)$$

Let $X = X(0)$ *and let* $Y = X(\kappa)$*. We will also denote the (relative) cell complex by* $j : X \to Y$ *and leaving the other structure implicit.* $\{X(n)\}_{n\leq\kappa}$ *is called a* **decomposition** *of* j *and* κ *is called the* **length** *of the decomposition.*

Let $(\kappa', \{X'(n)\}, \{K'(n)\}, \alpha', \{f'(r)\})$ *be another cell complex such that* $X(0) = X$*. If for every* $n < \kappa'$ *we have* $K'(n) \subset K(n)$*,* $\alpha' = \alpha|_{K'(n)}$ *for every* n *and for every* n *a map* $j(n) : X'(n) \to X(n)$ *such that* $j(n)f'(r) = f(r)$ *for every* $r \in K'(n)$ *then* $X \to X'(\kappa')$ *is called a* **(relative) sub(cell) complex** *of* $X \to Y$ *and the canonical map* $j : X'(\kappa') \to Y$ *is called the* **inclusion** *of a (relative) sub(cell) complex.*

Let $\mathcal{C}(I)$ *denote the class of maps that can be given the structure of a relative cell complex. Let* $K(Y)$ *denote* $\bigcup_{n\in\kappa} K(n)$*.* $K(Y)$ *is called the set of cells in* Y *and* $K(n)$ *the set of cells of filtration* n*. We call the* $f(r)$ *attaching maps.*

A relative cell complex is just any object which has had cells attached to it in some ordinal number of steps. For example in (Top_*, S), $* \to D^n$ is a cell complex and $* \to S^{n-1}$ is a subcomplex of it. $S^{n-1} \to D^n$ is a relative cell complex. In fact it is clear that in any cell category if $j : W \to X$

is a complex and $k : W \to Y$ is a subcomplex then there exist a complex $l : Y \to X$ such that $lk = j$.

Observe that $\mathcal{C}(I)$ is the closure of I under pushouts and direct limits. They will be the strong cofibrations in our closed model category. For a class $\mathcal{B}$ we will also allow ourselves to use the notation $\mathcal{C}(\mathcal{B})$ to denote this same closure.

Definition 4.3 *Let $j : X \to X(\kappa)$ be a cell complex. We define*

$$s(j) = |K(X(\kappa))|$$

We call $s(j)$ the **size** *of j. For $j \in \mathcal{C}(I)$ we take $s(j)$ to be the minimum of the sizes of all cell complex structures on j.*

The size of a cell complex is just the number (cardinality) of cells in it.

Definition 4.4 *Let $(\mathbf{C}, I)$ be a cell category and κ a cardinal. We say that $A \in \mathbf{C}$ is κ small if for every $j : X \to Y \in \mathcal{C}(I)$ and every map $f : A \to Y$ there exists a subcomplex of Y, $j' : X \to Y'$ with $s(j') < \kappa$ and a factorization $A \to Y' \to Y$. We say that $(\mathbf{C}, I)$ is κ* **small** *if for every $\alpha \in I$ $A(\alpha)$ is κ small.*

It is clear that if κ and δ are cardinals with $\kappa < \delta$ and $(\mathbf{C}, I)$ is κ small then $(\mathbf{C}, I)$ is δ small. This allows us to assume that if $(\mathbf{C}, I)$ is κ small then κ is regular since otherwise we can just take its successor cardinal. The next definition is some variant of the construction in Quillen's small object argument [15]. It is a transfinite gluing construction.

Definition 4.5 *Let $\mathbf{C}$ be any category and $\mathcal{G}$ a class of maps in $\mathbf{C}$. Fix a map $f : X \to Y$ and define*

$$S(f, F) = \{g, h : g \in \mathcal{G} \text{ and } h \circ g = f\}$$

Let $X(g)$ denote the range of g and if $S(f, \mathcal{G})$ is a set

$$G(f, \mathcal{G}) = colim_{(g,h) \in S(f,\mathcal{G})} X(g)$$

where the only non-identity maps in the system are $g : X \to X(g)$, one for every $(g, h) \in S(f, \mathcal{G})$. Also there are canonical maps

$$X \xrightarrow{i'(f)} G(f, \mathcal{G}) \xrightarrow{p'(f)} Y$$

Define

$$i(1) = i'(f)$$
$$p(1) = p'(f)$$
$$G(1) = G(f, \mathcal{G})$$
$$G(n+1) = G(p(n), \mathcal{G})$$
$$p(n+1) = p'(p(n))$$
$$i(n+1) = i'(p(n)) \circ i(n)$$

or if n *is a limit ordinal*

$$G(n) = colim_{m<n} G(m)$$

$$p(n) = colim_{m<n} p(m)$$

and let $i(n) : X \to G(n)$ *be the canonical map.*

5 κ small cell categories

For this section assume that $(\mathbf{C}, I)$ is a κ small cell category with κ a regular cardinal. The following example of the gluing construction gives us our first factorization. We use notation from definition 4.5.

Definition 5.1 *Let* $\mathcal{G}$ *be the class of all relative cell complexes of length and size* 1*. Let* $f : X \to Y$ *be any map. We then define*

$$CW(f) = G(\kappa)(f)$$

$$CW(i)(f) = G(i)(f)$$
$$p(f) = p(\kappa)(f)$$
$$i(f) = i(\kappa)(f)$$

Lemma 5.2 *For any* f*,* $i(f) \in \mathcal{C}(I)$

Proof: Clear. □

Lemma 5.3 *Any solid arrow diagram of the following form has an extension lift*

$$\begin{array}{ccc} A(\alpha) & \xrightarrow{g} & CW(f) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle p(f)} \\ B(\alpha) & \longrightarrow & Y \end{array}$$

Proof: $i(f) \to CW(f) \in \mathcal{C}(I)$ so there exists a factorization

$$\begin{array}{ccc} A(\alpha) & \xrightarrow{g'} & X' \\ & \searrow^{g} & \downarrow \\ & & CW(f) \end{array}$$

such that X' is a subcomplex of $CW(f)$ over X of size less than κ. Each cell of X' must be in $CW(\delta)$ for some $\delta < \kappa$. Since κ is regular and X' is of size less than κ all of its cells must be in $CW(\kappa')(f)$ for some $\kappa' < \kappa$. So we get a factorization

$$\begin{array}{ccc} A(\alpha) & \xrightarrow{h} & CW(\kappa') \\ & \searrow^{g} & \downarrow \\ & & CW(f) \end{array}$$

Therefore it follows that we get a commutative diagram

$$\begin{array}{ccc} A(\alpha) & \longrightarrow & CW(\kappa')(f) \\ \downarrow & & \downarrow \\ B(\alpha) & \longrightarrow & CW(\kappa'+1)(f) \end{array}$$

and we are done. □

Lemma 5.4 *For every map f, every $j \in \mathcal{C}(I)$ has the RLP with respect to $p(f)$.*

Proof: First prove for j such that $s(j) = 1$ using Lemma 5.3 and extend by taking colimits. □

Definition 5.5 *Let $(\mathbf{C}, I)$ be a cell category. We call a subclass $\mathcal{C}'$ of $\mathcal{C}(I)$* κ **factorable** *if for every map in $\mathcal{C}'$ of size at least 1 and every cell structure* *on that map there exists an $j \in \mathcal{C}'$ with the structure of a subcomplex such that $1 \leq s(j) < \kappa$.*

We call $\mathcal{C}'$ **strongly κ factorable** *if for every map in $\mathcal{C}'$ and every one of its subcomplexes, j', of size less than κ, there exists a subcomplex $j \in \mathcal{C}'$ containing j' and such that $s(j) < \kappa$.*

Clearly strongly κ factorable implies κ factorable and if $\mathcal{C}'$ is κ factorable then for any cardinal $\delta > \kappa$, $\mathcal{C}'$ is δ factorable. Again this allows us to deal with the problem relating to regular cardinals.

Definition 5.6 *Let $\mathcal{C}$ be a class of maps in a category $\mathbf{C}$. We say that $\mathcal{C}$ is* **closed under push outs** *if for every pushout*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow f & & \downarrow g \\ B & \longrightarrow & D \end{array}$$

$f \in \mathcal{C}$ implies that $g \in \mathcal{C}$.

We say that $\mathcal{C}$ is **closed under direct limits** *if for every direct system $\{X(\alpha)\}_{\alpha<\Gamma}$ such that every map is in $\mathcal{C}$ and such that for any limit ordinal $\beta < \Gamma$, $X(\beta) = colim_{\gamma<\beta} X(\gamma)$ we have that the canonical map $X(0) \to colim_{\alpha<\Gamma} X(\alpha)$ is in $\mathcal{C}$.*

Lemma 5.7 *Let $(\mathbf{C}, I)$ be a cell category. If $\mathcal{C}' = \mathcal{C}(I')$ for a set $I' \subset \mathcal{C}(I)$ then $\mathcal{C}'$ is strongly κ factorable for some κ and is closed under pushouts and direct limits.*

Proof: Easy. □

Lemma 5.8 *Let $\mathcal{B} \subset Hom_{\mathbf{C}}$ be a class closed under pushouts then the closure of $\mathcal{B}$ under retracts is closed under pushouts.*

Proof: If

$$\begin{array}{ccc} A & \longrightarrow & C \\ f\downarrow & & \downarrow g \\ B & \longrightarrow & D \end{array}$$

and

$$\begin{array}{ccc} A' & \longrightarrow & C \\ f'\downarrow & & \downarrow g' \\ B' & \longrightarrow & D' \end{array}$$

are pushouts and f is a retract of f' then g is a retract of g'. □

Lemma 5.9 *Let $\mathcal{B} \subset Hom_{\mathbf{C}}$ be a class closed under pushouts and direct limits then the closure of $\mathcal{B}$ under retracts is closed under direct limits.*

Proof: If A is a retract of A' and $f : A \to B$ is a retract of $f'' : A'' \to B''$ then with the obvious choice of map let f' and B' be defined by the pushout

$$\begin{array}{ccc} A'' & \longrightarrow & A' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} \\ B'' & \longrightarrow & B'. \end{array}$$

Then f is a retract of f'. So we see that composition preserves the closure of $\mathcal{B}$ under retracts. It is also clear that a direct limit of compatible retracts is a retract of the direct limit. The result follows. □

6 κ Factorable Classes

In this section for a regular cardinal we fix a κ small cell category $(\mathbf{C}, I)$ and a κ factorable class $\mathcal{C}' \subset \mathcal{C}(I)$ that is closed under pushouts and direct limits. For examples see Lemma 5.7. Most of the section is filled up by Lemma 6.3. The following example of the gluing construction gives us our second factorization. We use notation from Definition 4.5.

Definition 6.1 *Let $\mathcal{G}$ be the class of all relative cell complexes representing a map in $\mathcal{C}'$ and of size and length less than κ. Let $f : X \to Y$ be any map. Then we define*

$$\tilde{X}(f) = G(\kappa)(f)$$
$$\tilde{X}(i) = G(i)(f)$$
$$\tilde{p}(f) = p(\kappa)(f)$$
$$\tilde{i}(f) = i(\kappa)(f)$$

Lemma 6.2 *For every $f : X \to Y$ and for every $\delta < \kappa$, $\tilde{i}(\delta)(f)$ and $\tilde{i}(f)$ are in $\mathcal{C}(\mathcal{C}')$.*

Proof: Follows from the direct limit and pushout conditions. □

In the next lemma and its proof we will be using notation from Definition 6.1.

Lemma 6.3 *For every $f : X \to Y$, $\tilde{p}(f)$ has the RLP with respect to any $i \in \mathcal{C}(\mathcal{C}')$*

Proof: Let $j : E \to F \in \mathcal{C}(\mathcal{C}')$. Let $E(0) = E$ and assume that we have defined $E(n)$ and a map

$$j(n) : E(n) \to F \in \mathcal{C}(\mathcal{C}')$$

If $E(n) \neq F$ define $E(n+1)$ and

$$E(n) \xrightarrow{k} E(j+1) \xrightarrow{j(n+1)} F$$

in such a way that $j(n) = j(n+1) \circ k$, k is a weak equivalence and $1 \leq s(k) < \kappa$. We can do this since $\mathcal{C}'$ is κ factorable. If n is a limit ordinal let

$$E(n) = colim_{m<n} E(n)$$

and

$$j(n) = colim_{m<n} j(m).$$

Since $s(j)$ is some cardinal there must exist an ordinal r such that $E(r) = F$. So since we can lift a stage at a time and take colimits it is enough to consider the case $s(j) < \kappa$.

Let us take a commuting square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X}(f) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \tilde{p}(f)} \\ F & \longrightarrow & Y. \end{array}$$

Let the following pushout define W and j'

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X}(f) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ F & \longrightarrow & W. \end{array}$$

Then there exists a canonical map $p' : W \to Y$. We wish to define r such that $r \circ j' = id$ and the following diagram commutes

$$\begin{array}{ccc} \tilde{X}(f) & & \\ \uparrow{\scriptstyle r} & \searrow^{\tilde{p}(f)} & \\ W & \xrightarrow[p']{} & Y \end{array}$$

$s(j') < \kappa$ so j' can be decomposed in such a way that $W(0) = \tilde{X}$ and for every n

$$\begin{array}{ccc} \bigvee_{r\in K(n)} A(\alpha(r)) & \longrightarrow & W(n) \\ \downarrow & & \downarrow \\ \bigvee_{r\in K(n)} B(\alpha(r)) & \longrightarrow & W(n+1) \end{array}$$

is a pushout. These being the cells used in constructing F from E. $|K(n)| < \kappa$, and for n a limit ordinal

$$W(n) = colim_{m<n} W(m),$$

and $W = W(\gamma)$ for some $\gamma < \kappa$.

We let $t < \kappa$ and make the following induction hypotheses. Assume that for every $n < t$ there exists $k(n) < \kappa$, $W'(n)$ and a pushout

$$\begin{array}{ccc} \tilde{X}(k(n)) & \longrightarrow & \tilde{X}(f) \\ \downarrow & & \downarrow \\ W'(n) & \xrightarrow{q(n)} & W(n). \end{array}$$

The induction breaks naturally into two cases.

Case 1:

t is a successor ordinal. So we have pushouts

$$\begin{array}{ccc} \tilde{X}(k(t-1)) & \longrightarrow & \tilde{X}(f) \\ \downarrow & & \downarrow \\ W'(t-1) & \xrightarrow{q(t-1)} & W(t-1) \end{array}$$

and

$$\begin{array}{ccc} \bigvee_{r\in K(t-1)} A(\alpha(r)) & \longrightarrow & W(t-1) \\ \downarrow & & \downarrow \\ \bigvee_{r\in K(t-1)} B(\alpha(r)) & \longrightarrow & W(t) \end{array}$$

define $W'(t-1,l)$ by the pushout

$$\begin{array}{ccc} \tilde{X}(k(t-1)) & \longrightarrow & \tilde{X}(k(t-1)+l) \\ \downarrow & & \downarrow \\ W'(t-1) & \longrightarrow & W'(t-1,l) \end{array}$$

Then $W'(t-1,\kappa) = W(t-1)$ and because of κ smallness applied to the cell complex $W'(i-1) \to W(i-1)$ for every $r \in K(t-1)$ there exists $h(r) < \kappa$ such that the map $A(\alpha(r)) \to W(t-1)$ factors through $W'(t-1,h(r))$ as in the proof of 5.3. So

$$\bigvee_{r\in K(t-1)} A(\alpha(r)) \to W(t-1)$$

factors through $W'(t-1,h)$ for $h = sup_r(h(r))$. But since κ is regular $h < \kappa$. Let $k(t) = k(t-1) + h$ and define $W'(t)$ by the push out

$$\begin{array}{ccc} \bigvee_{r\in K(t-1)} A(\alpha(r)) & \longrightarrow & W'(t-1)(h) \\ \downarrow{\scriptstyle \vee\alpha(r)} & & \downarrow \\ \bigvee_{r\in K(t-1)} B(\alpha(r)) & \longrightarrow & W'(t). \end{array}$$

The map $W'(t) \to W(t)$ is determined by the map $W'(t-1)(h) \to W(t-1)$ and the inclusions of the cells $B(\alpha(r)) \to W(t)$.

Case 2:

Let t be a limit ordinal. Then

$$k(t) = sup_{m<t} k(m) < \kappa$$

since κ is regular. Let

$$W'(t) = \cup_{m<t} W'(m)$$

$\tilde{X}(k(t)) = \bigcup_{m<t} \tilde{X}(k(m))$ and it follows that

$$\begin{array}{ccc} \tilde{X}(k(t)) & \longrightarrow & \tilde{X}(f) \\ \downarrow & & \downarrow \\ W'(t) & \longrightarrow & W(t) \end{array}$$

is a pushout and we are finished the induction.

So there exists a pushout

$$\begin{array}{ccc} \tilde{X}(k(\gamma)) & \longrightarrow & \tilde{X} \\ \downarrow{\scriptstyle j''} & & \downarrow{\scriptstyle j'} \\ W'(\gamma) & \longrightarrow & W(\gamma) \end{array}$$

such that $j'' \in \mathcal{C}(I)$ and $s(j'') < \kappa$. The top map is an acyclic cofibration by Lemma 6.2 and so the bottom map is by the pushout condition. Also $j'' \in \mathcal{W}$ since the other three maps in the square are. Therefore there exists a map

$$r' : W(\gamma) \to \tilde{X}(k(\gamma) + 1)$$

such that $\tilde{p}|_{\tilde{X}(k(\gamma)+1)} \circ r' = p'|_{W'}$. Recalling that $W = W(\gamma)$ this determines a map $r : W \to \tilde{X}(f)$ such that $p' = \tilde{p} \circ r$ and by composition a lifting $h : F \to \tilde{X}(f)$. $\square$

7 Closed Model Structures

For this section let $(\mathbf{C}, I)$ be a small complete κ small cell category. Let $\mathcal{F}$ denote the closure under composition of all maps of the form $p(f)$ and $\tilde{p}(f)$ when both are defined.

In light of Lemma 5.7 if $\mathcal{C}' = \mathcal{C}(J)$ for some set $J \subset \mathcal{C}(I)$ then in the next theorem we have what is called a cofibrantly generated category [9] [11]. It is at times more convenient to instead specify a class of acyclic cofibrations. For example the cofibrations that are homology equivalences, as in Sections 8 and 9.

Theorem 7.1 *Let $\mathcal{W} \subset Hom_{\mathbf{C}}$ be a class satisfying the two out of three condition, closed under retracts and containing all isomorphisms. Assume $\mathcal{C}(I) \cap \mathcal{W}$ is a retract of a κ factorable class $\mathcal{C}'$. Then $(\mathbf{C}, \mathcal{C}(I), \mathcal{F}, \mathcal{W})$ is a model category if and only if $p(f)$ is a weak equivalence for every $f \in Hom_{\mathbf{C}}$ and $\mathcal{C}(I) \cap \mathcal{W}$ is closed under pushouts and direct limits.*

Proof: In any model category $\mathcal{C}(I) \cap \mathcal{W}$ is always closed under pushouts and direct limits and $p(f)$ is a weak equivalence since it has the RLP with respect to $\mathcal{C}(I)$.

First observe that we may assume κ is regular. Then the two factorizations are given by 5.1 and 6.1. The LLP of $\mathcal{C}(I) \cap \mathcal{W}$ with respect to $\mathcal{F}$ follows from 6.3, 5.4 and 3.3. We only have to verify the LLP of $\mathcal{C}$ with respect to $\mathcal{F} \cap \mathcal{W}$. It is sufficient to do the case for $j \in \mathcal{C}$ of size one. It is enough to show that every $j : A \to B \in \mathcal{C}(I)$ such that $s(j) = 1$ has the LLP with respect to every $p \in \mathcal{F} \cap \mathcal{W}$. It suffices to prove this in the case $j = \alpha$ for $\alpha \in I$. Let

$$\begin{array}{ccc} A(\alpha) & \longrightarrow & X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle p} \\ B(\alpha) & \longrightarrow & Y \end{array}$$

be a commutative diagram with i and p as above. Factor p

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \downarrow{\scriptstyle i(p)} & \nearrow{\scriptstyle h'} & \downarrow{\scriptstyle p} \\ CW(p) & \longrightarrow & Y \end{array}$$

$i(p) \in \mathcal{C}(I) \cap \mathcal{W}$ so there is a lift h', gotten by lifting over each map $p(g)$ or $\tilde{p}(g)$ that were composed together to get p. Also remembering the construction of $CW(p)$, there exists a commutative diagram

$$\begin{array}{ccccccc} A(\alpha) & \longrightarrow & X & \longrightarrow & X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B(\alpha) & \longrightarrow & CW(1)(p) & \longrightarrow & CW(p) & \longrightarrow & Y \end{array}$$

and so by composition we have our required lifting $h : B(\alpha) \to X$. $\square$

The next theorem is a variant of the previous one, though in practice it seems to be less usefull. For the theorem let $\tilde{\mathcal{F}}$ be all the maps in $\mathcal{F}$ that have the RLP with respect to $\mathcal{C}(I)$. Also let $\mathcal{C}' \subset \mathcal{C}$ be any κ factorable subclass. Let f be a weak equivalence if there exists a factorization of f, $f = pj$ such that $j \in \mathcal{C}(\mathcal{C}')$ and $p \in \tilde{\mathcal{F}}$. $\mathcal{W}$, as usual, will denote the class of weak equivalences.

Theorem 7.2 $(\mathbf{C}, \mathcal{C}(I), \mathcal{F}, \mathcal{W})$ *is a closed model category if and only if* $\mathcal{W}$ *satisfies the two out of three condition and is closed under retracts.*

Proof: First observe that $\tilde{\mathcal{F}} \subset \mathcal{W}$ and $\mathcal{C}(\mathcal{C}') \subset \mathcal{W}$ since we can take the factorization with one of the maps being the identity. So clearly all isomorphisms are also in $\mathcal{W}$. Next we show that $\mathcal{W} \cap \mathcal{C}(I)$ is in fact contained in the closure of $\mathcal{C}(\mathcal{C}')$ under retracts and $\mathcal{W} \cap \mathcal{F}$ is contained in the closure of $\tilde{\mathcal{F}}$ under retracts.

Let $f : X \to Y \in \mathcal{C}(I)$. Assume that we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that $j \in \mathcal{C}(\mathcal{C}')$ and $p \in \tilde{\mathcal{F}}$. Then by Lemmas 6.3 and 5.4 we get an extension lift h in the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[=]{} & Y \end{array}$$

and so we have realized f as a retract of i. The other case is similar. We can now apply Theorem 7.1 to get the result. $\square$

It should be remarked that if $\mathbf{C}$ has a closed model structure of the form of 7.1 and 7.2 then it is not too hard to give a natural closed model category structure for $\mathbf{C}^{\mathbf{D}}$, the category of functors $\mathbf{D} \to \mathbf{C}$. In the cofibrantly generated case this is done in [9] and [11]. Some other cases are looked at in [17].

Definition 7.3 *Define* $\mathbf{C}^{\mathbf{D}}$ *to be the category of* **diagrams** *from* $\mathbf{D}$ *to* $\mathbf{C}$. *So* $F \in \mathbf{C}^{\mathbf{D}}$ *is a functor* $F : \mathbf{D} \to \mathbf{C}$ *and* $\phi \in Hom(F, G)$ *is a natural transformation from* F *to* G.

We now define cells in $\mathbf{C}^{\mathbf{D}}$.

Definition 7.4 *For* $\alpha \in I$ *and* $x \in \mathbf{D}$ *define functors* $F(\alpha, x)$ *and* $G(\alpha, x)$ *and a natural transformation* $i(\alpha, x) : F(\alpha, x) \to G(\alpha, x)$ *as follows.*

$$F(\alpha, x)(y) = \bigvee_{f \in Hom(x,y)} A(\alpha)_f$$

where each $A(\alpha)_f$ *is a copy of* $A(\alpha)$ *and with the empty wedge interpreted as the initial object. For* $g \in Hom(y, z)$, $F(\alpha, x)$ *is determined by letting*

$$F(\alpha, x)(g)|_{A(\alpha)_f} = id : A(\alpha)_f \to A(\alpha)_{g \circ f}$$

$$G(\alpha, x)(y) = \bigvee_{f \in Hom(x,y)} B(\alpha)_f$$

where each $B(\alpha)_f$ *is a copy of* $B(\alpha)$. *For* $g \in Hom(y, z)$, $G(\alpha, x)$ *is determined by letting*

$$G(\alpha, x)(g)|_{B(\alpha)_f} = id : B(\alpha)_f \to B(\alpha)_{g \circ f}.$$

$i(\alpha, x)$ *is determined by the following equation*

$$i(\alpha, x)(y)|_{A(\alpha)_f} = i(\alpha) : A(\alpha)_f \to B(\alpha)_f.$$

Denote $\{F(\alpha, x) \to G(\alpha, x)\}_{(\alpha,x) \in I(\mathbf{C}) \times \mathbf{D}}$ *by* $I(\mathbf{C}^{\mathbf{D}})$.

Lemma 7.5 *For some cardinal* κ' $(\mathbf{C}^{\mathbf{D}}, I(\mathbf{C}^{\mathbf{D}}))$ *is a small complete* κ' *small cell category.*

Proof: Easy. □

Let $\mathcal{W} \subset Hom_{\mathbf{C}}$ be a class of weak equivalences. Let $\mathcal{W}' \subset Hom_{\mathbf{C}^{\mathbf{D}}}$ be the maps such that $F \in \mathcal{W}'$ if and only if for every $x \in \mathcal{D}$ $F(x) \in \mathcal{W}$.

Theorem 7.6 *Let* $\mathcal{C}(I) \cap \mathcal{W}$ *be strongly* κ *factorable. Then* $(\mathbf{C}^{\mathbf{D}}, \mathcal{C}(I(\mathcal{C}^{\mathbf{D}})), \mathcal{F}, \mathcal{W}')$ *is a closed model category.*

Proof: To apply Theorem 7.1 the only non-trivial thing to prove in that $\mathcal{C}(I(\mathbf{C}^{\mathbf{D}})) \cap \mathcal{W}'$ is κ' strongly factorable for cardinal κ'. We can take $\kappa' = \min\{\kappa, |Hom_{\mathbf{D}}|\}$. □

8 DGM(R)

The first example to which we apply our theorem is differential graded modules. We choose it because it allows the reader to become comfortable with the setting in a simple and familiar category.

Let $\mathbf{R}$ be any ring. For us a ring has an identity and is commutative. We let $DGM(\mathbf{R})$ be the category of differential graded modules. Note that $DGM(\mathbf{R})$ is complete and cocomplete with limits and colimits being taken in each grading. We define the following set of cells

$$I = \{i_n : \mathbf{R} < a_n > \to \mathbf{R} < a_n, b_{n+1} : db_{n+1} = a_n >\}$$

where the subscript denotes the degree and where $\mathbf{R} < x(1), \ldots, x(n) >$ denotes the free graded $\mathbf{R}$ module with basis $\{x(1), \ldots, x(n)\}$.

So we have determined our strong cofibrations $\mathcal{C}(I)$. We select our class $\mathcal{W}$ of weak equivalences to be the maps that induce isomorphisms on homology. We proceed to verify the four conditions of Theorem 7.1.

Lemma 8.1 *In $DGM(\mathbf{R})$ for a direct system $\{N(\alpha)\}$*

$$Hom(\mathbf{R} < a_n >, colim N(\alpha)) = colim Hom(\mathbf{R} < a_n >, N(\alpha)).$$

Proof: Follows since maps out of $\mathbf{R} < a_n >$ are determined by the set map on a_n, since the forgetful functor to sets commuted with direct limits and since the corresponding statement holds in the category of sets. $\square$

Corollary 8.2 *$(DGM(\mathbf{R}))$ is ω small.*

Lemma 8.3 *$\mathcal{W} \cap \mathcal{C}(I)$ is closed under direct limits.*

Proof: Follows since H_* commutes with direct limits. $\square$

Lemma 8.4 *$\mathcal{C}(I) \cap \mathcal{W}$ is closed under pushouts.*

Proof: Given a pushout in $DGM(\mathbf{R})$

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D, \end{array}$$

we get a long exact sequence

$$\to H_*(A) \to H_*(B) \oplus H_*(C) \to H_*(D) \to H_{*-1}(A) \to .$$

$\square$

Lemma 8.5 *$\mathcal{C}(I) \cap \mathcal{W}$ is strongly ω^+ factorable.*

Proof: Let $j : A \to B \in \mathcal{C}(I) \cap \mathcal{W}$ and $j(0) : A \to B(0)$ a subcomplex such that $s(j(0)) < \omega^+$ be given. For every $x \in cokerH_*(j(0))$ there exists a subcomplex of finite size $j_x : A \to B_x$ of j such that x is a boundary in B_x. Understand that $cokerH_*(j(0))$ has a countable basis. So we construct $j'(1) : B(0) \to B(1)$ a subcomplex of the complex determined by j, $B(0) \to B$ such that $s(j'(1)) < \omega^+$ and let $j(1) = j'(1)j(0) : A(0) \to B(1)$. Continuing in this way it is easy to get $j(\omega) : A \to B(\omega) \in \mathcal{C}(I)\mathcal{W}$ a subcomplex of j. □

This method is a basic technique in proofs of the factorability of classes of the form $\mathcal{C}(I) \cap \mathcal{W}$. As in the last section we now let $\mathcal{F}$ be the closure under composition of all maps of the form $p(f)$ and $\tilde{p}(f)$. The following theorem is well know [15] [10].

Theorem 8.6 *$(DGM(R), \mathcal{C}(I), \mathcal{F}, \mathcal{W})$ is a closed model category.*

Proof: 8.2, 8.3, 8.4, 8.5 and 7.1. □

9 CDGA(R)

We wish to demonstrate the usefulness of our theorem by applying it to an example. For any commutative ring $\mathbf{R}$ we give $CDGA(\mathbf{R})$ a closed model category structure such that the cofibrations are the semi free algebras, that is relative Sullivan algebras, and the weak equivalences are the homology equivalences. This model category structure is well know, and used, in the case that $\mathbf{R}$ is a field of characteristic 0. In the case of prime characteristic it was not known to exist.

Definition 9.1 *A commutative differential graded algebra over a commutative ring $\mathbf{R}$ (A, ϕ, e) is a differential graded module over $\mathbf{R}$ together with a unit $e : \mathbf{R} \to A$, where $\mathbf{R}$ is considered to be concentrated in degree 0 and have trivial differential, and an associative multiplication*

$$\phi : A \otimes_R A \to A$$

that is graded commutative. That is $\phi(a \otimes b) = (-1)^{|a||b|}\phi(b \otimes a)$. If $1/2 \notin R$ then we also require that $\phi(a \otimes a) = 0$ if $|a|$ is odd. We also require the graded Leibnitz law to hold.

$$d(\phi(a \otimes b)) = \phi(da \otimes b) + (-1)^{|a|}\phi(a \otimes db)$$

Maps of commutative differential graded algebras are maps of differential graded modules that commute with the multiplication. Let $CGDA(\mathbf{R})$ denote the category we have just described.

Again, $CDGA(\mathbf{R})$ is complete and cocomplete (an exercise left to the reader). It is traditional in $CDGA(\mathbf{R})$ to denote $\phi(a \otimes b)$ by ab and to let d be a map that raises degree by one. We will not break with tradition. We let $\mathcal{F} : CDGA(\mathbf{R}) \to DGM(\mathbf{R})$ denote the forgetful functor.

In $CDGA(\mathbf{R})$ we let our cells be the set

$$I = \{j_n : \Lambda(a_n) \to \Lambda(a_n, b_{n-1} : db = a)\} \cup \{j'_n : 0 \to \lambda(a_n)\}$$

where $\Lambda(a(1), ..., a(n))$ is the free graded commutative algebra on the $a(i)$ and the subscript denotes the dimension.

Lemma 9.2 *$(CDGA(\mathbf{R}))$ is a ω small cell category.*

Proof: This follows since maps out of $\Lambda(a)$ are determined by the map on the generator. □

Let $\mathcal{W}$ be the class of all maps such that $H_*\mathcal{F}$ is an isomorphism of graded modules. Clearly $\mathcal{W}$ is closed under retracts and satisfies the two out of three condition. We wish to see if we have determined a closed model category. We must demonstrate the four conditions of Theorem 7.1. First we show

Lemma 9.3 *In $(CDGA(\mathbf{R}), I)$, $\mathcal{C}(I) \cap \cap W$ is closed under direct limits.*

Proof: It follows since for a direct system $\{X_\alpha\}$, $\mathcal{F} colim X_\alpha = colim \mathcal{F}(X_\alpha)$ and since H_* commutes with direct limits.

Lemma 9.4 *For every $f \in Hom_{CDGA(\mathbf{R})}$, $p(f) \in \mathcal{W}$.*

Proof: For every Y, $Hom(\Lambda(a_n), Y) = Z_n(Y)$. Therefore $p(f)(1) : CW(f)(1) \to Y$ induces a surjection onto $Z_*(Y)$. So $p(f)$ is surjective on H_*. Let $x \in ker H_*(p(f)(n))$ and let $g : R < a_n > \to CW(f)(n)$ represent that element. There exists an extension

$$\begin{array}{ccc} R < a_n > & \xrightarrow{g} & CW(f)(n) \\ \downarrow & & \downarrow \\ R < a_n, b_{n-1} : db = a > & \longrightarrow & Y. \end{array}$$

So $x \in ker(CW(f)(n) \to CW(f)(n+1))$. Therefore $p(f)$ is injective also and we are done. □

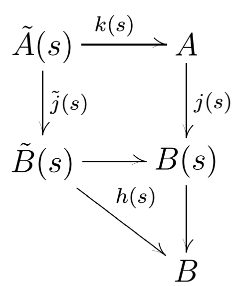
Lemma 9.5 *In $(CDGA(\mathbf{R}), I)$, $\mathcal{C}(I) \cap \mathcal{W}$ is closed under pushouts.*

Proof: Considering Lemmas 9.2 and 9.4 we can apply Chapter I Lemmas 1.14 and 8.13 of Baues [2]. □

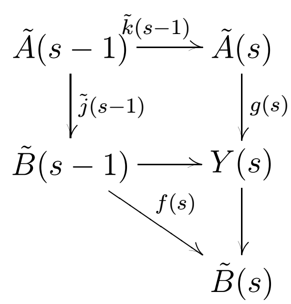
The proof of the following lemma is similar to the proof of Lemma 11.2 of [5]

Lemma 9.6 *$\mathcal{C}(I) \cap \mathcal{W}$ is strongly ω^+ factorable.*

Proof: Let $j : A \to B \in \mathcal{C}(I) \cap \mathcal{W}$. Let $j(0) : A \to B(0)$ be any subcomplex of j such that $s(j(1)) < \omega^+$. Fix $r < \omega$. Assume that for every $s < r$ we have diagrams



and



such that the squares are pushouts, $j(s)$ is a subcomplex of j of countable size, $\tilde{A}(s)$ is a countably generated subalgebra of A and $g(s)$ is a subcomplex of $\tilde{j}(s)$. Also assume that for every $x \in coker(H_*(\tilde{j}(s)))$ if x can be represented by an element of $ker H_*(h(s))$ then it can be represented by an element of $ker H_*(f(s))$. If $x \in coker H_*(\tilde{j}(s))/im(ker(H_*(h(s))))$ then its image in $\tilde{B}(s+1)$ can be represented by $x \in im\tilde{j}(s+1)$. Finally assume that if $x \in ker H_*(\tilde{j}(s-1))$ then $x \in ker H_*(k(s-1))$.

We will now construct the diagrams for $s = r+1$. Notice $H_*\tilde{B}(s)$ has a countable basis. So $coker H_*(\tilde{j}(s))$ has a countable basis $\{x_i\}_{i \in I \cup J}$ such that $\{x_i\}_{i \in J}$ is a basis for the image of $ker H_*(h(s))$ in $coker H_*(\tilde{j}(s))$. For every x_i, $i \in J$ there exists $y \in B$ such that $dy = x_i$ and y is an element of a countably generated subalgebra of B. Also for every $i \in I$ there is a countably generated subalgebra A' of A such that there exists $z \in A'$ and $h(s)(x_i)$ is homologous to $j(z)$. Finally if $x \in ker\tilde{j}(s)$ then since j is a weak equivalence $x \in ker(k(s))$ so again the element that bounds if lives in a countably generated subalgebra of A. So the induction step is easily completed.

We will now be able to construct our acyclic subcomplex of j. Let $\tilde{A}(\omega) = colim\tilde{A}(s)$ and $\tilde{B}(\omega) = colim\tilde{B}(r)$. Let the following diagram be a pushout

$$\begin{array}{ccc} \tilde{A}(\omega) & \longrightarrow & A \\ \downarrow & & \downarrow j' \\ \tilde{B}(\omega) & \longrightarrow & B'. \end{array}$$

Clearly j' is a subcomplex of j of countable size and if $\tilde{j}(\omega)$ is a weak equivalence j' is also by Lemma 9.5. We implicitly use the fact that H_* commutes with direct limits. $\tilde{j}(\omega)$ is surjective since all elements in the kernel of $H_*(\tilde{j}(s))$ are also in the kernel of $H_*(k(s))$. Next let $x \in coker H_*(\tilde{j}(\omega))$ and assume that $x \neq 0$. Then $x \in H_*(\tilde{j}(n))$ for some n and clearly $x \notin ker(H_*(f(n)))$ so there exists $y \in H_*(\tilde{A}(n+1))$ such that $H_*(\tilde{j}(n+1))(y) = x$. This contradicts our assumption. So $\tilde{j}(\omega)$ is a weak equivalence and we are done. □

We have proved the following theorem.

Theorem 9.7 *$(CDGA(\mathbf{R}), \mathcal{C}, \mathcal{F}, \mathcal{W})$ is a closed model category.*

Proof: Theorem 7.1 and Lemmas 9.2, 9.4, 9.5 and 9.6. □

If $\mathbf{R}$ is a field of characteristic 0 then the fibrations and surjections are the same. This fact allows for a direct proof of the existence of the closed model structure. If $\mathbf{R}$ is a field of characteristic p, not all fibrations are surjections and not all surjections are fibrations. For example the map $\Lambda(a_{2n}) \to \Lambda(a_{2n})/(a_{2n}^p)$ is a surjection that is not a fibration. If we factor $0 \to \Lambda(a_{2n})$ into a cofibration followed by an acyclic fibration the second map must be trivial and so it is a fibration that is not a surjection.

In $CGDA(\mathbf{R})$ the weak equivalences could also be defined as maps that induce bijections on homotopy classes of maps out of the $\Lambda(a_n)$, $[\Lambda(a_n), \_]$. This is an analogue of weak equivalences in Top_*. Carrying the analogy further we also get natural Hurewitz homomorphisms $[\Lambda(a_n), A] \to H_n(A)$.

They need not be equivalences [17]. In fact a map can be a H_* equivalence without being a $[\Lambda(a_n), \_]$ equivalence.

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