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Stability analysis of T-S fuzzy control systems by using set theory

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***Abstract*—This paper is concerned with the stability analysis for T-S fuzzy control systems. By exploiting the property of the structure of fuzzy inference engine, an equivalence relation on index set of the product of fuzzy rule weights is defined. Further, a new stability criterion is proposed by using the equivalence relation, and formulated into progressively less conservative sets of linear matrix inequalities. By using an extension of Pólya's Theorem, the new criterion is proved to be with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. A numerical example is given to illustrate the effectiveness of the proposed method.**

***Index Terms*—T-S fuzzy control systems, stability analysis, equivalence class, set theory, linear matrix inequalities (LMIs).**

I. Introduction

SINCE the terminology of the fuzzy set was proposed by Zadeh in 1965 [35], it has been found extensive applications in the areas of industrial and economical systems and so on. In particular, by constructing Takagi-Sugeno (T-S) fuzzy models of nonlinear control systems, various systematic mathematical techniques are successfully developed for guaranteeing the stability and performance of nonlinear systems. T-S fuzzy systems can be viewed as some locally linear time-invariant systems connected by IF-THEN rules. As a result, the conventional linear system theory can be applied for nonlinear control systems.

In recent years, stability analysis and synthesis of T-S fuzzy systems have been well studied [31], [6], [5], [32], [4], [34], [30], [37], [21], where quadratic Lyapunov function approaches [20], [9], [28], [18] are widely employed. Since a common Lyapunov matrix is used for all local models of fuzzy systems, the quadratic Lyapunov function approach often leads to conservative results. Then parameter dependent Lyapunov functions (or called fuzzy Lyapunov functions) [24], [11], [19], [17], [36], piecewise Lyapunov functions [13], [23] and k-sample variation Lyapunov functions [16] are respectively proposed for reducing the conservatism introduced by using quadratic Lyapunov functions. On the other hand, by sharing the same fuzzy rules with the fuzzy models, parallel distributed compensation (PDC) control schemes [29] are often used for designing fuzzy controllers in the existing literature. In addition, a number of alternative control schemes are also proposed for less conservative design, such as non-PDC control schemes [11], [33], switching constant controller gain schemes [10], local nonlinear feedback control schemes [7] and so on.

The above-mentioned results have made significant progress in stability analysis and synthesis of T-S fuzzy control systems, and they are applicable for the T-S fuzzy systems with any membership function and any fuzzy inference engine, which implies that they are independent of the actual membership shape and the choice of fuzzy inference engines. Hence, they might be conservative if specific knowledge of the fuzzy membership or fuzzy inference is available, then the properties of fuzzy membership shapes or fuzzy inference engines are exploited by many researchers, and some less conservative conditions for the stability analysis and synthesis of T-S fuzzy control systems are presented. For example, by incorporating shape information in the form of polynomial constraints, a stability and performance condition for polynomial-in-membership Takagi-Sugeno fuzzy systems is proposed in [27]. A stability analysis condition based on some inequalities in the form of a p-dimensional fuzzy summation is given in [25]. By using the property of pseudotrapezoid membership functions, a class of Lyapunov functions and fuzzy control schemes depending on dominant fuzzy membership functions are presented in [8]. By constructing tensor product T-S fuzzy models and using the property of the tensor product of membership functions, modelling and control based on a recursive algorithm are given in [2] and [1], respectively. By utilizing the extreme points in each partition to address the constraints of the fuzzy weights and their derivatives, a switching control law based on the partition is achieved in [15].

Motivated by the above works, where the properties about the shape of membership functions or the structure of fuzzy rule weights are exploited for less conservative conditions, we will further study the stability analysis problem for T-S fuzzy control systems by using some new properties of rule weights with a fuzzy product inference engine. By partitioning index set of the product of rule weights with the aid of an

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equivalence relation on the index set, a new stability analysis criterion is acquired and the new criterion is composed of a family of linear matrix inequalities with progressively less conservatism. In particular, by using an extension of the Pólya's Theorem, it is shown that the criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. Moreover, it is proved that the class of new approaches are not only with less conservatism but also with a lighter computational burden than the existing approaches in [20]. The comparisons with the existing approaches in [29], [20], [9], [24], [28] by a numerical example further illustrate that the new conditions have the potential to give less conservative results.

The rest of this paper is organized as follows. Section II gives some necessary preliminaries on set theory. T-S fuzzy models are given in Section III. By defining an equivalence relation on index set of the product of rule weights and using the equivalence relation, a new stability analysis condition is proposed in Section IV. In Section V, a numerical example is given to illustrate the effectiveness of the proposed methods. Section VI concludes the paper.

II. Preliminaries and Technical Lemmas

Set theory is one of the most fundamental branches of mathematics. In this section, some related notations and terminologies of elementary set theory are recalled. Further, some new technical lemmas are proposed, which are useful for obtaining a stability analysis criterion of T-S fuzzy control systems.

A. Notation, conception and some existing lemmas

$\mathbb{Z}_+$ denotes the positive integer set.
$\emptyset$ denotes empty set.
$|\mathbb{X}|$ denotes the number of elements (cardinality) of a set $\mathbb{X}$.
$\mathbb{X}_1$, $\mathbb{X}_2$, $\cdots$, $\mathbb{X}_n$ are sets,

$$\begin{aligned}\prod_{i=1}^{n}\mathbb{X}_i =&\mathbb{X}_1\times\cdots\times\mathbb{X}_n\\ =&\{(\mathrm{x}_1,\cdots,\mathrm{x}_n):\ \mathrm{x}_1\in\mathbb{X}_1\wedge\cdots\wedge\mathrm{x}_n\in\mathbb{X}_n\}\end{aligned} \tag{1}$$

where $(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_n)$ is an ordered n-tuple, $\wedge$ represents a classic logical operator "conjunction".

We also use the permutation $\mathrm{x}=\mathrm{x}_1\mathrm{x}_2\cdots\mathrm{x}_n$ to denote the ordered n-tuple $(\mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_n)$. Use $\mathrm{x}_{\langle i]}$ to represent the i-th element of x, i.e., an element τ belongs to $\prod\limits_{j=1}^{p}\mathbb{X}_j$, which means that $\tau=\tau_{\langle 1]}\tau_{\langle 2]}\cdots\tau_{\langle p]}$ and $\tau_{\langle j]}\in\mathbb{X}_j$, $j=1,\cdots,p$.

For $\sigma=\sigma_{\langle 1]}\sigma_{\langle 2]}\cdots\sigma_{\langle h_1+\cdots+h_p]}\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}$, where h_i, $i=1,\cdots,p$ are positive integers, we define two maps as follows:

$$\begin{aligned}&\chi_j:\prod_{i=1}^{p}\mathbb{S}_i^{h_i}\longrightarrow\mathbb{S}_j^{h_j}, && \text{for } j=1,\cdots,p\\ &\varrho_j:\prod_{i=1}^{p}\mathbb{S}_i^{h_i}\longrightarrow\prod_{i=1}^{p}\mathbb{S}_i, && \text{for } j=1,\cdots,g,\\ &g=\min\{h_i:1\le i\le p\}\end{aligned}$$

with

$$\begin{aligned}&\chi_1(\sigma)=\sigma_{\langle 1]}\sigma_{\langle 2]}\cdots\sigma_{\langle h_1]},\\ &\chi_2(\sigma)=\sigma_{\langle h_1+1]}\sigma_{\langle h_1+2]}\cdots\sigma_{\langle h_1+h_2]},\\ &\quad\vdots\\ &\chi_p(\sigma)=\\ &\sigma_{\langle h_1+\cdots+h_{p-1}+1]}\sigma_{\langle h_1+\cdots+h_{p-1}+2]}\cdots\sigma_{\langle h_1+\cdots+h_{p-1}+h_p]},\\ &\varrho_1(\sigma)=\sigma_{\langle 1]}\sigma_{\langle h_1+1]}\sigma_{\langle h_1+h_2+1]}\cdots\sigma_{\langle h_1+\cdots+h_{p-1}+1]}\\ &\varrho_2(\sigma)=\sigma_{\langle 2]}\sigma_{\langle h_1+2]}\sigma_{\langle h_1+h_2+2]}\cdots\sigma_{\langle h_1+\cdots+h_{p-1}+2]}\\ &\quad\vdots\\ &\varrho_g(\sigma)=\sigma_{\langle g]}\sigma_{\langle h_1+g]}\sigma_{\langle h_1+h_2+g]}\cdots\sigma_{\langle h_1+\cdots+h_{p-1}+g]}\end{aligned} \tag{2}$$

and denote $\chi_i(\sigma)$ by σ^{χ_i}, $\varrho_i(\sigma)$ by σ^{ϱ_i}.

For function $\mu_{ij}(v_i(t))$, $1\le i\le p$, $j\in\mathbb{S}_i\subset\mathbb{Z}_+$, we define

$$\begin{aligned}\mu_\tau=&\mu_\tau(v(t))=\prod_{j=1}^{p}\prod_{l=1}^{h_j}\mu_{j(\chi_j(\tau))_{\langle l]}}(v_j(t))\\ =&\prod_{j=1}^{p}\prod_{l=1}^{h_j}\mu_{j(\tau^{\chi_j})_{\langle l]}}(v_j(t))\end{aligned} \tag{3}$$

where $\tau\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}$.

B. Equivalence class and inequality

In this subsection, a relation on index set is defined, and it is proved to be an equivalence relation. By using the equivalence relation, a new condition is proposed for converting a parameter dependent inequality into parameter independent inequalities.

Let a set $\mathbb{S}_0\subset\mathbb{Z}_+$ with $|\mathbb{S}_0|<\infty$ ($|\mathbb{S}_0|$ denotes the cardinality of the set $\mathbb{S}_0$). If $(i_1,i_2,\cdots,i_{h_0})\in\mathbb{S}_0^{h_0}$, we can view the element $(i_1,i_2,\cdots,i_{h_0})$ of $\mathbb{S}_0^{h_0}$ as an h_0-ary permutation $i_1i_2\cdots i_{h_0}$. We define a map $st(\bullet)$ from $\mathbb{S}_0^{h_0}$ to $\mathbb{S}_0^{h_0}$

$$st(i_1i_2\cdots i_{h_0})=l_1l_2\cdots l_{h_0} \tag{4}$$

as an arrangement of the permutation $i_1i_2\cdots i_{h_0}$ with $l_1\le l_2\le\cdots\le l_{h_0}$.

Based on the mapping $st(\bullet)$, we define a binary relation on $\mathbb{S}_0^{h_0}$ as follows:

$$\begin{aligned}&\mathbb{R}_{0h_0}=\\ &\{(i_1i_2\cdots i_{h_0},j_1j_2\cdots j_{h_0}):st(j_1j_2\cdots j_{h_0})=st(i_1i_2\cdots i_{h_0})\}\end{aligned} \tag{5}$$

From the definition of the relation $\mathbb{R}_{0h_0}$, we can easily verify that $\mathbb{R}_{0h_0}$ is reflexive, symmetric, and transitive, i.e., $\mathbb{R}_{0h_0}$ is an equivalent relation over the set $\mathbb{S}_0^{h_0}$.

Denote $\mathbb{S}_0^{h_0}/\mathbb{R}_{0h_0}$ as the quotient of the equivalent relation $\mathbb{R}_{0h_0}$, i.e., $\mathbb{S}_0^{h_0}/\mathbb{R}_{0h_0}$ is formed of all equivalence classes of $\mathbb{R}_{0h_0}$. By Lemma 7 (see Appendix), we have the quotient set $\mathbb{S}_0^{h_0}/\mathbb{R}_{0h_0}$ is a partition of the set $\mathbb{S}_0^{h_0}$, i.e.,

$$\mathbb{S}_0^{h_0}=\bigcup_{\mathbb{s}_0\in\mathbb{S}_0^{h_0}/\mathbb{R}_{0h_0}}\mathbb{s}_0$$

with for all $\mathbb{x}\neq\mathbb{y}\in\mathbb{S}_0^{h_0}/\mathbb{R}_{0h_0}$, $\mathbb{x}\bigcap\mathbb{y}=\emptyset$.

For example, $\mathbb{X} = \{11, 12, 21, 22\}$, then $\sum_{\tau\in\mathbb{X}} M_\tau = M_{11} + M_{12} + M_{21} + M_{22}$. Definite a binary relation on $\mathbb{X}$ as

$$\mathbb{R} = \{(i_1 i_2, j_1 j_2) : st(j_1 j_2) = st(i_1 i_2)\}$$

where $st(\cdot)$ is the same as in (4).

Then the quotient set

$$\mathbb{X}/\mathbb{R} = \{[\![11]\!]_\mathbb{R}, [\![12]\!]_\mathbb{R}, [\![22]\!]_\mathbb{R}\}$$

with all the equivalence classes of $\mathbb{R}_{0h_0}$ as follows:

$$\begin{aligned}[\![11]\!]_\mathbb{R} =& \{11\}\\ [\![12]\!]_\mathbb{R} =& \{12, 21\} = [\![21]\!]_\mathbb{R}\\ [\![22]\!]_\mathbb{R} =& \{22\}\end{aligned}$$

The following fact can easily be obtained

$$\bigcup_{\mathbb{s}_0\in\mathbb{X}/\mathbb{R}} \mathbb{s}_0 = [\![11]\!]_\mathbb{R} \bigcup [\![12]\!]_\mathbb{R} \bigcup [\![22]\!]_\mathbb{R} = \mathbb{X}$$

which further validates Lemma 7, i.e., $\mathbb{X}/\mathbb{R}$ is a partition of $\mathbb{X}$.

Then

$$\begin{aligned}\sum_{\mathbb{s}\in\mathbb{X}/\mathbb{R}} \sum_{\tau\in\mathbb{s}} M_\tau =& \sum_{\tau\in[\![11]\!]_\mathbb{R}} M_\tau + \sum_{\tau\in[\![12]\!]_\mathbb{R}} M_\tau + \sum_{\tau\in[\![22]\!]_\mathbb{R}} M_\tau\\ =& M_{11} + M_{12} + M_{21} + M_{22} = \sum_{\tau\in\mathbb{X}} M_\tau\end{aligned}$$

Lemma 1: Let $\mathbb{S}_l \subset \mathbb{Z}_+$ with $|\mathbb{S}_l| < \infty$, $1 \le l \le p$, then $\mathbb{S}_1^{h_1}/\mathbb{R}_{1h_1} \times \mathbb{S}_2^{h_2}/\mathbb{R}_{2h_2} \times \cdots \times \mathbb{S}_p^{h_p}/\mathbb{R}_{ph_p}$ is a partition of $\mathbb{S}_1^{h_1} \times \mathbb{S}_2^{h_2} \times \cdots \times \mathbb{S}_p^{h_p}$, where

$$\begin{aligned}\mathbb{R}_{lh_l} =& \{(i_1 i_2 \cdots i_{h_l}, j_1 j_2 \cdots j_{h_l})|\\ & st(j_1 j_2 \cdots j_{h_l}) = st(i_1 i_2 \cdots i_{h_l})\}, \quad 1 \le l \le p \end{aligned} \tag{6}$$

and $st(\cdot)$ is the same as in (4).

Proof: See Appendix. ∎

Based on Lemma 1, the following useful lemma can be obtained

Lemma 2: Let $\mathbb{S}_l \subset \mathbb{Z}_+$ with $|\mathbb{S}_l| < \infty$, $1 \le l \le p$, and

$$\mu_{ji_j}(v_j(t)) \ge 0, \text{ and } \sum_{i_j\in\mathbb{S}_j} \mu_{ji_j}(v_j(t)) = 1, \text{ for } i_j \in \mathbb{S}_j,\\ j = 1, \cdots, p \tag{7}$$

if

$$\sum_{\sigma\in\bar{\mathbb{s}}} M_\sigma < 0, \text{ for } \bar{\mathbb{s}} \in \prod_{i=1}^{p} (\mathbb{S}_i^{h_i}/\mathbb{R}_{ih_i}) \tag{8}$$

then

$$\sum_{\sigma\in\prod_{i=1}^p \mathbb{S}_i^{h_i}} \mu_\sigma M_\sigma < 0 \tag{9}$$

where μ_σ and $\mathbb{R}_{lh_l}$ are the same as in (3) and (6), respectively.

Proof: See Appendix. ∎

III. System Description

T-S fuzzy system

The nonlinear system under consideration is described by the following fuzzy system model:

Plant Rule $(i_1 i_2 \cdots i_p)$**:**

$$\begin{aligned}&\text{IF } v_1(t) \text{ is } M_{1i_1} \text{ and } v_2(t) \text{ is } M_{2i_2}, \cdots, v_p(t) \text{ is } M_{pi_p}\\ &\text{THEN } \dot{x}(t) = A_{i_1 i_2\cdots i_p} x(t) + B_{i_1 i_2 \cdots i_p} u(t)\end{aligned} \tag{10}$$

$x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $v(t) = [v_1(t)\ v_2(t)\ \cdots\ v_p(t)\ ]^T \in \mathbb{R}^p$, $v_i(t)$, $i = 1, \cdots, p$ are the premise variables and assumed to be measurable, M_{ji_j}, $j = 1, \cdots, p$, $i_j = 1, \cdots, r_j$ denotes an $v_j(t)$-based fuzzy set and they are linguistic terms characterized by fuzzy membership functions $M_{ji_j}(v_j(t))$, where r_j is the number of $v_j(t)$-based fuzzy sets. Then, the fuzzy rule base consists of $r = \prod_{i=1}^{p} r_i$ IF-THEN rules.

By using a singleton fuzzifier, a product inference engine and a center average defuzzifier, the T-S fuzzy model is obtained as: Let

$$\mu_{ji_j}(v_j(t)) = \frac{M_{ji_j}(v_j(t))}{\sum_{l_j=1}^{r_j} M_{jl_j}(v_j(t))}, \quad \text{for } 1 \le j \le p,\ 1 \le i_j \le r_j \tag{12}$$

Combining it and (11), the fuzzy system can be written as follows:

$$\begin{aligned}\dot{x}(t) =& \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_p=1}^{r_p} \left(\prod_{j=1}^{p} \mu_{ji_j}(v_j(t)) \right) \times\\ & \Big(A_{i_1 i_2 \cdots i_p} x(t) + B_{i_1 i_2 \cdots i_p} u(t) \Big)\end{aligned} \tag{13}$$

From (12), it is resulted that

$$\sum_{i_j=1}^{r_j} \mu_{ji_j}(v_j(t)) = 1, \quad \text{for } 1 \le j \le p \tag{14}$$

By using set theory, (13) can be rewritten as follows:

$$\dot{x}(t) = \sum_{\tau\in\prod_{i=1}^p \mathbb{S}_i} \mu_\tau (A_\tau x(t) + B_\tau u(t)) \tag{15}$$

where μ_τ is the same as in (3) and

$$\mathbb{S}_i = \{1, 2, \cdots, r_i\}, \quad i = 1, 2\cdots, p \tag{16}$$

Fuzzy controller

In the existing literature, there are many fuzzy control schemes for T-S fuzzy systems, for example, parallel distributed compensation (PDC) control schemes [29], non-PDC control schemes [11], switching constant gain control schemes [10], dominant dependent fuzzy control schemes [8] and so on. This paper focuses on how to use the property of the product of rule weights based on the equivalence class in set theory for obtaining a better stability analysis condition, and any control scheme is applicable in this paper. In particular, the PDC controller is adopted in this paper as follows:

Control Rule $(i_1 i_2 \cdots i_p)$**:**

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$$\dot{x}(t)=\frac{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\left(\prod_{j=1}^{p}M_{ji_j}(v_j(t))\right)\left(A_{i_1i_2\cdots i_p}x(t)+B_{i_1i_2\cdots i_p}u(t)\right)}{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p}M_{ji_j}(v_j(t))} \tag{11}$$

IF $v_1(t)$ is M_{1i_1} and $v_2(t)$ is $M_{2i_2}, \cdots, v_p(t)$ is M_{pi_p}
THEN $u(t) = K_{i_1i_2\cdots i_p}x(t)$

By using a singleton fuzzifier, a product inference engine and a center average defuzzifier, the final output of the fuzzy controller is inferred as follows:

$$u(t)=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p}\mu_{ji_j}(v_j(t))K_{i_1i_2\cdots i_p}x(t) \tag{17}$$

Its substitutional description based on set theory is

$$u(t)=\sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\tau K_\tau x(t) \tag{18}$$

where μ_τ and $\mathbb{S}_i$ are the same as in (3) and (16), respectively.

Closed-loop fuzzy system

Now we substitute (18) into (15), then we have

$$\dot{x}(t)=\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\sigma A_\sigma x(t) + \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\sigma B_\sigma\left(\sum_{\eta\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\eta K_\eta x(t)\right) \tag{19}$$

where the definitions of μ_σ, μ_η refer to (3), $\mathbb{S}_i=\{1, 2, \cdots, r_i\}$, $i=1, 2, \cdots, p$.

Combining (14) and (19), it follows that

$$\dot{x}(t)=\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i}\sum_{\eta\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\sigma\mu_\eta(A_\sigma+B_\sigma K_\eta)x(t)$$

i.e.,

$$\dot{x}(t)=\sum_{\xi\in\prod_{i=1}^{p}\mathbb{S}_i^2}\mu_{\xi^{\varrho_1}}\mu_{\xi^{\varrho_2}}(A_{\xi^{\varrho_1}}+B_{\xi^{\varrho_1}}K_{\xi^{\varrho_2}})x(t) \tag{20}$$

where the relation of ξ and ξ^{ϱ_1} (or ξ^{ϱ_2}) is given in (2).

Let

$$\Lambda_\xi = A_{\xi^{\varrho_1}}+B_{\xi^{\varrho_1}}K_{\xi^{\varrho_2}} \tag{21}$$

then the closed-loop system (20) can be rewritten as:

$$\dot{x}(t)=\sum_{\xi\in\prod_{i=1}^{p}\mathbb{S}_i^2}\mu_{\xi^{\varrho_1}}\mu_{\xi^{\varrho_2}}\Lambda_\xi x(t) \tag{22}$$

Description of fuzzy system by using fuzzy basis functions

(13) can be further re-described by fuzzy basis functions

$$\begin{aligned}\mu_{i_1i_2\cdots i_p}(v(t)) &=\frac{\prod_{j=1}^{p}M_{ji_j}(v_j(t))}{\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\prod_{j=1}^{p}M_{ji_j}(v_j(t))}\\ &=\prod_{j=1}^{p}\mu_{ji_j}(v_j(t)),\quad i_1i_2\cdots i_p\in\prod_{i=1}^{p}\mathbb{S}_i\end{aligned}$$

as follows:

$$\begin{aligned}\dot{x}(t)&=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\sum_{i_p=1}^{r_p}\mu_{i_1i_2\cdots i_p}(v(t))\times\\ &\quad\left(A_{i_1i_2\cdots i_p}x(t)+B_{i_1i_2\cdots i_p}u(t)\right)\\ &=\sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i}\mu_\tau\left(A_\tau x(t)+B_\tau u(t)\right)\end{aligned} \tag{23}$$

where μ_τ is the same as in (3) and $v(t) = [v_1(t)\ v_2(t)\ \cdots\ v_p(t)]^T$.
Because $\mathbb{S}_l$, $1 \le l \le p$ is a set with finite elements (r_l elements), $\prod_{i=1}^{p}\mathbb{S}_i$ also consists of finite elements ($\prod_{i=1}^{p} r_i$ elements), which implies that the cardinality of the set $\prod_{i=1}^{p}\mathbb{S}_i$ is $\prod_{i=1}^{p} r_i$. Let $r = \prod_{i=1}^{p} r_i$, then from the definition of cardinality of set [26], there exists a $1-1$ mapping

$$q:\prod_{i=1}^{p}\mathbb{S}_i\longrightarrow\{1,2,\cdots,r\} \tag{24}$$

with $|\prod_{i=1}^{p}\mathbb{S}_i| = r$.

By virtue of the lexicographic order of the element $\tau_{\langle 1\rangle}\tau_{\langle 2\rangle}\cdots\tau_{\langle p\rangle}$ in the set $\prod_{i=1}^{p}\mathbb{S}_i$, a particular q can be chosen as follows:

$$\begin{aligned}q(\tau)=&\tau_{\langle 1\rangle}+(\tau_{\langle 2\rangle}-1)r_1+(\tau_{\langle 3\rangle}-1)r_1r_2+(\tau_{\langle 4\rangle}-1)r_1r_2r_3\\ &+\cdots+(\tau_{\langle p\rangle}-1)\prod_{j=1}^{p-1}r_j\\ =&\tau_{\langle 1\rangle}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}r_j(\tau_{\langle i\rangle}-1)\end{aligned}$$

i.e.,

$$q:\tau_{\langle 1\rangle}\tau_{\langle 2\rangle}\cdots\tau_{\langle p\rangle}\longmapsto\tau_{\langle 1\rangle}+\sum_{i=2}^{p}\prod_{j=1}^{i-1}r_j(\tau_{\langle i\rangle}-1) \tag{25}$$

Let

$$\alpha_{q(\tau)}(v(t))=\mu_\tau=\prod_{j=1}^{p}\mu_{j\tau_{\langle j\rangle}}(v_j(t)),\ \bar{A}_{q(\tau)}=A_\tau,$$

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$$\bar{B}_{q(\tau)} = B_\tau,\ \bar{K}_{q(\tau)} = K_\tau \tag{26}$$

then the closed-loop system (23) can be rewritten as follows:

$$\dot{x}(t) = \sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i} \alpha_{q(\tau)}(v(t))(\bar{A}_{q(\tau)}x(t) + \bar{B}_{q(\tau)}u(t))$$

which is equivalent to

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(v(t))(\bar{A}_i x(t) + \bar{B}_i u(t)) \tag{27}$$

Along the lines of the above technique, the fuzzy controller (17) can also be rewritten as follows:

$$u(t) = \sum_{i=1}^{r} \alpha_i(v(t))\bar{K}_i x(t) \tag{28}$$

Moreover, we can easily obtain $0 \le \alpha_i(v(t)) \le 1$, $i = 1, \cdots, r$, $\sum_{i=1}^{r} \alpha_i(v(t)) = 1$.

The fuzzy system description (27) with (28) is widely used in the existing literature, and there are various stability analysis conditions based on the description, see [29], [20], [28], and the reference therein, where the condition in [29] is with the least computational complexity based on LMIs, and the condition in [28] is asymptotically necessary and sufficient for quadratic stability analysis of T-S fuzzy control systems with any possible membership function and inference engine. In order to give the comparisons with the existing methods by theoretical proof, some existing conditions are recalled as follows:

Lemma 3: [29] If there exists a matrix $\bar{P} = \bar{P}^T > 0$ satisfying

$$\mathtt{He}(\bar{P}G_{ij} + \bar{P}G_{ji}) < 0,\ \text{for } 1 \le i \le j \le r \tag{29}$$

where

$$G_{ij} = \bar{A}_i + \bar{B}_i\bar{K}_j$$

then the fuzzy system (27) with (28) is asymptotically stable.

Lemma 4: [20] If there exist matrices $\bar{P} = \bar{P}^T > 0$, $\bar{Y}_{ij}$, $1 \le i \le j \le r$ satisfying

$$\mathtt{He}(\bar{P}G_{ij} + \bar{P}G_{ji}) \le \bar{Y}_{ij} + (\bar{Y}_{ij})^T,\ \text{for } 1 \le i \le j \le r \tag{30}$$

$$[\bar{Y}_{ij}] < 0 \tag{31}$$

then the fuzzy system (27) with (28) is asymptotically stable.

Lemma 5: [24] Assume that $\dot{\alpha}_i(v(t)) \le \phi_i$, $1 \le i \le r$, if there exist matrices $X = X^T$, $P_i = P_i^T$, $1 \le i \le r$, satisfying the following LMIs

$$\begin{aligned} &P_i > 0, && 1 \le i \le r \\ &P_i + X > 0, && 1 \le i \le r \\ &\tilde{P}_\phi + \frac{1}{2}\mathtt{He}(P_l G_{ij} + P_l G_{ji}) < 0, && 1 \le i \le l \le r,\ 1 \le j \le r \end{aligned}$$

where $\tilde{P}_\phi = \sum_{i=1}^{r} \phi_i(P_i + X)$, ϕ_i are scalars, then the fuzzy system (27) with (28) is asymptotically stable.

IV. Stability criterion

In this section, a new stability analysis criterion for T-S fuzzy systems is proposed with progressively less conservatism. It is proved that the new criterion is with less conservatism and complexity than Lemma 4. Moreover, by using an extension of Pólya's Theorem, it is shown that the criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. Before main results are presented, some propaedeutics are given as follows:

Since $\mathbb{S}_i$, $i = 1, \cdots, p$, are with finite elements, and $|\mathbb{S}_i| = r_i$, then $|\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}| = \prod_{i=1}^{p} r_j^{\bar{h}_i}$. Further, we can define a $1-1$ mapping from the set $\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}$ to the set $\{1, 2, \cdots, \tilde{r}\}$, where $\tilde{r} = \prod_{i=1}^{p} r_i^{\bar{h}_i}$.

A particular q can be chosen as

$$\begin{aligned} q(\tau) =& 1 + \sum_{i_1=1}^{\bar{h}_1}(\tau_{\langle i_1]} - 1)r_1^{i_1-1} + \sum_{i_2=1+\bar{h}_1}^{\bar{h}_1+\bar{h}_2}(\tau_{\langle i_2]} - 1)r_1^{\bar{h}_1}r_2^{i_2-1} \\ &+ \sum_{i_3=1+\bar{h}_1+\bar{h}_2}^{\bar{h}_1+\bar{h}_2+\bar{h}_3}(\tau_{\langle i_3]} - 1)\prod_{j=1}^{2} r_j^{\bar{h}_j} r_3^{i_3-1} + \cdots \\ &+ \sum_{i_p=1+\bar{h}_1+\cdots+\bar{h}_{p-1}}^{\sum_{m=1}^{p}\bar{h}_m}(\tau_{\langle i_p]} - 1)\prod_{j=1}^{p-1} r_j^{\bar{h}_j} r_p^{i_p-1} \end{aligned} \tag{32}$$

Let

$$\begin{aligned} \bar{\alpha}_{q(\tau)}(v(t)) =& \mu_\tau(v(t)) \\ =& \prod_{j=1}^{p}\prod_{l=1}^{\bar{h}_j} \mu_{j(\tau^{\times_j})_{\langle l]}}(v_j(t)),\ \text{for } \tau \in \prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i} \end{aligned} \tag{33}$$

Denote $\bar{\alpha}_{q(\tau)}(v(t))$ as $\bar{\alpha}_{q(\tau)}$, then

$$\sum_{i=1}^{\tilde{r}} \bar{\alpha}_i = \sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}} \mu_\tau \tag{34}$$

From (14), we have

$$\begin{aligned} 1 =& \prod_{j=1}^{p}\left(\sum_{i_j\in\mathbb{S}_j} \mu_{ji_j}(v_j(t))\right)^{\bar{h}_j} \\ =& \sum_{\tau_1\in\mathbb{S}_1^{\bar{h}_1}}\sum_{\tau_2\in\mathbb{S}_2^{\bar{h}_2}}\cdots\sum_{\tau_p\in\mathbb{S}_p^{\bar{h}_p}}\prod_{j=1}^{p}\prod_{l=1}^{\bar{h}_j}\mu_{j(\tau_j)_{\langle l]}}(v_j(t)) \\ =& \sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}}\prod_{j=1}^{p}\prod_{l=1}^{\bar{h}_j}\mu_{j(\tau^{\times_j})_{\langle l]}}(v_j(t)) \\ =& \sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}} \mu_\tau \end{aligned}$$

Combining it and (34), then we have

$$\sum_{i=1}^{r_s} \bar{\alpha}_i = 1,\quad 0 \le \bar{\alpha}_i \le 1 \tag{35}$$

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For $\sigma \in \prod_{i=1}^{p} \mathbb{S}_i^{2\bar{h}_i}$, define

$$\begin{aligned}\sigma^{\beta_1} =& \sigma_{\langle 1]}\sigma_{\langle 2]}\cdots\sigma_{\langle \bar{h}_1]}\sigma_{\langle 2\bar{h}_1+1]}\sigma_{\langle 2\bar{h}_1+2]}\cdots\sigma_{\langle 2\bar{h}_1+\bar{h}_2]}\\ &\cdots\sigma_{\langle 2\sum_{i=1}^{p-1}\bar{h}_i+1]}\sigma_{\langle 2\sum_{i=1}^{p-1}\bar{h}_i+2]}\cdots\sigma_{\langle 2\sum_{i=1}^{p-1}\bar{h}_i+\bar{h}_p]}\\ \sigma^{\beta_2} =& \sigma_{\langle \bar{h}_1+1]}\sigma_{\langle \bar{h}_1+2]}\cdots\sigma_{\langle 2\bar{h}_1]}\sigma_{\langle 2\bar{h}_1+\bar{h}_2+1]}\sigma_{\langle 2\bar{h}_1+\bar{h}_2+2]}\cdots\\ &\sigma_{\langle 2\sum_{i=1}^{2}\bar{h}_i]}\cdots\sigma_{\langle 2\sum_{i=1}^{p-1}\bar{h}_i+\bar{h}_p+1]}\sigma_{\langle 2\sum_{i=1}^{p-1}\bar{h}_i+\bar{h}_p+2]}\cdots\\ &\sigma_{\langle 2\sum_{i=1}^{p}\bar{h}_i]} \qquad (36)\end{aligned}$$

then σ^{β_1} and σ^{β_2} belong to $\prod_{i=1}^{p} \mathbb{S}_i^{\bar{h}_i}$.

Theorem 1: Given $h_j \in 2\mathbb{Z}_+$ ($2\mathbb{Z}_+$ denotes even set) with $h_j \geq 2$, $j = 1, \cdots, p$, binary relations $\mathbb{R}_{lh_l}$ over $\mathbb{S}_l^{h_l}$, $l = 1, \cdots, p$, which are the same as in Lemma 2. If there exist matrices $P = P^T > 0$, Y_σ, $\sigma \in \prod_{i=1}^{p} \mathbb{S}_i^{h_i}$, with $Y_\sigma = (Y_{\bar{\sigma}})^T$ for $\sigma^{\beta_1} = \bar{\sigma}^{\beta_2}$, $\sigma^{\beta_2} = \bar{\sigma}^{\beta_1}$, satisfying the following LMIs

$$\sum_{\sigma\in\bar{\mathbb{s}}} M_\sigma \leq \sum_{\sigma\in\bar{\mathbb{s}}} Y_\sigma, \text{ for } \bar{\mathbb{s}} \in \prod_{i=1}^{p}(\mathbb{S}_i^{d_i}/\mathbb{R}_{id_i}) \qquad (37)$$

$$[H_{ij}] < 0 \qquad (38)$$

where

$$M_\sigma = P\Lambda_\sigma + \Lambda_\sigma^T P, \text{ for } \sigma \in \prod_{i=1}^{p} \mathbb{S}_i^{h_i} \qquad (39)$$

and Λ_σ is the same as in (21), $H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} = Y_\sigma$, $q(\cdot)$ is the same as in (32), then the continuous time fuzzy system (13) is asymptotically stable.

Proof: Applying Lemma 2 to (37), then we have

$$\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma M_\sigma \leq \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma Y_\sigma \qquad (40)$$

Let $h_i = 2\bar{h}_i$, and define $q(\cdot)$ and α_i, $i = 1, \cdots, \tilde{r}$ by (32) and (33). It can be obtained from (38) that

$$\begin{bmatrix}\bar{\alpha}_1\\ \bar{\alpha}_2\\ \vdots\\ \bar{\alpha}_{\tilde{r}}\end{bmatrix}^T \begin{bmatrix}H_{11} & H_{12} & \cdots & H_{1\tilde{r}}\\ H_{21} & H_{22} & \cdots & H_{2\tilde{r}}\\ \vdots & \vdots & \ddots & \vdots\\ H_{\tilde{r}1} & H_{\tilde{r}2} & \cdots & H_{\tilde{r}\tilde{r}}\end{bmatrix} \begin{bmatrix}\bar{\alpha}_1\\ \bar{\alpha}_2\\ \vdots\\ \bar{\alpha}_{\tilde{r}}\end{bmatrix} < 0$$

i.e.,

$$\sum_{i=1}^{\tilde{r}}\sum_{j=1}^{\tilde{r}} \bar{\alpha}_i\bar{\alpha}_j H_{ij} < 0$$

Combining it and the definition of $q(\cdot)$, it yields that

$$\begin{aligned}&\sum_{i=1}^{\tilde{r}}\sum_{j=1}^{\tilde{r}} \bar{\alpha}_i\bar{\alpha}_j H_{ij}\\ =& \sum_{q(\sigma^{\beta_1})=1}^{\tilde{r}}\ \sum_{q(\sigma^{\beta_2})=1}^{\tilde{r}} \bar{\alpha}_{q(\sigma^{\beta_1})}\bar{\alpha}_{q(\sigma^{\beta_2})} H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}\\ =& \sum_{q(\sigma^{\beta_1})=1}^{\tilde{r}}\ \sum_{q(\sigma^{\beta_2})=1}^{\tilde{r}} \bar{\alpha}_{q(\sigma^{\beta_1})}\bar{\alpha}_{q(\sigma^{\beta_2})} Y_\sigma\\ =& \sum_{\sigma^{\beta_1}\in\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}}\ \sum_{\sigma^{\beta_2}\in\prod_{i=1}^{p}\mathbb{S}_i^{\bar{h}_i}} \mu_{\sigma^{\beta_1}}\mu_{\sigma^{\beta_2}} Y_\sigma\\ =& \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_{\sigma^{\beta_1}}\mu_{\sigma^{\beta_2}} Y_\sigma\\ =& \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma Y_\sigma\\ <&0\end{aligned}$$

Combining it and (40), we can obtain

$$\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma M_\sigma < 0 \qquad (41)$$

which is equivalent to

$$\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma \mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}}) < 0 \qquad (42)$$

Choose a quadratic Lyapunov function

$$V(t) = x^T(t)Px(t)$$

then it follows from (20) that

$$\begin{aligned}\dot{V}(t) =& 2x^T(t)P\dot{x}(t)\\ =& 2x^T(t)P \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2} \mu_{\sigma^{\varrho_1}}\mu_{\sigma^{\varrho_2}}(A_{\sigma^{\varrho_1}} + B_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})x(t)\\ =& x^T(t) \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2} \mu_{\sigma^{\varrho_1}}\mu_{\sigma^{\varrho_2}} \mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})\\ &\times x(t) \qquad (43)\end{aligned}$$

Consider

$$\sum_{\sigma\in\mathbb{S}_j^{h_j-2}} \prod_{l=1}^{h_j-2} \mu_{j\sigma_{\langle l]}}(v_j(t)) = \left(\sum_{i_j\in\mathbb{S}_j} \mu_{ji_j}(v_j(t))\right)^{h_j-2},$$

for $1 \leq j \leq p$

Combining it and (14), we have

$$\sum_{\sigma\in\mathbb{S}_j^{h_j-2}} \prod_{l=1}^{h_j-2} \mu_{j\sigma_{\langle l]}}(v_j(t)) = 1, \text{ for } 1 \leq j \leq p$$

From it and (43), we can obtain

$$\begin{aligned}\dot{V}(t) =& x^T(t)\left(\prod_{j=1}^{p}\left(\sum_{\sigma\in\mathbb{S}_j^{h_j-2}} \prod_{l=1}^{h_j-2} \mu_{j\sigma_{\langle l]}}(v_j(t))\right)\right)\times\\ &\left(\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2} \mu_{\sigma^{\varrho_1}}\mu_{\sigma^{\varrho_2}} \mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})\right)x(t)\\ =& x^T(t)\left\{\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i-2}} \left(\prod_{l=1}^{h_1-2} \mu_{1(\sigma^{\chi_1})_{\langle l]}}(v_1(t))\right)\times\right.\\ &\left.\left(\prod_{l=1}^{h_2-2} \mu_{2(\sigma^{\chi_2})_{\langle l]}}(v_2(t))\right)\cdots\left(\prod_{l=1}^{h_p-2} \mu_{p(\sigma^{\chi_p})_{\langle l]}}(v_p(t))\right)\right\}\\ &\times\left(\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2} \mu_{\sigma^{\varrho_1}}\mu_{\sigma^{\varrho_2}} \mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})\right)x(t)\end{aligned}$$

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$$= x^T(t) \sum_{\tau\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i-2}} \mu_\tau \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2} \mu_{\sigma^{\varrho_1}}\mu_{\sigma^{\varrho_2}}\mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})x(t)$$
$$= x^T(t) \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma \mathtt{He}(PA_{\sigma^{\varrho_1}} + PB_{\sigma^{\varrho_1}}K_{\sigma^{\varrho_2}})x(t)$$
$$= x^T(t) \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma \mathtt{He}(P\Lambda_\sigma)x(t) \quad (44)$$

From it and (42), we have that

$$\dot{V}(t) < 0, \text{ for } x(t) \neq 0$$

then by virtue of Lyapunov theory, it follows that the continuous time fuzzy system (13) is asymptotically stable. ■

Based on Theorem 1, the following corollary can easily be obtained.

Corollary 1: Given positive integers $h_j \geq 2$, binary relations $\mathbb{R}_{jh_j}$ over $\mathbb{S}_j^{h_j}$, $j = 1, 2, \cdots, p$, if there exists a matrix $P = P^T > 0$ satisfying the following LMIs

$$\sum_{\sigma\in\bar{\mathbb{s}}} M_\sigma < 0, \text{ for } \bar{\mathbb{s}} \in \prod_{i=1}^{p}(\mathbb{S}_i^{h_i}/\mathbb{R}_{jh_j}) \quad (45)$$

where M_σ and Λ_σ are respectively the same as in (39) and (21), then the fuzzy system (13) is asymptotically stable.

Proof: The proof is easily obtained from Theorem 1 and omitted. ■

Note that the condition (38) in Theorem 1 is dependent on the mapping $q(\cdot)$, however, the choice of the mapping $q(\cdot)$ does not affect the stability analysis results of Theorem 1, see Lemma 10 in Appendix. Moreover, the value of h_i, $1 \leq i \leq p$ of Theorem 1 is given in advance, if we increase the value of the positive integer h_i, $1 \leq i \leq p$, the conservatism of Theorem 1 will decrease. The fact is illustrated by the following theorem.

Theorem 2: If the condition of Theorem 1 holds for $h_i = 2d_i \in 2\mathbb{Z}_+$, $1 \leq i \leq p$, then the condition of Theorem 1 also holds for $h_i = 2\bar{d}_i \in 2\mathbb{Z}_+$ with $\bar{d}_i \geq d_i$, $1 \leq i \leq p$.

Proof: If the condition of Theorem 1 holds for $h_i = 2d_i$, $i = 1, 2, \cdots, p$. then there exists a scalar $\epsilon > 0$, such that

$$[H_{ij}] + \epsilon I < 0 \quad (46)$$

Choose

$$\tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} = \begin{cases} H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} + \epsilon I, & \sigma^{\beta_1} = \sigma^{\beta_2} \\ H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \text{others} \end{cases} \quad (47)$$

where $\sigma \in \prod_{i=1}^{p}\mathbb{S}_i^{2d_i}$ and σ^{β_1}, σ^{β_2} are the same as in (36). Then (46) can be written as

$$[\tilde{H}_{ij}] < 0 \quad (48)$$

Let $\mathbb{S}_1^{2d_1+2} \times \prod_{i=2}^{p}\mathbb{S}_i^{2d_i}$ is obtained from $\mathbb{S}_1^2$ and $\prod_{i=1}^{p}\mathbb{S}_i^{2d_i}$ by the following mapping,

$$\Psi(\tau,\sigma) = \sigma_{\langle 1]}\cdots\sigma_{\langle h_1]}\tau_{\langle 1]}\tau_{\langle 2]}\sigma_{\langle h_1+1]}\cdots\sigma_{\langle h_1+\cdots+h_p]} \in \mathbb{S}_1^{2d_1+2}\times\prod_{i=2}^{p}\mathbb{S}_i^{2d_i}$$

where $\tau \in \mathbb{S}_1^2$ and $\sigma \in \prod_{i=1}^{p}\mathbb{S}_i^{2d_i}$.

Let $\bar{\sigma} = \Psi(\tau,\sigma)$, and

$$\bar{H}_{q(\bar{\sigma}^{\beta_1})q(\bar{\sigma}^{\beta_2})} = \begin{cases} \tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} - \epsilon I, & \tau_{\langle 1]} = \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ \tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \text{others} \end{cases} \quad (49)$$

Choose $\bar{Y}_{\bar{\sigma}} = \bar{H}_{q(\bar{\sigma}^{\beta_1})q(\bar{\sigma}^{\beta_2})}$, then

$$\begin{aligned} \bar{Y}_{\bar{\sigma}} &= \bar{H}_{q(\bar{\sigma}^{\beta_1})q(\bar{\sigma}^{\beta_2})} \\ &= \begin{cases} \tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} - \epsilon I, & \tau_{\langle 1]} = \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ \tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \tau_{\langle 1]} \neq \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ \tilde{H}_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \text{others} \end{cases} \\ &= \begin{cases} H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \tau_{\langle 1]} = \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})} + \epsilon I, & \tau_{\langle 1]} \neq \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ H_{q(\sigma^{\beta_1})q(\sigma^{\beta_2})}, & \text{others} \end{cases} \\ &= \begin{cases} Y_\sigma, & \tau_{\langle 1]} = \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ Y_\sigma + \epsilon I, & \tau_{\langle 1]} \neq \tau_{\langle 2]}, \sigma^{\beta_1} = \sigma^{\beta_2} \\ Y_\sigma, & \text{others} \end{cases} \end{aligned} \quad (50)$$

For arbitrary $r_1^2\prod_{i=1}^{p}r_i^{h_i} = \bar{r}$-dimension vector $z = [z_1\ z_2\ \cdots\ z_{\bar{r}}]^T \neq 0$, pre- and post-multiplying $[\bar{H}_{ij}]$ by z^T and z, then it follows that

$$\begin{aligned} & z^T[\bar{H}_{ij}]z \\ &= \sum_{i=1}^{\bar{r}}\sum_{j=1}^{\bar{r}} z_i z_j \bar{H}_{ij} \\ &= \sum_{\bar{\sigma}^{\beta_1}\in\mathbb{S}_1^{d_1+1}\times\prod_{i=2}^{p}\mathbb{S}_i^{d_i}} \sum_{\bar{\sigma}^{\beta_2}\in\mathbb{S}_1^{d_1+1}\times\prod_{i=2}^{p}\mathbb{S}_i^{d_i}} z_{q(\bar{\sigma}^{\beta_1})} z_{q(\bar{\sigma}^{\beta_2})} \times \bar{H}_{q(\bar{\sigma}^{\beta_1})q(\bar{\sigma}^{\beta_2})} \\ &= \sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\tau_1\in\mathbb{S}_1}\sum_{\tau_2\in\mathbb{S}_1} z_{q(\sigma_1\diamond\tau_1)} z_{q(\sigma_2\diamond\tau_2)} \times \bar{H}_{q(\sigma_1\diamond\tau_1)q(\sigma_2\diamond\tau_2)} \end{aligned} \quad (51)$$

where $\sigma\diamond\tau = \sigma_{\langle 1]}\sigma_{\langle 2]}\cdots\sigma_{\langle d_1]}\tau\sigma_{\langle d_1+1]}\sigma_{\langle d_1+2]}\cdots\sigma_{\langle\sum_{i=1}^{p}d_i]} \in \mathbb{S}_1^{d_1+1}\times\prod_{i=2}^{p}\mathbb{S}_i^{d_i}$ with $\sigma \in \prod_{i=1}^{p}\mathbb{S}_i^{d_i}$ and $\tau \in \mathbb{S}_1$.

From (49) and (51), we have that

$$\begin{aligned} & z^T[\bar{H}_{ij}]z \\ &= \sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\tau_1\in\mathbb{S}_1}\sum_{\tau_2\in\mathbb{S}_1} z_{q(\sigma_1\diamond\tau_1)} z_{q(\sigma_2\diamond\tau_2)} \times \bar{H}_{q(\sigma_1\diamond\tau_1)q(\sigma_2\diamond\tau_2)} \\ &= \sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\tau_1\in\mathbb{S}_1}\sum_{\tau_2\in\mathbb{S}_1} z_{q(\sigma_1\diamond\tau_1)} z_{q(\sigma_2\diamond\tau_2)} \times \tilde{H}_{q(\sigma_1)q(\sigma_2)} - \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\sum_{\tau\in\mathbb{S}_1} z^2_{q(\sigma\diamond\tau)}\epsilon I \end{aligned} \quad (52)$$

Note that $z \neq 0$ means that

$$\| z \|^2 = \sum_{\bar{\sigma}\in S_1^{d_1+1}\prod_{i=2}^{p}\mathbb{S}_i^{d_i}} z^2_{q(\bar{\sigma})} = \sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\sum_{\tau\in\mathbb{S}_1} z^2_{q(\sigma\diamond\tau)} \neq 0$$

Combining it and (52), yields that

$$z^T[\bar{H}_{ij}]z < \sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}} \sum_{\tau_1\in\mathbb{S}_1}\sum_{\tau_2\in\mathbb{S}_1} z_{q(\sigma_1\diamond\tau_1)} z_{q(\sigma_2\diamond\tau_2)} \times$$

$$\begin{aligned}&\tilde{H}_{q(\sigma_1)q(\sigma_2)}\\ =&\sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\left(\sum_{\tau_1\in\mathbb{S}_1}\sum_{\tau_2\in\mathbb{S}_1}z_{q(\sigma_1\diamond\tau_1)}z_{q(\sigma_2\diamond\tau_2)}\right)\times\\ &\tilde{H}_{q(\sigma_1)q(\sigma_2)}\\ =&\sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\left(\sum_{\tau_1\in\mathbb{S}_1}z_{q(\sigma_1\diamond\tau_1)}\right)\times\\ &\left(\sum_{\tau_2\in\mathbb{S}_1}z_{q(\sigma_2\diamond\tau_2)}\right)\tilde{H}_{q(\sigma_1)q(\sigma_2)}\end{aligned}\tag{53}$$

Let $Z_{q(\sigma)}=\sum_{\tau\in\mathbb{S}_1}z_{q(\sigma\diamond\tau)}$ and $\prod_{i=1}^{p}r_i^{d_i}=\tilde{r}$, then from (53), we have that

$$\begin{aligned}z^T[\bar{H}_{ij}]z<&\sum_{\sigma_1\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}\sum_{\sigma_2\in\prod_{i=1}^{p}\mathbb{S}_i^{d_i}}Z_{q(\sigma_1)}Z_{q(\sigma_2)}\tilde{H}_{q(\sigma_1)q(\sigma_2)}\\ =&\sum_{q(\sigma_1)=1}^{\tilde{r}}\sum_{q(\sigma_2)=1}^{\tilde{r}}Z_{q(\sigma_1)}Z_{q(\sigma_2)}\tilde{H}_{q(\sigma_1)q(\sigma_2)}\\ =&\sum_{i=1}^{\tilde{r}}\sum_{j=1}^{\tilde{r}}Z_iZ_j\tilde{H}_{ij}\\ =&\begin{bmatrix}Z_1\\Z_2\\\vdots\\Z_{\tilde{r}}\end{bmatrix}^T\begin{bmatrix}\tilde{H}_{11}&\tilde{H}_{12}&\cdots&\tilde{H}_{1\tilde{r}}\\\tilde{H}_{21}&\tilde{H}_{22}&\cdots&\tilde{H}_{2\tilde{r}}\\\vdots&\vdots&\ddots&\vdots\\\tilde{H}_{\tilde{r}1}&\tilde{H}_{\tilde{r}2}&\cdots&\tilde{H}_{\tilde{r}\tilde{r}}\end{bmatrix}\begin{bmatrix}Z_1\\Z_2\\\vdots\\Z_{\tilde{r}}\end{bmatrix}\end{aligned}$$

Combining it and (48), it follows that

$$z^T[\bar{H}_{ij}]z<0$$

which implies that $[\bar{H}_{ij}]<0$ for $z\neq 0$. Further, we have that (38) with (50) holds for $h_1=2d_1+2$, $h_i=2d_i$, d_i, $i=2,3,\cdots,p$.

On the other hand, let $\mathbb{s}_1$ is an equivalence class of $\mathbb{S}_1^{2d_1+2}$ with the equivalence relation $\mathbb{R}_{1(2d_1+2)}$, and $\mathbb{s}_i$, $i=2, 3, \cdots$, p are respectively the equivalence class of $\mathbb{S}_i^{2d_i}$, $i=2, 3, \cdots, p$ with the equivalence relation $\mathbb{R}_{i(2d_i)}$, where $\mathbb{R}_{ih_i}$ is the same as in (6). Further, we define a relation over the set $\mathbb{s}_1$ as follows:

$$\begin{aligned}\bar{\mathbb{R}}_1=\Big\{&(\eta,\tau):st(\eta_{\langle 1]}\eta_{\langle 2]}\cdots\eta_{\langle 2d_1]})=st(\tau_{\langle 1]}\tau_{\langle 2]}\cdots\tau_{\langle 2d_1]}),\\ &\eta_{\langle 2d_1+1]}=\tau_{\langle 2d_1+1]},\eta_{\langle 2d_1+2]}=\tau_{\langle 2d_1+2]},\eta,\tau\in\mathbb{s}_1\Big\}\end{aligned}$$

It is easily obtained that $\bar{\mathbb{R}}_1$ is an equivalence relation on the set $\mathbb{s}_1$, then it follows from Lemma 9 that $\mathbb{s}_1/\bar{\mathbb{R}}_1$ is a partition of the set $\mathbb{s}_1$, which implies that

$$\sum_{\tau\in\mathbb{s}_1\times\prod_{i=2}^{p}\mathbb{s}_i}(M_\tau-\hat{Y}_\tau)=\sum_{\mathscr{S}_1\in\mathbb{s}_1/\bar{\mathbb{R}}_1}\sum_{\tau\in\mathscr{S}_1\times\prod_{i=2}^{p}\mathbb{s}_i}(M_\tau-\hat{Y}_\tau)\tag{54}$$

where

$$\begin{aligned}&\hat{Y}_\tau=Y_{\tau_{\langle 1]}\cdots\tau_{\langle 2d_1]}\tau_{\langle 2d_1+3]}\cdots\tau_{\langle 2d_1+2+\cdots+2d_p]}},\\ &\tau=\tau_{\langle 1]}\cdots\tau_{\langle 2d_1]}\tau_{\langle 2d_1+1]}\tau_{\langle 2d_1+2]}\tau_{\langle 2d_1+3]}\cdots\tau_{\langle 2d_1+2+\cdots+2d_p]}\\ &\in\mathbb{s}_1\subseteq\mathbb{S}_1^{2d_1+2}\end{aligned}$$

It follows from (37), (50) and (54) that

$$\begin{aligned}&\sum_{\bar{\sigma}\in\bar{\mathbb{s}}}M_{\bar{\sigma}}\leq\sum_{\bar{\sigma}\in\bar{\mathbb{s}}}\bar{Y}_{\bar{\sigma}},\ \text{for }\bar{\mathbb{s}}\in(\mathbb{S}_1^{2d_1+2}/\mathbb{R}_{1(2d_1+2)})\times\\ &\prod_{i=2}^{p}(\mathbb{S}_i^{2d_i}/\mathbb{R}_{i(2d_i)})\end{aligned}$$

i.e., (37) holds for $h_1=2d_1+2$, $h_i=2d_i$, d_i, $i=2,3,\cdots,p$.

We have proved that if the condition of Theorem 1 holds for $h_i=2d_i$, $i=1,2,\cdots,p$, then the condition of Theorem 1 also holds for $h_1=2d_1+2$, $h_i=2d_i$, $i=2,\cdots,p$. Further, it is easily obtained that the condition of Theorem 1 also holds for $h_1=2\bar{d}_1\geq 2d_1$, $h_i=2d_i$, $i=2,\cdots,p$.

Adopt the same technique for only h_i increasing for $i=2,\cdots,p$. Finally, we can obtain that the condition of Theorem 1 holds for $h_i=2\bar{d}_i\geq 2d_i$, $i=1,\cdots,p$. Thus the proof is complete. ■

Remark 1: Theorem 1 collects the interactions of the product of membership functions in a single matrix. The similar technique for dealing with the interactions of the fuzzy rule weights has been proposed in [20]. What it follows, it is proved that the condition of Theorem 1 is more relaxed than Lemma 4 and with a lighter computational burden, see the following theorem and Remark 2.

Theorem 3: If the condition of Lemma 4 holds, then the condition of Theorem 1 holds.

Proof: If there exists a matrix $\bar{P}=\bar{P}^T>0$, satisfying (30) and (31), then we have that

$$\begin{aligned}&\mathtt{He}(\bar{P}A_{\sigma^{\varrho_1}}+\bar{P}B_{\sigma^{\varrho_1}}\bar{K}_{\sigma^{\varrho_2}}+\bar{P}A_{\sigma^{\varrho_2}}+\bar{P}B_{\sigma^{\varrho_2}}\bar{K}_{\sigma^{\varrho_1}})\\ &<Y_\sigma+(Y_\sigma)^T,\ \text{for }\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^2\end{aligned}\tag{55}$$

$$[H_{ij}]<0\tag{56}$$

where $\bigstar^{\varrho_1}$ and $\bigstar^{\varrho_2}$ are the same as in (2), $H_{q(\sigma^{\varrho_1})q(\sigma^{\varrho_2})}=Y_\sigma=\bar{Y}_{q(\sigma^{\varrho_1})q(\sigma^{\varrho_2})}$, $q(\cdot)$ is defined in (25).

Define a binary relation $\tilde{\mathbb{R}}$ over the set $\prod_{i=1}^{p}\mathbb{s}_i\in\prod_{i=1}^{p}(\mathbb{S}_i^2/\mathbb{R}_{i2})$, where $\tilde{\mathbb{R}}$ is given as follows:

$$\begin{aligned}\tilde{\mathbb{R}}=\Big\{&(\pi,\vartheta):(\pi^{\varrho_1}=\vartheta^{\varrho_2}\text{ and }\pi^{\varrho_2}=\vartheta^{\varrho_1})or(\pi=\vartheta),\\ &\pi,\vartheta\in\prod_{i=1}^{p}\mathbb{s}_i\Big\}\end{aligned}\tag{57}$$

It is easily obtained that the relation $\tilde{\mathbb{R}}$ is reflexive, symmetric, and transitive, i.e, it is an equivalence relation. Further, we have that the set $(\prod_{i=1}^{p}\mathbb{s}_i)/\tilde{\mathbb{R}}=\{[\![\vartheta]\!]_{\tilde{\mathbb{R}}}:\vartheta\in\prod_{i=1}^{p}\mathbb{s}_i\}$ is a partition of the set $\prod_{i=1}^{p}\mathbb{s}_i$.

Therefore,

$$\begin{aligned}&\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{s}_i}\mathtt{He}(\bar{P}A_{\sigma^{\varrho_1}}+\bar{P}B_{\sigma^{\varrho_1}}\bar{K}_{\sigma^{\varrho_2}}+\bar{P}A_{\sigma^{\varrho_2}}+\\ &\bar{P}B_{\sigma^{\varrho_2}}\bar{K}_{\sigma^{\varrho_1}}-Y_\sigma)\\ =&\sum_{\mathrm{S}\in(\prod_{i=1}^{p}\mathbb{s}_i)/\tilde{\mathbb{R}}}\sum_{\sigma\in\mathrm{S}}\mathtt{He}(\bar{P}A_{\sigma^{\varrho_1}}+\bar{P}B_{\sigma^{\varrho_1}}\bar{K}_{\sigma^{\varrho_2}}+\bar{P}A_{\sigma^{\varrho_2}}+\\ &\bar{P}B_{\sigma^{\varrho_2}}\bar{K}_{\sigma^{\varrho_1}}-Y_\sigma)\end{aligned}\tag{58}$$

On the other hand, if $\mathtt{S} \in (\prod_{i=1}^{p} \mathbb{s}_i)/\tilde{\mathbb{R}}$, for any $\vartheta, \pi \in \mathtt{S}$, we have that $\vartheta^{\varrho_1} = \pi^{\varrho_2}$, $\vartheta^{\varrho_2} = \pi^{\varrho_1}$ or $\vartheta = \pi$, which implies that $|\mathtt{S}| = 1$ or 2.

For all $\mathtt{S}$, assume some $\vartheta \in \mathtt{S}$, from $\mathtt{S} \subseteq \prod_{i=1}^{p} \mathbb{s}_i \subseteq \prod_{i=1}^{p} \mathbb{S}_i^2$, we have that $\vartheta \in \prod_{i=1}^{p} \mathbb{S}_i^2$. For the ϑ, by virtue of (55), we can obtain

$$\begin{aligned}&\mathtt{He}(PA_{\vartheta^{\varrho_1}} + PB_{\vartheta^{\varrho_1}} K_{\vartheta^{\varrho_2}} + PA_{\vartheta^{\varrho_2}} + PB_{\vartheta^{\varrho_2}} K_{\vartheta^{\varrho_1}} - Y_\sigma)\\ &< 0\end{aligned}$$

which implies that

$$\sum_{\vartheta \in \mathtt{S}} \mathtt{He}(PA_{\vartheta^{\varrho_1}} + PB_{\vartheta^{\varrho_1}} K_{\vartheta^{\varrho_2}}) < \sum_{\vartheta \in \mathtt{S}} Y_\vartheta, \text{ for } \mathtt{S} \in (\prod_{i=1}^{p} \mathbb{s}_i)/\tilde{\mathbb{R}}$$

then

$$\begin{aligned}\sum_{\sigma \in \prod_{i=1}^{p} \mathbb{s}_i} M_\sigma &= \sum_{\mathtt{S} \in (\prod_{i=1}^{p} \mathbb{s}_i)/\tilde{\mathbb{R}}} \sum_{\sigma \in \mathtt{S}} M_\sigma \\ &< \sum_{\mathtt{S} \in (\prod_{i=1}^{p} \mathbb{s}_i)/\tilde{\mathbb{R}}} \sum_{\sigma \in \mathtt{S}} Y_\sigma = \sum_{\sigma \in \prod_{i=1}^{p} \mathbb{s}_i} Y_\sigma \end{aligned} \quad (59)$$

Combining it and (56), we have that (37) and (38) hold for $h_1 = h_2 = \cdots = h_p = 2$. Further, by virtue of Theorem 2, we have that the condition of Theorem 1 with $h_i \geq 2$, $i = 1, 2, \cdots, p$ holds. Thus, the proof is complete. ■

Remark 2: Note that Theorem 3 shows that the condition of Theorem 1 is more relaxed than one of Lemma 4. In particular, the number of LMIs in Theorem 1 is $\prod_{i=1}^{p} \begin{pmatrix} h_i + r_i - 1 \\ h_i \end{pmatrix} + 2$ (see Theorem 3.5.1 in [3], i.e., computing formula of combinatorial numbers for multiple set) and the number of LMIs in Lemma 4 is $\begin{pmatrix} 1 + \prod_{i=1}^{p} r_i \\ 2 \end{pmatrix} + 2$. For the case of $h_i = 2$, the number of LMIs in Theorem 1 is $\prod_{i=1}^{p} \begin{pmatrix} 1 + r_i \\ 2 \end{pmatrix} + 2$ and we can prove that $\prod_{i=1}^{p} \begin{pmatrix} 1 + r_i \\ 2 \end{pmatrix} \leq \begin{pmatrix} 1 + \prod_{i=1}^{p} r_i \\ 2 \end{pmatrix}$ (see Lemma 8 (ii)), which implies that the number of LMIs of Theorem 1 is smaller than Lemma 4. On the other hand, the number and size of variables in Theorem 1 with $h_i = 2$ are the same in Lemma 4, therefore, Theorem 1 with $h_i = 2$ is with a lighter computational burden than Lemma 4.

Note that we have shown that the conservatism of Theorem 1 becomes less along with increasing h_i, $i = 1, \cdots, p$. In fact, if the h_i is sufficiently large, the conditions of Theorem1 is with no conservatism for any possible membership. The fact will be illustrated in Theorem 4. In order to obtain the proof of Theorem 4, the useful knowledge about standard r_q-simplex is necessary.

We write Δ_q for the standard r_q-simplex

$$\begin{aligned}&\Delta_q = \\ &\left\{ [\mu_{q1}, \mu_{q2}, \cdots, \mu_{qr_q}] \in R^{r_q} : \sum_{i=1}^{r_q} \mu_{qi} = 1, 0 \leq \mu_{qi} \leq 1 \right\}, \\ &\text{for } q = 1, \cdots, p\end{aligned}$$

The following Lemma is an extension as the Pólya's Theorem.

Lemma 6: [14] Let $M(\mu) = M(\mu_{11}, \mu_{12}, \cdots, \mu_{1r_1}, \mu_{21}, \mu_{22}, \cdots, \mu_{2r_2}, \cdots, \mu_{p1}, \mu_{p2}, \cdots, \mu_{pr_p})$ is a homogeneous matrix-valued polynomial on $\Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_p}$, then $M(\mu) > 0$ for $\mu \in \Delta_{r_1} \times \Delta_{r_2} \times \cdots \times \Delta_{r_p}$ if and only if there exists a sufficiently large positive integer d, such that

$$\prod_{i=1}^{p} \left(\sum_{j=1}^{r_j} \mu_{ij} \right)^d M(\mu)$$

has all its coefficients positive.

Based on Lemma 6, we can obtain the following theorem.

Theorem 4: For arbitrary possible membership function $\mu_{ji_j}(v_j(t))$, $j = 1, \cdots, p$, $i_j = 1, \cdots, r_j$

$$M(\mu) = \sum_{\sigma \in \prod_{i=1}^{p} \mathbb{S}_i^2} \mu_\sigma M_\sigma < 0$$

if and only if there exists a sufficiently large positive integer d, such that

$$\sum_{\sigma \in \bar{\mathbb{s}}} M_\sigma < 0, \text{ for } \bar{\mathbb{s}} \in \prod_{i=1}^{p} (\mathbb{S}_i^{d+2}/\mathbb{R}_{i(d+2)})$$

Proof: If we consider the membership functions μ_{ji_j}, $j = 1, \cdots, p$, $i_j = 1, \cdots, r_j$, as the variables of the matrix-value polynomial

$$M(\mu) = \sum_{\sigma \in \prod_{i=1}^{p} \mathbb{S}_i^2} \mu_\sigma M_\sigma$$

where M_σ are matrices, and μ_σ is the same as in (3) and from the property of membership function, we have that $\mu_\sigma \in \Delta_{r_1}^2 \times \Delta_{r_2}^2 \times \cdots \times \Delta_{r_p}^2$ is a monomial with variables $\mu_{j\sigma_{\langle 2j-1 \rangle}}$, $\mu_{j\sigma_{\langle 2j \rangle}} \in \Delta_{r_j}$, $j = 1, 2, \cdots, p$.

From (14), it follows that

$$M(\mu) = \prod_{j=1}^{p} \left(\sum_{i_j=1}^{r_j} \mu_{ji_j} \right)^d M(\mu) = \sum_{\bar{\sigma} \in \prod_{i=1}^{p} \mathbb{S}_i^{d+2}} \mu_{\bar{\sigma}} M_{\bar{\sigma}} \quad (60)$$

Note that the like terms in (60) are not collected, in fact, if the term $\mu_{\bar{\sigma}} M_{\bar{\sigma}}$ and the term $\mu_\eta M_\eta$ are like terms, which implies that $\mu_{\bar{\sigma}} = \mu_\eta$. Because $\prod_{i=1}^{p} (\mathbb{S}_i^{d+2}/\mathbb{R}_{i(d+2)})$ is a partition of $\prod_{i=1}^{p} \mathbb{S}_i^{d+2}$ by Lemma 1, there exists $\bar{\mathbb{s}} \in \prod_{i=1}^{p} (\mathbb{S}_i^{d+2}/\mathbb{R}_{i(d+2)})$ such that $\bar{\sigma} \in \bar{\mathbb{s}}$. From the definition of the equivalence relations $\mathbb{R}_{ih_i}$, we have that $\eta \in \bar{\mathbb{s}}$. On the other hand, if some element $\varpi \in \bar{\mathbb{s}}$, it follows that $\mu_\varpi = \mu_{\bar{\sigma}}$, therefore, the coefficients of like terms of $\mu_{\bar{\sigma}}$ is $\sum_{\sigma \in \bar{\mathbb{s}}} M_\sigma$. By virtue of Lemma 6, $-M(\mu) = -\sum_{\sigma \in \prod_{i=1}^{p} \mathbb{S}_i^2} \mu_\sigma M_\sigma > 0$ if and only if there exists a sufficiently large positive integer d, such that $-\sum_{\sigma \in \bar{\mathbb{s}}} M_\sigma > 0$, for $\bar{\mathbb{s}} \in \prod_{i=1}^{p} (\mathbb{S}_i^{d+2}/\mathbb{R}_{i(d+2)})$. Thus, the proof is complete. ■

Remark 3: From Theorem 4, it follows that if h_i, $i = 1, 2 \cdots, p$ are sufficiently large, the condition of Theorem 1 is sufficient and necessary for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. We should point out that if the properties of the shape of membership function or the firing probability of fuzzy rules are considered, then less conservative results can be obtained, however this paper focuses on how to use the property of fuzzy product inference engine for less conservative and lighter computational burden conditions, then these properties about the shape and the firing probability are not used in this paper.

V. Example

In this section, a numerical example is given, the conditions of Theorem 1, Corollary 1 and the ones in [29], [20], [9], [28] are applied for illustrating the effectiveness of the new methods. All experiments are implemented in MATLAB, version 7.0.0 ($R14$) using the packages Yalmip [22] and SeDuMi $1.1R3$. The computer used is an Intel (R) Core (TM)2 Quad CPU Q9400 (2.66 GHz), 3.5GB RAM, Windows XP Professional 2002 SP3.

Consider a continuous-time T-S fuzzy system (10) with $p = 2$, $r_1 = r_2 = 2$, where

$$A_{11} = \begin{bmatrix} a & -10 \\ 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & -10 \\ 1 & 2 \end{bmatrix} \quad A_{21} = \begin{bmatrix} 2 & -10 \\ 1 & 1 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} 2 & -10 \\ 1 & 0 \end{bmatrix} \quad B_{11} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1 \\ -0.1 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$$

The local feedback gains K_τ, $\tau \in \{(11), (12), (21), (22)\}$ are determined by selecting $[-2, -2]$ as the eigenvalues of the subsystems in the PDC controller (17). Figs. 1-10 show the feasible areas of a and b satisfying the conditions of Lemmas 3 and 4 in this paper, Theorem 5 in [9], Theorem 5 in [28] and Lemma 5 with $A_1 = A_{11}$, $A_2 = A_{12}$, $A_3 = A_{21}$, $A_4 = A_{22}$, $B_1 = B_{11}$, $B_2 = B_{12}$, $B_3 = B_{21}$, $B_4 = B_{22}$, Theorem 1 with $h_1 = h_2 = 2, 4$, Corollary 1 with $h_1 = h_2 = 2, 3, 4$, respectively.

It can be seen from Figs. 8 and 9 that the condition of Theorem 1 becomes more relaxed along with increasing h_1, h_2, which verifies Theorem 2. Note that Lemma 5 is based on fuzzy Lyapunov functions, and Fig. 10 shows the stability area obtained by Lemma 5 with the assumption of $\dot{\alpha}_i(v(t)) \leq 0.85$, $1 \leq i \leq 4$. Comparing Figs. 2-4, 8, 9 with Fig. 10, it can be seen that the stability areas obtained by Theorem 1 and Corollary 1 are larger than the one by Lemma 5, though Theorem 1 and Corollary 1 are based on a single Lyapunov function. The numerical complexity of LMI conditions is closely related to the number of lines $\mathcal{L}$ and decision variables $\mathcal{D}$ in the LMIs to be solved, and LMI conditions can be solved in polynomial time with complexity proportional $\mathcal{C} = \mathcal{D}^3\mathcal{L}$ [7]. The numerical values of $\mathcal{L}$, $\mathcal{D}$, $\mathcal{C}$ and the CPU time of the different methods are collected in Table I for illustrating the numerical complexity of different LMI conditions.

From Table I, it can be seen that the condition in Corollary 1 with $h_1 = h_2 = 2$ is of the least numerical complexity among these methods and has larger feasible area than Lemma 3. For $7 \leq a \leq 10$, $b = 3.4$, the conditions of Lemmas 3, 4, 5, Theorem 5 in [9], Theorem 5 in [28] are unfeasible, however, the condition of Corollary 1 is feasible. It implies that the condition of Corollary 1 may give less conservative

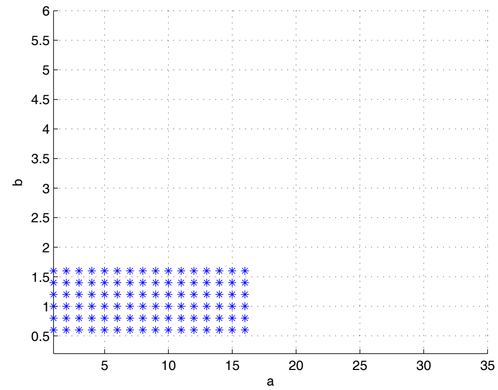

Fig. 1: Stability area by Lemma 3

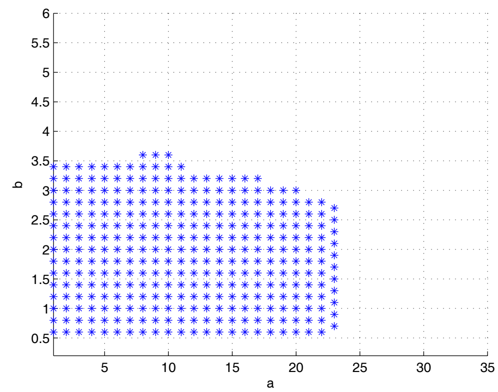

Fig. 2: Stability area by Corollary 1 with $h_1 = h_2 = 2$

results than the existing conditions and with less numerical complexity.

Moreover, it can also be seen that the condition of Theorem 1 are with larger feasible area than the existing conditions and Corollary 1, which implies that the condition of Theorem 1 is more relaxed than the existing ones.

Compare Fig. 2 with Fig. 8, Fig. 4 with Fig. 9, it can be found that the feasible area of Corollary 1 is smaller than one of Theorem 1 for the same h_i, which implies that Theorem 1 can effectively reduce conservatism than Corollary 1.

VI. Conclusion

In this paper, we have addressed the problem of the stability analysis for T-S fuzzy control systems. By constructing an equivalence relation on the index set of the product of fuzzy rule weights, a new stability analysis criterion of T-S fuzzy systems is proposed based on equivalence classes in set theory and the new criterion is stated as progressively less conservative sets of linear matrix inequalities. Further, it is proved that the new criterion is with no conservatism for quadratic stability analysis of T-S fuzzy control systems with a product inference engine and any possible fuzzy membership functions. A numerical example has been given to illustrate the effectiveness of the proposed method. Dynamic output feedback control problem of T-S fuzzy control systems will

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TABLE I: $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{C} = \mathcal{D}^3\mathcal{L}$

Methods	Lemma 3	Corollary 1 with $h_1 = h_2 = 2$	Corollary 1 with $h_1 = h_2 = 3$	Corollary 1 with $h_1 = h_2 = 4$
$\mathcal{L}$	22	20	34	52
$\mathcal{D}$	3	3	3	3
$\mathcal{C}$	594	540	918	1404
CPU time	0.0469	0.0313	0.0625	0.0938

Methods	Lemma 4	Lemma 5 with $\phi_i = 0.85$	Theorem 5 in [9]	Theorem 5 in [28] with $n = 4$	Theorem 1 with $h_1 = h_2 = 2$	Theorem 1 with $h_1 = h_2 = 3$
$\mathcal{L}$	30	96	74	182	28	84
$\mathcal{D}$	39	15	147	2051	39	175
$\mathcal{C}$	1779570	324000	235062702	1.5702×10^{12}	1660932	450187500
CPU time	0.1250	0.0938	0.1406	1.7344	0.1094	0.9375

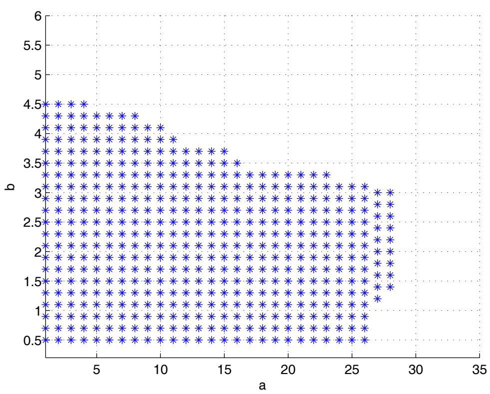

Fig. 3: Stability area by Corollary 1 with $h_1 = h_2 = 3$

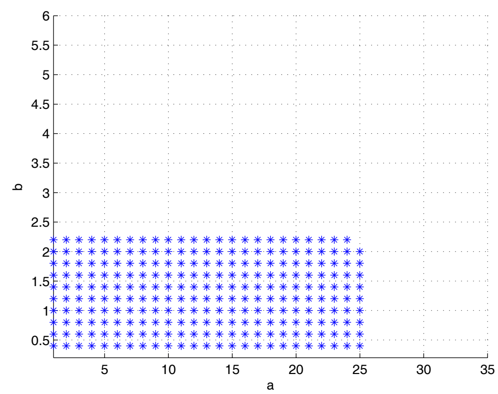

Fig. 5: Stability area by Lemma 4

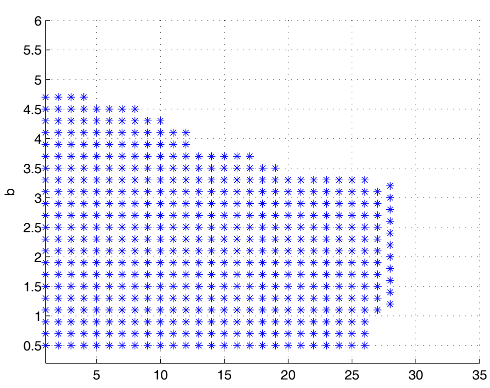

Fig. 4: Stability area by Corollary 1 with $h_1 = h_2 = 4$

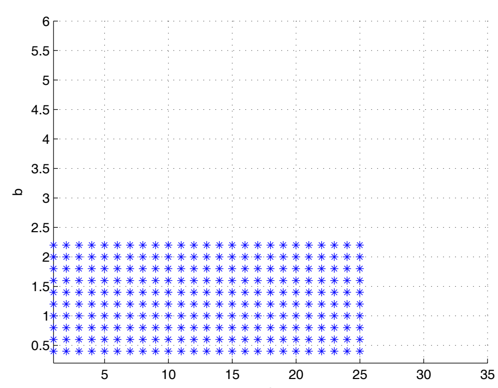

Fig. 6: Stability area by Theorem 5 in [9]

be exploited by using set theory in the future. We also plan to apply set theory to fuzzy fault tolerant control problems.

APPENDIX

Definition 1: [12], [26],

- A n**-ary relation** $\mathbb{R}$ is a set of ordered n-tuples, denoted by $(\mathrm{x}_1, \cdots, \mathrm{x}_n)$ is the ordered collection of elements that has x_1 as its first element, x_2 as its second element,..., and x_n as its nth element. Two n-tuples are equal, if each corresponding pair of their elements is equal. $\mathbb{R}$ is a n-ary relation on $\mathbb{X}$ if $\mathbb{R} \subseteq \mathbb{X}^n$. It is customary to write $\mathbb{R}(\mathrm{x}_1, \cdots, \mathrm{x}_n)$ instead of $(\mathrm{x}_1, \cdots, \mathrm{x}_n) \in \mathbb{R}$ and in case that $\mathbb{R}$ is binary, then we also use $\mathrm{x}\mathbb{R}\mathrm{y}$ instead of $(\mathrm{x}, \mathrm{y}) \in \mathbb{R}$.
- A binary relation $\mathbb{R}$ on $\mathbb{X}$ is **reflexive** if $\mathrm{x}\mathbb{R}\mathrm{x}$ for every element x of $\mathbb{X}$, i.e.,

$$\mathbb{R} \text{ is reflexive} \Longleftrightarrow \forall \mathrm{x}(\mathrm{x} \in \mathbb{X} \longrightarrow \mathrm{x}\mathbb{R}\mathrm{x})$$

- A binary relation on $\mathbb{X}$ is **symmetric**, if $\mathrm{x}\mathbb{R}\mathrm{y}$, then $\mathrm{y}\mathbb{R}\mathrm{x}$, i.e.,

$$\mathbb{R} \text{ is symmetric} \Longleftrightarrow$$
$$\forall \mathrm{x} \forall \mathrm{y}(\mathrm{x} \in \mathbb{X} \wedge \mathrm{y} \in \mathbb{X} \wedge \mathrm{x}\mathbb{R}\mathrm{y} \longrightarrow \mathrm{y}\mathbb{R}\mathrm{x})$$

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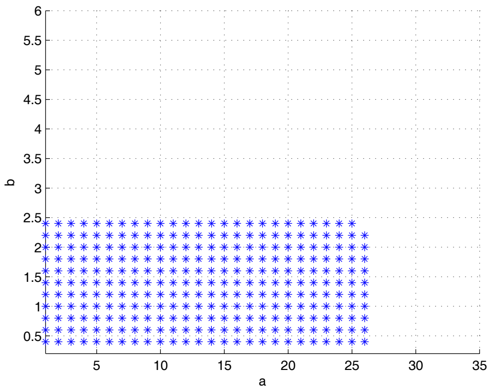

Fig. 7: Stability area by Theorem 5 with $n = 4$ in [28]

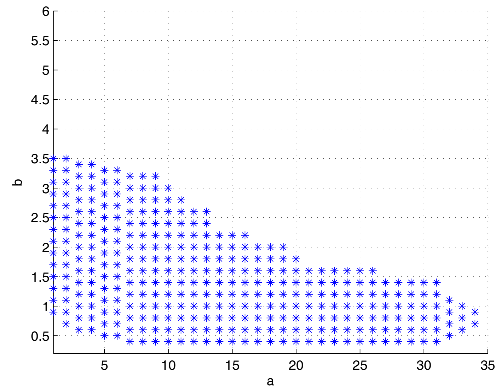

Fig. 10: Stability area by Lemma 5 with $\phi_i = 0.85$

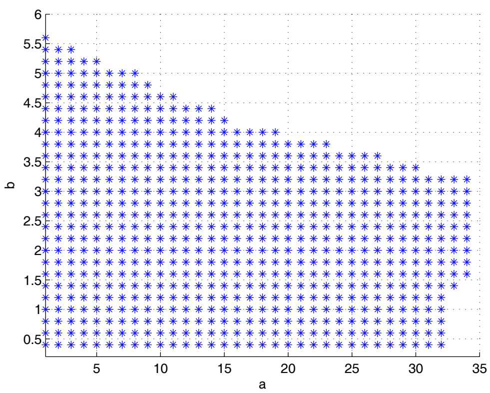

Fig. 8: Stability area by Theorem 1 with $h_1 = h_2 = 2$

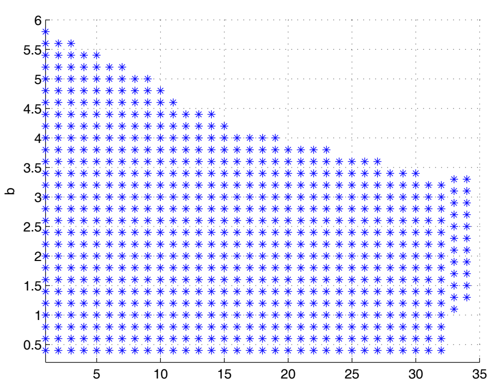

Fig. 9: Stability area by Theorem 1 with $h_1 = h_2 = 4$

- A binary relation $\mathbb{R}$ on $\mathbb{X}$ is **transitive** if $\forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathbb{X}$ and $\mathrm{x}\mathbb{R}\mathrm{y}$ and $\mathrm{y}\mathbb{R}\mathrm{z}$, then $\mathrm{x}\mathbb{R}\mathrm{z}$, i.e.,

  $$\mathbb{R} \text{ is transitive} \Longleftrightarrow$$
  $$\forall \mathrm{x}\forall \mathrm{y}\forall \mathrm{z}(\mathrm{x} \in \mathbb{X} \wedge \mathrm{y} \in \mathbb{X} \wedge \mathrm{z} \in \mathbb{X} \wedge \mathrm{x}\mathbb{R}\mathrm{y} \wedge \mathrm{y}\mathbb{R}\mathrm{z} \longrightarrow \mathrm{x}\mathbb{R}\mathrm{z})$$
- A binary relation $\mathbb{R}$ on $\mathbb{X}$ is an **equivalence relation** if it is reflexive, symmetric, and transitive.
- Let $\mathbb{R}$ be an equivalence relation on $\mathbb{X}$. For every $\mathrm{x} \in \mathbb{X}$, let $[\![x]\!]_{\mathbb{R}} = \{\mathrm{y} \in \mathbb{X} : \ \mathrm{y}\mathbb{R}\mathrm{x}\}$. The set $[\![\mathrm{x}]\!]_{\mathbb{R}}$ is the **equivalence class** of x, x is the representative element of the equivalence class.
- A **partition** $\mathbb{P}$ of a non-empty set $\mathbb{X}$ is a set of non-empty subsets of $\mathbb{X}$ such that: (a) For each element $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{P}$, either $\mathbb{S}_1 = \mathbb{S}_2$ or $\mathbb{S}_1 \cap \mathbb{S}_2 = \phi$. (b) $\mathbb{X} = \bigcup\limits_{\mathbb{S} \in \mathbb{P}} \mathbb{S}$

Lemma 7: [12] (pp. 12) If $\mathbb{R}$ is an equivalence relation on $\mathbb{X}$, then the set $\mathbb{X}/\mathbb{R} = \{[\![\mathrm{x}]\!]_{\mathbb{R}} : \ \mathrm{x} \in \mathbb{X}\}$ is a partition of $\mathbb{X}$. Conversely, for each partition of $\mathbb{X}$, there exists an equivalence relation $\mathbb{R}_o$ on $\mathbb{X}$, such that $\mathbb{X}/\mathbb{R}_o = \{[\![\mathrm{x}]\!]_{\mathbb{R}_o} : \ \mathrm{x} \in \mathbb{X}\}$ is the partition.

Lemma 8: (i): Let a, $b \in \mathbb{Z}_+$, then

$$\frac{(a+1)(b+1)}{2} \leq ab + 1 \tag{61}$$

(ii) Let $r_i \in \mathbb{Z}_+$, $i = 1, \cdots, p$, then

$$\prod_{i=1}^{p} \left(\frac{1 + r_i}{2} \right) \leq \left(\frac{1 + \prod_{i=1}^{p} r_i}{2} \right) \tag{62}$$

Proof: (i): Consider two cases: (1) one of a,b is 1, (2) $a \geq 2$, $b \geq 2$.

For the case one of a,b is 1, then it is easily obtained that (61) holds. For the case $a \geq 2$, $b \geq 2$, we have that $ab \geq \max\{2a, 2b\} \geq a + b$, which implies that

$$ab + a + b + 1 \leq 2ab + 2$$

i.e.,

$$\frac{(a+1)(b+1)}{2} \leq ab + 1$$

Thus, the proof is complete.

(ii): We use mathematical induction, it is easily obtained that (62) holds for $p = 2$ from (i). Assume (62) holds for $p = k$, then we have

$$\prod_{i=1}^{k} \left(\frac{1 + r_i}{2} \right) \leq \left(\frac{1 + \prod_{i=1}^{k} r_i}{2} \right)$$

which implies that

$$\prod_{i=1}^{k} \frac{1 + r_i}{2} \leq \frac{1 + \prod_{i=1}^{k} r_i}{2}$$

Multiplying both sides of the above inequality by $\frac{1+r_{k+1}}{2}$, it follows that

$$\prod_{i=1}^{k+1}\frac{1+r_i}{2}\le\frac{1+\prod_{i=1}^{k}r_i}{2}\frac{1+r_{k+1}}{2} \tag{63}$$

Let $a=\prod_{i=1}^{k}r_i$, $b=r_{k+1}$, from (i), it yields that

$$\begin{aligned}\frac{1+\prod_{i=1}^{k}r_i}{2}\frac{1+r_{k+1}}{2}&=\frac{(1+a)(1+b)}{4}\\ \le\frac{1+ab}{2}&=\frac{1+\prod_{i=1}^{k+1}r_i}{2}\end{aligned}$$

Combining it and (63), then (62) holds for $p=k+1$. Thus, by virtue of mathematical induction, the proof is complete. ■

Lemma 9: Let $\mathbb{S}\subset\mathbb{Z}_+$ with $|\mathbb{S}|<\infty$, $[\![\xi]\!]_{\mathbb{R}}$ is an equivalence class of $\mathbb{S}^{h+1}$ with $\mathbb{R}=\{(i_1i_2\cdots i_{h+1},j_1j_2\cdots j_{h+1})|st(j_1j_2\cdots j_{h_l})=st(i_1i_2\cdots i_{h_l})\}$, where $st(\cdot)$ is the same as in (4). For the set $[\![\xi]\!]_{\mathbb{R}}$, we define a binary relation as

$$\begin{aligned}\bar{R}=\{&(\eta_1\eta_2\cdots\eta_{h+1},\gamma_1\gamma_2\cdots\gamma_{h+1})\in(\mathbb{S}^{h+1})^2|\\ &st(\eta_1\eta_2\cdots\eta_h)=st(\gamma_1\gamma_2\cdots\gamma_h),\eta_{h+1}=\gamma_{h+1}\}\end{aligned}$$

Then the relation $\bar{\mathbb{R}}$ is an equivalence relation and $[\![\xi]\!]_{\mathbb{R}}/\bar{\mathbb{R}}$ is a partition of the set $[\![\xi]\!]_{\mathbb{R}}$.

Proof: The proof is easily obtained and omitted. ■

The proof of Lemma 1

Proof: For any element $(i_1,i_2,\cdots,i_p)\in\mathbb{S}_1^{h_1}\times\mathbb{S}_2^{h_2}\times\cdots\times\mathbb{S}_p^{h_p}$, we have $i_l\in\mathbb{S}_l^{h_l}$, $1\le l\le p$. Because $\mathbb{S}_l^{h_l}/\mathbb{R}_{lh_l}$ is a partition of set $\mathbb{S}_l^{h_l}$, then there exists an equivalence class $[\![i_l]\!]_{\mathbb{R}_{lh_l}}$, such that $i_l\in[\![i_l]\!]_{\mathbb{R}_{lh_l}}$. Therefore, $(i_1,i_2,\cdots,i_p)\in[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}$. So we have

$$\begin{aligned}&\mathbb{S}_1^{h_1}\times\mathbb{S}_2^{h_2}\times\cdots\times\mathbb{S}_p^{h_p}\\ \subseteq&\bigcup_{\substack{[\![i_1]\!]_{\mathbb{R}_{1h_1}}\in\mathbb{S}^{h_1}/\mathbb{R}_{1h_1}\\ \vdots\\ [\![i_p]\!]_{\mathbb{R}_{ph_p}}\in\mathbb{S}^{h_p}/\mathbb{R}_{ph_p}}}[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\end{aligned} \tag{64}$$

Since $[\![i_l]\!]_{\mathbb{R}_{lh_l}}\subseteq\mathbb{S}_l^{h_l}$, $1\le l\le p$,

$$\begin{aligned}&\mathbb{S}_1^{h_1}\times\mathbb{S}_2^{h_2}\times\cdots\times\mathbb{S}_p^{h_p}\\ \supseteq&\bigcup_{\substack{[\![i_1]\!]_{\mathbb{R}_{1h_1}}\in\mathbb{S}^{h_1}/\mathbb{R}_{1h_1}\\ \vdots\\ [\![i_p]\!]_{\mathbb{R}_{ph_p}}\in\mathbb{S}^{h_p}/\mathbb{R}_{ph_p}}}[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\end{aligned}$$

Combining it and (64), it follows that

$$\begin{aligned}&\mathbb{S}_1^{h_1}\times\mathbb{S}_2^{h_2}\times\cdots\times\mathbb{S}_p^{h_p}\\ =&\bigcup_{\substack{[\![i_1]\!]_{\mathbb{R}_{1h_1}}\in\mathbb{S}^{h_1}/\mathbb{R}_{1h_1}\\ \vdots\\ [\![i_p]\!]_{\mathbb{R}_{ph_p}}\in\mathbb{S}^{h_p}/\mathbb{R}_{ph_p}}}[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\end{aligned} \tag{65}$$

On the other hand, note that $[\![i_l]\!]_{\mathbb{R}_{lh_l}}$ and $[\![j_l]\!]_{\mathbb{R}_{lh_l}}$ are both equivalence classes on $\mathbb{S}_l^{h_l}$, then $[\![i_l]\!]_{\mathbb{R}_{lh_l}}=[\![j_l]\!]_{\mathbb{R}_{lh_l}}$ or $[\![i_l]\!]_{\mathbb{R}_{lh_l}}\cap[\![j_l]\!]_{\mathbb{R}_{lh_l}}=\emptyset$.

There are the following two possible cases for sets $[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}$ and $[\![j_1]\!]_{\mathbb{R}_{1h_1}}\times[\![j_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![j_p]\!]_{\mathbb{R}_{ph_p}}$.

- Case 1: If there exits some l satisfying $[\![i_l]\!]_{\mathbb{R}_{lh_l}}\cap[\![j_l]\!]_{\mathbb{R}_{lh_l}}=\emptyset$, then

$$\begin{aligned}&[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\cap\\ &[\![j_1]\!]_{\mathbb{R}_{1h_1}}\times[\![j_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![j_p]\!]_{\mathbb{R}_{ph_p}}=\emptyset\end{aligned}$$

- Case 2: If there doesn't exit l satisfying $[\![i_l]\!]_{\mathbb{R}_{lh_l}}\cap[\![j_l]\!]_{\mathbb{R}_{lh_l}}=\emptyset$, which implies that $[\![i_l]\!]_{\mathbb{R}_{lh_l}}=[\![j_l]\!]_{\mathbb{R}_{lh_l}}$ for all l, $1\le l\le p$. It means that

$$\begin{aligned}&[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\\ =&[\![j_1]\!]_{\mathbb{R}_{1h_1}}\times[\![j_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![j_p]\!]_{\mathbb{R}_{ph_p}}\end{aligned}$$

Therefore, it follows from the Cases 1 and 2 that $[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}\cap[\![j_1]\!]_{\mathbb{R}_{1h_1}}\times[\![j_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![j_p]\!]_{\mathbb{R}_{ph_p}}=\emptyset$ or $[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}=[\![j_1]\!]_{\mathbb{R}_{1h_1}}\times[\![j_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![j_p]\!]_{\mathbb{R}_{ph_p}}$. From the fact and (65), we can obtain that set $\{[\![i_1]\!]_{\mathbb{R}_{1h_1}}\times[\![i_2]\!]_{\mathbb{R}_{2h_2}}\times\cdots\times[\![i_p]\!]_{\mathbb{R}_{ph_p}}:[\![i_l]\!]_{\mathbb{R}_{lh_l}}\subseteq\mathbb{S}_l^{h_l},1\le l\le p\}$ is a partition of the set $\mathbb{S}_1^{h_1}\times\mathbb{S}_2^{h_2}\times\cdots\times\mathbb{S}_p^{h_p}$. Thus, the proof is complete. ■

The proof of Lemma 2

Proof: From Lemma 1, it follows that

$$\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}}\mu_\sigma M_\sigma=\sum_{\bar{\mathbb{s}}\in\prod_{i=1}^{p}(\mathbb{S}_i^{h_i}/\mathbb{R}_{ih_i})}\sum_{\sigma\in\bar{\mathbb{s}}}\mu_\sigma M_\sigma \tag{66}$$

where $\bar{\mathbb{s}}=\prod_{i=1}^{p}\mathbb{s}_i$ with $\mathbb{s}_i\in\mathbb{S}_i^{h_i}/\mathbb{R}_{ih_i}$.

From the property of equivalence class in set theory, we can choose an arbitrary element in the equivalence class as its representative element. Let $\varsigma_j\in\mathbb{s}_j$, then we choose ς_j as the representative element of the equivalence class $\mathbb{s}_j$, and denote $\mathbb{s}_j$ as $[\![\varsigma_j]\!]_{\mathbb{R}_{jh_j}}$. Further, it follows from the definition of the equivalence relation $\mathbb{R}_{jh_j}$ that

$$\prod_{i_j=1}^{h_j}\mu_{j\tau_{\langle i_j]}}=\prod_{i_j=1}^{h_j}\mu_{j\varsigma_{j\,\langle i_j]}},\ \text{for all }\tau\in\mathbb{s}_j=[\![\varsigma_j]\!]_{\mathbb{R}_{jh_j}}$$

Then for $\sigma\in\bar{\mathbb{s}}=\prod_{i=1}^{p}\mathbb{s}_i$,

$$\begin{aligned}&\sum_{\sigma\in\bar{\mathbb{s}}=\prod_{i=1}^{p}\mathbb{s}_i}\mu_\sigma M_\sigma\\ =&\sum_{\sigma\in\bar{\mathbb{s}}=\prod_{i=1}^{p}([\![\varsigma_i]\!]_{\mathbb{R}_{ih_i}})}\mu_\sigma M_\sigma\\ =&\sum_{\sigma\in\bar{\mathbb{s}}=\prod_{i=1}^{p}([\![\varsigma_i]\!]_{\mathbb{R}_{ih_i}})}\prod_{j=1}^{p}\prod_{i_j=1}^{h_j}\mu_{j\varsigma_{j\,\langle i_j]}}M_\sigma\\ =&\prod_{j=1}^{p}\prod_{i_j=1}^{h_j}\mu_{j\varsigma_{j\,\langle i_j]}}\sum_{\sigma\in\bar{\mathbb{s}}=\prod_{i=1}^{p}([\![\varsigma_i]\!]_{\mathbb{R}_{ih_i}})}M_\sigma\\ =&\mu_{\bar{\mathbb{s}}}\sum_{\sigma\in\bar{\mathbb{s}}}M_\sigma\end{aligned} \tag{67}$$

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where

$$\mu_{\bar{\mathbb{s}}} = \prod_{j=1}^{p} \prod_{i_j=1}^{h_j} \mu_{j\varsigma_{j\langle i_j\rangle}}, \text{ with } \bar{\mathbb{s}} = \prod_{i=1}^{p} \mathbb{s}_i = \prod_{i=1}^{p} [\![\varsigma_i]\!]_{\mathbb{R}_{ih_i}} \quad (68)$$

From (66) and (67), yields that

$$\begin{aligned} &\sum_{\sigma\in\prod_{i=1}^{p}\mathbb{S}_i^{h_i}} \mu_\sigma M_\sigma \\ =& \sum_{\bar{\mathbb{s}}\in\prod_{i=1}^{p}(\mathbb{S}_i^{h_i}/\mathbb{R}_{ih_i})} \sum_{\sigma\in\bar{\mathbb{s}}} \mu_\sigma M_\sigma \\ =& \sum_{\bar{\mathbb{s}}\in\prod_{i=1}^{p}(\mathbb{S}_i^{h_i}/\mathbb{R}_{ih_i})} \mu_{\bar{\mathbb{s}}} \sum_{\sigma\in\bar{\mathbb{s}}} M_\sigma \end{aligned}$$

Combining it and (7), (8), it follows that (9) holds. Thus, the proof is complete. ■

Lemma 10: If the 1-1 mapping $q(\cdot)$ in (32) is respectively chosen as $q_a(\cdot)$ and $q_b(\cdot)$, then (38) in Theorem 1 respectively becomes

$$[H_{ij}^a] < 0, \text{ with } H^a_{q_a(\sigma^{\beta_1})q_a(\sigma^{\beta_2})} = Y_\sigma \quad (69)$$

and

$$[H_{ij}^b] < 0, \text{ with } H^b_{q_b(\sigma^{\beta_1})q_b(\sigma^{\beta_2})} = Y_\sigma \quad (70)$$

then (69) is equivalent to (70).

Proof: Define a mapping ϖ from the set $\{1, 2, \cdots, r\}$ to itself with $\varpi(\cdot) = q_b\left(q_a^{-1}(\cdot)\right)$. Since $q_a(\cdot)$ and $q_b(\cdot)$ are both 1-1 mappings, the inverse mapping of q_a exists and ϖ is also a 1-1 mapping. From (69) and (70), we have that

$$H_{ij}^a = H^b_{\varpi(i)\varpi(j)}$$

Then (69) can be rewritten as

$$\begin{bmatrix} H^b_{\varpi(1)\varpi(1)} & H^b_{\varpi(1)\varpi(2)} & \cdots & H^b_{\varpi(1)\varpi(r)} \\ H^b_{\varpi(2)\varpi(1)} & H^b_{\varpi(2)\varpi(2)} & \cdots & H^b_{\varpi(2)\varpi(r)} \\ \vdots & \vdots & \ddots & \vdots \\ H^b_{\varpi(r)\varpi(1)} & H^b_{\varpi(r)\varpi(2)} & \cdots & H^b_{\varpi(r)\varpi(r)} \end{bmatrix} < 0 \quad (71)$$

Since ϖ is also a 1-1 mapping, there exists a permutation matrix T, such that

$$\begin{bmatrix} \varpi(1) & \varpi(2) & \cdots & \varpi(r) \end{bmatrix} T = \begin{bmatrix} 1 & 2 & \cdots & r \end{bmatrix}$$

Let $\mathcal{T} = T \otimes I_{n_x\times n_x}$, then

$$\begin{aligned} &\mathcal{T} \begin{bmatrix} H^b_{11} & H^b_{12} & \cdots & H^b_{1r} \\ H^b_{21} & H^b_{22} & \cdots & H^b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ H^b_{r1} & H^b_{r2} & \cdots & H^b_{rr} \end{bmatrix} \mathcal{T}^T \\ =& \begin{bmatrix} H^b_{\varpi(1)\varpi(1)} & H^b_{\varpi(1)\varpi(2)} & \cdots & H^b_{\varpi(1)\varpi(r)} \\ H^b_{\varpi(2)\varpi(1)} & H^b_{\varpi(2)\varpi(2)} & \cdots & H^b_{\varpi(2)\varpi(r)} \\ \vdots & \vdots & \ddots & \vdots \\ H^b_{\varpi(r)\varpi(1)} & H^b_{\varpi(r)\varpi(2)} & \cdots & H^b_{\varpi(r)\varpi(r)} \end{bmatrix} < 0 \end{aligned}$$

which implies that

$$\begin{bmatrix} H^b_{11} & H^b_{12} & \cdots & H^b_{1r} \\ H^b_{21} & H^b_{22} & \cdots & H^b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ H^b_{r1} & H^b_{r2} & \cdots & H^b_{rr} \end{bmatrix} = [H^b_{ij}] < 0$$

Then we have that (69) is equivalent to (70). ■

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